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1 The action and covariant equations

Our starting point is the action [3]

$$S = \int \left\{ \frac{1}{16\pi G} R - \frac{1}{2} \left[g^{\alpha\beta} \nabla_\alpha \bar{\varphi} \nabla_\beta \varphi + V(\varphi) \right] \right\} \sqrt{-g}\, d^4x\,, \tag{1.1}$$

where φ denotes a complext scalar field $\varphi = \varphi_\mathsf{r} + \mathrm{i}\varphi_\mathsf{i}$, g is the determinant of the spacetime metric $g_{\alpha\beta}$ and V a potential function which we leave free for now.

Variation of the action with respect to the metric and the scalar field give us (after some calculation) the Einstein-Klein-Gordon equations which we write in the form

$$\begin{aligned} G_{\alpha\beta} &= 8\pi G T_{\alpha\beta} \\ \text{with} \qquad T_{\alpha\beta} &= \frac{1}{2} \left[\partial_\alpha \bar{\varphi}\, \partial_\beta \varphi + \partial_\alpha \varphi\, \partial_\beta \bar{\varphi} - g_{\alpha\beta} \big(g^{\mu\nu} \partial_\mu \bar{\varphi}\, \partial_\nu \varphi + V(\varphi) \big) \right]\,, \end{aligned} \tag{1.2}$$

$$\nabla^\mu \nabla_\mu \varphi = \frac{\mathrm{d}V}{\mathrm{d}|\varphi|^2} \varphi =: V' \varphi =: V_{,|\varphi|^2} \varphi\,. \tag{1.3}$$

Here, the scalar potential V is chosen to be a function of the amplitude only of the scalar field. In practice, we typically use a potential of the form

$$V(\varphi) = \mu |\varphi|^2 + \lambda_4 |\varphi|^4 + \lambda_6 |\varphi|^6\,, \tag{1.4}$$

but for now we let the potential to be an arbitrary function of $|\varphi|$.

In numerical time evolutions of complex scalar fields, we typically use the real and imaginary parts,

$$\varphi = \mathsf{Re}(\varphi) + \mathrm{i}\, \mathsf{Im}(\varphi) =: \varphi_\mathsf{r} + \mathrm{i}\varphi_\mathsf{i}\,. \tag{1.5}$$

Alternatively, we can write a complex variable in terms of amplitude and phase,

$$\varphi = A e^{\mathrm{i}\phi}\,. \tag{1.6}$$

I admit that the use of the two variations of the letter "phi" may be suboptimal, but let's keep it for now. In LaTeXsyntax, $\varphi =$ `\varphi` and $\phi =$ `\phi`.

Single, stationary boson stars in spherical symmetry are solutions to the Einstein-Klein-Gordon equations where the scalar field has the form

$$\varphi(t, r) = A(r) e^{\mathrm{i}\omega t}\,, \tag{1.7}$$

where (t, r) denote the time and radial coordinates (e.g. in radial gauge and polar slicing) and ω is a constant frequency that yields a regular (i.e. finite-mass) solution only for a set of countably infinite, discrete values. Translated into Eqs. (1.5) and (1.6), we immediately see the benefits of a formulation in terms of amplitude and phase, since

$$\varphi_\mathsf{r} = A(r) \cos \omega t\,, \qquad \varphi_\mathsf{i} = A(r) \sin \omega t\,, \tag{1.8}$$

$$A = A(r)\,, \qquad \phi = \omega t\,. \tag{1.9}$$

Clearly, the latter formulation is a good deal simpler. In numerical simulations, in particular, we would greatly benefit from the trivially simple time evolution of the phase and amplitude. Our goal is to find such a formulation and a functional set of differential equations that governs the time evolution of A and ϕ. As simple as this may appear, it turns out to be a considerable challenge and I am not sure there is a solution. Anyway, let's have a go...

2 The complex wave equation in $1D$ flat space

The complete set of the Einstein-Klein-Gordon equations are highly complex and will make our investigation very difficult. On the other hand, we will likely encounter most (if not all) conceptual problems in our attempts to reformulate the complex scalar field variables in the much simpler case of the complex wave equation in one dimension and in flat, Minkowski spacetime. In that case, we simply use the Minkowski metric $g_{\alpha\beta} = \eta_{\alpha\beta}$, use Cartesian coordinates $x^\alpha = (t, x, y, z)$ and drop the y and z dependence. In that case, $\nabla_\alpha = \partial_\alpha$ and our wave equation (1.3) reduces to

$$\partial_t^2 \varphi = \partial_x^2 \varphi + V' \varphi \,. \tag{2.1}$$

If we furthermore ignore the potential, we have the simple wave equation

$$\partial_t^2 \varphi = \partial_x^2 \varphi \,, \tag{2.2}$$

with $\varphi = \varphi(t, x)$; the only difference from the text book wave equation is that φ is complex. If you prefer the $3D$ viewpoint, think of φ as a plane wave in three dimensions. But whatever viewpoint we may choose, it doesn't matter for the maths we'll be doing next. Of course, solving our problem for the simple case (2.2) does not guarantee a solution in the more complicated case of the Einstein-Klein-Gordon equations (1.2), (1.3), but it may point us in the direction and, if we are lucky, a solution for the toy problem might even directly translate into a generic method without much ado. In any case, let us cross that bridge when we come to it. The toy problem is hard enough...

2.1 Separation of variables

The standard approach to solve the wave equation (2.2) – whether complex or not – consists in the spearation of variables. Writing

$$\varphi(t, x) = \xi(x)\tau(t) \,, \tag{2.3}$$

and using the usual short-hand notation $\xi' = \partial_x \xi$, $\dot{\tau} = \partial_t \tau$, we directly obtain

$$\ddot{\tau}\xi = \xi''\tau \qquad \Rightarrow \qquad \frac{\ddot{\tau}}{\tau} = \frac{\xi''}{\xi} = \text{const} =: -\omega^2 \,. \tag{2.4}$$

For the spatial part, we obtain

$$\xi'' + \omega^2 \xi = 0 \qquad \Rightarrow \qquad \xi(x) = A \sin \omega x + B \cos \omega x = \tilde{A} e^{\mathrm{i}\omega x} + \tilde{B} e^{-\mathrm{i}\omega x} \,. \tag{2.5}$$

For now, let's keep the sin/cos version and see whereto it leads us. In order to proceed further, we need to specify boundary conditions. For a boson-star spacetime, we usually demand a vanishing scalar field at infinty, but all indications are that the location where we impose the boundary condition is of no relevance for the problem we wish to solve. So let us just choose our conditions as

$$\varphi(t, 0) = \varphi(t, 1) = 0 \,. \tag{2.6}$$

For our spatial solution (2.5), this clearly implies

$$B = 0 \qquad \wedge \qquad A \sin\omega = 0 \quad \Rightarrow \quad \omega = n\pi\,, \quad n \in \mathbb{Z}\,. \tag{2.7}$$

$n = 0$ is the trivial solution and negative n add no new solutions, so that we can restrict ourselves to $\omega = \pi,\ 2\pi,\ 3\pi, \ldots$ without loss of generality.

For the temporal part of Eq. (2.4), we likewise find

$$\ddot{\tau} + \omega^2 \tau = 0 \qquad \Rightarrow \qquad \tau(t) = C \sin\omega t + D \cos\omega t\,. \tag{2.8}$$

We now have to specify initial data. Say, we start with

$$\xi(0, x) = \sin(n\pi x)\,, \qquad \dot{\xi}(0, x) = 0\,, \tag{2.9}$$

then the complete solution is a simple standing wave

$$\varphi(t, x) = \sin(n\pi x)\ \cos(n\pi t)\,. \tag{2.10}$$

As it turns out, the standing wave solution is a key indicator for all the trouble we are facing in our attempts to reformulate the complex wave equation. Speaking of which, how does the complex analysis of Eq. (2.4) look like? In that case, we write the product Ansatz in the form, guessing $\omega = \text{const}$,

$$\varphi(t, x) = A(x) e^{\mathrm{i}\omega t}\,, \tag{2.11}$$

and the wave equation becomes

$$(\partial_t^2 - \partial_x^2)\varphi = -\omega^2 A e^{\mathrm{i}\omega t} - \partial_x^2 A e^{\mathrm{i}\omega t} \overset{!}{=} 0 \tag{2.12}$$

$$\Rightarrow \quad \partial_x^2 A = -\omega^2 A \tag{2.13}$$

$$\Rightarrow \quad A(x) = E e^{\mathrm{i}\omega x} + F e^{-\mathrm{i}\omega x}\,, \qquad E,\ F = \text{const}\,. \tag{2.14}$$

For the initial data (2.9), we would recover the standing wave (2.10). Initial data of the form

$$\xi(0, x) = \sin(n\pi x)\,, \qquad \dot{\xi}(0, x) = \mathrm{i}n\pi \sin(n\pi x)\,, \tag{2.15}$$

however, resemble more closely the situation we face in the modeling of boson stars and lead to the solution

$$\boxed{\xi(t, x) = \sin(\omega x) e^{\mathrm{i}\omega t}\,, \qquad \omega = n\pi\,,\ n \in \mathbb{N}\,.} \tag{2.16}$$

Let us therefore bear this complex standing wave solution in mind as the prototypical example that we would like to model in our toy problem.

Before proceeding with the maths, it may be helpful to visualize a bit in our minds, how this solution looks like. At every point on our one-dimensional domain, we have harmonic time dependence. Furthermore, at every point x, the amplitude is contant. So at every point, we can depict the solution in the complex plane as a point moving on a circle of radius $\sin(\omega x)$ with phase ωt. Note that the radius of this circle is zero at the nodes where $\sin(\omega x) = 0$. For $n = 1$, this only happens on the boundary, but for $n > 1$, this also happens at some points inside the domain. This is analogous to the stationary boson star models: $n = 1$ corresponds to the ground-state boson star with no zero crossings of the scalar field amplitude (except at infinity) while $n > 1$ are the excited states with $(n-1)$ zero crossings. Finally, note that at the zero crossings x_0,

$$\varphi(t, x_0) = 0 \qquad \text{but} \qquad \partial_x \varphi(t, x_0) \neq 0 \,. \tag{2.17}$$

This seemingly innocent behaviour, that is inherent to the sin function, will haunt us with a vengeance throughout a good part of these notes...

2.2 A first order formulation of the wave equation

In numerical studies one often recasts partial differential equations in a form that contains only first derivatives. Typically this is done either in time or in both space and time, but, to the best of my knowledge, not in space only. An entirely first-order form is particularly convenient for analysing the structure of the characteristics, but for the moment, we will not enter this point in more detail.

The massless wave equation (2.2) is easily converted into a purely first-order form by defining

$$F := \partial_t \varphi \,, \qquad G := \partial_x \varphi$$

$$\Rightarrow \quad \partial_t F = \partial_x G \qquad \wedge \qquad \partial_t G = \partial_x F \,, \tag{2.18}$$

or, in matrix form

$$\partial_t \begin{pmatrix} F \\ G \end{pmatrix} = \partial_x \begin{pmatrix} G \\ F \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x \begin{pmatrix} F \\ G \end{pmatrix} \,. \tag{2.19}$$

The latter expression gives us the wave equation in flux conservative form and the characteristic speeds would follow from solving for the eigenvalues of the 2×2 matrix. Of course, these Eigenvalues are ± 1, but our discussion is independent of the propagation speeds. We have introduced the first-order version of the wave equation, because it provides us with more freedom to reformulate the wave equation and will allow us to overcome some (though unfortunately not all) of the singular behaviour we encounter in an amplitude-phase formulation.

3 Amplitude and phase

If we regard the complex scalar field of a stationary boson star at a given point in space, the scalar field will simply trace out a circle in the complex plane, whereas its real and imaginary parts oscillate sinusoidally. The first, and most evident, approach to avoid this highly oscillatory behaviour of the real and imaginary parts of a complex scalar field consists in switching to amplitude and phase according to

$$\varphi(t, x) = A(t, x) e^{\mathrm{i}\phi(t,x)} \,. \tag{3.1}$$

Note that this differs from the product Ansatz (2.11) by the general dependence of both, amplitude and phase, on time and space. We insert (3.1) into $\partial_t^2 \varphi - \partial_x^2 \varphi = 0$, using (for either t or x derivatives),

$$\partial \varphi = (\partial A) e^{\mathrm{i}\phi} + \mathrm{i} A (\partial \phi)\, e^{\mathrm{i}\phi} \,, \tag{3.2}$$

$$\partial^2 \varphi = (\partial^2 A) e^{\mathrm{i}\phi} + 2\mathrm{i} (\partial A)(\partial \phi) e^{\mathrm{i}\phi} - A (\partial \phi)^2 e^{\mathrm{i}\phi} + \mathrm{i} A (\partial^2 \phi) e^{\mathrm{i}\phi} \,. \tag{3.3}$$

The real and imaginary parts of the resulting complex equation result in one wave-like equation for the amplitude A and one for the phase ϕ,

$$
\begin{aligned}
\partial_t^2 A - \partial_x^2 A &= A[(\partial_t\phi)^2 - (\partial_x\phi)^2]\,, & (3.4)\\
\partial_t^2 \phi - \partial_x^2 \phi &= 2\left(\frac{\partial_x A}{A}\partial_x\phi - \frac{\partial_t A}{A}\partial_t\phi\right). & (3.5)
\end{aligned}
$$

The ill defined nature of the phase at the origin of the complex plane manifests itself in the latter of these equations: the right-hand side becomes singular as $A \to 0$. This is not surprising; as the scalar field moves by an infinitesimal amount across $\varphi = 0$, the phase can change by a finite amount.

Interestingly, the $1/A$ terms can be eliminated by switching to a convenient first-order version of Eqs. (3.4), (3.5). For this purpose, we define

$$
\begin{aligned}
F &:= \frac{\partial_t A}{A}\,, & G &:= \frac{\partial_x A}{A}\,,\\
\psi &:= \partial_t\phi\,, & \eta &:= \partial_x\phi\,. \qquad (3.6)
\end{aligned}
$$

Using

$$
\begin{aligned}
&\partial_t F = \frac{1}{A}(\partial_t^2 A - F\partial_t A) = \frac{\partial_t^2 A}{A} - F^2\,,\\
&\partial_x G = \frac{1}{A}(\partial_x^2 A - G\partial_x A) = \frac{\partial_x^2 A}{A} - G^2\\
\Rightarrow\ &\frac{\partial_x^2 A}{A} + (\partial_t\phi)^2 - (\partial_x\phi)^2 - F^2 = \partial_x G + G^2 - F^2 + \psi^2 - \eta^2\,,\\
\wedge\ &\partial_t G = \frac{\partial_x\partial_t A}{A} - \partial_x A\frac{\partial_t A}{A^2} = \partial_x\left(\frac{\partial_t A}{A}\right) = \partial_x F\,, \qquad (3.7)
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\partial_t F &= \partial_x G + G^2 - F^2 + \psi^2 + \eta^2\,,\\
\partial_t \psi &= \partial_x \eta + 2(G\eta - F\psi)\,,\\
\partial_t G &= \partial_x F\,,\\
\partial_t \eta &= \partial_x \psi\,. \qquad (3.8)
\end{aligned}
$$

The problem with this approach arises in the construction of initial data for a standing wave. At the nodes, the initial data has $A = 0$ and $\partial_x A \neq 0$, so that $G = \partial_x A/A$ diverges. This happens, for example, for our standing wave (2.16) which corresponds to

$$A = \sin(n\pi x)\,, \qquad \phi = \omega t\,. \qquad (3.9)$$

I do not know how a numerical evolution using the system (3.8) would react if a standing wave is reached through a dynamical evolution, but somehow, the variable G must blow up. Nonetheless, Eq. (2.16) is the closest I have come to a satisfactory solution to our problem. One should be able to evolve ground state boson stars in this approach, since these do not have zero crossings except at spatial infinity.

A similar problem is encountered if we eliminate the $\frac{1}{A}$ terms in Eqs. (3.4), (3.5) by introducing a logarithmic variable

$$\boxed{\alpha = \ln A \,. \qquad (3.10)}$$

Using

$$A = e^{\alpha} \qquad \Rightarrow \qquad \partial A = e^{\alpha}\partial\alpha \qquad \Rightarrow \qquad \partial^2 A = e^{\alpha}\partial^2\alpha + (\partial\alpha)^2 e^{\alpha}\,, \tag{3.11}$$

we obtain

$$\boxed{\begin{aligned} \partial_t^2\alpha - \partial_x^2\alpha &= (\partial_x\alpha)^2 - (\partial_t\alpha)^2 + (\partial_t\phi)^2 - (\partial_x\phi)^2 \\ \partial_t^2\phi - \partial_x^2\phi &= 2(\partial_x\alpha\,\partial_x\phi - \partial_t\alpha\,\partial_t\phi)\,. \qquad (3.12) \end{aligned}}$$

Again, there is no explicit singularity in the equations, but obviously $A = 0$ implies $\alpha \to -\infty$, so we could not evolve a vanishing scalar field, clearly an unacceptable limitation.

4 Energy momentum

From a physical point of view, it may be more natural to evolve the energy and momentum density of the scalar field rather than the field components. After all, it is the energy, momentum and stress that directly affect the spacetime.

4.1 Free scalar fields

Let us first consider the simplest scenario which is a free scalar field ($V = 0$) in one-dimension. In that case, the scalar field obeys the wave equation (2.2)

$$\partial_t^2\varphi = \partial_x^2\varphi\,,$$

and its energy and momentum density are given by

$$\rho = \frac{1}{2}\partial_t\varphi\,\partial_t\bar{\varphi} + \frac{1}{2}\partial_x\varphi\,\partial_x\bar{\varphi}\,, \tag{4.1}$$

$$j = -\frac{1}{2}\partial_t\varphi\,\partial_x\bar{\varphi} - \frac{1}{2}\partial_x\varphi\,\partial_t\bar{\varphi}\,. \tag{4.2}$$

We will present the general derivation of the energy, momentum density and stress in a moment, so for now we take these expressions for granted. This simple case illustrates the progress we have made and also the problems we are still facing.

By differentiating Eqs. (4.1), (4.2) and using the wave equation (2.2), we obtain

$$\begin{aligned}
2\partial_t\rho &= \partial_t^2\varphi\,\partial_t\bar{\varphi} + \partial_t\varphi\,\partial_t^2\bar{\varphi} + \partial_t\partial_x\varphi\,\partial_x\bar{\varphi} + \partial_x\varphi\,\partial_t\partial_x\bar{\varphi} \\
2\partial_x j &= -\partial_x\partial_t\varphi\,\partial_x\bar{\varphi} - \partial_t\varphi\,\partial_x^2\bar{\varphi} - \partial_x^2\varphi\,\partial_t\bar{\varphi} - \partial_x\varphi\,\partial_x\partial_t\bar{\varphi} \\
&\overset{(2.2)}{=} -\partial_x\partial_t\varphi\,\partial_x\bar{\varphi} - \partial_t\varphi\,\partial_t^2\bar{\varphi} - \partial_t^2\varphi\,\partial_t\bar{\varphi} - \partial_x\varphi\,\partial_x\partial_t\bar{\varphi} \\
&= -2\partial_t\rho\,. \qquad (4.3)
\end{aligned}$$

Likewise, we obtain

$$\begin{aligned}
2\partial_t j &= -\partial_t^2\varphi\,\partial_x\bar{\varphi} - \partial_t\varphi\,\partial_t\partial_x\bar{\varphi} - \partial_t\partial_x\varphi\,\partial_t\bar{\varphi} - \partial_x\varphi\,\partial_t^2\bar{\varphi} \\
2\partial_x\rho &= \partial_x\partial_t\varphi\,\partial_t\bar{\varphi} + \partial_t\varphi\,\partial_x\partial_t\bar{\varphi} + \partial_x^2\varphi\,\partial_x\bar{\varphi} + \partial_x\varphi\,\partial_x^2\bar{\varphi} \\
&\overset{(2.2)}{=} \partial_x\partial_t\varphi\,\partial_t\bar{\varphi} + \partial_t\varphi\,\partial_x\partial_t\bar{\varphi} + \partial_t^2\varphi\,\partial_x\bar{\varphi} + \partial_x\varphi\,\partial_t^2\bar{\varphi} \\
&= -2\partial_t j\,. \qquad (4.4)
\end{aligned}$$

So we have the remarkably simple evolution system

$$\partial_t\rho = -\partial_x j\,, \qquad \partial_t j = -\partial_x\rho\,. \qquad (4.5)$$

For our standing wave (2.16), for example, we find after a short and straightforward calculation that

$$\rho = \omega^2\,, \qquad j = 0\,, \qquad (4.6)$$

which gives us a nice and well-behaved solution that naturally satisfies Eq. (4.5). So far so good. But two important questions remain.

1. How about a non-zero potential? Can we generalize this formalism to somehow include the potential or can we reconstruct φ or, at least, $|\varphi|^2$ from j and ρ?
2. Can we generalize Eq. (4.5) to more spatial dimensions and/or curved spacetimes?

4.2 1D wave equation with potential $V(|\varphi|^2)$

Let us first try to generalize this approach to the 1D wave equation for a complex scalar with a potential of the form $V(|\varphi|^2)$. We will see that a non-zero potential breaks the symmetry of ρ and j, and the above method of only evolving these variables no longer works.

Compared to the free scalar field, we have two main differences. First, the scalar wave equation (1.3) now reduces to the 1D flat spacetime limit

$$\boxed{\partial_t^2\varphi = \partial_x^2\varphi - \varphi V_{,|\varphi|^2}\,. \qquad (4.7)}$$

Note the minus sign which arises from rearranging terms in Eq. (1.3) using $\nabla^\alpha\nabla_\alpha\varphi = -\partial_0^2\varphi \pm \ldots$. The second change is that the energy density ρ now picks up a potential term, while j remains unchanged,

$$\rho = T_{00} \quad = \quad \frac{1}{2}\partial_0\bar{\varphi}\,\partial_0\varphi + \frac{1}{2}\partial_x\bar{\varphi}\,\partial_x\varphi + \frac{1}{2}V\,. \tag{4.8}$$

$$j = -T_{0x} \quad = \quad -\frac{1}{2}\partial_0\bar{\varphi}\,\partial_x\varphi - \frac{1}{2}\partial_0\varphi\,\partial_x\bar{\varphi} \tag{4.9}$$

As before, we can employ the wave equation to trade second derivatives and thus obtain

$$\begin{aligned}
2\partial_t\rho &= \partial_t^2\varphi\,\partial_t\bar{\varphi} + \partial_t\varphi\,\partial_t^2\bar{\varphi} + \partial_t\partial_x\varphi\,\partial_x\bar{\varphi} + \partial_x\varphi\,\partial_t\partial_x\bar{\varphi} + \partial_t V \\
2\partial_x j &= -\partial_x\partial_t\varphi\,\partial_x\bar{\varphi} - \partial_t\varphi\,\partial_x^2\bar{\varphi} - \partial_x^2\varphi\,\partial_t\bar{\varphi} - \partial_x\varphi\,\partial_x\partial_t\bar{\varphi} \\
&\overset{(4.7)}{=} -\partial_x\partial_t\varphi\,\partial_x\bar{\varphi} - \partial_t\varphi\,\partial_t^2\bar{\varphi} - \partial_t^2\varphi\,\partial_t\bar{\varphi} - \partial_x\varphi\,\partial_x\partial_t\bar{\varphi} - \bar{\varphi}V_{,|\varphi|^2}\partial_t\varphi - \varphi V_{,|\varphi|^2}\partial_t\bar{\varphi} \\
&= -2\partial_t\rho + \partial_t V - \bar{\varphi}V_{,|\varphi|^2}\partial_t\varphi - \varphi V_{,|\varphi|^2}\partial_t\bar{\varphi}\,.
\end{aligned} \tag{4.10}$$

For a potential of the form $V = V(|\varphi|^2) = V(\varphi\bar{\varphi})$, however, we always have

$$\partial_t V = V_{,|\varphi|^2}\partial_t|\varphi|^2 = V_{,|\varphi|^2}\partial_t(\varphi\bar{\varphi}) = V_{,|\varphi|^2}(\varphi\partial_t\bar{\varphi} + \bar{\varphi}\partial_t\varphi)\,, \tag{4.11}$$

and we indeed recover $\partial_t\rho = -\partial_x j$.

So far so good. Unfortunately it doesn't quite work the other way round. To see this, we compute the spatial derivative of ρ and the time derivative of j using their definitions in Eqs. (4.8), (4.9),

$$-2\partial_t j \quad = \quad \partial_t^2\bar{\varphi}\,\partial_x\varphi + \partial_t\bar{\varphi}\,\partial_t\partial_x\varphi + \partial_t^2\varphi\,\partial_x\bar{\varphi} + \underline{\partial_t\varphi\,\partial_t\partial_x\bar{\varphi}}\,, \tag{4.12}$$

$$2\partial_x\rho \quad = \quad \partial_x\partial_t\bar{\varphi}\,\partial_t\varphi + \partial_t\bar{\varphi}\,\partial_x\partial_t\varphi + \partial_x^2\bar{\varphi}\,\partial_x\varphi + \partial_x\bar{\varphi}\,\partial_x^2\varphi + \partial_x V \tag{4.13}$$

$$\overset{(4.7)}{=} \quad \underline{\partial_x\partial_t\bar{\varphi}\,\partial_t\varphi} + \partial_t\bar{\varphi}\,\partial_x\partial_t\varphi + \partial_x\varphi\,\partial_t^2\bar{\varphi} + V'\bar{\varphi}\partial_x\varphi + \partial_x\bar{\varphi}\,\partial_t^2\varphi + V'\varphi\partial_x\bar{\varphi} + \partial_x V \tag{4.14}$$

$$= \quad -2\partial_t j + V'(\varphi\partial_x\bar{\varphi} + \bar{\varphi}\partial_x\varphi) + \partial_x V \tag{4.15}$$

$$= \quad -2\partial_t j + V'\partial_x(\varphi\bar{\varphi}) + \partial_x V \quad = \quad -2\partial_t j + 2\partial_x V\,. \tag{4.16}$$

To summarize, we have now the system of equations

$$\partial_t\rho \quad = \quad -\partial_x j\,, \tag{4.17}$$

$$\partial_t j \quad = \quad -\partial_x\rho + \partial_x V = -\partial_x P \quad \text{with} \quad P \coloneqq \rho - V\,. \tag{4.18}$$

The presence of the pressure P is indeed a manifestation of the perfect fluid analogy for scalar fields developed my Madsen [4, 5]; see also [1]. We will explore this analogy in some more detail below, but first summarize the 3+1 evolution equations of the energy momentum tensor in general relativity.

5 The energy momentum

5.1 The 3+1 formalism for the energy momentum tensor

We start with the standard 3+1 decomposition of the energy momentum tensor; see for example [2] whose derivations we largely follow here. In general, the energy momentum tensor can be decomposed into energy- and momentum density and stress using the timelike unit normal n_μ and the spatial projection operator $\perp^\alpha{}_\mu = \delta^\alpha{}_\mu + n^\alpha n_\mu$ according to

$$\rho := T_{\mu\nu} n^\mu n^\nu \,, \qquad j_\alpha := -T_{\mu\nu} \perp^\mu{}_\alpha n^\nu \,, \qquad S_{\alpha\beta} := T_{\mu\nu} \perp^\mu{}_\alpha \perp^\nu{}_\beta \tag{5.1}$$

$$\Leftrightarrow \quad T_{\alpha\beta} = \rho n_\alpha n_\beta + j_\alpha n_\beta + j_\beta n_\alpha + S_{\alpha\beta} \,. \tag{5.2}$$

Given that $\boldsymbol{n} = -\alpha \mathbf{d}t = -\mathbf{d}t/||\mathbf{d}t||$ and $\perp^\mu{}_\alpha n_\mu = 0$ by construction, we see that

$$j^0 = \boldsymbol{j}(\mathbf{d}t) = j^\mu \frac{-n_\mu}{\alpha} = 0 \,, \qquad S^{0\beta} = \boldsymbol{S}(\mathbf{d}t, \mathbf{d}x^\beta) = S^{\mu\nu} \frac{-n_\mu}{\alpha} (\mathbf{d}x^\beta)_\nu = 0 \,. \tag{5.3}$$

In adapted coordinates, $\boldsymbol{j}$ and $\boldsymbol{S}$ are therefore entirely determined by their purely spatial components. This is in agreement with the number of degrees of freedom, 10 for $T_{\alpha\beta}$ versus 1 for ρ, 3 for j^μ and 6 for $S_{\alpha\beta}$.

Of course, we only have 4 evolution equations from $\nabla_\mu T^{\mu\nu} = 0$ which, in general, means that the matter evolution in time requires some further information, as for example in the form of an equation of state for the pressure P. For the moment, however, we are only interested in translating the conservation law $\nabla_\mu T^{\mu\nu} = 0$ into the 3+1 formalism. We start with the time projection,

$$\begin{aligned}
& \nabla_\mu T^\mu{}_\alpha = 0 && \Big|\; \cdot n^\alpha \\
\Rightarrow\; & n^\alpha \nabla_\mu T^\mu{}_\alpha = \nabla_\mu (T^\mu{}_\alpha n^\alpha) - T^\mu{}_\alpha \nabla_\mu n^\alpha = 0 && \Big| \nabla_\mu n_\beta = -K_{\mu\beta} - n_\mu a_\beta \,,\; a_\beta = n^\rho \nabla_\rho n_\beta \\
\Rightarrow\; & \nabla_\mu \left[(\rho n_\alpha n^\mu + j^\mu n_\alpha + j_\alpha n^\mu + S^\mu{}_\alpha)\, n^\alpha \right] = (\rho n^\mu n_\alpha + j^\mu n_\alpha + j_\alpha n^\mu + S^\mu{}_\alpha)(-K_\mu{}^\alpha - n_\mu a^\alpha) \\
\Rightarrow\; & \nabla_\mu (-\rho n^\mu - j^\mu) = -S^{\mu\alpha} K_{\mu\alpha} - n_\mu a^\alpha (j_\alpha n^\mu) \\
\Rightarrow\; & \nabla_\mu j^\mu + n^\mu \nabla_\mu \rho + \rho \nabla_\mu n^\mu = S^{\mu\beta} K_{\mu\beta} - j^\alpha a_\alpha \,.
\end{aligned} \tag{5.4}$$

Next, we need

$$\begin{aligned}
D_\mu j^\mu &= \perp^\alpha{}_\mu \perp^\mu{}_\beta \nabla_\alpha j^\beta = \perp^\alpha{}_\beta \nabla_\alpha j^\beta = \nabla_\mu j^\mu + n^\alpha n_\beta \nabla_\alpha j^\beta = \nabla_\mu j^\mu - n^\alpha j^\beta \nabla_\alpha n_\beta \\
&= \nabla_\mu j^\mu - j^\beta a_\beta \,.
\end{aligned} \tag{5.5}$$

Combining with (5.4) and recalling the Lie derivative $\mathcal{L}_{\boldsymbol{n}} \rho = n^\mu \nabla_\mu \rho$ for a scalar, we obtain

$$D_\mu j^\mu + j^\beta a_\beta + \mathcal{L}_{\boldsymbol{n}} \rho + \rho(-K_\mu{}^\mu - n_\mu a^\mu) = S^{\mu\nu} K_{\mu\nu} - a^\alpha j_\alpha \,, \tag{5.6}$$

so that

$$\boxed{\mathcal{L}_{\boldsymbol{n}}\rho + D_\mu j^\mu + 2j^\mu a_\mu - \rho K - S^{\mu\nu}K_{\mu\nu} = 0 \quad \text{with} \quad \mathcal{L}_{\boldsymbol{n}}\rho = \mathcal{L}_{\frac{1}{\alpha}(\boldsymbol{\partial}_t - \boldsymbol{\beta})}\rho = \frac{1}{\alpha}\partial_t\rho - \frac{1}{\alpha}\beta^\mu\partial_\mu\rho\,. \quad (5.7)}$$

We also recall that $a_\mu = \frac{\partial_\mu \alpha}{\alpha}$.

We next address the time evolution of the momentum density j_α. Using once again $\nabla_\mu n_\alpha = -K_{\mu\alpha} - n_\mu a_\alpha$, we can write

$$\begin{aligned}
\nabla_\mu T^\mu{}_\alpha &= \nabla_\mu(\rho n^\mu n_\alpha + j^\mu n_\alpha + n^\mu j_\alpha + S^\mu{}_\alpha) \\
&= \nabla_\mu S^\mu{}_\alpha + n^\mu n_\alpha \nabla_\mu\rho + \rho n_\alpha \nabla_\mu n^\mu + \rho n^\mu \nabla_\mu n_\alpha + n_\alpha \nabla_\mu j^\mu + j^\mu \nabla_\mu n_\alpha + j_\alpha \nabla_\mu n^\mu + n^\mu \nabla_\mu j_\alpha \\
&= \nabla_\mu S^\mu{}_\alpha + n_\alpha n^\mu \nabla_\mu \rho + \rho n_\alpha(-K_\mu{}^\mu) + \rho n^\mu(-K_{\mu\alpha} - n_\mu a_\alpha) + n_\alpha \nabla_\mu j^\mu + j^\mu(-K_{\mu\alpha} - n_\mu a_\alpha) \\
&\quad - j_\alpha K + n^\mu \nabla_\mu j_\alpha \\
&= \nabla_\mu S^\mu{}_\alpha + n_\alpha n^\mu \nabla_\mu\rho - \rho K n_\alpha + \rho a_\alpha + n_\alpha \nabla_\mu j^\mu - j^\mu K_{\mu\alpha} - K j_\alpha + n^\mu \nabla_\mu j_\alpha \overset{!}{=} 0 \qquad \Big| \cdot \perp^\alpha{}_\beta \\
\Rightarrow 0 &= \perp^\alpha{}_\beta \nabla_\mu S^\mu{}_\alpha + \rho a_\beta - j^\mu K_{\mu\beta} - K j_\beta + \perp^\alpha{}_\beta n^\mu \nabla_\mu j_\alpha\,.
\end{aligned} \tag{5.8}$$

Now we would like to replace the spacetime covariant derivatives with spatial or time derivatives. Starting with the first term, we can write

$$\begin{aligned}
D_\mu S^\mu{}_\alpha &= \perp^\rho{}_\mu \perp^\mu{}_\sigma \perp^\gamma{}_\alpha \nabla_\rho S^\sigma{}_\gamma = \perp^\rho{}_\sigma \perp^\gamma{}_\alpha \nabla_\rho S^\sigma{}_\gamma = \perp^\gamma{}_\alpha\Big(\nabla_\sigma S^\sigma{}_\gamma + n^\rho n_\sigma \nabla_\rho S^\sigma{}_\gamma\Big) \\
&= \perp^\gamma{}_\alpha\Big(\nabla_\sigma S^\sigma{}_\gamma - n^\rho S^\sigma{}_\gamma \nabla_\rho n_\sigma\Big) = \perp^\gamma{}_\alpha\Big[\nabla_\mu S^\mu{}_\gamma - n^\rho S^\sigma{}_\gamma(-K_{\rho\sigma} - n_\rho a_\sigma)\Big] \\
&= \perp^\gamma{}_\alpha \nabla_\mu S^\mu{}_\gamma - a_\sigma S^\sigma{}_\alpha\,.
\end{aligned} \tag{5.9}$$

In order to rewrite the covariant derivative of j_α, we need the Lie derivative and its spatial projection

$$\begin{aligned}
\perp^\alpha{}_\beta \mathcal{L}_{\boldsymbol{n}} j_\alpha &= (\delta^\alpha{}_\beta + n^\alpha n_\beta)(n^\mu \nabla_\mu j_\alpha + j_\mu \nabla_\alpha n^\mu) = \mathcal{L}_{\boldsymbol{n}} j_\beta + n_\beta n^\alpha n^\mu \nabla_\mu j_\alpha + n^\alpha n_\beta j_\mu \nabla_\alpha n^\mu \\
&= \mathcal{L}_{\boldsymbol{n}} j_\beta \underbrace{- n_\beta n^\mu j_\alpha \nabla_\mu n^\alpha + n_\beta n^\alpha j_\mu \nabla_\alpha n^\mu}_{=0} = \mathcal{L}_{\boldsymbol{n}} j_\beta\,.
\end{aligned} \tag{5.10}$$

Then,

$$\perp^\alpha{}_\beta n^\mu \nabla_\mu j_\alpha = \perp^\alpha{}_\beta(\mathcal{L}_{\boldsymbol{n}} j_\alpha - j_\mu \nabla_\alpha n^\mu) = \mathcal{L}_{\boldsymbol{n}} j_\beta - \perp^\alpha{}_\beta j_\mu(-K_\alpha{}^\mu - n_\alpha a^\mu) = \mathcal{L}_{\boldsymbol{n}} j_\beta + j_\mu K_\beta{}^\mu\,. \tag{5.11}$$

Plugging Eqs. (5.9) and (5.11) into (5.8), we obtain

$$D_\mu S^\mu{}_\beta + a_\mu S^\mu{}_\beta + \rho a_\beta + \mathcal{L}_{\boldsymbol{n}} j_\beta - K j_\beta = 0\,. \tag{5.12}$$

In summary, the evolution of j_β is given by

$$\boxed{\begin{aligned}
&\mathcal{L}_{\boldsymbol{n}} j_\beta + D_\mu S^\mu{}_\beta + S^\mu{}_\beta a_\mu - K j_\beta + \rho a_\beta = 0\,, \\
&\text{with} \quad \mathcal{L}_{\boldsymbol{n}} j_\beta = \frac{1}{\alpha}\partial_t j_\beta - \frac{1}{\alpha}\mathcal{L}_{\boldsymbol{\beta}} j_\beta = \frac{1}{\alpha}\partial_t j_\beta - \frac{\beta^\mu}{\alpha}\partial_\mu j_\beta - \frac{1}{\alpha} j_\mu \partial_\beta \beta^\mu\,. \qquad (5.13)
\end{aligned}}$$

5.2 The energy momentum tensor of a complex scalar field

We can also derive "3+1" expressions for the projections of the energy momentum tensor themselves. I am not sure these will be helpful, but it doesn't hurt, so let's get them. For this purpose, we start with the generic relations between the spacetime metric $g_{\alpha\beta}$ on the one side and the ADM variables α, β^i, γ_{ij} as well as the unit time like normal $n_\mu = -\alpha \mathbf{d}t$ on the other. We write these in the form

$$\gamma_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta \,, \qquad n_\alpha = (-\alpha,\ 0)\,, \qquad n^\alpha = \left(\frac{1}{\alpha},\ -\frac{\beta^i}{\alpha}\right)\,,$$

$$g_{\alpha\beta} = \begin{pmatrix} -\alpha^2 + \beta_m\beta^m & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \qquad \Leftrightarrow \qquad g^{\alpha\beta} = \begin{pmatrix} -\alpha^{-2} & \alpha^{-2}\beta^j \\ \alpha^{-2}\beta^i & \gamma^{ij} - \alpha^{-2}\beta^i\beta^j \end{pmatrix}$$

$$\gamma_{\alpha\beta} = \begin{pmatrix} \beta_m\beta^m & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \qquad \Leftrightarrow \qquad \gamma^{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & \gamma^{ij} \end{pmatrix}\,. \tag{5.14}$$

The energy-momentum tensor of the complex scalar field is given by Eq. (1.2) which we repeat here,

$$T_{\alpha\beta} = \frac{1}{2}\left[\partial_\alpha\bar{\varphi}\,\partial_\beta\varphi + \partial_\alpha\varphi\,\partial_\beta\bar{\varphi} - g_{\alpha\beta}\left(g^{\mu\nu}\partial_\mu\bar{\varphi}\,\partial_\nu\varphi + V\right)\right]\,.$$

Using the definition

$$\Pi := -\frac{1}{2}n^\mu\partial_\mu\varphi = -\frac{1}{2\alpha}(\partial_t\varphi - \beta^m\partial_m\varphi) \qquad \Leftrightarrow \qquad n^\mu\partial_\mu\varphi = -2\Pi\,, \tag{5.15}$$

We can then write the energy density as

$$\begin{aligned}
\rho &= n^\alpha n^\beta T_{\alpha\beta} = n^\alpha n^\beta \partial_\alpha\varphi\partial_\beta\bar{\varphi} + \frac{1}{2}g^{\mu\nu}\partial_\mu\bar{\varphi}\,\partial_\nu\varphi + \frac{V}{2} \\
&= n^\alpha n^\beta \partial_\alpha\varphi\partial_\beta\bar{\varphi} + \frac{1}{2}(\gamma^{\mu\nu} - n^\mu n^\nu)\partial_\mu\bar{\varphi}\,\partial_\nu\varphi + \frac{V}{2} = \frac{1}{2}n^\alpha n^\beta\partial_\alpha\varphi\,\partial_\beta\bar{\varphi} + \frac{1}{2}\gamma^{\mu\nu}\partial_\mu\bar{\varphi}\,\partial_\nu\varphi + \frac{V}{2} \\
&= 2\Pi\,\bar{\Pi} + \frac{1}{2}\gamma^{mn}\partial_m\bar{\varphi}\,\partial_n\varphi + \frac{V}{2}\,.
\end{aligned} \tag{5.16}$$

For the momentum density, we obtain

$$\begin{aligned}
j^0 &= -\frac{1}{2}\underbrace{\gamma^{\mu 0}}_{=0} n^\nu\left[\partial_\mu\bar{\varphi}\,\partial_\nu\varphi + \partial_\mu\varphi\,\partial_\nu\bar{\varphi}\right] = 0\,, \\
j^i &= -\frac{1}{2}\gamma^{\mu i}n^\nu\left[\partial_\mu\bar{\varphi}\,\partial_\nu\varphi + \partial_\mu\varphi\,\partial_\nu\bar{\varphi}\right] = -\frac{1}{2}(-2\Pi\gamma^{mi}\partial_m\bar{\varphi} - 2\bar{\Pi}\gamma^{mi}\partial_m\varphi) = \Pi\gamma^{im}\partial_m\bar{\varphi} + \bar{\Pi}\gamma^{im}\partial_m\varphi
\end{aligned} \tag{5.17}$$

Finally, using

$$\perp^i{}_\alpha \perp^j{}_\beta g^{\alpha\beta} = \perp^i{}_\alpha \perp^j{}_\beta(\gamma^{\alpha\beta} - n^\alpha n^\beta) = \gamma^{ij}\,, \tag{5.18}$$

the stress tensor $S^{\alpha\beta} = \gamma^{\alpha\mu}\gamma^{\beta\nu}T_{\mu\nu}$ can be written as

$$\begin{aligned}
S^{00} &= S^{0j} = S^{i0} = 0\,,\\
S^{ij} &= \frac{1}{2}\gamma^{i\mu}\gamma^{j\nu}\left[\partial_\mu\bar\varphi\,\partial_\nu\varphi + \partial_\mu\varphi\,\partial_\nu\bar\varphi - g_{\mu\nu}(g^{\rho\sigma}\partial_\rho\bar\varphi\,\partial_\sigma\varphi + V)\right]\\
&= \frac{1}{2}\gamma^{im}\gamma^{jn}(\partial_m\bar\varphi\,\partial_n\varphi + \partial_m\varphi\,\partial_n\bar\varphi) - \frac{1}{2}\gamma^{ij}\left[(\gamma^{\rho\sigma} - n^\rho n^\sigma)\partial_\rho\bar\varphi\,\partial_\sigma\varphi + V\right]\\
&= \frac{1}{2}\gamma^{im}\gamma^{jn}(\partial_m\bar\varphi\,\partial_n\varphi + \partial_m\varphi\,\partial_n\bar\varphi) - \frac{1}{2}\gamma^{ij}\left(\gamma^{mn}\partial_m\bar\varphi\,\partial_n\varphi - 4\Pi\,\bar\Pi + V\right)\,. \qquad (5.19)
\end{aligned}$$

We can also compute some traces,

$$\begin{aligned}
S &= g^{\mu\nu}S_{\mu\nu} = (\gamma^{\mu\nu} - n^\mu n^\nu)S_{\mu\nu} = \gamma^{mn}S_{mn} = \gamma_{mn}S^{mn} = \gamma_{ij}S^{ij}\\
&= \frac{1}{2}\gamma^{mn}(\partial_m\bar\varphi\,\partial_n\varphi + \partial_m\varphi\,\partial_n\bar\varphi) - \frac{3}{2}\left(\gamma^{mn}\partial_m\bar\varphi\,\partial_n\varphi - 4\Pi\,\bar\Pi + V\right)\\
&= -\frac{1}{2}\gamma^{mn}\partial_m\bar\varphi\,\partial_n\varphi + 6\Pi\,\bar\Pi - \frac{3}{2}V\,. \qquad (5.20)
\end{aligned}$$

$$\begin{aligned}
T &= g^{\mu\nu}\partial_\mu\bar\varphi\,\partial_\nu\varphi + \frac{1}{2}(-4)[(\gamma^{\mu\nu} - n^\mu n^\nu)\partial_\mu\bar\varphi\,\partial_\nu\varphi + V]\\
&= \gamma^{mn}\partial_m\bar\varphi\,\partial_n\varphi - 4\Pi\,\bar\Pi - 2(\gamma^{mn}\partial_m\bar\varphi\,\partial_n\varphi - 4\Pi\,\bar\Pi + V)\\
&= -\gamma^{mn}\partial_m\bar\varphi\,\partial_n\varphi + 4\Pi\,\bar\Pi - 2V = S - \rho\,, \qquad (5.21)
\end{aligned}$$

where the last equality is expected from

$$T = g^{\mu\nu}T_{\mu\nu} = (\gamma^{\mu\mu} - n^\mu n^\nu)T_{\mu\nu} = \gamma^{\mu\nu}T_{\mu\nu} - \rho = g^{\alpha\beta}\perp^\mu{}_\alpha\perp^\nu{}_\beta T_{\mu\nu} - \rho = g^{\alpha\beta}S_{\alpha\beta} - \rho = S - \rho\,. \qquad (5.22)$$

We also consider the matter Lagrangian itself which is given by

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{2}(\gamma^{\alpha\beta} - n^\alpha n^\beta)\partial_\alpha\bar\varphi\,\partial_\beta\varphi - \frac{V}{2} = -\frac{1}{2}\gamma^{mn}\partial_m\bar\varphi\,\partial_n\varphi + \frac{1}{2}(-2\bar\Pi)(-2\Pi) - \frac{V}{2}\\
&= -\frac{1}{2}\gamma^{mn}\partial_m\bar\varphi\,\partial_n\varphi + 2\bar\Pi\,\Pi - \frac{V}{2}\,. \qquad (5.23)
\end{aligned}$$

Let us summarize these results in one box,

$$\begin{aligned}
\rho &= \frac{1}{2}\gamma^{mn}\partial_m\bar{\varphi}\,\partial_n\varphi + 2\Pi\,\bar{\Pi} + \frac{V}{2}\,, && (5.24)\\
j^i &= \Pi\gamma^{im}\partial_m\bar{\varphi} + \bar{\Pi}\gamma^{im}\partial_m\varphi\,, && (5.25)\\
S^{ij} &= \frac{1}{2}\gamma^{im}\gamma^{jn}(\partial_m\bar{\varphi}\,\partial_n\varphi + \partial_m\varphi\,\partial_n\bar{\varphi}) - \frac{1}{2}\gamma^{ij}(\gamma^{mn}\partial_m\bar{\varphi}\,\partial_n\varphi - 4\Pi\,\bar{\Pi} + V)\,, && (5.26)\\
S &= -\frac{1}{2}\gamma^{mn}\partial_m\bar{\varphi}\,\partial_n\varphi + 6\Pi\,\bar{\Pi} - \frac{3}{2}V\,, && (5.27)\\
T &= -\gamma^{mn}\partial_m\bar{\varphi}\,\partial_n\varphi + 4\Pi\,\bar{\Pi} - 2V = S - \rho\,, && (5.28)\\
\mathcal{L} &= -\frac{1}{2}\gamma^{mn}\partial_m\bar{\varphi}\,\partial_m\varphi + 2\Pi\,\bar{\Pi} - \frac{V}{2}\,. && (5.29)
\end{aligned}$$

Here we have included in the last line the matter part of the Lagrangian (1.1) which satisfies the relation $\rho - V = \gamma^{mn}\partial_m\varphi\partial_n\bar{\varphi} + \mathcal{L}$.

In spacetime notation, we can also write these equations as

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{2}g^{\alpha\beta}\partial_\alpha\varphi\,\partial_\beta\bar{\varphi} - \frac{1}{2}V\,,\\
T_{\alpha\beta} &= \partial_{(\alpha}\varphi\,\partial_{\beta)}\bar{\varphi} + g_{\alpha\beta}\mathcal{L}\,,\\
\rho &= n^\alpha n^\beta T_{\alpha\beta} = n^\alpha n^\beta\partial_\alpha\varphi\,\partial_\beta\bar{\varphi} - \mathcal{L}\,,\\
&= n^\alpha n^\beta\partial_\alpha\varphi\,\partial_\beta\bar{\varphi} + \frac{1}{2}g^{\alpha\beta}\partial_\alpha\varphi\,\partial_\beta\bar{\varphi} + \frac{1}{2}V\\
&= \gamma^{\alpha\beta}\partial_\alpha\varphi\,\partial_\beta\bar{\varphi} + V + \mathcal{L}\,. && (5.30)
\end{aligned}$$

Now, we would expect that we recover Eq. (10) in Madsen's work [4] if our scalar field is real, $\varphi = \bar{\varphi}$, and we set $n^\alpha = u^\alpha = \partial_\alpha\varphi/||\partial_\alpha\varphi||$, i.e. use the fluid's rest frame. Bearing in mind that Madsen has a factor 1/2 in his potential[1] and using a tilde on his fluid variables to mark them as measured in the fluid's rest frame, his Eq. (10) in our variables is $\tilde{\rho} = \mathcal{L} + V$, but the term $\gamma^{\alpha\beta}\partial_\alpha\varphi\,\partial_\beta\bar{\varphi}$ is missing. This omission is not an error in Madsen's calculation as we can check ourselves. If we equate the energy-momentum tensor of a perfect fluid and that of the scalar field, using our sign- and potential conventions, we obtain for a real scalar field

$$T_{\alpha\beta} = (\tilde{\rho} + \tilde{p})u_\alpha u_\beta + \tilde{p}g_{\alpha\beta} = \partial_\alpha\varphi\,\partial_\beta\varphi + g_{\alpha\beta}\mathcal{L}\,. \tag{5.31}$$

Here we construct the 4-velocity from the scalar field assuming a timelike $\partial_\alpha\varphi$,

$$g^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi < 0\,, \tag{5.32}$$

[1] Note that Madsen also uses a $+---$ signature for the metric which makes the comparison pretty much a horror show.

so that

$$u_\alpha := \frac{\partial_\alpha \varphi}{\sqrt{-g^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi}}\,, \tag{5.33}$$

is a perfectly honorable 4-velocity with $u_\mu u^\mu = -1$. Finally, if we also define

$$\tilde{\rho} := -g^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi - \mathcal{L} = -\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi + \frac{1}{2}V = \mathcal{L} + V\,, \tag{5.34}$$

then our energy momentum tensor becomes

$$T_{\alpha\beta} = \partial_\alpha\varphi\,\partial_\beta\varphi + g_{\alpha\beta}\mathcal{L} = (-g^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi)u_\alpha u_\beta + \mathcal{L}g_{\alpha\beta} = (\tilde{\rho}+\mathcal{L})u_\alpha u_\beta + \mathcal{L}g_{\alpha\beta}\,, \tag{5.35}$$

and we have indeed recovered the energy-momentum tensor of a perfect fluid with pressure $\tilde{p} = \mathcal{L}$. But is Eq. (5.34) in agreement with the energy density in Eq. (5.30) which has an extra term $\gamma^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi$ on the right-hand side? In fact the two are compatible: In the frame comoving with the fluid, the timelike unit normal *is* the fluid velocity u^α which *in this frame* satisfies $\gamma^{\alpha\beta}u_\alpha u_\beta = 0$ by construction. Finally, we recall that $u_\alpha \propto \partial_\alpha\varphi$ by Eq. (5.33) with non-zero $||\partial_\alpha\varphi||$ and we conclude that $\gamma^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi = 0$.

5.3 Perfect fluids

For comparison, we discuss in more detail the energy momentum tensor and its space-time decomposition for a perfect fluid. For this purpose, we first need to reinvestigate some subtleties of the standard 3+1 split, namely the proper time measured by observers traveling with some 4-velocity vector u^α. Let us assume that we have foliated the spacetime in the usual way using hypersurfaces Σ_t of constant t with timelike gradient $\mathbf{d}t$. We furthermore assume that the 4-velocity u^α is future pointing everywhere, so that $\langle \mathbf{d}t, \boldsymbol{u}\rangle > 0$. We can therefore parametrize the observers world line using our time coordinate t and the proper time measured by the observer along the worldline is given by

$$\Delta\tau = \int_{t_1}^{t_2}\sqrt{-g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\,\frac{\mathrm{d}x^\beta}{\mathrm{d}t}}\,\mathrm{d}t \qquad \Rightarrow \qquad \mathrm{d}\tau = \sqrt{-g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\,\frac{\mathrm{d}x^\beta}{\mathrm{d}t}}\,\mathrm{d}t\,. \tag{5.36}$$

Of course, the latter expression is what we typically write in the form $\mathrm{d}\tau^2 = -g_{\alpha\beta}\mathrm{d}x^\alpha\,\mathrm{d}x^\beta$. We keep the $\mathrm{d}t$ factor in, however, to highlight that this $\mathrm{d}\tau$ represents the amount of proper time that the observer measures along the particular piece of the world line from t to $t+\mathrm{d}t$. In the literature this is often described as the "proper time measured over the coordinate time interval $\mathrm{d}t$". People then even start drawing vectors from Σ_t to $\Sigma_{t+\delta t}$ in blatant violation of the fact that vectors live in the tangent spaces of individual points of the manifold and certainly *do not* connect points of the manifold. Makes one wonder what comes next... Tax on breathing? Anyway, here we prefer the description of the proper time along that segment of the wordline that covers the parameter range $[t, t+\mathrm{d}t]$.

Now, it is important to note that the 4-velocity is in general *not* equal to the curve's tangent vector $dx^\alpha/\mathrm{d}t$. The two are proportional, but only agree if we use the observer's proper time as our coordinate time. In general, we do not do this. In coordinates adapted to the foliation, however, we have the simplifying relation $x^0 = t$, so that

$$\frac{\mathrm{d}x^\alpha}{\mathrm{d}t} = \left(1,\ \frac{\mathrm{d}x^i}{\mathrm{d}t}\right) =: m^\alpha\,. \tag{5.37}$$

Two specific choices for the observer's 4-velocity are of particular interest for our dicsussion.

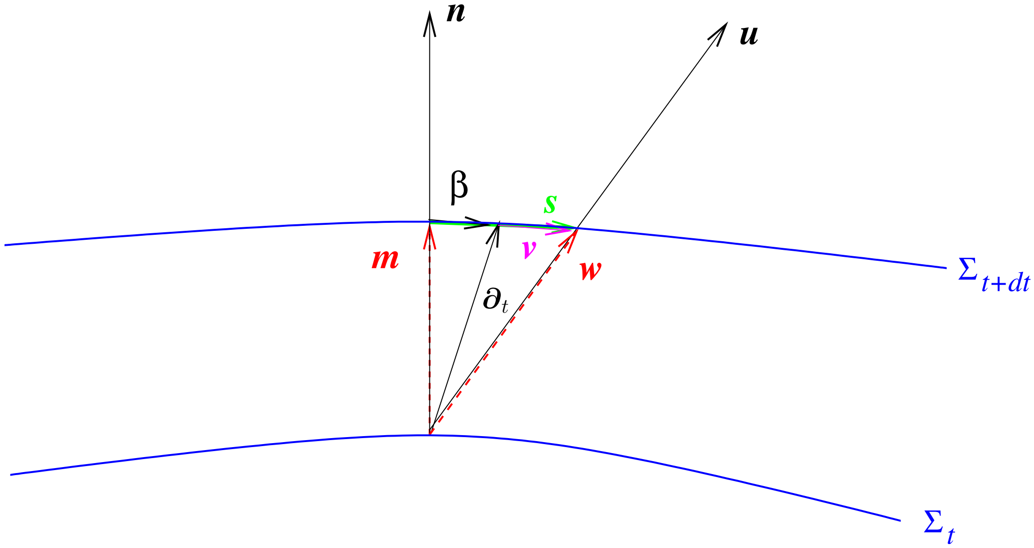

Figure 1: An Eulerian observer moves with a 4-velocity given by the timelike unit normal n^α. For the evaluation of this observer's proper time along the world line segment with parameter range $(t,\, t+\mathrm{d}t)$, we need the tangent vector $m^\alpha := \mathrm{d}x^\alpha/\mathrm{d}t$. For Lagrangian observers comoving with the fluid's 4-velocity u^α, we likewise need the tangent vector $w^\alpha := \mathrm{d}x^\alpha/\mathrm{d}t$. The proportionality between these two pairs of vectors is given by $m^\alpha = \alpha n^\alpha$ and $w^\alpha = \frac{\alpha}{\Gamma} u^\alpha$. The spatial vector $s^\alpha := w^\alpha - m^\alpha$ measures the displacement between Eulerian and Lagrangian observer whose relative velocity we define as $U^\alpha := \alpha s^\alpha$.

Case 1: Eulerian observers whose 4-velocity equals the timelike unit normal, i.e. $m^\alpha \propto n^\alpha$. Then we can conclude

$$n^\alpha = \left(\frac{1}{\alpha},\ -\frac{\beta^i}{\alpha}\right)$$

$$\Rightarrow \quad m^\alpha = \left(1,\ -\beta^i\right) = \alpha n^\alpha \qquad \text{or} \qquad \boldsymbol{m} = \alpha \boldsymbol{n} \tag{5.38}$$

$$\Rightarrow \quad g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}t} \frac{\mathrm{d}x^\nu}{\mathrm{d}t} = g_{\mu\nu} m^\mu m^\nu = -\alpha^2 \tag{5.39}$$

$$\Rightarrow \quad \mathrm{d}\tau_{\mathrm{E}} = \sqrt{-g_{\mu\nu} m^\mu m^\nu} = \alpha \mathrm{d}t\,. \tag{5.40}$$

This is the well known result that the lapse function α determines the flow of proper time $\mathrm{d}\tau$ as an Eulerian observer moving along n^μ travels from hypersurface Σ_t to hypersurface $\Sigma_{t+\mathrm{d}t}$.

Case 2: An observer comoving with the fluid elements. Sometimes these are called Lagrangian observers. In analogy to the rescaled normal m^α, we define the tangent vector to the world lines of this observer as parametrized by coordinate time $w^\alpha := \mathrm{d}x^\alpha/\mathrm{d}t$. The 4-velocity of this observer is just that of the fluid, so that now $w^\alpha \propto u^\alpha$ or $w^\alpha = \lambda u^\alpha$. In adapted coordinates $w^0 = 1$, so that $\lambda = \frac{1}{u^0}$ and, hence,

$$u^0 = \boldsymbol{u}(\mathbf{d}t) = \boldsymbol{u}\left(-\alpha^{-1}\boldsymbol{n}\right) = -\frac{1}{\alpha}\boldsymbol{u}\cdot\boldsymbol{n} \tag{5.41}$$

We now define the Lorentz factor and find

$$\Gamma := -\boldsymbol{u}\cdot\boldsymbol{n} \quad \Rightarrow \quad u^0 = \frac{\Gamma}{\alpha} \quad \Rightarrow \quad \lambda = \frac{\alpha}{\Gamma}$$
$$\Rightarrow \quad w^\mu = \frac{\alpha}{\Gamma}u^\mu \qquad \text{or} \qquad \boldsymbol{w} = \frac{\alpha}{\Gamma}\boldsymbol{u}$$
$$\Rightarrow \quad \mathrm{d}\tau_{\mathrm{F}} = \sqrt{-g_{\mu\nu}w^\mu w^\nu}\,\mathrm{d}t = \frac{\alpha}{\Gamma}\sqrt{-g_{\mu\nu}u^\mu u^\nu}\,\mathrm{d}t = \frac{\alpha}{\Gamma}\mathrm{d}t\,. \tag{5.42}$$

In summary the proper times measured by the Eulerian and Lagrangian observers as they move with 4-velocities n^μ and u^μ, respectively, over a parameter interval $\mathrm{d}t$ along their respective world lines are related by

$$\frac{\mathrm{d}\tau_{\mathrm{E}}}{\mathrm{d}\tau_{\mathrm{F}}} = \Gamma\,. \tag{5.43}$$

In words, Γ is the Lorentz factor of an observer looking at a fluid moving with some velocity relative to this observer. Note that $\Gamma = \boldsymbol{u}\cdot\boldsymbol{n}$ is symmetric under exchange of $\boldsymbol{u}$ and $\boldsymbol{n}$ as it should be; each observer sees the other time dilated according to the same Γ. Finally, we can also express the Lorentz factor according to

$$\Gamma = -\boldsymbol{n}\cdot\boldsymbol{u} = \alpha\mathbf{d}t(\boldsymbol{u}) = \alpha u^0\,. \tag{5.44}$$

We display the various velocity vectors of the two observers in Fig. 1. Note that n^μ and u^μ are normalized 4-velocities and that m^α and w^α are proportional tangent vectors to these curves as parameterized by coordinate time t, i.e. with time components $m^0 = 1$, $w^0 = 1$.

We next define the spatial velocity of the fluid relative to the Eulerian observer

$$\boldsymbol{V} := \frac{1}{\Gamma}\boldsymbol{u} - \boldsymbol{n} = \frac{1}{\alpha}(\boldsymbol{w} - \boldsymbol{m})\,. \tag{5.45}$$

This is a purely spatial vector since

$$\boldsymbol{n}\cdot\boldsymbol{V} = \frac{1}{\Gamma}\boldsymbol{n}\cdot\boldsymbol{u} - \boldsymbol{n}\cdot\boldsymbol{n} = \frac{-\Gamma}{\Gamma} + 1 = 0\,. \tag{5.46}$$

This spatial velocity is related to the displacement $s^\alpha := w^\alpha - m^\alpha$ by

$$\alpha\boldsymbol{V} = \boldsymbol{w} - \boldsymbol{m} = \boldsymbol{s}\,. \tag{5.47}$$

Using $\boldsymbol{m} = \boldsymbol{\partial}_t - \boldsymbol{\beta}$, we furthermore obtain for the spatial components

$$\alpha V^i = w^i + \beta^i\,. \tag{5.48}$$

The definition of the spatial velocity V^α may appear a bit convoluted, but it has the nice property that

$$\boldsymbol{u}\cdot\boldsymbol{u} = -1 \quad \Rightarrow \quad \Gamma^2(\boldsymbol{n}+\boldsymbol{V})(\boldsymbol{n}+\boldsymbol{V}) = -1 \quad \Rightarrow \quad \Gamma^2(-1+\boldsymbol{V}^{\mathbf{2}}) = -1$$
$$\Rightarrow \quad \Gamma = \frac{1}{\sqrt{1-\boldsymbol{V}^2}}\,. \tag{5.49}$$

We are not quite done yet with velocities; we still need the fluid's coordinate velocity, i.e. the fluid's velocity relative to worldlines of constant spatial coordinates $x^i = \text{const}$. We denote this velocity by $\boldsymbol{v}$ (magenta color) in Fig. 1 and it is given by

$$\boldsymbol{v} = \boldsymbol{w} - \boldsymbol{\partial}_t \,, \tag{5.50}$$

where $\boldsymbol{\partial}_t$ is the tangent vector to curves $x^i = \text{const}$. In coordinates, this vector trivially has components

$$(\partial_t)^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}t} = (1,\, 0,\, 0,\, 0)\,, \tag{5.51}$$

so like the other rescaled vectors $\boldsymbol{m}$ and $\boldsymbol{w}$, it satisfies $\mathbf{d}t(\boldsymbol{\partial}_t) = \mathbf{d}t(\boldsymbol{m}) = \mathbf{d}t(\boldsymbol{w}) = 1$. Recalling $\boldsymbol{m} = \alpha\boldsymbol{n} = \boldsymbol{\partial}_t - \boldsymbol{\beta}$, the fluid velocity relative to $\boldsymbol{\partial}_t$ is given by

$$\boldsymbol{v} = \boldsymbol{w} - \boldsymbol{\partial}_t = \alpha\boldsymbol{V} + \boldsymbol{m} - \boldsymbol{\partial}_t = \alpha\boldsymbol{V} - \boldsymbol{\beta}\,, \tag{5.52}$$

This velocity is clearly tangent to Σ_t, so that $v^0 = \boldsymbol{v}(\mathbf{d}t) = 0$ and its spatial components can also be written as

$$v^i := \frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{u^i}{u^0}\,. \tag{5.53}$$

Let us summarize the key definitions for a perfect fluid,

$\boldsymbol{n} :=$	unit timelike normal = 4-velocity of Eulerian observer
$\tilde{\rho} :=$	Energy density as measured in the fluid's rest frame
$\tilde{p} :=$	Pressure measured in the fluid's rest frame
$\boldsymbol{u} :=$	4-velocity of the fluid elements

$\boldsymbol{T} := (\tilde{\rho} + \tilde{p})\boldsymbol{u}\otimes\boldsymbol{u} + \tilde{p}\boldsymbol{g}$	Energy-momentum tensor
$\rho := \boldsymbol{T}(\boldsymbol{n}, \boldsymbol{n})$	Energy density measured by an Eulerian observer
$\boldsymbol{j} := \perp\boldsymbol{T}(\boldsymbol{n},\, .\,)$	Momentum density measured by an Eulerian observer
$\boldsymbol{S} := \perp\boldsymbol{T}(\,.\,,\,.\,)$	Stress tensor in Eulerian frame
$\Gamma := -\boldsymbol{u}\cdot\boldsymbol{n}$	Lorentz factor
$\boldsymbol{w} := \frac{\alpha}{\Gamma}\boldsymbol{u}\,, \quad \boldsymbol{m} := \alpha\boldsymbol{n}$	Rescaled versions of $\boldsymbol{u}$ and $\boldsymbol{n}$ such that $\mathbf{d}t(\boldsymbol{u}) = \mathbf{d}t(\boldsymbol{n}) = 1$
$\boldsymbol{V} := \frac{1}{\Gamma}\boldsymbol{u} - \boldsymbol{n} = \frac{1}{\alpha}(\boldsymbol{w} - \boldsymbol{m})$	Spatial fluid velocity relative to Eulerian observer
$\boldsymbol{v} := \boldsymbol{w} - \boldsymbol{\partial}_t = \alpha\boldsymbol{V} - \boldsymbol{\beta}$	Spatial fluid velocity relative to observer at fixed x^i

With these definitions, we can establish the following relations,

$$\begin{aligned}
&\perp\boldsymbol{u} = \boldsymbol{u} + (\boldsymbol{n}\cdot\boldsymbol{u})\boldsymbol{n} = \boldsymbol{u} - \Gamma\boldsymbol{n} = \Gamma\left(\frac{1}{\Gamma}\boldsymbol{u} - \boldsymbol{n}\right) = \Gamma\boldsymbol{V}\,, \\
&\rho = \boldsymbol{T}(\boldsymbol{n}, \boldsymbol{n}) = (\tilde{\rho} + \tilde{p})(\boldsymbol{u}\cdot\boldsymbol{n})^2 - \tilde{p} = \Gamma^2(\tilde{\rho} + \tilde{p}) - \tilde{p}\,, \\
&\boldsymbol{j} = -\perp\boldsymbol{T}(\boldsymbol{n},\, .\,) = -(\tilde{\rho} + \tilde{p})(\boldsymbol{u}\cdot\boldsymbol{n})\perp\boldsymbol{u} - \tilde{p}\perp\boldsymbol{g}(\boldsymbol{n},\, .\,) = \Gamma^2(\tilde{\rho} + \tilde{p})\boldsymbol{V} = (\rho + \tilde{p})\boldsymbol{V}\,, \\
&\boldsymbol{S} = \perp\boldsymbol{T}(\,.\,,\,.\,) = (\tilde{\rho} + \tilde{p})\perp\boldsymbol{u}\otimes\perp\boldsymbol{u} + \tilde{p}\perp\boldsymbol{g} = \Gamma^2(\tilde{\rho} + \tilde{p})\boldsymbol{V}\otimes\boldsymbol{V} + \tilde{p}\boldsymbol{\gamma} = (\rho + \tilde{p})\boldsymbol{V}\otimes\boldsymbol{V} + \tilde{p}\boldsymbol{\gamma}\,.
\end{aligned} \tag{5.54}$$

5.3.1 Baryon conservation

With this zoo of velocities dealt with, we can finally start addressing the dynamics of perfect fluids. For this purpose, we consider the baryon number density $n_{\rm B}$ which defines a baryon flow $J^\alpha_{\rm B} = n_{\rm B} u^\alpha$. Assuming that baryons are neither destroyed nor created out of nothing, the baryon flow is conserved,

$$\nabla_\mu J^\mu_{\rm B} = 0\,. \tag{5.55}$$

The baryon number density measured by the Eulerian observer is

$$\mathcal{N}_{\rm B} := -\boldsymbol{J}_{\rm B}\cdot\boldsymbol{n} = -n_{\rm B}\boldsymbol{u}\cdot\boldsymbol{n} \overset{(5.44)}{=} \Gamma n_{\rm B}\,. \tag{5.56}$$

As we would expect, the observed baryon density is amplified by one Lorentz factor Γ representing the Lorentz contraction in the direction of motion.

We likewise define the baryon number current measured by the Eulerian observer as the spatial projection of $\boldsymbol{J}_{\rm B}$ onto Σ_t,

$$\boldsymbol{j}_{\rm B} := \perp\boldsymbol{J}_{\rm B} \qquad \text{or} \qquad j^\alpha_{\rm B} := \perp^\alpha{}_\mu J^\mu_{\rm B}\,. \tag{5.57}$$

Before we convert the Baryon conservation law $\boldsymbol{\nabla}\boldsymbol{J}_{\rm B} = 0$ into 3+1 form, we recall two useful relations. First,

$$\nabla_\alpha n_\beta = -K_{\alpha\beta} - n_\alpha a_\beta \qquad \Rightarrow \qquad \nabla_\mu n^\mu = -K - n_\mu a^\mu = -K\,, \tag{5.58}$$

and, second, for any spatial vector $\boldsymbol{A}$ (i.e. $\boldsymbol{A}\cdot\boldsymbol{n} = 0$), we have

$$\begin{aligned} D_\mu A^\mu &= \perp^\rho{}_\mu \perp^\mu{}_\sigma \nabla_\rho A^\sigma = \perp^\rho{}_\sigma \nabla_\rho A^\sigma = (\delta^\rho{}_\sigma + n^\rho n_\sigma)\nabla_\rho A^\sigma = \nabla_\rho A^\rho - n^\rho A^\sigma \nabla_\rho n_\sigma = \nabla_\rho A^\rho - A^\sigma a_\sigma \\ \Rightarrow \nabla_\sigma A^\sigma &= D_\sigma A^\sigma + A^\sigma \frac{\partial_\rho \alpha}{\alpha}\,. \end{aligned} \tag{5.59}$$

With these results, we can write

$$\begin{aligned} & 0 = \nabla_\mu(n_{\rm B}u^\mu) = \nabla_\mu\left[n_{\rm B}\Gamma(n^\mu + V^\mu)\right] = \nabla_\mu(\mathcal{N}_{\rm B}n^\mu + \mathcal{N}_{\rm B}V^\mu) \\ \Rightarrow\quad & n^\mu\nabla_\mu\mathcal{N}_{\rm B} + \mathcal{N}_{\rm B}\underbrace{\nabla_\mu n^\mu}_{=-K} + \nabla_\mu(\mathcal{N}_{\rm B}V^\mu) = 0 \qquad \Big|\ \boldsymbol{n}\cdot\boldsymbol{V} = 0 \\ \Rightarrow\quad & \mathcal{L}_{\boldsymbol{n}}\mathcal{N}_{\rm B} - K\mathcal{N}_{\rm B} + D_\mu(\mathcal{N}_{\rm B}V^\mu) + \mathcal{N}_{\rm B}V^\mu\frac{\partial_\mu\alpha}{\alpha} \\ \Rightarrow\quad & \partial_t\mathcal{N}_{\rm B} - \beta^\mu\partial_\mu\mathcal{N}_{\rm B} + D_\mu(\alpha\mathcal{N}_{\rm B}V^\mu) - \alpha K\mathcal{N}_{\rm B} = 0 \qquad \Big|\ \alpha\boldsymbol{V} = \boldsymbol{v} + \boldsymbol{\beta} \\ \Rightarrow\quad & \partial_t\mathcal{N}_{\rm B} - \beta^\mu\partial_\mu\mathcal{N}_{\rm B} + D_\mu(v^\mu\mathcal{N}_{\rm B}) + D_\mu(\beta^\mu\mathcal{N}_{\rm B}) - \alpha K\mathcal{N}_{\rm B} \\ \Rightarrow\quad & \partial_t\mathcal{N}_{\rm B} + D_\mu(v^\mu\mathcal{N}_{\rm B}) + \mathcal{N}_{\rm B}(D_\mu\beta^\mu - \alpha K) = 0\,. \end{aligned} \tag{5.60}$$

In practice, we might find the 3rd line from bottom the most convenient of these expressions, since $\boldsymbol{V}$ is the velocity variable we will mostly be working with in the conservation laws for energy and momentum further below.

5.3.2 Energy conservation

The evolution of the energy density for a generic energy-momentum tensor is given by Eq. (5.7) above, which we now write in terms of purely spatial indices in the form

$$\partial_t \rho - \beta^m \partial_m \rho + \alpha D_m j^m + 2 j^m \partial_m \alpha - \alpha \rho K - \alpha S^{mn} K_{mn} \,. \tag{5.61}$$

The momentum density and stress tensor are given in terms of the perfect fluid variables by the equations in box (5.54). We can straightforwardly plug these in and obtain after some minor term management

$$\boxed{\partial_t \rho - \beta^m \partial_m \rho + \alpha D_m[(\rho+\tilde{p})V^m] + 2(\rho+\tilde{p})V^m \partial_m \alpha - \alpha(\rho+\tilde{p})(K + K_{mn}V^m V^n) = 0 \,. \quad (5.62)}$$

5.3.3 Conservation of momentum and the Euler equation

For the Euler equation that determines the conservation of linear momentum, we have to do a bit more work than for the energy conservation. We start by recalling the generic evolution equation for the momentum density (5.13) which we write with spatial indices as

$$\partial_t j_i - \beta^m \partial_m j_i - j_m \partial_i \beta^m + \alpha D_m S^m{}_i + S^m{}_i \partial_m \alpha - \alpha K j_i + \rho \partial_i \alpha \,. \tag{5.63}$$

Using the relations

$$j_i = (\rho+\tilde{p})V_i \,, \qquad S_{ij} = (\rho+\tilde{p})V_i V_j + \tilde{p}\gamma_{ij} \,, \tag{5.64}$$

together with Eq. (5.62), we can eliminate the time derivative of ρ by writing

$$\begin{aligned}
(\partial_t - \beta^m \partial_m) j_i &= V_i(\partial_t - \beta^m \partial_m)(\rho+\tilde{p}) + (\rho+\tilde{p})(\partial_t - \beta^m\partial_m)V_i \\
&= (\rho+\tilde{p})(\partial_t - \beta^m \partial_m)V_i + V_i(\partial_t - \beta^m\partial_m)\tilde{p} \\
&\quad + V_i\left\{-\alpha D_m[(\rho+\tilde{p})V^m] - 2(\rho+\tilde{p})V^m\partial_m\alpha + \alpha(\rho+\tilde{p})(K + K_{mn}V^mV^n)\right\} \,.
\end{aligned}$$

For the evolution of the momentum density, we thus find

$$\begin{aligned}
0 &= (\rho+\tilde{p})(\partial_t - \beta^m\partial_m)V_i + V_i(\partial_m - \beta^m\partial_m)\tilde{p} - \alpha V_i D_m[(\rho+\tilde{p})V^m] - 2V_i(\rho+\tilde{p})V^m\partial_m\alpha \\
&\quad + \alpha V_i(\rho+\tilde{p})(K + K_{mn}V^mV^n) - (\rho+\tilde{p})V_m\partial_i\beta^m + \alpha D_m[(\rho+\tilde{p})V^mV_i + \tilde{p}\gamma^m{}_i] \\
&\quad + (\rho+\tilde{p})V^mV_i\partial_m\alpha + \tilde{p}\gamma^m{}_i\partial_m\alpha - \alpha K(\rho+\tilde{p})V_i + \rho\partial_i\alpha \\
\Rightarrow \quad 0 &= \partial_t V_i - \beta^m\partial_m V_i + \frac{V_i}{\rho+\tilde{p}}(\partial_m - \beta^m\partial_m)\tilde{p} + \alpha V^m D_m V_i - V_iV^m\partial_m\alpha + \alpha K_{mn}V^mV^nV_i \\
&\quad - V_m\partial_i\beta^m + \frac{\alpha}{\rho+\tilde{p}}D_i\tilde{p} + \partial_i\alpha \\
\Rightarrow \quad 0 &= \partial_t V_i - \beta^m\partial_m V_i + \frac{1}{\rho+\tilde{p}}\left[\alpha D_i\tilde{p} + V_i(\partial_t\tilde{p} - \beta^m\partial_m\tilde{p})\right] - V_m\partial_i\beta^m + \partial_i\alpha \\
&\quad + \alpha V^m D_m V_i - V_iV^m\partial_m\alpha + \alpha K_{mn}V^mV^n
\end{aligned} \tag{5.65}$$

We can further manipulate this equation by realizing

$$\begin{aligned}
\boldsymbol{v} &= \alpha \boldsymbol{V} - \boldsymbol{\beta} \\
\Rightarrow \alpha V^m D_m V_i &= v^m D_m V_i + \beta^m D_m V_i \\
\Rightarrow -\beta^m \partial_m V_i - V_m \partial_i \beta^m + \alpha V^m {}_{D_m} V_i &= -\beta^m \partial_m V_i - V_m \partial_i \beta^m + v^m D_m V_i + \beta^m D_m V_i \\
&= -\beta^m D_m V_i - V_m D_i \beta^m + v^m D_m V_i + \beta^m D_m V_i \\
&= v^m D_m V_i - V_m D_i \beta^m \,, \qquad (5.66)
\end{aligned}$$

whence,

$$\partial_t V_i + v^m D_m V_i - V_m D_i \beta^m + \alpha K_{mn} V^m V^n V_i + \partial_i \alpha - V_i V^m \partial_m \alpha + \frac{1}{\rho + \tilde{p}} \left[\alpha D_i \tilde{p} + V_i (\partial_t \tilde{p} - \beta^m \partial_m \tilde{p}) \right] = 0 \,. \qquad (5.67)$$

I am not sure this is an improvement over the first version which I am therefore inclined to keep for now,

$$\boxed{\begin{aligned}
0 = {} & \partial_t V_i - \beta^m \partial_m V_i + \frac{1}{\rho + \tilde{p}} \left[\alpha D_i \tilde{p} + V_i (\partial_t \tilde{p} - \beta^m \partial_m \tilde{p}) \right] - V_m \partial_i \beta^m + \partial_i \alpha \\
& + \alpha V^m D_m V_i - V_i V^m \partial_m \alpha + \alpha K_{mn} V^m V^n \qquad (5.68)
\end{aligned}}$$

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