Notes on simulating core collapse in scalar-tensor theories

December 4, 2023

Contents

1 A tale of three formulations

This document summarizes the formulation of scalar-tensor theories of gravity and their application in numerical relativity simulations of supernova core collapse. For now, the main focus will be on such collapse in spherical symmetry along the lines of O'Connor & Ott [?]. For this purpose, we will first review the formulation of scalar-tensor theories of gravity. In scalar-tensor theories, gravity is mediated by the standard metric $g_{\alpha\beta}$ known from general relativity as well as a scalar field φ. The coupling of the scalar field to the metric depends on the particular way the theory is formulated, but in some form or other, there exists a non-minimal coupling. Three different formulations of the theory are commonly encountered in the literature.

1.1 Einstein Frame

In the Einstein frame, the spacetime geometry is described in terms of a conformal metric

$$g_{\alpha\beta} = a^2 \bar{g}_{\alpha\beta} = \frac{1}{F}\bar{g}_{\alpha\beta}, \tag{1.1}$$

where the conformal factors a or F are functions of the scalar field which we typically denote by φ in the Einstein frame or ϕ in the Jordan frame. Note that φ and ϕ are non-trivial functions of each other and their precise relation depends on the choice of the conformal factor; cf. below. In the following, we will set $G = 1$ and only explicitly write it in the action S. The key advantage resulting from this conformal transformation is a minimal coupling between the conformal metric and the scalar field. On the other hand, the entire spacetime geometry is that corresponding to the physical metric $g_{\alpha\beta}$. In particular, matter couples to the physical metric and spacetime geodesics are those corresponding to $g_{\alpha\beta}$ instead of $\bar{g}_{\alpha\beta}$. These subtleties become clear when we consider the action in the Einstein frame

$$S = \frac{1}{16\pi G}\int dx^4 \sqrt{-\bar{g}}\left[\bar{R} - 2\bar{g}^{\mu\nu}(\partial_\mu\varphi)(\partial_\nu\varphi) - 4W(\varphi)\right] + S_m[\psi_m, a^2(\varphi)\bar{g}_{\mu\nu}]. \tag{1.2}$$

Here $\bar{R}$ is the Riemann scalar associated with $\bar{g}_{\alpha\beta}$, ψ_m is the set of fields describing the matter variables and $W(\varphi)$ the potential of the scalar field. Note that these matter fields couple to the physical metric $g_{\alpha\beta}$. The field equations obtained from the action (1.2) are (e. g. [?, ?]

$$\begin{aligned} \bar{G}_{\alpha\beta} &= 2\partial_\alpha\varphi\partial_\beta\varphi - \bar{g}_{\alpha\beta}\bar{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + 8\pi\bar{T}_{\alpha\beta} - 2W(\varphi)\bar{g}_{\alpha\beta}\,, & (1.3)\\ \bar{\Box}\varphi &= -4\pi\alpha(\varphi)\bar{T} + W_{,\varphi}\,. & (1.4) \end{aligned}$$

Here all barred quantities are associated with the conformal metric $\bar{g}_{\alpha\beta}$ and

$$\begin{aligned} \bar{T}^{\alpha\beta} &\equiv \frac{2}{\sqrt{-\bar{g}}}\frac{\delta S_m}{\delta\bar{g}_{\alpha\beta}} = a^6(\varphi)\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g_{\alpha\beta}} \equiv a^6(\varphi)T^{\alpha\beta}, & (1.5)\\ \alpha(\varphi) &\equiv \frac{\partial \ln a}{\partial\varphi} = -\frac{1}{2}\frac{\partial \ln F}{\partial\varphi}. & (1.6) \end{aligned}$$

Conservation of the energy-momentum tensor $\nabla_\mu T^{\mu\alpha} = 0$ in the Einstein frame is given by

$$\bar{\nabla}_\mu\bar{T}^{\mu\alpha} = \frac{a_{,\varphi}}{a}\bar{T}\bar{g}^{\alpha\mu}\bar{\nabla}_\mu\varphi. \tag{1.7}$$

We see from Eq. (1.3) that the conformal metric is merely minimally coupled to the scalar field. On the other hand, the evolution of the matter terms given by Eq. (1.7) is now sourced by the scalar field. The scalar field, in turn, is driven by the matter as evident in Eq. (1.4). Note that the alternative theory has two functions of freedom, the coupling function $A(\varphi)$ and the scalar potential $W(\varphi)$.
We finally note that the derivatives of the coupling function evaluated at the asymptotic value φ_0 of the scalar field at infinity

$$\alpha_0 = \alpha(\varphi_0), \tag{1.8}$$
$$\beta_0 = \frac{\partial\alpha}{\partial\varphi}(\varphi_0), \tag{1.9}$$

are related to the Eddington parameters $\gamma_{\rm Edd}$, $\beta_{\rm Edd}$ by

$$\gamma_{\rm Edd} - 1 = \frac{-2\alpha_0^2}{1+\alpha_0^2}, \tag{1.10}$$
$$\beta_{\rm Edd} - 1 = \frac{\beta_0\alpha_0^2}{2(1+\alpha_0^2)^2}. \tag{1.11}$$

For more details, see [?].

1.2 Jordan-Fierz frame

In the Jordan-Fierz frame we formulate the Einstein equations in terms of the physical metric $g_{\alpha\beta}$. It is convenient, for this purpose, to represent the scalar field in terms of the "new" function ϕ related to the above φ by [?, ?]

$$\frac{\partial\varphi}{\partial\phi} = \frac{1}{a}\sqrt{3a_{,\phi}{}^2 + 4\pi a^4} = \sqrt{\frac{3}{4}\frac{F_{,\phi}{}^2}{F^2} + \frac{4\pi}{F}}. \tag{1.12}$$

Using $a_{,\phi} = a_{,\varphi}\partial\varphi/\partial\phi$, we can invert this relation and obtain

$$\frac{\partial\phi}{\partial\varphi} = \sqrt{\frac{a^2 - 3a_{,\varphi}{}^2}{4\pi a^4}} = \sqrt{\frac{4F^2 - 3F_{,\varphi}{}^2}{16\pi F}}. \tag{1.13}$$

A straightforward calculation[1] shows that the action (1.2) can be written as

$$S = \int dx^4\sqrt{-g}\left[\frac{F(\phi)}{16\pi G}R - \frac{1}{2}g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi)\right] + S_m(\psi_m, g_{\mu\nu}). \tag{1.14}$$

The equations for the metric and the scalar field then become

$$G_{\alpha\beta} = \frac{8\pi}{F}\left(T^F_{\alpha\beta} + T^\phi_{\alpha\beta} + T_{\alpha\beta}\right), \tag{1.15}$$
$$T^F_{\alpha\beta} = \frac{1}{8\pi}(\nabla_\alpha\nabla_\beta F - g_{\alpha\beta}\nabla^\mu\nabla_\mu F), \tag{1.16}$$
$$T^\phi_{\alpha\beta} = \partial_\alpha\phi\partial_\beta\phi - g_{\alpha\beta}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right], \tag{1.17}$$
$$\Box\phi = -\frac{1}{16\pi}F_{,\phi}R + V_{,\phi} \tag{1.18}$$

[1] taking into acount some boundary terms which do not contribute in the variation

Here the potential is represented by

$$V(\phi) = \frac{F^2}{4\pi} W(\varphi). \tag{1.19}$$

Combining the Bianchi identities with the field equations can be shown to imply that the "standard" energy momentum tensor is conserved on its own, i. e.

$$\nabla_\mu T^{\mu\alpha} = 0. \tag{1.20}$$

It is this feature that makes the Jordan frame quite convenient for the physical systems we wish to study; the matter equations do not need to be modified from their GR version. The price we pay for this are the additional terms appearing in the evolution of the metric in Eq. (1.15). From the system (1.15)-(1.18) we further note that in the absence of "standard" matter, an initially constant scalar field with vanishing potential V cannot source itself, i. e. a GR solution is a solution to the scalar tensor theory with constant scalar field.

1.3 Brans-Dicke style

The Brans-Dicke formulation is a special case in the Jorda-Fierz frame obtained by a rescaling of the scalar field according to

$$\Phi \equiv F(\phi). \tag{1.21}$$

The action then becomes

$$S = \frac{1}{16\pi G} \int dx^4 \sqrt{-g} \left[\Phi R - \frac{\omega_{\text{BD}}}{\Phi} g^{\mu\nu} (\partial_\mu \Phi)(\partial_\nu \Phi) + 2\Phi\lambda \right] + S_m[\psi_m, g_{\mu\nu}], \tag{1.22}$$

where

$$\omega_{\text{BD}} \equiv \frac{8\pi\Phi}{F_{,\phi}^2}, \tag{1.23}$$

$$\lambda \equiv -\frac{8\pi V(\phi)}{\Phi}. \tag{1.24}$$

In their original work, Brans & Dicke [?] considered the special case of a constant ω_{BD}

$$\omega_{\text{BD}} = \omega = \text{const}, \tag{1.25}$$

$$\Rightarrow \quad F_{,\phi}^2 = \frac{8\pi F}{\omega}, \tag{1.26}$$

$$\Rightarrow \quad F = \frac{2\pi}{\omega}(\phi - \phi_0)^2. \tag{1.27}$$

Addition from November 2018 to clarify some points

We have a considerable amount of conventions to choose from which are largely related to the sign ambiguities. This can lead to serious confusion, so that it is worth clarifying here once and for all our conventions.

First, we notice that in the relation (1.13) between ϕ and φ, we have a choice for the sign. If we wish to keep the + sign, that implies for an increasing φ, we also have a growing ϕ. There is no problem with that, but from Eqs. (1.27) and (2.2), this would imply $\alpha_0 \leq 0$ because

$$F = a^{-2}\,. \tag{1.28}$$

In our massless paper, however, we have chosen the convention $\alpha_0 \geq 0$ and we have also used this convention in all our practical studies. Furthermore, it is standard notation in the literature to set

$$a(\varphi) = e^{\alpha_0(\varphi-\varphi_0)+\frac{1}{2}\beta_0(\varphi-\varphi_0)^2} \quad \Leftrightarrow \quad F(\varphi) = e^{-2\alpha_0(\varphi-\varphi_0)-\beta_0(\varphi-\varphi_0)^2}\,, \tag{1.29}$$

and we would like to keep this convention as well. If we also keep our choice $\partial\phi/\partial\varphi > 0$, then Eq. (1.27) cannot hold in its present form: For a positive α_0, F decreases for a growing φ while on the other hand, by convention ϕ grows with φ. In consequence, we have a decreasing F for an increasing ϕ. This is only compatible with Eq. (1.27) if we actually sit on the negative branch of $\phi-\phi_0$, i.e. we have $\phi-\phi_0 \leq 0$ throughout.

This makes it rather tricky to relate the Brans-Dicke parameter to the α_0 in the Einstein frame formulation. For Brans-Dicke theory we have the conformal factor given by Eq. (1.29). At the same time, we know that for Brans-Dicke theory,

$$F(\varphi) = e^{-2\alpha_0\varphi}\,. \tag{1.30}$$

The two scalar fields are related by Eq. (1.13) and from those we straightforwardly obtain (plugging in the known $F(\phi)$) that

$$\frac{\partial\phi}{\partial\varphi} = \sqrt{\frac{3}{(\phi-\phi_0)^2} + \frac{2\omega_{\rm BD}}{(\phi-\phi_0)^2}} \stackrel{!}{=} (3+2\omega_{\rm BD})^{-1/2}(\phi_0-\phi)\,, \tag{1.31}$$

where in the last equality we used $\sqrt{x} = -x$ for $x < 0$ and our knowledge that $\phi-\phi_0 < 0$. The solution to this differential equation is

$$\phi - \phi_0 = Ce^{-(3+2\omega_{\rm BD})^{-1/2}\varphi}\,, \tag{1.32}$$

where $C < 0$ is a constant. Next, we plug this result into (1.29) and obtain

$$F = \frac{2\pi}{\omega_{\rm BD}}C^2 e^{-2(\omega_{\rm BD})^{-1/2}\varphi} \stackrel{!}{=} e^{-2\alpha_0\varphi}\,. \tag{1.33}$$

This implies that $C = -\sqrt{\omega_{\rm BD}/(2\pi)}$ and

$$\alpha_0 = (3+2\omega_{\rm BD})^{-1/2} \quad \Leftrightarrow \quad \omega_{\rm BD} = \frac{1-3\alpha_0^2}{2\alpha_0^2}\,. \tag{1.34}$$

Note that we have a minor typo in the journal version of [**?**], where on page 12, 2nd line, we wrongly write $-6\alpha_0^2$ in the numerator.

A further issue arises from Eq. (1.13) for some special cases. Consider, for example,

$$F(\varphi) = e^{-2\alpha_0\varphi}\,, \tag{1.35}$$

and insert into the right-hand side of (1.13). The numerator inside the root then becomes $(4 - 12\alpha_0^2)e^{-2\alpha_0\varphi}$. This becomes negative if $\alpha_0^2 > 1/3$ which would then imply an imginary $\partial\phi/\partial\varphi$. I am not sure what this means physically; in practice, we seem to able to evolve such large α_0 cases just fine. Possibly, the formulation in terms of the variable ϕ is just no longer suitable.

2 Choices for the coupling function

For the case of a vanishing or negligible potential $V(\varphi) = 0$, there remains one free function to specify the scalar-tensor theory, the coupling function a. From Eqs. (1.13) and (1.27) we see that for Brans & Dicke's constant ω

$$\phi = e^{\alpha_0 \varphi}, \tag{2.1}$$

where the exponent $\alpha_0 = \sqrt{\frac{2}{6+\omega}}$ is that defined in Eq. (1.8). Post-Newtonian experiments only probe the low-order perturbative structure of scalar-tensor theories in that they only depend on $\alpha(\varphi)$ and its gradient $\beta(\varphi) = \partial\alpha/\partial\varphi$ at the cosmologically determined value of the scalar field φ_0 [?]. A common choice for the coupling function is therefore given by

$$a(\varphi) = e^{\alpha_0(\varphi-\varphi_0)+\beta_0(\varphi-\varphi_0)^2/2} \tag{2.2}$$

which represents a direct parameterization in terms of the coefficients α_0, β_0 defined in Eqs. (1.8), (1.9). In particular, such a coupling function with setting $\alpha_0 = 0$ has been used by Damour & Esposito-Farese in their discovery of the *spontaenous scalarization* [?] for sufficiently negative $\beta_0 \lesssim -4$. Brans-Dicke theory is recovered by setting $\beta_0 = 0$.

3 Spherically symmetric collapse: Einstein frame

Here we review the equations for a spherically symmetric fluid as derived by Novak [?, ?] and then case in flux conservative form in [?]. Their starting point is the Einstein-frame metric

$$d\bar{s}^2 = \bar{g}_{\alpha\beta}dx^\alpha dx^\beta = -\bar{N}^2dt^2 + \bar{A}^2dr^2 + r^2d\Omega, \tag{3.1}$$

Note that this metric employs radial gauge in the *Einstein frame.*
The physical energy momentum tensor is given by

$$\begin{aligned} T_{\alpha\beta} &= (e+p)u_\alpha u_\beta + p g_{\alpha\beta}, && (3.2)\\ u^\alpha &= \frac{1}{a\sqrt{1-v^2}}\left[\frac{1}{\bar{N}},\ \frac{v}{\bar{A}},\ 0,\ 0\right], && (3.3)\end{aligned}$$

and is related to its Einstein-frame counter part by

$$\bar{T}_{\alpha\beta} = a^2 T_{\alpha\beta} = \frac{1}{F}T_{\alpha\beta}. \tag{3.4}$$

We also introduce the baryonic flow

$$J^\alpha = n_{\rm B}u^\alpha. \tag{3.5}$$

The spacetime field equations in the Einstein frame are

$$\bar{G}_{\alpha\beta} = 8\pi\bar{T}_{\alpha\beta} + 2\partial_\alpha\varphi\partial_\beta\varphi - \bar{g}_{\alpha\beta}\partial^\mu\varphi\partial_\mu\varphi. \tag{3.6}$$

The matter equations, the conservation of the baryon density $n_{\rm B}$ and the wave equation for the scalar field follow from

$$\begin{aligned} \bar{\nabla}_\mu\bar{T}^\mu{}_\nu &= \frac{a_{,\varphi}}{a}\bar{T}\bar{\nabla}_\mu\varphi, && (3.7)\\ \bar{\Box}\varphi &= -4\pi\frac{a_{,\varphi}}{a}\bar{T}, && (3.8)\\ \nabla_\mu J^\mu &= 0, && (3.9)\end{aligned}$$

note that even Novak expresses baryon conservation in terms of the Jordan frame variables; cf. Eq. (2.30) in [?].

The resulting equations are formulated conveniently after introducing the variables

$$m = \frac{r}{2}\left(1-\frac{1}{\bar{A}^2}\right), \tag{3.10}$$

$$\nu = \ln \bar{N}. \tag{3.11}$$

Novak *et al.* also use the following variables to represent derivatives of the scalar field

$$\eta = \frac{1}{\bar{A}}\partial_r\varphi, \tag{3.12}$$

$$\psi = \frac{1}{\bar{N}}\partial_t\varphi, \tag{3.13}$$

$$\Xi = \eta^2+\psi^2 = \frac{1}{\bar{A}^2}(\partial_r\varphi)^2+\frac{1}{\bar{N}^2}(\partial_t\varphi)^2, \tag{3.14}$$

and describe the matter using additional auxiliary variables

$$E = \frac{e+p}{1-v^2}-p, \tag{3.15}$$

$$\bar{D} = \bar{A}\frac{n_{\mathrm{B}}a^4}{\sqrt{1-v^2}}, \tag{3.16}$$

$$\bar{\mu} = a^4(E+p)v, \tag{3.17}$$

$$\bar{\tau} = a^4E-\bar{D} \tag{3.18}$$

Having the notation in place now, we list the resulting equations. Einstein's field equations result in

$$\partial_r\bar{\nu} = 4\pi\bar{A}^2\left\{\frac{m}{4\pi r^2}+a^4r\left[p+(E+p)v^2\right]+\frac{r\Xi}{8\pi}\right\}, \tag{3.19}$$

$$\partial_r m = \frac{r^2}{2}\Xi+4\pi r^2a^4E, \tag{3.20}$$

$$\partial_t m = r^2\frac{\bar{N}}{\bar{A}}\left[\psi\eta-4\pi a^4(E+p)v\right]. \tag{3.21}$$

The evolution of the scalar field is given by

$$\begin{aligned}\partial_t{}^2\varphi = {} & \frac{\bar{N}^2}{\bar{A}^2}\left[\partial_r{}^2\varphi+\frac{2}{r}\partial_r\varphi+\left(\frac{\partial_r\bar{N}}{\bar{N}}-\frac{\partial_r\bar{A}}{\bar{A}}\right)\partial_r\varphi\right]+\left(\frac{\partial_t\bar{N}}{\bar{N}}-\frac{\partial_t\bar{A}}{\bar{A}}\right)\partial_t\varphi \\ & -4\pi\frac{a_{,\varphi}}{a}a^4\bar{N}^2\left[E-3p-(E+p)v^2\right].\end{aligned} \tag{3.22}$$

Baryon conservation leads to

$$\partial_t\bar{D}+\frac{a}{r^2}\partial_r\left(\frac{r^2}{a}\frac{\bar{N}}{\bar{A}}\bar{D}v\right) = \bar{D}\frac{a_{,\varphi}}{a}\partial_t\varphi. \tag{3.23}$$

Finally, conservation of energy momentum gives

$$\begin{aligned}
\partial_t \bar{\mu} + \frac{1}{r^2}\partial_r \left[(\bar{\mu} v + a^4 p) r^2 \frac{\bar{N}}{\bar{A}}\right] &= \bar{N}\bar{A}(\bar{\mu} v - \bar{\tau} - \bar{D})\left(8\pi r a^4 p + \frac{m}{r^2} + \frac{a_{,\varphi}}{a}\frac{\eta}{\bar{A}}\right) \\
&\quad + \bar{N}\bar{A} a^4 p \frac{m}{r^2} + 2\frac{\bar{N}}{\bar{A}} a^4 \frac{p}{r} - 2r\bar{\mu}\partial_r\varphi\partial_t\varphi + 3\frac{\bar{N}}{\bar{A}} a^4 p \frac{a_{,\varphi}}{a}\partial_r\varphi \\
&\quad - \frac{r}{2}\bar{N}\bar{A}(\eta^2 + \psi^2)(\bar{\tau} + \bar{D} + a^4 p)(1 + v^2), \qquad (3.24) \\
\partial_t \bar{\tau} + \frac{1}{r^2}\partial_r \left[r^2 \frac{\bar{N}}{\bar{A}}(\bar{\mu} - \bar{D} v)\right] &= -(\bar{\tau} + \bar{D} + a^4 p)\bar{N}\bar{A} r \left[(1 + v^2)\eta\psi + v\Xi\right] \\
&\quad - \bar{N}\frac{a_{,\varphi}}{a}\left[\bar{D} v\eta + (\bar{\mu} v - \bar{\tau} + 3a^4 P)\psi)\right] \qquad (3.25)
\end{aligned}$$

Here, the dots represent a few terms involving the scalar field where we still differ from Eqs. (8-10) in [?]. We will return to these terms later on.

4 Spherically symmetric collapse: Jordan frame

Next, we look at the equations in the Jordan frame. We will first list the equations in the form of primitive variables and then consider the choice of evolution variables and fluxes. We start with the line element

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = -\alpha^2 dt^2 + X^2 dr^2 + \frac{r^2}{F} d\Omega^2 \,. \qquad (4.1)$$

As before, we introduce the metric functions

$$\begin{aligned}
m &= \frac{r}{2}\left(1 - \frac{1}{FX^2}\right), \qquad (4.2) \\
\Phi &= \ln(\sqrt{F}\alpha). \qquad (4.3)
\end{aligned}$$

Note the factor of $1/F$ as a result of which we are imposing radial gauge in the Einstein frame (as Novak *et al.*) which implies that we are not quite in radial gauge in the physical (Jordan) frame. Following O'Connor & Ott [?], the energy-momentum tensor is given by

$$\begin{aligned}
T_{\alpha\beta} &= \rho h u_\alpha u_\beta + P g_{\alpha\beta}, \qquad (4.4) \\
u^\alpha &= \frac{1}{\sqrt{1 - v^2}}\left[\frac{1}{\alpha},\ \frac{v}{X},\ 0,\ 0\right]. \qquad (4.5)
\end{aligned}$$

and the baryonic flow by

$$J^\alpha = \rho u^\alpha, \qquad (4.6)$$

We will describe the baryon density on terms of the same variable $\bar{D}$ as used in the Einstein frame

$$\bar{D} = \frac{\rho X}{F\sqrt{F}\sqrt{1 - v^2}}. \qquad (4.7)$$

These definitions imply the following relation between our Jordan frame variables and their (mostly) Einstein-frame analogues used by Novak *et al.*:

$$\begin{aligned}
\alpha^2 &= \frac{\bar{N}^2}{F}, \quad X^2 = \frac{\bar{A}^2}{F}, \quad \Phi = \nu, \qquad (4.8) \\
\rho &= n_{\mathrm{B}}, \quad P = p, \quad \rho h = e + p. \qquad (4.9)
\end{aligned}$$

The scalar field is now represented by ϕ related to the above φ by

$$\frac{\partial \varphi}{\partial \phi} = \sqrt{\frac{3}{4}\frac{{F_{,\phi}}^2}{F} + \frac{4\pi}{F}}. \tag{4.10}$$

The field equations, evolution of the scalar field, conservation of energy momentum and baryon density are obtained from

$$\begin{aligned}
G_{\alpha\beta} &= \frac{8\pi}{F}(T_{\alpha\beta} + T^F_{\alpha\beta} + T^\phi_{\alpha\beta}), && (4.11)\\
\nabla^\mu \nabla \mu\phi &= -\frac{1}{16\pi}F_{,\phi}R, && (4.12)\\
\nabla_\mu {T^\mu}_\alpha &= 0, && (4.13)\\
\nabla_\mu J^\mu &= 0. && (4.14)
\end{aligned}$$

The metric evolution is thus given by

$$\begin{aligned}
\partial_r \Phi &= 4\pi X^2 F\left[\frac{m}{4\pi r^2} + \frac{r}{F^2}\left(\frac{\rho h v^2}{1-v^2} + P\right) + \frac{r}{8\pi F}\left(\frac{\partial \varphi}{\partial \phi}\right)^2\left(\frac{(\partial_r\phi)^2}{X^2} + \frac{(\partial_t\phi)^2}{\alpha^2}\right)\right], && (4.15)\\
\partial_r m &= 4\pi\frac{r^2}{F^2}\left(\frac{\rho h}{1-v^2} - P\right) + \frac{r^2}{2F}\left(\frac{\partial \varphi}{\partial \phi}\right)^2\left[\frac{(\partial_r\phi)^2}{X^2} + \frac{(\partial_t\phi)^2}{\alpha^2}\right], && (4.16)\\
\partial_t m &= r^2\left(\frac{1}{FX^2}\left(\frac{\partial \varphi}{\partial \phi}\right)^2\partial_r\phi\partial_t\phi - 4\pi\frac{\alpha}{F^2 X}\frac{\rho h v}{1-v^2}\right). && (4.17)
\end{aligned}$$

The scalar field evolves via

$$\begin{aligned}
&\left(\frac{\partial \varphi}{\partial \phi}\right)^2 {\partial_t}^2\phi + \frac{1}{2}\partial_t\left[\left(\frac{\partial \varphi}{\partial \phi}\right)^2\right]\partial_t\phi\\
&= \frac{\alpha^2}{X^2}\left[\left(\frac{\partial \varphi}{\partial \phi}\right)^2\left({\partial_r}^2\phi + \frac{2}{r}\partial_r\phi\right) + \frac{1}{2}\partial_r\left(\frac{\partial \varphi}{\partial \phi}\right)^2\partial_r\phi + \left(\frac{\partial_r\alpha}{\alpha} - \frac{\partial_r X}{X}\right)\left(\frac{\partial \varphi}{\partial \phi}\right)^2\partial_r\phi\right]\\
&\quad + \left(\frac{\partial_t\alpha}{\alpha} - \frac{\partial_t X}{X}\right)\left(\frac{\partial \varphi}{\partial \phi}\right)^2\partial_t\phi + 2\pi\alpha^2\frac{F_{,\phi}}{F^2}(\rho h - 4P). \qquad (4.18)
\end{aligned}$$

Baryon conservation implies

$$\partial_t\bar{D} + \frac{1}{r^2\sqrt{F}}\partial_r\left(r^2\sqrt{F}\frac{\alpha}{X}\bar{D}v\right) + \bar{D}\frac{F_{,\phi}}{2F}\partial_t\phi = 0 \tag{4.19}$$

Finally, the matter equations are

$$\begin{aligned}
&\partial_t\left[\frac{\rho h v}{F^2(1-v^2)}\right]+\frac{1}{r^2}\partial_r\left[r^2\frac{\alpha}{F^2X}\left(\frac{\rho h v^2}{1-v^2}+P\right)\right]\\
&=(-\rho h+P)\frac{\alpha X}{F}\left(8\pi r\frac{P}{F^2}+\frac{m}{r^2}-\frac{F_{,\phi}}{2F^2X^2}\partial_r\phi\right)+\frac{\alpha X}{F}P\frac{m}{r^2}+2\frac{\alpha}{X}\frac{P}{F^2 r}\\
&\quad-2\left(\frac{\partial\varphi}{\partial\phi}\right)^2\partial_r\phi\partial_t\phi\frac{r}{F^2}\frac{\rho h v}{1-v^2}-\frac{3\alpha}{2X}P\frac{F_{,\phi}}{F^3}\partial_r\phi-\frac{r}{2}\frac{\alpha X}{F^2}\left(\frac{\partial\varphi}{\partial\phi}\right)^2\left[\frac{(\partial_r\phi)^2}{X^2}+\frac{(\partial_t\phi)^2}{\alpha^2}\right]\frac{\rho h(1+v^2)}{1-v^2}
\end{aligned} \tag{4.20}$$

$$\begin{aligned}
&\partial_t\left[\frac{\rho h}{F^2(1-v^2)}-\frac{P}{F^2}-\bar D\right]+\frac{1}{r^2}\partial_r\left[\frac{\alpha r^2}{X}\left(\frac{\rho h v}{F^2(1-v^2)}-\bar D v\right)\right]\\
&=-\frac{\rho h}{F^2(1-v^2)}\left\{r(1+v^2)\left(\frac{\partial\varphi}{\partial\phi}\right)^2\partial_r\phi\partial_t\phi+rv\left(\frac{\partial\varphi}{\partial\phi}\right)^2\left[\frac{\alpha}{X}(\partial_r\phi)^2+\frac{X}{\alpha}(\partial_t\phi)^2\right]\right\}\\
&\quad+\frac{\bar D}{2}\frac{F_{,\phi}}{F}\partial_t\phi+\frac{\alpha}{2X}\bar D v\frac{F_{,\phi}}{F}\partial_r\phi+\left(2P-\frac{\rho h}{2}\right)\frac{F_{,\phi}}{F^3}\partial_t\phi.
\end{aligned} \tag{4.21}$$

5 Converging to the final set of equations

We will now generalize the variables used in O'Connor & Ott [?] to the case of scalar-tensor theory of gravity to write the above Eqs. (4.15)-(4.21) in a more compact form. In particular, the structure of the wave equation for the scalar field (4.18) suggests that we should use the Einstein frame version φ instead. We also carry over Novak's notation for the derivatives of the scalar field

$$\eta = \frac{1}{X}\partial_r\varphi, \tag{5.1}$$

$$\psi = \frac{1}{\alpha}\partial_t\varphi. \tag{5.2}$$

The flux-conservative variables used in O'Connor & Ott are generalized according to

$$D=\frac{\rho X}{\sqrt{1-v^2}} \quad\to\quad \bar D=\frac{\rho X}{F\sqrt{F}\sqrt{1-v^2}}, \tag{5.3}$$

$$S^r=\frac{\rho h v}{1-v^2} \quad\to\quad \bar S^r=\frac{\rho h v}{F^2(1-v^2)}, \tag{5.4}$$

$$\tau=\frac{\rho h}{1-v^2}-P-D \quad\to\quad \bar\tau=\frac{\rho h}{F^2(1-v^2)}-\frac{P}{F^2}-\bar D. \tag{5.5}$$

With that, the metric equations become

$$\partial_r\Phi = X^2F\left[\frac{m}{r^2}+4\pi r\left(\bar S^r v+\frac{P}{F^2}\right)+\frac{r}{2F}(\eta^2+\psi^2)\right]-rFX^2W\,, \tag{5.6}$$

$$\partial_r m = 4\pi r^2(\bar\tau+\bar D)+r^2W+\frac{r^2}{2F}(\eta^2+\psi^2), \tag{5.7}$$

$$\partial_t m = r^2\frac{\alpha}{X}\left(\frac{1}{F}\eta\psi-4\pi\bar S^r\right). \tag{5.8}$$

For completeness, we also write the metric equations in terms of the variables α and X which may be useful when we wish to replace derivatives of these functions. We obtain

$$\frac{\partial_r \alpha}{\alpha} = X^2 F\left[\frac{m}{r^2} + 4\pi r\left(\bar{S}^r v + \frac{P}{F^2}\right) + \frac{r}{2F}(\eta^2+\psi^2)\right] - \frac{F_{,\varphi}}{2F}X\eta - rFX^2 W\,, \tag{5.9}$$

$$\frac{\partial_r X}{X} = 4\pi r F X^2(\bar{\tau}+\bar{D}) + \frac{rX^2}{2}(\eta^2+\psi^2) - FX^2\frac{m}{r^2} - \frac{F_{,\varphi}}{2F}X\eta + rFX^2W\,, \tag{5.10}$$

$$\frac{\partial_t X}{X} = rX\alpha(\eta\psi - 4\pi F\bar{S}^r) - \frac{F_{,\varphi}}{2F}\alpha\psi. \tag{5.11}$$

Note that the lapse function α (or Φ) is determined only up to a multiplicative (or additive) constant. In GR, we can fix this by matching to an external Schwarzschild metric. Because in scalar-tensor theory we do not have a direct analogue of the Birkhoff theorem, we cannot fix this constant in that manner. Novak assumes instead that the product

$$\bar{A}(R)\bar{N}(R) = F(R)\alpha X(R) = e^{\Phi(R)}\frac{1}{\sqrt{1-\frac{2m(R)}{R}}} = K_{AN} \tag{5.12}$$

at the outer edge of the grid R, is roughly constant and can therefore be evaluated using the initial (static) profile. The boundary condition on Φ reads

$$\Phi_{\text{bound}}(R) = \frac{1}{2}\log\left[K_{AN}^2\left(1-\frac{2m(R)}{R}\right)\right]\,. \tag{5.13}$$

Novak reports variations of $\Delta\bar{A}(R) \lesssim 10^{-5}$ during the evolution [?], indicating that this approximation does not introduce a significant error. The GR Birkhoff theorem corresponds to $K_{AN} = 1$.
The wave equation for the scalar field is

$$\begin{aligned}\partial_t\partial_t\varphi &= \frac{\alpha^2}{X^2}\left[\partial_r\partial_r\varphi + \frac{2}{r}\partial_r\varphi + \left(\frac{\partial_r\alpha}{\alpha} - \frac{\partial_r X}{X}\right)\partial_r\varphi\right] + \left(\frac{\partial_t\alpha}{\alpha} - \frac{\partial_t X}{X}\right)\partial_t\varphi \\ &\quad + 2\pi\alpha^2\left(\bar{\tau} - \bar{S}^r v + \bar{D} - 3\frac{P}{F^2}\right)F_{,\varphi} - \alpha^2 F W_{,\varphi}.\end{aligned} \tag{5.14}$$

We write this equation as a first-order system using the definitions (5.1), (5.2) and the identity $\partial_t\partial_r\eta = \partial_r\partial_t\eta$ to obtain

$$\partial_t\varphi = \alpha\,\psi\,, \tag{5.15}$$

$$\partial_t\eta = -\eta\frac{\partial_t X}{X} + \frac{\alpha}{X}\left(\partial_r\psi + \psi\frac{\partial_r\alpha}{\alpha}\right)\,, \tag{5.16}$$

$$\partial_t\psi = \frac{\alpha}{X}\left[\partial_r\eta + \frac{2}{r}\eta + \eta\frac{\partial_r\alpha}{\alpha}\right] - \psi\frac{\partial_t X}{X} + 2\pi\alpha\left(\bar{\tau} - \bar{S}^r v + \bar{D} - 3\frac{P}{F^2}\right)F_{,\varphi} - \alpha F W_{,\varphi}\,. \tag{5.17}$$

For numerical purposes, it turns out to be more convenient to apply the finite differencing to a version of these equations where we gather all radial derivatives in a flux term. This version is given by

$$\partial_t\varphi = \alpha\psi, \tag{5.18}$$

$$\partial_t\eta = -\eta\frac{\partial_t X}{X} + \frac{1}{X}\partial_r\left(\alpha\psi\right)\,, \tag{5.19}$$

$$\partial_t\psi = \frac{1}{r^2X}\partial_r\left(\alpha r^2\eta\right) - \psi\frac{\partial_t X}{X} + 2\pi\alpha\left(\bar{\tau} - \bar{S}^r v + \bar{D} - 3\frac{P}{F^2}\right)F_{,\varphi}i - \alpha F W_{,\varphi}\,. \tag{5.20}$$

By finite differencing the compound terms in parentheses following the ∂_r operator instead of summing up the individual derivatives as in Eqs. (5.16), (5.17), we reduce numerical noise near the origin.
The scalar field should satisfy an outgoing boundary condition at spatial infinity:

$$\varphi(t,r) \quad \overset{r\to\infty}{\longrightarrow} \quad \varphi_o + \frac{F(t-r)}{r} + \mathcal{O}(r^{-2})\,. \tag{5.21}$$

This, taking into account the asymptotic behaviour of the different quantities near spatial infinity, can be translated into the following differential condition:

$$\partial_t\varphi + \partial_r\varphi + \frac{\varphi-\varphi_o}{r} = 0\,. \tag{5.22}$$

In the case we want to use the first-order formulation for the scalar field wave equation, the outgoing boundary conditions for the variables (φ,ψ,η) are:

$$\partial_t\psi + \partial_r\psi + \frac{\psi}{r} = 0\,, \tag{5.23}$$

$$\partial_t\eta + \partial_r\eta + \frac{\eta}{r} - \frac{\varphi-\varphi_o}{r^2} = 0\,, \tag{5.24}$$

and equation (5.15) for φ.
Finally, the matter evolution is determined by

$$\partial_t\bar{D} + \frac{1}{\sqrt{F}r^2}\partial_r\left(r^2\frac{\alpha}{X}\sqrt{F}f_{\bar{D}}\right) = s_{\bar{D}}, \tag{5.25}$$

$$\partial_t\bar{S}^r + \frac{1}{r^2}\partial_r\left(r^2\frac{\alpha}{X}f_{\bar{S}^r}\right) = s_{\bar{S}^r}, \tag{5.26}$$

$$\partial_t\bar{\tau} + \frac{1}{r^2}\partial_r\left(r^2\frac{\alpha}{X}f_{\bar{\tau}}\right) = s_{\bar{\tau}}, \tag{5.27}$$

where

$$f_{\bar{D}} = \bar{D}v, \tag{5.28}$$

$$f_{\bar{S}^r} = \bar{S}^r v + \frac{P}{F^2}, \tag{5.29}$$

$$f_{\bar{\tau}} = \bar{S}^r - \bar{D}v, \tag{5.30}$$

$$s_{\bar{D}} = -\bar{D}\frac{F_{,\varphi}}{2F}\alpha\psi, \tag{5.31}$$

$$\begin{aligned} s_{\bar{S}^r} = {} & (\bar{S}^r v - \bar{\tau} - \bar{D})\alpha X F\left(8\pi r\frac{P}{F^2} + \frac{m}{r^2} - \frac{F_{,\varphi}}{2F^2X}\eta - rW\right) + \frac{\alpha X}{F}P\frac{m}{r^2} + 2\frac{\alpha P}{rXF^2} \\ & -r\alpha X\frac{PW}{F} - 2r\alpha X\bar{S}^r\eta\psi - \frac{3}{2}\alpha\frac{P}{F^2}\frac{F_{,\varphi}}{F}\eta - \frac{r}{2}\alpha X(\eta^2+\psi^2)\left(\bar{\tau} + \frac{P}{F^2} + \bar{D}\right)(1+v^2), \end{aligned} \tag{5.32}$$

$$s_{\bar{\tau}} = -\left(\bar{\tau} + \frac{P}{F^2} + \bar{D}\right)r\alpha X\left[(1+v^2)\eta\psi + v(\eta^2+\psi^2)\right] + \frac{\alpha}{2}\frac{F_{,\varphi}}{F}\left[\bar{D}v\eta + \left(\bar{S}^r v - \bar{\tau} + 3\frac{P}{F^2}\right)\psi\right] \tag{5.33}$$

Let us now analyze some aspects of this system of conservation laws that determines the evolution of the matter variables. Although the first equation looks a bit different than the other two because of the factors of $\sqrt{F}$ it can actually be written as:

$$\partial_t\bar{D} + \frac{1}{r^2}\partial_r\left(r^2\frac{\alpha}{X}f_{\bar{D}}\right) + \frac{F_{,\varphi}}{2F}\alpha\eta f_{\bar{D}} = s_{\bar{D}}\,, \tag{5.34}$$

which looks like the other two. The extra term does not modify the principal part of the system of equations, not even when we take into account that it is coupled to the system of evolution equations for the scalar field [Eqs. (5.15)-(5.17)] since it is a lower-order derivative term. Therefore we could absorb this term with a redefinition of the source $s_{\bar{D}}$. The conclusion of this discussion is that in order to analyze its hyperbolic structure we can assume that it can be written in the following vector form:

$$\partial_t \mathbf{U} + \frac{1}{r^2}\partial_r \left(r^2 \frac{\alpha}{X} \mathbf{f}(\mathbf{U}) \right) = \mathbf{s}(\mathbf{U}) , \tag{5.35}$$

where

$$\begin{aligned} \mathbf{U} &= (\bar{D}, \bar{S}^r, \bar{\tau}) , && (5.36) \\ \mathbf{f}(\mathbf{U}) &= (f_{\bar{D}}, f_{\bar{S}^r}, f_{\bar{\tau}}) , && (5.37) \\ \mathbf{s}(\mathbf{U}) &= (\tilde{s}_{\bar{D}}, s_{\bar{S}^r}, s_{\bar{\tau}}) , && (5.38) \end{aligned}$$

with

$$\tilde{s}_{\bar{D}} = -\bar{D} \frac{F_{,\varphi}}{2F} \alpha \left(\psi + v\eta \right) . \tag{5.39}$$

Then, looking at the system of equation (5.35), it is clear that the hyperbolic structure of these equations is dictated by the Jacobian matrix:

$$\mathbf{J}_{\mathbf{U}} = \frac{\partial \mathbf{f}(\mathbf{U})}{\partial \mathbf{U}} . \tag{5.40}$$

A practical way of computing this Jacobian is by making use of the primitive variables:

$$\mathbf{W} = (\rho, v, \epsilon) , \tag{5.41}$$

where ϵ is the specific internal energy, related to the specific enthalpy by $h = 1 + \epsilon + P(\epsilon, \rho)/\rho$. The reason for using $\mathbf{W}$ is that the following two Jacobians are easy to be computed:

$$\mathbf{J}_{\mathbf{W}} = \frac{\partial \mathbf{f}(\mathbf{U}(\mathbf{W}))}{\partial \mathbf{W}} , \qquad \mathbf{J}_{\mathbf{W}\to\mathbf{U}} = \frac{\partial \mathbf{U}(\mathbf{W})}{\partial \mathbf{W}} . \tag{5.42}$$

Then,

$$\mathbf{J}_{\mathbf{U}} = \mathbf{J}_{\mathbf{W}} \cdot \mathbf{J}^{-1}_{\mathbf{W}\to\mathbf{U}} . \tag{5.43}$$

By making the computation we find that the three eigenvalues, which are the characteristic speeds associated with the propagation of the matter fields, are:

$$\begin{aligned} \lambda_0 &= v , && (5.44) \\ \lambda_{\pm} &= \frac{v \pm c_s}{1 \pm v\, c_s} , && (5.45) \end{aligned}$$

where c_s is given by

$$h\, c_s^2 = \frac{\partial P}{\partial \rho} + \frac{P}{\rho^2} \frac{\partial P}{\partial \epsilon} , \tag{5.46}$$

and represents the local matter sound speed. The main conclusion of this computation is that the characteristic speeds are exactly the same as in General Relativity, since they do not contain any dependence on the conformal factor F.

6 The static limit

We obtain the static limit of the equations of the previous section by setting all time derivatives and the velocity v to zero. The matter variables now simplify to

$$\bar{D} = \frac{\rho X}{F\sqrt{F}}, \tag{6.1}$$

$$\bar{S}^r = 0, \tag{6.2}$$

$$\bar{\tau} = \frac{\rho h - P}{F^2} - \bar{D}. \tag{6.3}$$

For the metric, Eqs. (5.6), (5.7) survive and yield

$$\partial_r \Phi = X^2 F\left(\frac{m}{r^2} + 4\pi r\frac{P}{F^2} + \frac{r}{2F}\eta^2\right) - rFX^2 W\,, \tag{6.4}$$

$$\partial_r m = 4\pi r^2(\bar{\tau} + \bar{D}) + \frac{r^2}{2F}\eta^2 + r^2 W. \tag{6.5}$$

Similarly, their analogs (5.9), (5.10) in terms of α and X become

$$\frac{\partial_r \alpha}{\alpha} = X^2 F\left(\frac{m}{r^2} + 4\pi r\frac{P}{F^2} + \frac{r}{2F}\eta^2\right) - \frac{F_{,\varphi}}{2F}X\eta - rFX^2 W\,, \tag{6.6}$$

$$\frac{\partial_r X}{X} = 4\pi r F X^2(\bar{\tau} + \bar{D}) + \frac{rX^2}{2}\eta^2 - FX^2\frac{m}{r^2} - \frac{F_{,\varphi}}{2F}X\eta + rFX^2 W\,. \tag{6.7}$$

The scalar field equation (5.16) vanishes identically while Eq. (5.17) has the static limit

$$\partial_r \varphi = X\eta, \tag{6.8}$$

$$\partial_r \eta + \frac{2}{r}\eta + \eta\frac{\partial_r \alpha}{\alpha} + 2\pi X\left(\bar{\tau} + \bar{D} - 3\frac{P}{F^2}\right)F_{,\varphi} - FXW_{,\varphi} = 0\,. \tag{6.9}$$

Of the matter equations (5.25)-(5.27), only (5.26) survives and gives

$$\frac{1}{r^2}\partial_r\left(r^2\frac{\alpha}{X}f_{\bar{S}^r}\right) = s_{\bar{S}^r}, \tag{6.10}$$

with

$$f_{\bar{S}^r} = \frac{P}{F^2}, \tag{6.11}$$

$$\begin{aligned} s_{\bar{S}^r} = {} & -(\bar{\tau} + \bar{D})\alpha X F\left(8\pi r\frac{P}{F^2} + \frac{m}{r^2} - \frac{F_{,\varphi}}{2F^2 X}\eta\right) + \frac{\alpha X}{F}P\frac{m}{r^2} + 2\frac{\alpha P}{rXF^2} \\ & - \frac{3}{2}\alpha\frac{P}{F^2}\frac{F_{,\varphi}}{F}\eta - \frac{r}{2}\alpha X\eta^2\left(\bar{\tau} + \frac{P}{F^2} + \bar{D}\right) - rF\alpha XW\left(\bar{\tau} + \bar{D} + \frac{3P}{F^2}\right)\,. \end{aligned} \tag{6.12}$$

We will see whether we want to rewrite the static equations in a form closer to the numerical implementation.
The boundary conditions for the scalar field follow from regularity at the origin

$$\partial_r \varphi|_{r=0} \quad \Rightarrow \quad \eta|_{r=0} = 0, \tag{6.13}$$

while at the outer boundary, a *cosmological background value* φ_0 is imposed. Novak typically chooses

$$\varphi_0 = 10^{-5}, \tag{6.14}$$

see Table I in [**?**].
Finally we would like to express the static limit of the equations in the form of Eq. (7) of [**?**]. This give us

$$\begin{aligned}
\partial_r \Phi &= FX^2\left(\frac{m}{r^2} + 4\pi r\frac{P}{F^2} + \frac{r}{2F}\eta^2\right) - rFX^2W\,, && (6.15)\\
\frac{\partial_r \alpha}{\alpha} &= FX^2\left(\frac{m}{r^2} + 4\pi r\frac{P}{F^2} + \frac{r}{2F}\eta^2\right) - \frac{F_{\varphi}}{2F}X\eta - rFX^2W\,, && (6.16)\\
\partial_r m &= 4\pi r^2\frac{\rho h - P}{F^2} + \frac{r^2}{2F}\eta^2 + r^2W\,, && (6.17)\\
\frac{\partial_r X}{X} &= 4\pi rFX^2\frac{\rho h - P}{F^2} + \frac{r}{2}X^2\eta^2 - FX^2\frac{m}{r^2} - \frac{F_{,\varphi}}{2F}X\eta + rFX^2W\,, && (6.18)\\
\partial_r P &= -\rho hFX^2\left(\frac{m}{r^2} + 4\pi r\frac{P}{F^2} + \frac{r}{2F}\eta^2 - rW\right) + \rho h\frac{F_{,\varphi}}{2F}X\eta, && (6.19)\\
\partial_r \varphi &= X\eta, && (6.20)\\
\partial_r \eta &= -2\frac{\eta}{r} - 2\pi X\frac{\rho h - 4P}{F^2}F_{,\varphi} - F\eta X^2\frac{m}{r^2} - 4\pi rX^2\eta\frac{P}{F} - \frac{r}{2}X^2\eta^3 + \frac{X}{2}\frac{F_{,\varphi}}{F}\eta^2\\
&\quad + rFX^2W\eta + FXW_{,\varphi}\,. && (6.21)
\end{aligned}$$

If we want to compactify the vacuum region exterior to the star, we switch to a radial coordinate $y = 1/r$, so that

$$\partial_r = -y^2\partial_y \qquad \Leftrightarrow \qquad \partial_y = -r^2\partial_r\,, \tag{6.22}$$

and change the variable $X\eta = \partial_r\varphi$ to $X\tilde{\eta} = \partial_y\varphi$ which implies

$$\eta = -y^2\tilde{\eta} \qquad \Leftrightarrow \qquad \tilde{\eta} = -r^2\eta\,. \tag{6.23}$$

The static equations (in vacuum) then become

$$\begin{aligned}
\frac{\partial_y X}{X} &= FX^2m - \frac{1}{2}yX^2\tilde{\eta}^2 - \frac{1}{2}\frac{F_{,\varphi}}{F}X\tilde{\eta} - \frac{1}{y^3}FX^2W\,, && (6.24)\\
\partial_y \Phi &= -FX^2m - \frac{y}{2}X^2\tilde{\eta}^2 + \frac{1}{y^3}FX^2W\,, && (6.25)\\
\partial_y \tilde{\eta} &= FX^2m\tilde{\eta} + \frac{y}{2}X^2\tilde{\eta}^3 + \frac{1}{2}X\tilde{\eta}\frac{F_{,\varphi}}{F} - \frac{1}{y^3}FX^2\tilde{\eta}W + \frac{1}{y^4}FXW_{,\varphi}\,. && (6.26)
\end{aligned}$$

Note that Damour uses variables related to ours via

$$\mu = m = \frac{r}{2}\left(1 - \frac{1}{FX^2}\right), \tag{6.27}$$
$$\nu = 2\Phi, \tag{6.28}$$
$$\psi = X\eta, \tag{6.29}$$
$$\tilde{p} = P, \tag{6.30}$$
$$\tilde{e} = \rho h - P = e, \tag{6.31}$$
$$A(\varphi) = \frac{1}{\sqrt{F(\varphi)}}. \tag{6.32}$$

In particular, note that his ψ has nothing to do with our variable of the same name defined in terms of the time derivative of the scalar field in Eq. (5.2).
The total baryonic mass of the star is given by

$$m_{\mathrm{B}} = m_b \int d^3x \sqrt{-g}\, n_b u^t \tag{6.33}$$

where m_b is the atomic mass unit and n_b is the physical baryon number density (see below). Taking into account that $-g = \alpha^2 X^2 r^4/F^2$, $u^t = 1/\alpha$ (in the static limit), $X = \bar{A}/\sqrt{F}$, and $\bar{A} = 1/\sqrt{1-2m(r)/r}$, we get,integrating over angles, the following

$$m_{\mathrm{B}} = 4\pi m_b \int_0^{r_s} dr\, r^2 \frac{n_b}{F^{3/2}\sqrt{1-2m(r)/r}}. \tag{6.34}$$

An ingredient we may require for imposing an outer boundary condition is how the value of the scalar field at the stellar surface (or, in our case, at the outer edge of the stellar atmosphere) is related to the cosmological background value. We shall label the value of the scalar field at the surface by φ_s and call the cosmological background value φ_0 as before. Given that we have a much larger computational domain than Damour & Esposito-Farese [**?**], we might just get away with the simple option

$$\varphi_0 = \varphi_s. \tag{6.35}$$

A more elaborate contruction involves the use of an exterior metric with vanishing matter sources [**?**, **?**, **?**]. The matching of the interior to the exterior solution gives a relation between the scalar field at the stellar surface and at infinity, Eq. (8) in [**?**] which translates into our variables as

$$\varphi_0 = \varphi_s + \frac{X_s \eta_s}{\sqrt{(\partial_r \Phi_s)^2 + X_s^2 \eta_s^2}} \operatorname{arctanh} \frac{\sqrt{(\partial_r \Phi_s)^2 + X_s^2 \eta_s^2}}{\partial_r \Phi_s + 1/r_s}. \tag{6.36}$$

The solution of the static equations then consists in a shooting method. At the inner boundary we have

$$m(0) = 0, \tag{6.37}$$
$$\Phi(0) = 0, \tag{6.38}$$
$$\eta(0) = 0, \tag{6.39}$$
$$P(0) = P_c, \tag{6.40}$$

where P_c parametrizes the family of stars. We also need some central value for the scalar field

$$\varphi(0) = \varphi_c, \tag{6.41}$$

which, after integration will result in some value for the cosmological background value φ_0 via Eq. (6.35) or (6.36), depending on which options we choose. We then iteratively determine the correct φ_c that leads to the specified φ_0 (10^{-5} if we follow Novak) via a shooting algorithm.
From the grid function m the gravitational mass can be calculated by using the matching to an exterior solution mentioned above. The result is given by Eq. (9) of [?] and translates into our notation as

$$m_{\text{grav}} = r^2 \partial_r \Phi_s \sqrt{1 - \frac{2m}{r}} \exp\left\{ -\frac{\partial_r \Phi_s}{\sqrt{(\partial_r \Phi_s)^2 + X^2 \eta^2}} \operatorname{arctanh}\left[\frac{\sqrt{(\partial_r \Phi_s)^2 + X^2 \eta^2}}{\partial_r \Phi_s + 1/r} \right] \right\}. \tag{6.42}$$

This should be the mass to be compared with M_G in Novak's Table I [?].

6.1 The coupling function

We still need to specify the coupling function $F(\varphi) = 1/a(\varphi)^2$. Following various articles in the literature, e. g. [?, ?], we will use

$$F(\varphi) = \frac{1}{a^2(\varphi)} = e^{-2\alpha_0(\varphi - \varphi_0) - \beta_0(\varphi - \varphi_0)^2}, \tag{6.43}$$

which implies a logarithmic derivative

$$\frac{F_{,\varphi}}{F} = -2\frac{a_{,\varphi}}{a} = -2\left[\alpha_0 + \beta_0(\varphi - \varphi_0)\right]. \tag{6.44}$$

Novak's models are described in more detail in Table I in [?]; α_0 is constrained to be small from Solar system tests while large negative values for β_0 result in *spontaneous scalarization* [?] but are excluded from binary pulsar tests [?].

6.2 Generating Polytropic Neutron Star Initial Data

Novak [?] considers various polytropic neutron stars as initial data. He uses a polytropic EOS of the form

$$P = K m_b n_0 \left(\frac{n_b}{n_0}\right)^{\gamma}, \tag{6.45}$$

where K is a constant, γ is the adiabatic exponent, $m_b = 1.66 \times 10^{-27}\,\text{kg}$ is the atomic mass unit, $n_0 = 0.1\,\text{fm}^{-3}$ is the assumed baryon number density scale, and n_b is the baryon number density in units of fm^{-3}. Novak uses two different choices of K and γ:

$$\begin{aligned} &\text{EOS}\,1: \quad K = 0.0195 \quad \gamma = 2.34\,, \\ &\text{EOS}\,2: \quad K = 0.1 \qquad\;\; \gamma = 2.0\,. \end{aligned} \tag{6.46}$$

Unfortunately, the constant K does depend on the choice of units in the code *and* on the choice of the units for m_b and n_0. Novak does not say what units he uses.
Using EOS 1 and assuming MKS (since m_b is given in kg) and converting $1\,\text{fm}^{-3} = 10^{45}\,\text{m}^{-3}$, one obtains a pressure at $n_b = n_0$ of $\sim 3.2 \times 10^{15}\,\text{N}\,\text{m}^{-2}$, which is about 20 orders of magnitude too

small for a neutron star (see, e.g., [?]). More sensible values are obtained when assuming that Novak uses $c = G = 1$ units, converting to cgs by $P_{\rm cgs} = P_{c=G=1}c^2$, and taking m_b and n_0 in cgs: $P \sim 3 \times 10^{33}\,{\rm dyn\,cm^{-2}}$ at $n_b = n_0$, which is just fine. Now, polytropic EOS in stellar astrophysics are usually written as

$$P = K^\star \rho^\gamma \ . \tag{6.47}$$

Here we have introduced the superscript * to differentiate $K^\star$ from Novak's K. In order to convert Novak's polytropic EOS into this common form, we use $\rho = m_b n_b$ and write

$$P_{\rm cgs} = K c^2_{\rm cgs} * (m_b n_0)_{\rm cgs}\, n_{0,{\rm cgs}}^{-\gamma}\, m_{b,{\rm cgs}}^{-\gamma}\, \rho^\gamma \ , \tag{6.48}$$

thus we have

$$K^\star_{\rm cgs} = K c^2_{\rm cgs} * (m_b n_0)_{\rm cgs}\, n_{0,{\rm cgs}}^{-\gamma}\, m_{b,{\rm cgs}}^{-\gamma} \ . \tag{6.49}$$

We must now convert to $c = G = M_\odot = 1$ units used in `GR1D`. This is straightforward:

$$P_{c=G=M_\odot=1} = c_P K^\star_{\rm cgs} \left(\frac{\rho_{c=G=M_\odot=1}}{c_\rho}\right)^\gamma \ , \tag{6.50}$$

where $c_P = 1.802 \times 10^{-39}$ and $c_\rho = 1.6193 \times 10^{-18}$ are the conversion factors connecting cgs and $c = G = M_\odot = 1$: $P_{c=G=M_\odot=1} = c_P P_{\rm cgs}$, $\rho_{c=G=M_\odot=1} = c_\rho \rho_{\rm cgs}$. So, finally,

$$K^\star_{c=G=M_\odot=1} = c_P K^\star_{\rm cgs} (c_\rho)^{-\gamma} \ . \tag{6.51}$$

With this, we now have:

$$\begin{array}{lllll} {\rm EOS\,1:} & K = 0.0195 & K_{\rm cgs} = 1.543446 & K_{c=G=M_\odot=1} = 1186.783 & \gamma = 2.34 \ , \\ {\rm EOS\,2:} & K = 0.1 & K_{\rm cgs} = 5.41384 \times 10^5 & K_{c=G=M_\odot=1} = 372.592 & \gamma = 2.0 \ . \end{array} \tag{6.52}$$

6.3 Initial data format

The GR version of the dynamical code takes *.short* files as input. The first line contains the number of zones; the following lines list 8 values in **CGS** units: (1) zone index from 1, (2) enclosed mass, (3) radial coordinate, (4) temperature, (5) mass density, (6) radial velocity, (7) electron fraction, (8) angular velocity if one wants rotation.

The ST version needs, of course, more inputs since the variables describing the scalar field must be specified. These are φ and its derivatives. Moreover, the boundary condition of the metric potential Φ is now specified using the initial profile [cf. Eq. (5.12)]. We therefore modify the *.short* input format as follows. The number of zones is still indicated in the first line; the following lines list 12 values in **CGS** units: (1) zone index from 1, (2) enclosed mass, (3) radial coordinate, (4) temperature, (5) mass density, (6) radial velocity, (7) electron fraction, (8) angular velocity (can't be used, because ST is not implemented with rotation), (9) φ, which is dimensionless, (10) η in ${\rm cm}^{-1}$ and (11) ψ in ${\rm s}^{-1}$, (12) K_{AN} (dimensionless).

7 Dimensions and Units

We provide some important expressions in scalar-tensor theories in general units/dimensions (with G and c not equal to unity) from where we derive the dimensions of the main quantities in these theories.

The action in the Einstein frame is:

$$S[\bar{g}_{\alpha\beta}, \varphi, \psi_m] = \frac{c^4}{16\pi G} \int \frac{d^4x}{c} \sqrt{-\bar{g}} \left(\bar{R} - 2\bar{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \right) + S_m[\psi_m, a^2(\varphi)\bar{g}_{\alpha\beta}] , \tag{7.1}$$

where G is the bare gravitational coupling constant. The dimensions (mass $\equiv M$, length $\equiv L$, time $\equiv T$) of all the quantities here are

$$[c] = LT^{-1} , \quad [G] = M^{-1}L^3T^{-2} , \quad [S] = ML^2T^{-1} , \quad [x^\mu] = L , \quad [d^4x] = L^4 , \quad [g_{\alpha\beta}] = 1 , \tag{7.2}$$

$$[\bar{R}] = L^{-2} , \quad [a] = 1 , \quad [\varphi] = 1 . \tag{7.3}$$

The field equations are

$$\bar{R}_{\mu\nu} = 2\partial_\mu \varphi \partial_\nu \varphi + \frac{8\pi G}{c^4} \left(\bar{T}_{\mu\nu} - \frac{1}{2} \bar{T} \bar{g}_{\mu\nu} \right) , \tag{7.4}$$

$$\Box_{\bar{g}} \varphi = -\frac{4\pi G}{c^4} \alpha(\varphi) \bar{T} , \tag{7.5}$$

from where

$$[\bar{T}_{\mu\nu}] = ML^{-1}T^{-2} , \quad [\alpha] = 1 . \tag{7.6}$$

On the other hand, the expansion of the scalar field near spatial infinity has the form

$$\varphi(r) = \varphi_0 + \frac{G\omega_A}{c^2 \, r} + \mathcal{O}(1/r^2) , \tag{7.7}$$

where the scalar charge of a body (e.g. a neutron star), labeled A, has dimensions of mass

$$\omega_A = -\frac{\partial m_A}{\partial \varphi_0} , \quad \Rightarrow \quad [\omega_A] = [m_A] = M , \tag{7.8}$$

where m_A is the body's total mass-energy.
On the other hand, when the matter fields behave as a perfect fluid:

$$\bar{T}_{\alpha\beta} = \left(\bar{e} + \frac{\bar{p}}{c^2} \right) \bar{u}_\alpha \bar{u}_\beta + \bar{p} \bar{g}_{\alpha\beta} , \quad \bar{u}^\alpha = \frac{dx^\alpha}{d\tau} \quad \left(\bar{g}_{\alpha\beta} \bar{u}^\alpha \bar{u}^\beta = -c^2 \right) , \tag{7.9}$$

where τ is proper time ($[\tau] = T$) and thus $[\bar{u}^\mu] = LT^{-1}$. Then, as expected,

$$[\bar{e}] = ML^{-3} , \quad [\bar{p}] = ML^{-1}T^{-2} . \tag{7.10}$$

Then, typical equations of state have the form

$$\frac{\bar{p}}{c^2} = K \bar{m}_b \bar{n}_0 \left(\frac{\bar{n}_b}{\bar{n}_0} \right)^\gamma , \quad \bar{e} = \bar{n}_b \bar{m}_b + \frac{K \bar{n}_0 \bar{m}_b}{\gamma - 1} \left(\frac{\bar{n}_b}{\bar{n}_0} \right)^\gamma , \tag{7.11}$$

so that the dimensions of the different quantities are

$$[K] = [\gamma] = 1 , \quad [\bar{m}_b] = M , \quad [\bar{n}_0] = [\bar{n}_b] = L^{-3} . \tag{7.12}$$

8 Conversion between primitive and conserved variables

Eqs. (5.3)-(5.5) express the conserved hydrodynamic variables $\bar{D}$, $\bar{S}^r$ and $\bar{\tau}$ in terms of the primitive variables ρ, ϵ and v. While the relation is straightforward in this direction, the reverse conversion is a non-trivial operation because of the presence of the pressure P which is an intrisic function of ρ and ϵ given by the equation of state. In practice, we convert conserved to primitive variables using a Newton-Raphson procedure as we will illustrate in detail in this section.
The inversion of eqs. (5.3)-(5.5) would be straightforward if we knew the exact value of the pressure P. In that case, we could calculate in this order

$$\begin{aligned} v &= \frac{\bar{S}^r}{\bar{\tau} + P/F^2 + \bar{D}}, && (8.1) \\ \rho &= \frac{F^{3/2}\bar{D}}{XW}, && (8.2) \\ \epsilon &= h - \frac{P}{\rho} - 1 = \frac{F^2(\bar{\tau}+\bar{D}) + P(1-W^2)}{\rho W^2} - 1, && (8.3) \end{aligned}$$

where

$$W = \frac{1}{\sqrt{1-v^2}}. \tag{8.4}$$

Note that the primitive variables determine a pressure $\hat{P}$ via the equation of state and if that value agrees with the value P used in the calculation of the primitive variables, we have obtained a self-consistent solution for the primitive variables. In other words, we need to find the root of the equation

$$f(P) = \hat{P}[\rho(P), \epsilon(P)] - P = 0, \tag{8.5}$$

where we have explicitly written that the values of ρ and ϵ depend on our "choice" for P. The Newton-Raphson algorithm solves this equation iteratively via

$$P_{\rm new} = P_{\rm old} - \Delta f \left(\frac{df}{dP}\right)^{-1}, \tag{8.6}$$

with

$$\frac{df}{dP} = \frac{\partial\hat{P}}{\partial\rho}\left.\frac{\partial\rho}{\partial P}\right|_{\bar{D},\bar{S}^r,\bar{\tau}} + \frac{\partial\hat{P}}{\partial\epsilon}\left.\frac{\partial\epsilon}{\partial P}\right|_{\bar{D},\bar{S}^r,\bar{\tau}} - 1. \tag{8.7}$$

Here, the vertical bars indicate that the derivatives need to be taken while keeping the conserved variables $\bar{D}$, $\bar{S}^r$ and $\bar{\tau}$ constant (their values are fixed from the time evolution). The derivatives $\partial\hat{P}/\partial\rho$ and $\partial\hat{P}/\partial\epsilon$ are provided directly by the equation of state in the code, but we still need to find the derivatives of ϵ and ρ with respect to P. For this purpose, it is helpful to first consider a set of auxiliary relations. Note that the velocity v is regarded as dependent on P in this context. Specifically, we have from (8.1)

$$\begin{aligned} \frac{\partial W}{\partial v} &= W^3 v, && (8.8) \\ \frac{\partial v}{\partial P} &= \frac{-\bar{S}^r}{F^2(\bar{\tau}+P/F^2+\bar{D})^2}, && (8.9) \\ \frac{\partial W}{\partial P} &= \frac{-W^3(\bar{S}^r)^2}{F^2(\bar{\tau}+P/F^2+\bar{D})^3}. && (8.10) \end{aligned}$$

Note that we have a considerable amount of freedom in writing these equations. For example, Eq. (8.1) allows us to trade the velocity v for the more complex expression on the right hand side involving conserved variables and the pressure. An auxiliary variable used in the code is given by

$$T1 = \left(\bar{\tau} + \frac{P}{F^2} + \bar{D}\right)^2 - (\bar{S}^r)^2. \tag{8.11}$$

An easy calculation shows that

$$\left(\bar{\tau} + \frac{P}{F^2} + \bar{D}\right) = W\sqrt{T1}. \tag{8.12}$$

With these relations we find

$$\left.\frac{\partial\rho}{\partial P}\right|_{\bar{D},\bar{S}^r,\bar{\tau}} = \frac{\partial}{\partial P}\left[\frac{F^{3/2}\bar{D}}{XW}\right] = \ldots = \frac{\bar{D}(\bar{S}^r)^2}{X\sqrt{F\,T1}(\bar{\tau} + P/F^2 + \bar{D})^2} \tag{8.13}$$

The internal energy ϵ is given in terms of h, P and ρ by Eq. (8.3) and we consider the individual contributions to its derivative. The enthalpy can be written as

$$h = \frac{F^2(\bar{\tau} + \bar{D}) + P}{F^{3/2}W\bar{D}/X} = \frac{(\bar{\tau} + P/F^2 + \bar{D})\sqrt{F}X}{W\bar{D}}, \tag{8.14}$$

which after some manipulation gives its derivative as

$$\left.\frac{\partial h}{\partial P}\right|_{\bar{D},\bar{S}^r,\bar{\tau}} = \ldots = \frac{WX}{F^{3/2}\bar{D}} = \frac{1}{\rho}. \tag{8.15}$$

For ϵ this implies

$$\left.\frac{\partial\epsilon}{\partial P}\right|_{\bar{D},\bar{S}^r,\bar{\tau}} = \left.\frac{\partial}{\partial P}\right|_{\bar{D},\bar{S}^r,\bar{\tau}}\left(h - \frac{P}{\rho} - 1\right) = \frac{P}{\rho^2}\left.\frac{\partial\rho}{\partial P}\right|_{\bar{D},\bar{S}^r,\bar{\tau}} = \ldots = \frac{(\bar{S}^r)^2 P}{\rho F^2\,T1(\bar{\tau} + P/F^2 + \bar{D})}. \tag{8.16}$$

With these expressions we are able to calculate the derivative in Eq. (8.7) and use that in the Newton-Raphson iteration (8.6). In practice, we initialize the iteration by using for P the pressure on the preceding timeslice as a first guess.

9 Numerical dissipation

Our evolution system for the scalar field (5.18)-(5.20) turns out to be susceptible to numerical noise, especially at the origin. Even though the finite differencing of the total fluxes in this system (rather than summing up individual derivatives) helps reducing the noise, we need to add artificial dissipation of Berger-Olliger type [**?**, **?**] to the evolution system in order to obtain long-term stable evolutions. Our main evolution uses second-order finite differencing and we consequently add a dissipation term of the from

$$D\hat{u}_i = \epsilon h^4 u'''' = \frac{\epsilon}{16}\left(\hat{u}_{i-2} - 4\hat{u}_{i-1} + 6\hat{u}_i - 4\hat{u}_{i+1} + \hat{u}_{i+2}\right), \tag{9.1}$$

to the right-hand-side of Eqs. (5.18)-(5.20), where $\hat{u}$ stands for either of our scalar-field variables. In general, we use a non-uniform grid, so that the rightmost expansion of the fourth spatial derivative in

Model	EOS	φ_0	α_0	β_0	ρ_c $[10^{15}\,\mathrm{g\,cm^{-3}}]$	φ_c	$R_{\star,\mathrm{E}}$ [km]	$R_{\star,\mathrm{J}}$ [km]	M_G $[M_\odot]$	M_B $[M_\odot]$	ω
A	1	1.0^{-5}	5×10^{-5}	-5	1.7264	0.0681	11.2 (11.244)	11.205	1.97 (1.965)	2.26 (2.256)	0.204 (0.200)
B	1	1.0^{-5}	2.5×10^{-2}	-5	1.7264	-0.1505	11.8 (11.796)	11.559	2.07 (2.067)	2.41 (2.407)	0.484 (-0.484)
C	2	1.0^{-5}	5×10^{-5}	-5	0.6640	0.1823	21.5 (21.54)	21.166	3.31 (3.313)	3.68 (3.684)	0.921 (0.920)
D	2	1.0^{-5}	2.5×10^{-2}	-5	0.6640	-0.2154	22.2 (22.206)	21.578	3.41 (3.406)	3.82 (3.821)	1.16 (-1.159)
AB0	1	0	0	0	1.7264	0	11.127	11.127	1.943	2.223	0
CD0	2	0	0	0	0.1818	0	20.343	20.343	3.130	3.432	0
fig1	1	1.0^{-5}	0	-6	1.826	0.236	13.128	12.285	2.365	2.867	0.931
Sol1	1	0	0	-6	0.6581	0	13.2 (13.128)	13.467t	1.378 (1.382)	1.500 (1.506)	0 (0)
Sol2	1	0	0	-6	0.8833	0.201	13 (13.261)	12.834	1.373 (1.377)	1.500 (1.504)	0.781 (0.788)
Sol3	1	0	0	-6	0.8833	0.201	13 (13.261)	12.834	1.373 (1.377)	1.500 (1.504)	-0.781 (-0.788)
Sol4	1	0	10^{-2}	-6	0.6581	0.016	13.2 (13.462)	13.461	1.378 (1.382)	1.500 (1.505)	-0.0591 (0.0584)
Sol5	1	0	10^{-2}	-6	0.8910	-0.205	13 (13.285)	12.822	1.372 (1.375)	1.500 (1.504)	0.803 (-0.810)
Sol6	1	0	10^{-2}	-6	0.8743	0.197	13 (13.234)	12.848	1.374 (1.378)	1.500 (1.504)	-0.757 (0.763)

Table 1: Novak's Neutron Star Models compared to our Results for the Static Case. Our results are highlighted in red color. We uses 300 Km grids as Novak, with 10^5 equispaced zones. On the sign of ω, there's an ambiguity here because two scalarized solutions exist. Moreover, Novak has a bug in the sign of α_0, which changes the sign of ω if α_0 is not tiny. Models A-D and "fig1" refer to [?], while models Sol.1-6 refer to [?]. Beside the sign, the small errors in Sol1-6 are due to the unit conversions: the central mass density is not specified with enough digits in the original paper.

(9.1) is not valid for our case. The generalization to a non-uniform grid is given as follows. We let i label the grid point in question and define $h_j \equiv x_j - x_i$ for $j = -2, -1, 0, 1, 2$. With the coefficients

$$A = -6\frac{h_2 h_1 h_{-1}}{(h_2 - h_{-2})(h_1 - h_{-2})(h_{-1} - h_{-2})}, \tag{9.2}$$

$$B = -6\frac{h_2 h_1 h_{-2}}{(h_2 - h_{-1})(h_1 - h_{-1})(h_{-2} - h_{-1})}, \tag{9.3}$$

$$C = 6, \tag{9.4}$$

$$D = -6\frac{h_2 h_{-1} h_{-2}}{(h_2 - h_1)(h_{-1} - h_1)(h_{-2} - h_1)}, \tag{9.5}$$

$$E = -6\frac{h_1 h_{-1} h_{-2}}{(h_1 - h_2)(h_{-1} - h_2)(h_{-2} - h_2)}, \tag{9.6}$$

$$\tag{9.7}$$

we write the above dissipation term as

$$D\hat{u}_i = \frac{\epsilon}{16}\left(A\hat{u}_{-2} + B\hat{U}_{i-1} + C\hat{u}_i + D\hat{U}_{i+1} + E\hat{U}_{i+2}\right). \tag{9.8}$$

Note that we add this term to the right-hand side of the evolution equations so that the actual dissipation in the numerical system gets multiplied by the Courant factor C_{CFL}. In practice, we have obtained good results using $\epsilon \times C_{\mathrm{CFL}} = 0.5$.

10 Results

10.1 Comparison with previous results

Here we compare our results with [?, ?]. We have been in touch with Novak, which confirmed he has a bug in the sign of α_0, or equivalently

Figure 1: Static profiles for the density n_b, the metric potentials $\bar{A}$ and $\bar{N}$, and the scalar field φ. Left panel is from Novak's paper [**?**], right panel is obtained with our code. I had to fix $n_b = 1.1$ fm^{-3} by visually inspecting that figure, because Novak doesn't say which value he is using.

Figure 2: Evolution of various quantities for Novak's profile A. Left panel is from Novak's paper, right panel is from us. Time goes from red (initial profile), through yellow, to green (lapse freezing and BH formation). Note that the first line in Novak's paper is **not** at t=0, but it's already evolved. You can check this by comparing $\phi(r = 0, t = 0)$ between his Figs. 2 and 3. Finally, we can go much closer to the horizon formation (see e.g. the final stages of the lapse α).

Figure 3: Physical quantities vs. time, for Novak's profile A. Left panel is from Novak's paper, right panel is from us. In the top-left panel, we used different extraction radii $R_{\rm ext}$ to check convergence towards the end of the grid (300 Km). We *suspect* that Novak is actually extracting at 80 Km, despite of his caption: from his Fig. 2, φ appears to be ~ 0.4 at 80 km. In any case, φ cannot be so large at 300 km because it goes as $\sim 1/r$ at large radii. We cannot efficiently reproduced the bottom-right panel (velocity of the star's surface) because the surface can only be determined with the grid precision in our finite-difference scheme.

Figure 4: Wave extraction, for Novak's profile B. The sign difference is due to Novak's bug on the sign of α_0, which is not tiny for model B. We correctly get the wave profile, but we differ in the normalization as in the previous figure. Note that we plot retarded time rather than time on the x axis, to test our extraction procedure. The extracted wave is rather independent of $R_{\rm ext}$.

Figure 5: Transition from the unstable GR-like solution to the scalarized one. This is going from Novak's 4th solution to the 6th in his table. Profiles

Figure 6: Transition from the unstable GR-like solution to the scalarized one. This is going from Novak's 4th solution to the 6th in his table. Wave extraction

Figure 7: Convergence test, following [?].

10.2 Convergence test

Following [?], we perform a self-convergence test for a fiducial collapse in scalar-tensor theories. We try to stay as close as possible to their Fig.4. In particular, we use their very same EOS and we also generate a $n = 3$ polytope as initial profile. Here we show results for a ST theory with $\alpha_0 = 10^{-2}$, $\beta_0 = -5$ and $\phi_0 = 0$. We run three simulations with increasing resolution: $N_{\rm zone} = 3000, 6000, 12000$. We note that we can't efficiently run lower resolution in ST as they did in GR.
The self-convergence factor for a quantity q is

$$Q = \frac{q_c - q_m}{q_m - q_f} = \frac{dx_c^n - dx_m^n}{dx_m^n - dx_f^n} \tag{10.1}$$

where q_i and dx_i are the values of q and the grid spacing for the coarse/medium/fine resolution runs and n is the expected convergence order. For our runs $dx_c = 2dx_m = 4dx_f$. GR1D should show $n = 2$ (Q=4) in the pre-shock region and $n = 1$ (Q=2) in the post-shock regions. Our results are reported in Fig. 10.2. Top panels show the evolution of ρ and ϕ at different times (with time evolving from red, pre-shock, to green, post-shock). Middle panels show the convergence test analysis for both $M_{\rm grav}$ (cf. Fig.4 in [?]) and the scalar field. Solid lines shows $q_c - q_m$, dashed (dotted) lines show $Q(q_m - q_f)$ for $Q = 2$ ($Q = 4$). Solid lines should stay in between the dashed and the dotted lines of the same color. The bottom panels show the actual Q. The bad red thing on the right is ok: from the middle-right panel you can see that we are comparing zeroes.

10.3 GW sensitivity curves

Here we report the conventions we use to plot scalar monopolar GW signals and the detector sensitivity curve.
The output of a GW detector $s(t)$ is the sum of noise and signal: $s(t) = n(t) + h(t)$. The signal $h(t)$ is related to the metric perturbation in the transverse traceless gauge $h_{\mu\nu}$ through the beam pattern functions $h = A_+h_+ + A_\times h_\times$. Let's denote with $\tilde{h}(f)$ and $\tilde{n}(f)$ the Fourier transform of $h(t)$ and $n(t)$ respectively. The (one-sided) noise power spectral density $S_n(f)$ is defined as

$$\langle \tilde{n}(f)\tilde{n}^*(f')\rangle = \frac{1}{2}\delta(f - f')S_n(f) \tag{10.2}$$

where $\langle\cdot\rangle$ denotes time average for stationary stochastic noise. The signal-to-noise ratio is defined as (see calculations in [?] where numerical factors is derived; see also [?])

$$\rho^2 = \int_0^\infty \frac{4|\tilde{h}(f)|^2}{S_n(f)} df \tag{10.3}$$

The characteristic strain for noise and signal are defined as

$$h_n(f) = \sqrt{fSn(f)} \tag{10.4}$$

$$h_c(f) = 2f|\tilde{h}(f)| \tag{10.5}$$

such that ρ^2 can be manifestly written as the squared ratio between signal and noise:

$$\rho^2 = \int_{-\infty}^{+\infty} \left[\frac{h_c(f)}{h_n(f)}\right]^2 d\log f \tag{10.6}$$

When plotting h_c and h_n on a log-log scale, the area between the source and detector curves is related to the SNR. The more common convention used for detector sensitivity curves is the square root of the power-spectral density:

$$\sqrt{S_n(f)} = \frac{h_n(f)}{\sqrt{f}}\,. \tag{10.7}$$

By analogy, one defines

$$\sqrt{S_h(f)} = \frac{h_c(f)}{\sqrt{f}} = 2\sqrt{f}|\tilde{h}(f)|\,. \tag{10.8}$$

GW plots typically compare $\sqrt{S_n(f)}$ and $\sqrt{S_h(f)}$. With some confusion, $\sqrt{S_n(f)}$ and $\sqrt{S_h(f)}$ are both called *GW strain*; sometimes the label n is replaced with h even for the detector. In this kind of plots, the area between two curves is not related to the SNR in a simple way. Note that $h_n(f)$ and $h_c(f)$ are dimensionless, while $\sqrt{S_n(f)}$ and $\sqrt{S_h(f)}$ have dimension $\mathrm{Hz}^{-1/2}$. The LIGO collaboration provides the expected $\sqrt{S_n(f)}$ for Advanced LIGO: datafiles are available at [**?**]; we will use their `Zero Det, High Power` configuration.
In Ref. [**?**], Damour and Esposito-Farese shows that the analogous expression to $h(t)$ for a monopolar scalar wave is

$$h(t) = \frac{2}{D}\alpha_0 r(\varphi - \varphi_0) \tag{10.9}$$

where D is the distance between the detector and the source. See Eq.(5.6) in Ref. [**?**], also Eq. (3.4) in [**?**] and Eq. (4.2) in [**?**]. Note that a (typically small) factor α_0 is present; this is due to the coupling between the scalar field and the matter of which the detector is built. Note that Eq. (3.6) in Novak's Ref. [**?**] misses a factor 2 when compared to our Eq. (10.8).
At a given extraction radius $r_{\rm ext}$, we extract $\varphi(t)$ from our simulations and we compute $h(t)$ from Eq. (10.9); we then obtain $\tilde{h}(f)$ using a FFT algorithm and finally compute $\sqrt{S_h(f)}$ from Eq.(10.8). This is compared with $\sqrt{S_n(f)}$, as provided by the LIGO collaboration.

11 Multi-scalar fields

11.1 General formalism

We now consider the case of gravity mediated by a tensorial plus several scalar fields, *tensor-multi-scalar theories of gravity*. We will eventually focus on the case of 2 scalar fields, but present as much of the formalism as possible for n fields. The general formalism is presented in Sec. 2 of [**?**].
We use here the Einstein frame and start with the action

$$S = \frac{c^4}{4\pi\bar{G}} \int \frac{dx^4}{c} \sqrt{-\bar{g}} \left[\frac{\bar{R}}{4} - \frac{1}{2} \bar{g}^{\mu\nu} \gamma_{ab} (\partial_\mu \varphi^a)(\partial_\nu \varphi^b) - W(\varphi^a) \right] + S_m[\psi_m, a^2(\varphi^a)\bar{g}_{\mu\nu}] \,. \tag{11.1}$$

Here the scalar fields span their own manifold which is sometimes referred to as the *target space* and γ_{ab} denotes its metric. In this space, the standard formalism of differential geometry applies and we have in particular the inverse target metric and Christoffel symbols

$$\gamma^{ac}\gamma_{cb} = \delta^a{}_b \,, \tag{11.2}$$

$$\gamma^a_{bc} = \frac{1}{2} \gamma^{ad} \left(\partial_b \gamma_{cd} + \partial_c \gamma_{db} - \partial_d \gamma_{bc} \right) \,. \tag{11.3}$$

We shall discuss below a particular example for the target metric.
The Jordan-Fierz or physical metric is related to the Einstein metric by

$$g_{\mu\nu} = a^2(\varphi)\bar{g}_{\mu\nu} \,, \tag{11.4}$$

where we use the notation of Sec. 3. As an example for the matter sources, we may consider a minimally coupled scalar field (not to be confused with those mediating the gravitational interaction) Ψ where

$$S_m = -\frac{1}{2} \int \frac{d^4x}{c} \sqrt{-g} \left[g^{\mu\nu} (\partial_\mu \Psi)(\partial_\nu \Psi) + m^2 \Psi^2 \right] \,. \tag{11.5}$$

Note that this action contains the physical metric $g_{\mu\nu}$, not the conformal Einstein metric $\bar{g}_{\mu\nu}$.
For convenience, we define the following auxiliary variables

$$\bar{T}^{\mu\nu} \equiv \frac{2c}{\sqrt{-\bar{g}}} \frac{\delta S_m[\psi_m, a^2\bar{g}_{\mu\nu}]}{\delta \bar{g}_{\mu\nu}} \tag{11.6}$$

$$\bar{\Phi}_{\mu\nu} \equiv \gamma_{ab} \left[(\partial_\mu \varphi^a)(\partial_\nu \varphi^b) - \frac{1}{2} \bar{g}_{\mu\nu} \bar{g}^{\rho\sigma} (\partial_\rho \varphi^a)(\partial_\sigma \varphi^b) \right] - W(\varphi^a)\bar{g}_{\mu\nu} \tag{11.7}$$

$$\alpha_a(\varphi^a) \equiv \frac{\partial \ln a}{\partial \varphi^a} = \frac{1}{a} \frac{\partial a}{\partial \varphi^a} \tag{11.8}$$

$$W_a(\varphi^a) \equiv \frac{\partial W}{\partial \varphi^a} \tag{11.9}$$

$$\Phi^a \equiv \bar{\Box}\varphi^a + \bar{g}^{\mu\nu} \gamma^a_{bc} (\partial_\mu \varphi^b)(\partial_\nu \varphi^c) - W^a \tag{11.10}$$

$$\bar{G}_{\mu\nu} \equiv \bar{R}_{\mu\nu} - \frac{1}{2} \bar{R} \bar{g}_{\mu\nu} \,. \tag{11.11}$$

Note that we move indices here by

$$\bar{T}_{\mu\nu} = \bar{g}_{\mu\alpha}\bar{g}_{\nu\beta}\bar{T}^{\alpha\beta} \tag{11.12}$$

$$W^a = \gamma^{ab}W_b\,, \tag{11.13}$$

etc. Variation of the action gives the equations of motion [cf. Eqs.(2.8)-(2.10) in [?]]

$$\bar{G}^{\mu\nu} = 2\bar{\Phi}^{\mu\nu} + \frac{8\pi\bar{G}}{c^4}\bar{T}^{\mu\nu}\,, \tag{11.14}$$

$$\Phi^a = -\frac{4\pi\bar{G}}{c^4}\alpha^a\bar{T}\,, \tag{11.15}$$

$$\frac{\delta S_m[\psi_m, a^2\bar{g}_{\mu\nu}]}{\delta\psi_m} = 0\,. \tag{11.16}$$

These equations can be written in the following form which we shall use for our further calculations

$$\bar{R}_{\mu\nu} = 2\gamma_{ab}(\partial_\mu\varphi^a)(\partial_\nu\varphi^b) + 2W(\varphi^a)\bar{g}_{\mu\nu} + \frac{8\pi\bar{G}}{c^4}\left(\bar{T}_{\mu\nu} - \frac{1}{2}\bar{T}\bar{g}_{\mu\nu}\right)\,, \tag{11.17}$$

$$\bar{\Box}\varphi^a = -\gamma^a_{bc}\bar{g}^{\mu\nu}(\partial_\mu\varphi^a)(\partial_\nu\varphi^b) - \frac{4\pi\bar{G}}{c^4}\gamma^{ab}\frac{1}{a}\frac{\partial a}{\partial\varphi^b}\bar{T} + W^a\,, \tag{11.18}$$

$$\bar{\nabla}_\nu\bar{T}^{\mu\nu} = \frac{1}{a}\frac{\partial a}{\partial\varphi^a}\bar{T}\,\bar{\nabla}^\mu\varphi^a\,. \tag{11.19}$$

Note that the physical energy-momentum tensor $T_{\mu\nu}$ is defined and related to the conformal $\bar{T}_{\mu\nu}$ by

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_m[\psi_m, g_{\mu\nu}]}{\delta g_{\mu\nu}}\,, \tag{11.20}$$

$$\bar{T}^{\mu\nu} = a^6 T^{\mu\nu}\,, \tag{11.21}$$

$$\sqrt{-\bar{g}}\bar{T}_\mu{}^\nu = \sqrt{-g}T_\mu{}^\nu\,, \tag{11.22}$$

where the latter equality follows from the variation of the matter action S_m

$$\delta_{\bar{g}}S_m = \frac{1}{2}\int\frac{d^4x}{c}\sqrt{-\bar{g}}\bar{T}^{\mu\nu}\delta\bar{g}_{\mu\nu} = \frac{1}{2}\int\frac{d^4x}{c}\sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu}\,, \tag{11.23}$$

and $\delta g_{\mu\nu} = a^2\,\delta\bar{g}_{\mu\nu}$; note that the indices are moved up and down in Eq. (11.22) with $\bar{g}_{\mu\nu}$ on the left-hand side and with $g_{\mu\nu}$ on the right-hand side.

11.2 The dynamic equations (version 1)

Here we derive the equations using the same variables as in Sec. 3, i.e. the notation also used in Novak's papers [?, ?, ?]. We will later translate them into the notation used in our core collapse code. The starting point is the Einstein-frame metric

$$d\bar{s}^2 = \bar{g}_{\alpha\beta}dx^\alpha dx^\beta = -\bar{N}^2dt^2 + \bar{A}^2dr^2 + r^2d\Omega\,, \tag{11.24}$$

which employs radial gauge in the *Einstein frame*.
The Einstein field, scalar field and matter equations are then obtained from Eqs. (11.17)-(11.19). Taking appropriate linear combinations of all these equations (done in practice in MAPLE using GRTENSOR), we obtain the following generalizations of Eqs. (3.10)-(3.25).
Auxialiary variables are defined as

$$\bar{m} = \frac{r}{2}\left(1 - \frac{1}{\bar{A}^2}\right), \tag{11.25}$$

$$\bar{\nu} = \ln \bar{N}. \tag{11.26}$$

$$\bar{\eta}^a = \frac{1}{\bar{A}}\partial_r \varphi^a, \tag{11.27}$$

$$\bar{\psi}^a = \frac{1}{\bar{N}}\partial_t \varphi^a, \tag{11.28}$$

$$\bar{\Xi} = \gamma_{ab}(\bar{\eta}^a\bar{\eta}^b + \bar{\psi}^a\bar{\psi}^b) = \gamma_{ab}\left[\frac{1}{\bar{A}^2}(\partial_r\varphi^a)(\partial_r\varphi^b) + \frac{1}{\bar{N}^2}(\partial_t\varphi^a)(\partial_t\varphi^b)\right], \tag{11.29}$$

and describe the matter using additional auxiliary variables

$$E = \frac{e+p}{1-v^2} - p = \frac{\rho h}{1-v^2} - P, \tag{11.30}$$

$$\bar{D} = \bar{A}\frac{n_{\mathrm{B}} a^4}{\sqrt{1-v^2}} = \bar{A}\frac{\rho a^4}{\sqrt{1-v^2}}, \tag{11.31}$$

$$\bar{\mu} = a^4(E+p)v = a^4\frac{\rho h}{1-v^2}v, \tag{11.32}$$

$$\bar{\tau} = a^4 E - \bar{D} = a^4\left(\frac{\rho h}{1-v^2} - P\right) - \bar{D}\,, \tag{11.33}$$

where we also included their expression in terms of the primitive matter vairables used in GR1D. The Einstein's field equations result in

$$\partial_r\bar{\nu} = 4\pi\bar{A}^2\left\{\frac{\bar{m}}{4\pi r^2} + a^4 r\left[p + (E+p)v^2\right] + \frac{r\bar{\Xi}}{8\pi}\right\}, \tag{11.34}$$

$$\partial_r\bar{m} = \frac{r^2}{2}\bar{\Xi} + 4\pi r^2 a^4 E, \tag{11.35}$$

$$\partial_t\bar{m} = r^2\frac{\bar{N}}{\bar{A}}\left[\gamma_{ab}\bar{\psi}^a\bar{\eta}^b - 4\pi a^4(E+p)v\right]. \tag{11.36}$$

The evolution equations for the scalar fields can be written in the form

$$\begin{aligned}\partial_t{}^2\varphi^a = {} & \frac{\bar{N}^2}{\bar{A}^2}\left[\partial_r\partial_r\varphi^a + \frac{2}{r}\partial_r\varphi^a + \left(\frac{\partial_r\bar{N}}{\bar{N}} - \frac{\partial_r\bar{A}}{\bar{A}}\right)\partial_r\varphi^a\right] + \left(\frac{\partial_t\bar{N}}{\bar{N}} - \frac{\partial_t\bar{A}}{\bar{A}}\right)\partial_t\varphi^a \\ & -4\pi\gamma^{ab}\frac{\partial_b a}{a}a^4\bar{N}^2\left[E - 3p - (E+p)v^2\right] - \bar{N}^2\gamma^a_{bc}(\bar{\psi}^b\bar{\psi}^c - \bar{\eta}^b\bar{\eta}^c)\,,\end{aligned} \tag{11.37}$$

where $\partial_b a \equiv \partial a/\partial\varphi^b$. Baryon conservation leads to

$$\partial_t\bar{D} + \frac{a}{r^2}\partial_r\left(\frac{r^2}{a}\frac{\bar{N}}{\bar{A}}\bar{D}v\right) = \bar{D}\frac{\partial_b a}{a}\partial_t\varphi^b\,. \tag{11.38}$$

Finally, conservation of energy momentum gives

$$\begin{aligned}
\partial_t\bar{\mu} + \frac{1}{r^2}\partial_r\left[(\bar{\mu}v + a^4p)r^2\frac{\bar{N}}{\bar{A}}\right] &= \bar{N}\bar{A}(\bar{\mu}v - \bar{\tau} - \bar{D})\left(8\pi r a^4 p + \frac{\bar{m}}{r^2} + \frac{\partial_b a}{a}\frac{\bar{\eta}^b}{\bar{A}}\right) \\
&\quad + \bar{N}\bar{A}a^4p\frac{\bar{m}}{r^2} + 2\frac{\bar{N}}{\bar{A}}a^4\frac{p}{r} - 2r\bar{\mu}\gamma_{ab}\partial_r\varphi^a\partial_t\varphi^b + 3\bar{N}a^4p\frac{\partial_b a}{a}\bar{\eta}^b \\
&\quad - \frac{r}{2}\bar{N}\bar{A}\bar{\Xi}(\bar{\tau} + \bar{D} + a^4p)(1+v^2), \qquad (11.39) \\
\partial_t\bar{\tau} + \frac{1}{r^2}\partial_r\left[r^2\frac{\bar{N}}{\bar{A}}(\bar{\mu} - \bar{D}v)\right] &= -(\bar{\tau} + \bar{D} + a^4p)\bar{N}\bar{A}r\left[(1+v^2)\gamma_{ab}\bar{\eta}^a\bar{\psi}^b + v\bar{\Xi}\right] \\
&\quad - \bar{N}\frac{\partial_b a}{a}\left[\bar{D}v\bar{\eta}^b + (\bar{\mu}v - \bar{\tau} + 3a^4P)\bar{\psi}^b)\right]. \qquad (11.40)
\end{aligned}$$

11.3 The dynamic equations (version 2)

We now rewrite the equations of the previous subsection in the form of Sec. 5 which is also the version used in the code. The line element we are using now is in the Jordan frame and given by

$$ds^2 = -\alpha^2 dt^2 + X^2 dr^2 + \frac{r^2}{F}d\Omega^2\,, \qquad (11.41)$$

where $F \equiv 1/a^2$ plays the role of the conformal factor. The relations between the Einstein and the physical metric variables are

$$\begin{aligned}
\alpha &= \frac{\bar{N}}{\sqrt{F}} & & \bar{N} = \sqrt{F}\alpha \\
& & \Leftrightarrow & \\
X &= \frac{\bar{A}}{\sqrt{F}} & & \bar{A} = \sqrt{F}X
\end{aligned} \qquad (11.42)$$

We will also use the metric variables already employed in the Einstein frame which are defined by

$$\bar{m} = \frac{r}{2}\left(1 - \frac{1}{\bar{A}^2}\right) = \frac{r}{2}\left(1 - \frac{1}{FX^2}\right)\,,$$

$$\Leftrightarrow \quad \bar{A}^2 = \left(1 - \frac{2\bar{m}}{r}\right)^{-1}\,, \qquad \frac{1}{FX^2} = 1 - \frac{2\bar{m}}{r}\,, \qquad X = \frac{1}{\sqrt{1 - \frac{2\bar{m}}{r}}}\frac{1}{\sqrt{F}}\,, \qquad (11.43)$$

$$\bar{\Phi} = \ln\bar{N} = \ln\left(\sqrt{F}\alpha\right). \qquad (11.44)$$

We describe the scalar fields φ^a in terms of the auxiliary variables

$$\eta^a = \frac{\partial_r\varphi^a}{X} = \sqrt{F}\frac{\partial\varphi^a}{\bar{A}} = \sqrt{F}\bar{\eta}^a\,, \qquad (11.45)$$

$$\psi^a = \frac{\partial_t\varphi^a}{\alpha} = \sqrt{F}\frac{\partial_t\varphi^a}{\bar{N}} = \sqrt{F}\bar{\psi}^a\,, \qquad (11.46)$$

$$\Xi = \gamma_{ab}(\eta^a\eta^b + \psi^a\psi^b) = F\bar{\Xi}\,. \qquad (11.47)$$

Finally, we have the matter variables

$$\bar{D} = \frac{\rho X}{F\sqrt{F}\sqrt{1-v^2}}\,, \tag{11.48}$$

$$\bar{S}^r = \frac{\rho h v}{F^2(1-v^2)} = \bar{\mu}\,, \tag{11.49}$$

$$\bar{\tau} = \frac{\rho h}{F^2(1-v^2)} - \frac{P}{F^2} - \bar{D}\,. \tag{11.50}$$

With that, the metric equations become

$$\partial_r \bar{\Phi} = X^2 F\left[\frac{\bar{m}}{r^2} + 4\pi r\left(\bar{S}^r v + \frac{P}{F^2}\right) + \frac{r}{2F}\Xi\right], \tag{11.51}$$

$$\partial_r \bar{m} = 4\pi r^2(\bar{\tau}+\bar{D}) + \frac{r^2}{2F}\Xi, \tag{11.52}$$

$$\partial_t \bar{m} = r^2\frac{\alpha}{X}\left(\gamma_{ab}\frac{\eta^a\psi^b}{F} - 4\pi\bar{S}^r\right)\,. \tag{11.53}$$

It will be convenient to also write the metric equations in terms of the variables α and X so that we can use them to replace derivatives of these functions. We obtain

$$\frac{\partial_r\alpha}{\alpha} = X^2 F\left[\frac{\bar{m}}{r^2} + 4\pi r\left(\bar{S}^r v + \frac{P}{F^2}\right) + \frac{r}{2F}\Xi\right] - \frac{\partial_b F}{2F}X\eta^b\,, \tag{11.54}$$

$$\frac{\partial_r X}{X} = 4\pi r F X^2(\bar{\tau}+\bar{D}) + \frac{rX^2}{2}\Xi - FX^2\frac{\bar{m}}{r^2} - \frac{\partial_b F}{2F}X\eta^b\,, \tag{11.55}$$

$$\frac{\partial_t X}{X} = rX\alpha(\gamma_{ab}\eta^a\psi^b - 4\pi F\bar{S}^r) - \frac{\partial_b F}{2F}\alpha\psi^b\,. \tag{11.56}$$

The wave equations for the scalar fields are

$$\begin{aligned}\partial_t\partial_t\varphi^a = {} & \frac{\alpha^2}{X^2}\left[\partial_r\partial_r\varphi^a + \frac{2}{r}\partial_r\varphi^a + \left(\frac{\partial_r\alpha}{\alpha} - \frac{\partial_r X}{X}\right)X\eta^a\right] + \left(\frac{\partial_t\alpha}{\alpha} - \frac{\partial_t X}{X}\right)\alpha\psi^a \\ & + 2\pi\alpha^2\left(\bar{\tau} - \bar{S}^r v + \bar{D} - 3\frac{P}{F^2}\right)\gamma^{ab}\partial_b F - \alpha^2\gamma^a_{bc}(\psi^b\psi^c - \eta^b\eta^c)\,.\end{aligned} \tag{11.57}$$

We write this equation as a first-order system using the definitions (11.45)-(11.47) and the identity $\partial_t\partial_r\eta^a = \partial_r\partial_t\eta^a$ to obtain

$$\partial_t\varphi^a = \alpha\,\psi^a\,, \tag{11.58}$$

$$\partial_t\eta^a = -\eta^a\frac{\partial_t X}{X} + \frac{\alpha}{X}\left(\partial_r\psi^a + \psi^a\frac{\partial_r\alpha}{\alpha}\right)\,, \tag{11.59}$$

$$\begin{aligned}\partial_t\psi^a = {} & \frac{\alpha}{X}\left[\partial_r\eta^a + \frac{2}{r}\eta^a + \eta^a\frac{\partial_r\alpha}{\alpha}\right] - \psi^a\frac{\partial_t X}{X} + 2\pi\alpha\left(\bar{\tau} - \bar{S}^r v + \bar{D} - 3\frac{P}{F^2}\right)\gamma^{ab}\partial_b F \\ & - \alpha\gamma^a_{bc}(\psi^b\psi^c - \eta^b\eta^c)\,.\end{aligned} \tag{11.60}$$

For numerical purposes, it turns out to be more convenient to apply the finite differencing to a version of these equations where we gather all radial derivatives in a flux term. This version is given by

$$\partial_t \varphi^a = \alpha\psi^a\,, \tag{11.61}$$

$$\partial_t \eta^a = -\eta^a \frac{\partial_t X}{X} + \frac{1}{X}\partial_r\left(\alpha\psi^a\right), \tag{11.62}$$

$$\partial_t \psi^a = \frac{1}{r^2 X}\partial_r\left(\alpha r^2 \eta^a\right) - \psi^a\frac{\partial_t X}{X} + 2\pi\alpha\left(\bar{\tau} - \bar{S}^r v + \bar{D} - 3\frac{P}{F^2}\right)\gamma^{ab}\partial_b F - \alpha\gamma^a_{bc}(\psi^b\psi^c - \eta^b\eta^c)\,. \tag{11.63}$$

By finite differencing the compound terms in parentheses following the ∂_r operator instead of summing up the individual derivatives as in Eqs. (5.16), (5.17), we reduce numerical noise near the origin.
The scalar field should satisfy an outgoing boundary condition at spatial infinity:

$$\varphi^a(t,r) \quad \overset{r\to\infty}{\longrightarrow} \quad \varphi^a_o + \frac{f^a(t-r)}{r} + \mathcal{O}(r^{-2})\,. \tag{11.64}$$

This, taking into account the asymptotic behaviour of the different quantities near spatial infinity, can be translated into the following differential condition:

$$\partial_t\varphi^a + \partial_r\varphi^a + \frac{\varphi^a - \varphi^a_o}{r} = 0\,. \tag{11.65}$$

In the case we want to use the first-order formulation for the scalar field wave equation, the outgoing boundary conditions for the variables $(\varphi^a, \psi^a, \eta^a)$ are:

$$\partial_t\psi^a + \partial_r\psi^a + \frac{\psi^a}{r} = 0\,, \tag{11.66}$$

$$\partial_t\eta^a + \partial_r\eta^a + \frac{\eta^a}{r} - \frac{\varphi^a - \varphi^a_o}{r^2} = 0\,, \tag{11.67}$$

and equation (11.58) for φ.
Finally, the matter evolution is determined by

$$\partial_t\bar{D} + \frac{1}{\sqrt{F}r^2}\partial_r\left(r^2\frac{\alpha}{X}\sqrt{F}f_{\bar{D}}\right) = s_{\bar{D}}, \tag{11.68}$$

$$\partial_t\bar{S}^r + \frac{1}{r^2}\partial_r\left(r^2\frac{\alpha}{X}f_{\bar{S}^r}\right) = s_{\bar{S}^r}, \tag{11.69}$$

$$\partial_t\bar{\tau} + \frac{1}{r^2}\partial_r\left(r^2\frac{\alpha}{X}f_{\bar{\tau}}\right) = s_{\bar{\tau}}, \tag{11.70}$$

where

$$
\begin{aligned}
f_{\bar{D}} &= \bar{D}v, && (11.71)\\
f_{\bar{S}^r} &= \bar{S}^r v + \frac{P}{F^2}, && (11.72)\\
f_{\bar{\tau}} &= \bar{S}^r - \bar{D}v, && (11.73)\\
s_{\bar{D}} &= -\bar{D}\frac{\partial_b F}{2F}\alpha\psi^b, && (11.74)\\
s_{\bar{S}^r} &= (\bar{S}^r v - \bar{\tau} - \bar{D})\alpha X F\left(8\pi r\frac{P}{F^2} + \frac{\bar{m}}{r^2} - \frac{\partial_b F}{2F^2 X}\eta^b\right) + \frac{\alpha X}{F}P\frac{\bar{m}}{r^2} + 2\frac{\alpha P}{rXF^2} \\
&\quad -2r\alpha X\bar{S}^r\gamma_{ab}\eta^a\psi^b - \frac{3}{2}\alpha\frac{P}{F^2}\frac{\partial_b F}{F}\eta^b - \frac{r}{2}\alpha X\Xi\left(\bar{\tau} + \frac{P}{F^2} + \bar{D}\right)(1+v^2), && (11.75)\\
s_{\bar{\tau}} &= -\left(\bar{\tau} + \frac{P}{F^2} + \bar{D}\right) r\alpha X\left[(1+v^2)\gamma_{ab}\eta^a\psi^b + v\Xi\right] + \frac{\alpha}{2}\frac{\partial_b F}{F}\left[\bar{D}v\eta^b + \left(\bar{S}^r v - \bar{\tau} + 3\frac{P}{F^2}\right)\psi^b\right] && (11.76)
\end{aligned}
$$

11.4 The static limit

We now consider the static limit of Eqs. (11.51)-(11.56), (11.58)-(11.60), (11.68)-(11.70). For this purpose, we first note that forstatic stars

$$v = 0 = \bar{S}^r\,. \tag{11.77}$$

Our TOV integrator furthermore uses a variable ϵ for the internal energy of the matter which is related to our variables by

$$\epsilon = h - \frac{P}{\rho} - 1 \qquad \Leftrightarrow \qquad h = \epsilon + \frac{P}{\rho} + 1\,, \tag{11.78}$$

and implies that

$$\bar{\tau} = \frac{1}{F^2}(\rho + \rho\epsilon) - \bar{D} \qquad \Rightarrow \qquad \bar{\tau} + \bar{D} + \frac{P}{F^2} = \frac{\rho h}{F^2}\,. \tag{11.79}$$

The multi-scalar version of the static Eqs. (6.15)-(6.21) then becomes with $\Xi = \gamma_{bc}\eta^b\eta^c$

$$
\begin{aligned}
\partial_r\Phi &= FX^2\left(\frac{\bar{m}}{r^2} + 4\pi r\frac{P}{F^2} + \frac{r}{2F}\Xi\right), && (11.80)\\
\frac{\partial_r\alpha}{\alpha} &= FX^2\left(\frac{\bar{m}}{r^2} + 4\pi r\frac{P}{F^2} + \frac{r}{2F}\Xi\right) - \frac{\partial_b F}{2F}X\eta^b, && (11.81)\\
\partial_r\bar{m} &= 4\pi r^2\frac{\rho h - P}{F^2} + \frac{r^2}{2F}\Xi, && (11.82)\\
\frac{\partial_r X}{X} &= 4\pi rFX^2\frac{\rho h - P}{F^2} + \frac{r}{2}X^2\Xi - FX^2\frac{\bar{m}}{r^2} - \frac{\partial_b F}{2F}X\eta^b, && (11.83)\\
\partial_r P &= -\rho hFX^2\left(\frac{\bar{m}}{r^2} + 4\pi r\frac{P}{F^2} + \frac{r}{2F}\Xi\right) + \rho h\frac{\partial_b F}{2F}X\eta^b, && (11.84)\\
\partial_r\varphi^a &= X\eta^a, && (11.85)\\
\partial_r\eta^a &= -2\frac{\eta^a}{r} - 2\pi X\frac{\rho h - 4P}{F^2}\gamma^{ab}\partial_b F - F\eta^a X^2\frac{\bar{m}}{r^2} - 4\pi rX^2\eta^a\frac{P}{F} - \frac{r}{2}X^2\eta^a\Xi + \frac{X}{2}\frac{\partial_b F}{F}\eta^b\eta^a - X\gamma^a_{bc}\eta^b\eta^c\,. && (11.86)
\end{aligned}
$$

We furthermore rewrite Eq. (6.34) for the baryon mass in our variables in the form

$$\partial_r m_{\rm B} = 4\pi r^2 \rho \frac{X}{F}\,. \tag{11.87}$$

In order to impose outer boundary conditions on the scalar field, we may have to integrate in the vacuum exterior to large radii. This is done using the vacuum version of Eqs. (11.80)-(11.86) which is obtained by setting $\rho = 0$, $P = 0$ and gives

$$\begin{aligned}
\partial_r \Phi &= F X^2 \left(\frac{\bar{m}}{r^2} + \frac{r}{2F}\Xi\right), && (11.88)\\
\frac{\partial_r \alpha}{\alpha} &= F X^2 \left(\frac{\bar{m}}{r^2} + \frac{r}{2F}\Xi\right) - \frac{\partial_b F}{2F} X \eta^b, && (11.89)\\
\partial_r \bar{m} &= \frac{r^2}{2F}\Xi, && (11.90)\\
\frac{\partial_r X}{X} &= \frac{r}{2} X^2 \Xi - F X^2 \frac{\bar{m}}{r^2} - \frac{\partial_b F}{2F} X \eta^b, && (11.91)\\
\partial_r P &= 0, && (11.92)\\
\partial_r \varphi^a &= X \eta^a, && (11.93)\\
\partial_r \eta^a &= -2\frac{\eta^a}{r} - F \eta^a X^2 \frac{\bar{m}}{r^2} - \frac{r}{2} X^2 \eta^a \gamma_{bc} \eta^b \eta^c + \frac{X}{2}\frac{\partial_b F}{F} \eta^b \eta^a - X \gamma^a_{bc} \eta^b \eta^c\,. && (11.94)
\end{aligned}$$

Finally, we would like to integrate the equations towards infinity which requires compactification of the radial domain. We achieve that by introducing a radial coordinate

$$y = \frac{1}{r} \quad \Rightarrow \quad \frac{\partial}{\partial r} = -y^2 \frac{\partial}{\partial y}\,. \tag{11.95}$$

Furthermore, we change to a variable $\tilde{\eta}^a$ describing the radial derivative of the scalar fields which are given by

$$\tilde{\eta}^a = \frac{\partial_y \phi^a}{X} = -\frac{\eta^a}{y^2} \quad \Rightarrow \quad \eta^a = -y^2 \tilde{\eta}^a\,. \tag{11.96}$$

With these choices, all singular terms at $y = 0$ in the compactified equations can be regularized and the system (11.88)-(11.94) becomes

$$\partial_y \Phi = -FX^2 \left(\bar{m} + \frac{y}{2F} \gamma_{ab} \tilde{\eta}^a \tilde{\eta}^b \right) , \tag{11.97}$$

$$\frac{\partial_y \alpha}{\alpha} = -FX^2 \left(\bar{m} + \frac{y}{2F} \gamma_{ab} \tilde{\eta}^a \tilde{\eta}^b \right) - \frac{\partial_b F}{2F} X \tilde{\eta}^b , \tag{11.98}$$

$$\partial_y \bar{m} = -\frac{1}{2F} \gamma_{ab} \tilde{\eta}^a \tilde{\eta}^b , \tag{11.99}$$

$$X = \frac{1}{\sqrt{F}\sqrt{1 - 2\bar{m}y}} , \tag{11.100}$$

$$\frac{\partial_y X}{X} = -\frac{y}{2} X^2 \gamma_{ab} \tilde{\eta}^a \tilde{\eta}^b + FX^2 \bar{m} - \frac{\partial_b F}{2F} X \tilde{\eta}^b , \tag{11.101}$$

$$\partial_y \varphi^a = X \tilde{\eta}^a , \tag{11.102}$$

$$\partial_y \tilde{\eta}^a = F \tilde{\eta}^a X^2 \bar{m} + \frac{y}{2} X^2 \tilde{\eta}^a \gamma_{bc} \tilde{\eta}^b \tilde{\eta}^c + \frac{X}{2} \frac{\partial_b F}{F} \tilde{\eta}^b \tilde{\eta}^a - X \gamma^a_{bc} \tilde{\eta}^b \tilde{\eta}^c . \tag{11.103}$$

11.5 Diagnostics

Equation (6.42) gave us the gravitational mass for a single scalar field. Following a private communication by Michael Horbatsch, this generalizes to multiple scalar field by replacing η^2 with $\gamma_{ab}\eta^a\eta^b$, i.e.

$$m_{\rm grav} = r^2 \partial_r \bar{\Phi}_s \sqrt{1 - \frac{2m}{r}} \exp\left\{ -\frac{\partial_r \bar{\Phi}_s}{\sqrt{(\partial_r \bar{\Phi}_s)^2 + X^2 \gamma_{ab} \eta^a \eta^b}} \operatorname{arctanh}\left[\frac{\sqrt{(\partial_r \bar{\Phi}_s)^2 + X^2 \gamma_{ab} \eta^a \eta^b}}{\partial_r \bar{\Phi}_s + 1/r} \right] \right\} . \tag{11.104}$$

12 A biscalar example

Emanuele and Hector consider the specific example of an action where in Eq. (11.1) two scalar fields φ^a, $a = 1, 2$ are combined into one complex field φ such that

$$\bar{g}^{\mu\nu} \gamma_{ab} (\partial_\mu \varphi^a)(\partial_\nu \varphi^b) = \left[1 + \frac{\varphi^* \varphi}{4\mathcal{R}^2} \right]^{-2} \bar{g}^{\mu\nu} (\partial_\mu \varphi^*)(\partial_\nu \varphi) , \tag{12.1}$$

where the asterisk denotes the complex conjugate. Writing

$$\varphi = \varphi^1 + i\varphi^2 , \tag{12.2}$$

a straightoforward calculation shows that Eq. (12.1) implies

$$\gamma_{ab} = \begin{pmatrix} \gamma_{11} & 0 \\ 0 & \gamma_{11} \end{pmatrix} , \qquad \gamma_{11} = \left[1 + \frac{(\varphi^1)^2 + (\varphi^2)^2}{4\mathcal{R}^2} \right]^{-2} . \tag{12.3}$$

Regarding γ_{ab} as a metric in the target space, we can calculate its Ricci scalar and obtain $R = 2/\mathcal{R}^2$ in agreement with Emanuele's notes.
We can now evaluate the individual scalar field terms appearing in our evolution equations for this special case and obtain

$$\gamma_{ab} = \begin{pmatrix} \gamma_{11} & 0 \\ 0 & \gamma_{11} \end{pmatrix} \quad \Leftrightarrow \quad \gamma^{ab} = \begin{pmatrix} \frac{1}{\gamma_{11}} & 0 \\ 0 & \frac{1}{\gamma_{11}} \end{pmatrix}, \tag{12.4}$$

$$\gamma^1_{11} = \gamma^2_{12} = \gamma^2_{21} = -\gamma^1_{22} = \frac{1}{2}\frac{\partial_1\gamma_{11}}{\gamma_{11}} = -\frac{2\varphi^1}{4\mathcal{R}^2 + (\varphi^1)^2 + (\varphi^2)^2}, \tag{12.5}$$

$$\gamma^2_{22} = \gamma^1_{12} = \gamma^1_{21} = -\gamma^2_{11} = \frac{1}{2}\frac{\partial_2\gamma_{11}}{\gamma_{11}} = -\frac{2\varphi^2}{4\mathcal{R}^2 + (\varphi^1)^2 + (\varphi^2)^2}, \tag{12.6}$$

and

$$\gamma_{ab}\psi^a\eta^b = \gamma_{11}(\psi^1\eta^1 + \psi^2\eta^2), \tag{12.7}$$

$$\gamma^{1b}\partial_b a = \frac{\partial_1 a}{\gamma_{11}}, \tag{12.8}$$

$$\gamma^{2b}\partial_b a = \frac{\partial_2 a}{\gamma_{11}}, \tag{12.9}$$

$$\gamma^1_{bc}(\psi^b\psi^c + \eta^b\eta^c) = \gamma^1_{11}\left[(\psi^1)^2 + (\eta^1)^2 - (\psi^2)^2 - (\eta^2)^2\right] + 2\gamma^2_{22}(\psi^1\psi^2 + \eta^1\eta^2), \tag{12.10}$$

$$\gamma^2_{bc}(\psi^b\psi^c + \eta^b\eta^c) = \gamma^2_{22}\left[(\psi^2)^2 + (\eta^2)^2 - (\psi^1)^2 - (\eta^1)^2\right] + 2\gamma^1_{11}(\psi^1\psi^2 + \eta^1\eta^2), \tag{12.11}$$

$$\partial_b a\,\psi^b = \partial_1 a\,\psi^1 + \partial_2 a\,\psi^2, \tag{12.12}$$

$$\partial_b a\,\eta^b = \partial_1 a\,\psi^1 + \partial_2 a\,\psi^2, \tag{12.13}$$

$$\Xi = \gamma_{ab}(\eta^a\eta^b + \psi^a\psi^b) = \gamma_{11}\left[(\eta^1)^2 + (\psi^1)^2 + (\eta^2)^2 + (\psi^2)^2\right], \tag{12.14}$$

$$\gamma_{ab}\eta^a\psi^b = \gamma_{11}(\eta^1\psi^1 + \eta^2\psi^2). \tag{12.15}$$

The specific coupling function we will be using is

$$F = e^{-(\beta_0+\beta_1)(\varphi^1)^2 - (\beta_0-\beta_1)(\varphi^2)^2}, \tag{12.16}$$

$$\frac{\partial_{\varphi^1}F}{F} = -2(\beta_0+\beta_1)\varphi^1, \tag{12.17}$$

$$\frac{\partial_{\varphi^2}F}{F} = -2(\beta_0-\beta_1)\varphi^2. \tag{12.18}$$