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1 Coordinates

We consider a D dimensional spacetime with Cartesian coordinates

$$x^A = (t,\, x,\, y,\, z,\, u,\, v,\, w, \ldots) = (t,\, x,\, y,\, z,\, w^a)\,, \tag{1.1}$$

where capital latin indices run from 0 to $D-1$ and lower case early latin indices from 4 to $D-1$.
We assume $SO(D-3)$ symmetry and therefore have rotational Killing vectors in the planes spanned by z, w^a or w^a, w^b. We consider coordinate transformations which replace two of the Cartesian coordinates $(z,\, w^a)$ by polar coordinates denoted by $(\rho,\, \varphi)$.
Notation: For simplicity, we will use the following set of coordinates which covers all possible cases of the general system.

- The coordinate x is used to represent either x or y. Any result obtained for x is valid for y after replacing any occurrence of x by y.
- z exclusively stands for itself.
- $w,\, u,\, v$ stand for any but different ones of the coordinates w^a, i.e. $w \neq u$, $u \neq v$ and $v \neq w$.
- None of the transformation involves the time coordinate, so that for most purposes we can ignore t.

Assuming that the transformation to polar coordinates involved z and w^n (n fixed now), the polar coordinates are denoted by

$$x^{\bar{M}} = (t,\, x,\, y,\, \rho,\, w^4,\, \ldots,\, w^{\bar{n}-1},\, \varphi,\, w^{\bar{n}+1},\, \ldots,\, w^{D-1})\,. \tag{1.2}$$

This notation becomes quite cumbersome and this is the main reason for our introduction of the w, v and u which we shall use in the remainder of these notes. The relevant parts of the two coordinate systems are then given by

$$x^A = (x,\, z,\, w,\, u,\, v) \quad \leftrightarrow \quad x^{\bar{M}} = (x,\, \rho,\, \varphi,\, u,\, v)\,, \tag{1.3}$$

and likewise if we consider the zu or zv plane.
The coordinate transformations are given by

$$\begin{aligned} \rho &= \sqrt{z^2 + w^2}\,, \qquad & z &= \rho\cos\varphi\,, \\ \varphi &= \arctan\tfrac{w}{z}\,, \qquad & w &= \rho\sin\varphi\,, \end{aligned} \tag{1.4}$$

and the Jacobian matrices by

$$\begin{pmatrix} \frac{\partial z}{\partial \rho} = \frac{z}{\rho} & \frac{\partial z}{\partial \varphi} = -w \\ \frac{\partial w}{\partial \rho} = \frac{w}{\rho} & \frac{\partial w}{\partial \varphi} = z \end{pmatrix}, \qquad \begin{pmatrix} \frac{\partial \rho}{\partial z} = \frac{z}{\sqrt{w^2+z^2}} & \frac{\partial \rho}{\partial w} = \frac{w}{\sqrt{w^2+z^2}} \\ \frac{\partial \varphi}{\partial z} = -\frac{w}{z^2+w^2} & \frac{\partial \varphi}{\partial w} = \frac{z}{z^2+w^2} \end{pmatrix}. \tag{1.5}$$

2 Lie derivatives and spherical symmetry

Spherical symmetry in N dimensions implies the existence of $N(N-1)/2$ rotational Killing vectors, one for each plane in the N dimensions. For Killing vector ξ, we have

$$\mathcal{L}_\xi g_{AB} = 0\,, \tag{2.1}$$

which also holds in the "barred" coordinate system, i.e. $\mathcal{L}_\xi g_{\bar{M}\bar{N}} = 0$.
Specifically, for the rotational symmetry in the zw plane, $\xi = \partial_\varphi \quad \Leftrightarrow \quad \xi^{\bar{M}} = \delta^{\bar{M}}{}_{\bar{n}}$, where $\bar{n}$ is the (fixed) index of the coordinate φ in the system $x^{\bar{M}}$; cf. Eq. (1.2). Plugging this into the definition of the Lie derivative of the metric, we obtain

$$\mathcal{L}_\xi g_{\bar{M}\bar{N}} = \xi^{\bar{K}}\partial_{\bar{K}} g_{\bar{M}\bar{N}} + \underbrace{(\partial_{\bar{M}}\xi^{\bar{K}})}_{=0} g_{\bar{K}\bar{N}} + \underbrace{(\partial_{\bar{N}}\xi^{\bar{K}})}_{=0} g_{\bar{M}\bar{K}} = \partial_\varphi g_{\bar{M}\bar{N}} = 0\,. \tag{2.2}$$

This relation holds separately for all metric components. From

$$\partial_\varphi g^{\bar{A}\bar{B}} = -g^{\bar{A}\bar{M}} g^{\bar{B}\bar{N}} \partial_\varphi g_{\bar{M}\bar{N}} \stackrel{!}{=} 0\,, \tag{2.3}$$

we immediately obtain the same result for the inverse metric. All ADM variables can be constructed sequentially from the metric, its derivative and its inverse, using at every stage exclusively variables whose φ derivative has already been shown to vanish,

$$\begin{aligned}
\gamma_{IM} &= g_{IM}\,,\\
\beta^I &= \gamma^{IM} g_{0M}\,,\\
\alpha &= \frac{1}{\sqrt{-g^{00}}}\,,\\
K_{IJ} &= -\frac{1}{2\alpha}\left(\partial_t\gamma_{IJ} - \beta^M\partial_{\bar{M}}\gamma_{\bar{I}\bar{J}} - \gamma_{MJ}\partial_I\beta^M - \gamma_{IM}\partial_J\beta^M\right)\,.
\end{aligned} \tag{2.4}$$

This construction is valid in any coordinates adapted to the space-time decomposition and we therefore ommit the "bars" over the indices. Using also the fact that ∂_φ commutes with all other partial derivative operators and repeating the procedure (2.3) for the inverse spatial metric $\gamma^{\bar{I}\bar{J}}$, we conclude that the ϕ derivative of all ADM variables vanishes.
Furthermore, all BSSN variables are constructed directly from the ADM variables, as well as the derivative of the spatial metric and its inverse,

$$\begin{aligned}
\phi &= \tfrac{1}{4(D-1)}\ln\gamma, & K &= \gamma^{MN}K_{MN},\\
\tilde{\gamma}_{IJ} &= e^{-4\phi}\gamma_{IJ} & \Leftrightarrow\quad \tilde{\gamma}^{IJ} &= e^{4\phi}\gamma^{IJ},\\
\tilde{A}_{IJ} &= e^{-4\phi}\left(K_{IJ} - \tfrac{1}{D-1}\gamma_{IJ}K\right) & \Leftrightarrow\quad K_{IJ} &= e^{4\phi}\left(\tilde{A}_{IJ} + \frac{1}{D-1}\tilde{\gamma}_{IJ}K\right)\,,\\
\tilde{\Gamma}^i &= \tilde{\gamma}^{MN}\tilde{\Gamma}^I_{MN} = \tilde{\gamma}^{MN}\tilde{\gamma}^{IK}\left(\partial_M\tilde{\gamma}_{NK} - \tfrac{1}{2}\partial_K\tilde{\gamma}_{MN}\right)\,,
\end{aligned} \tag{2.5}$$

Again, this construction is valid in any coordinate system adapted to the space-time split. In particular, we have now obtained the result that the φ derivative of all BSSN variables vanishes.
We need one further ingredient to derive the relations imposed upon the Cartesian components and derivatives of the BSSN variables. For the case of rotational symmetry in the φ direction, we can always choose coordinates such that the off-diagonal metric components $g_{\bar{M}\varphi}$ vanish. This property is carried through to the BSSN variables $\tilde{\gamma}_{IJ}$ and $\tilde{A}_{IJ}$ in the same manner as described above for the φ derivatives. The construction of the shift vector in Eq. (2.4) implies that $\beta^\varphi = 0$ because $\gamma^{\varphi M} = 0$. Likewise, the vanishing of $\tilde{\gamma}^{\varphi K}$ in the construction of $\tilde{\Gamma}^I$ in Eq. (2.5) implies that $\tilde{\Gamma}^\varphi = 0$.

3 Components and derivatives in the w direction: An example

We shall illustrate for the case of the ww component of a tensor density of weight $\mathcal{W}$, how we obtain relations between the different tensor components and their derivatives from rotational symmetry. We first note that a tensor density of weight transforms under a coordinate transformation (1.4) according to

$$T_{\bar{A}\bar{B}} = D^{\mathcal{W}} \frac{\partial x^M}{\partial x^{\bar{A}}} \frac{\partial x^N}{\partial x^{\bar{B}}} T_{MN} \,, \tag{3.1}$$

where

$$D \equiv \det\left(\frac{\partial x^M}{\partial x^{\bar{A}}}\right) = \begin{vmatrix} \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \varphi} \\ \frac{\partial w}{\partial \rho} & \frac{\partial w}{\partial \varphi} \end{vmatrix} = \rho \,, \tag{3.2}$$

is the determinant of the Jacobian.

Our goal now is to trade derivatives in directions off the xyz hyperplane, i.e. derivatives with respect to w, u, $v, \ldots$, for derivatives in the x, y or z direction. The procedure to obtain such relations is similar for all types of variables and we will illustrate it for the case of the T_{ww} component. For this purpose, we write

$$\partial_w T_{ww} = \frac{\partial \rho}{\partial w} \partial_\rho T_{ww} + \frac{\partial \varphi}{\partial w} \partial_\varphi T_{ww} \,, \tag{3.3}$$

and substitute for T_{ww} using

$$T_{ww} = D^{-\mathcal{W}} \left(\frac{\partial \rho}{\partial w}\frac{\partial \rho}{\partial w} T_{\rho\rho} + 2\frac{\partial \rho}{\partial w}\frac{\partial \varphi}{\partial w} \underbrace{T_{\rho\varphi}}_{=0} + \frac{\partial \varphi}{\partial w}\frac{\partial \varphi}{\partial w} T_{\varphi\varphi} \right) . \tag{3.4}$$

This is the first point where we have used a symmetry property: $T_{\rho\varphi} = 0$. Inserting (3.4) in (3.3) gives us, among plenty of other terms, derivatives of the $\rho\rho$ and $\varphi\varphi$ components of T with respect to the coordinates ρ and φ. The vanishing of the latter is the second point where rotational symmetry enters whereas the ρ derivatives are obtained from the standard transformation,

$$\begin{aligned}
\partial_\rho T_{\rho\rho} &= \left(\frac{\partial z}{\partial \rho}\partial_z + \frac{\partial w}{\partial \rho}\partial_w\right) \left[D^{\mathcal{W}} \left(\frac{\partial z}{\partial \rho}\frac{\partial z}{\partial \rho} T_{zz} + 2\frac{\partial z}{\partial \rho}\frac{\partial w}{\partial \rho} T_{zw} + \frac{\partial w}{\partial \rho}\frac{\partial w}{\partial \rho} T_{ww} \right) \right] , \\
\partial_\rho T_{\varphi\varphi} &= \left(\frac{\partial z}{\partial \rho}\partial_z + \frac{\partial w}{\partial \rho}\partial_w\right) \left[D^{\mathcal{W}} \left(\frac{\partial z}{\partial \varphi}\frac{\partial z}{\partial \varphi} T_{zz} + 2\frac{\partial z}{\partial \varphi}\frac{\partial w}{\partial \varphi} T_{zw} + \frac{\partial w}{\partial \varphi}\frac{\partial w}{\partial \varphi} T_{ww} \right) \right] , \\
\partial_\varphi T_{\rho\rho} &= 0 \,, \\
\partial_\varphi T_{\varphi\varphi} &= 0 \,.
\end{aligned} \tag{3.5}$$

Gathering all terms is best done with a computational algebra package such as `Maple` or `Mathematica`. The final step is to set $w = 0$ in the resulting expression because eventually, we only need the relations in the computational domain, i.e. the xyz hyperplane. After plugging everything into (3.3) and setting $w = 0$, we obtain $\partial_w T_{ww} = 0$. We can apply additional Cartesian derivative operators to the complete expression for $\partial_w T_{ww}$ (setting $w = 0$ only after having taken the derivative!) and thus obtain the following relations

$$\begin{aligned}
\partial_w T_{ww} &= \partial_z \partial_w T_{ww} = \partial_x \partial_w T_{ww} = \partial_u \partial_w T_{ww} = 0 \,, \\
\partial_w \partial_w T_{ww} &= \frac{\partial_z T_{ww}}{z} + 2\frac{T_{zz} - T_{ww}}{z^2} \,.
\end{aligned} \tag{3.6}$$

We still need to calculate the u derivative of T_{ww} to complete this analysis. For this purpose we start with expression (3.4) for T_{ww} and replace therein

$$T_{\rho\rho} = D^{\mathcal{W}} \left(\frac{\partial z}{\partial \rho}\frac{\partial z}{\partial \rho} T_{zz} + 2\frac{\partial z}{\partial \rho}\frac{\partial w}{\partial \rho} T_{zw} + \frac{\partial w}{\partial \rho}\frac{\partial w}{\partial \rho} T_{ww} \right), \tag{3.7}$$

$$T_{\varphi\varphi} = D^{\mathcal{W}} \left(\frac{\partial z}{\partial \varphi}\frac{\partial z}{\partial \varphi} T_{zz} + 2\frac{\partial z}{\partial \varphi}\frac{\partial w}{\partial \varphi} T_{zw} + \frac{\partial w}{\partial \varphi}\frac{\partial w}{\partial \varphi} T_{ww} \right). \tag{3.8}$$

This gives us an expression for T_{ww} exclusively in terms of Cartesian components and coordinates which is constrained under the rotational symmetry in the zw plane; having obtained this expression, we can forget about the coordinate system (ρ, φ).
Instead, we now consider a separate coordinate transformation to accomodate the rotation in the zu plane

$$\begin{aligned} r &= \sqrt{z^2+u^2}\,, & z &= r\cos\psi\,, \\ \psi &= \arctan\tfrac{u}{z}\,, & w &= r\sin\psi\,, \end{aligned} \tag{3.9}$$

and the Jacobian matrices by

$$\begin{pmatrix} \frac{\partial z}{\partial r} = \frac{z}{r} & \frac{\partial z}{\partial \psi} = -u \\ \frac{\partial u}{\partial r} = \frac{u}{r} & \frac{\partial u}{\partial \psi} = z \end{pmatrix}, \qquad \begin{pmatrix} \frac{\partial r}{\partial z} = \frac{z}{\sqrt{u^2+z^2}} & \frac{\partial r}{\partial u} = \frac{u}{\sqrt{u^2+z^2}} \\ \frac{\partial \psi}{\partial z} = -\frac{u}{z^2+u^2} & \frac{\partial \psi}{\partial u} = \frac{z}{z^2+u^2} \end{pmatrix}. \tag{3.10}$$

Note that rotational symmetry now implies that all ψ derivatives vanish, so that we can write

$$\partial_u T_{ww} = \frac{\partial r}{\partial u}\partial_r T_{ww} + \frac{\partial \psi}{\partial u}\underbrace{\partial_\psi T_{ww}}_{=0} = \frac{\partial r}{\partial u}\left(\frac{\partial u}{\partial r}\partial_u T_{ww} + \frac{\partial z}{\partial r}\partial_z T_{ww} \right). \tag{3.11}$$

Gathering all terms and setting $u = 0$, $w = 0$ we find that $\partial_u T_{ww}$ vanishes. As before, we can directly take additional derivatives of $\partial_u T_{ww}$ and afterwards set $u = w = 0$. This gives us the

$$\begin{aligned} \partial_u T_{ww} &= \partial_v \partial_u T_{ww} = \partial_w \partial_u T_{ww} = \partial_x \partial_u T_{ww} = \partial_z \partial_u T_{ww} = 0\,, \\ \partial_u \partial_u T_{ww} &= \frac{\partial_z T_{ww}}{z}\,. \end{aligned} \tag{3.12}$$

Equations (3.6) and (3.12) allow us to replace all derivatives of T_{ww} off the computational domain in terms of expressions inside the xyz hyperplane. We therefore need to introduce additonal grid functions, such as T_{ww}, but do not need to extend the computational domain beyond the "3+1" case.
We finally note that spherical symmetry also gives us constraints on the components of T. In analogy to (3.4) we construct

$$T_{zw} = D^{-\mathcal{W}} \left(\frac{\partial \rho}{\partial w}\frac{\partial \rho}{\partial z} T_{\rho\rho} + \frac{\partial \varphi}{\partial w}\frac{\partial \varphi}{\partial z} T_{\varphi\varphi} \right), \tag{3.13}$$

$$T_{zz} = D^{-\mathcal{W}} \left(\frac{\partial \rho}{\partial z}\frac{\partial \rho}{\partial z} T_{\rho\rho} + \frac{\partial \varphi}{\partial z}\frac{\partial \varphi}{\partial z} T_{\varphi\varphi} \right), \tag{3.14}$$

where we have used that from rotational symmetry $T_{\rho\varphi} = 0$. Substituting in Eqs. (3.4), (3.13), (3.14) the expressions (3.7) and (3.8) for $T_{\rho\rho}$ and $T_{\varphi\varphi}$ and setting $w = 0$, we obtain $T_{zw} = 0$. In place of z we could have chosen any other Cartesian coordinate that forms a rotationally symmetric plane with

w such that the construction giving $T_{zw} = 0$ likewise gives $T_{uw} = 0$. For T_{ww} and T_{zz} we merely obtain identities with no further information, but we note that we can always rescale the Cartesian coordinates such that $T_{uu} = T_{ww}$. Finally, we can write

$$T_{xw} = \frac{\partial \rho}{\partial w} T_{x\rho} + \frac{\partial \varphi}{\partial w} \underbrace{T_{x\varphi}}_{=0} = \frac{w}{\rho} T_{x\rho} \overset{w \to 0}{=} 0 \,. \tag{3.15}$$

4 Summary of all expressions

Applying the procedure described in the previous section to scalar, vector and rank-2 tensors or densities thereof, we obtain the following relations for derivatives in Cartesian coordinates.
For a scalar function Ψ, we have

$$\begin{aligned} \partial_w \Psi &= \partial_x \partial_w \Psi = \partial_z \partial_w \Psi = \partial_u \partial_w \Psi = 0 \,, \\ \partial_w \partial_w \Psi = \frac{\partial_z \Psi}{z} \,. \end{aligned} \tag{4.1}$$

For a vector or vector density, we find

$$\begin{aligned} V^w &= \partial_x V^w = \partial_z V^w = \partial_u V^w = 0 \,, \\ \partial_w V^w &= \frac{V^z}{z} \,, \\ \partial_w \partial_w V^w &= 2\frac{V^w}{z^2} + 2\frac{\partial_w V^z}{z} = 0 + 0 = 0 \,, \\ \partial_u \partial_w V^w &= \frac{\partial_u V^z}{z} = 0 \,, \\ \partial_x \partial_w V^w &= \frac{\partial_x V^z}{z} \,, \\ \partial_z \partial_w V^w &= \frac{\partial_z V^z}{z} - \frac{V^z}{z^2} \,, \\ \partial_w V^z &= \partial_x \partial_w V^z = \partial_z \partial_w V^z = \partial_u \partial_w V^z = 0 \,, \\ \partial_w \partial_w V^z &= \frac{\partial_z V^z}{z} - \frac{V^z}{z^2} \,, \\ \partial_w V^x &= \partial_x \partial_w V^x = \partial_z \partial_w V^x = \partial_u \partial_w V^x = 0 \,, \\ \partial_w \partial_w V^x &= \frac{\partial_z V^x}{z} \,. \end{aligned} \tag{4.2}$$

Finally, for rank-2 tensors and denstities, we get

$$
\begin{aligned}
T_{ww} &= T_{uu}\,, \\
T_{xw} &= T_{zw} = T_{uw} = 0\,, \\
\partial_w T_{ww} &= \partial_x\partial_w T_{ww} = \partial_z\partial_w T_{ww} = \partial_u\partial_w T_{ww} = 0\,, \\
\partial_w\partial_w T_{ww} &= 2\frac{T_{zz}-T_{ww}}{z^2} + \frac{\partial_z T_{ww}}{z}\,, \\
\partial_u T_{ww} &= \partial_x\partial_u T_{ww} = \partial_z\partial_u T_{ww} = \partial_w\partial_u T_{ww} = \partial_v\partial_u T_{ww} = 0\,, \\
\partial_u\partial_u T_{ww} &= \frac{\partial_z T_{ww}}{z}\,, \\
\partial_u T_{uw} &= \partial_x\partial_u T_{uw} = \partial_z\partial_u T_{uw} = \partial_u\partial_u T_{uw} = \partial_v\partial_u T_{uw} = 0\,, \\
\partial_u\partial_w T_{uw} &= \frac{\partial_u T_{zu}}{z} = \frac{T_{zz}-T_{ww}}{z^2}\,, \\
\partial_v T_{uw} &= \partial_x\partial_v T_{uw} = \partial_z\partial_v T_{uw} = \partial_{\tilde{v}}\partial_v T_{uw} = 0\,, \\
\partial_w T_{zw} &= \frac{T_{zz}-T_{ww}}{z}\,, \\
\partial_x\partial_w T_{zw} &= \frac{\partial_x T_{zz}-\partial_x T_{ww}}{z}\,, \\
\partial_z\partial_w T_{zw} &= -\frac{T_{zz}-T_{ww}}{z^2} + \frac{\partial_z T_{zz}-\partial_z T_{ww}}{z}\,, \\
\partial_w\partial_w T_{zw} &= 2\frac{\partial_w T_{zz}}{z} + 8\frac{T_{zw}}{z^2} - 2\frac{\partial_w T_{ww}}{z} = 0\,, \\
\partial_u\partial_w T_{zw} &= \frac{\partial_u T_{zz}}{z} - \frac{\partial_u T_{ww}}{z} = 0\,, \\
\partial_u T_{zw} &= \partial_x\partial_u T_{zw} = \partial_z\partial_u T_{zw} = \partial_u\partial_u T_{zw} = \partial_v\partial_u T_{zw} = 0\,, \\
\partial_w T_{zz} &= \partial_x\partial_w T_{zz} = \partial_z\partial_w T_{zz} = \partial_u\partial_w T_{zz} = 0\,, \\
\partial_w\partial_w T_{zz} &= -2\frac{T_{zz}-T_{ww}}{z^2} + \frac{\partial_z T_{zz}}{z}\,, \\
\partial_w T_{xz} &= \partial_x\partial_w T_{xz} = \partial_z\partial_w T_{xz} = \partial_u\partial_w T_{xz} = 0\,, \\
\partial_w\partial_w T_{xz} &= \frac{\partial_z T_{xz}}{z} - \frac{T_{xz}}{z^2}\,, \\
\partial_w T_{xw} &= \frac{T_{xz}}{z}\,, \\
\partial_x\partial_w T_{xw} &= \frac{\partial_x T_{xz}}{z}\,, \\
\partial_z\partial_w T_{xw} &= \frac{\partial_z T_{xz}}{z} - \frac{T_{xz}}{z^2}\,, \\
\partial_w\partial_w T_{xw} &= \frac{\partial_w T_{xz}}{z} + \frac{T_{xw}}{z^2} = 0\,, \\
\partial_u T_{xw} &= \partial_x\partial_u T_{xw} = \partial_z\partial_u T_{xw} = \partial_w\partial_u T_{xw} = \partial_u\partial_u T_{xw} = \partial_v\partial_u T_{xw} = 0\,, \\
\partial_w T_{xx} &= \partial_x\partial_x T_{xx} = \partial_z\partial_w T_{xx} = \partial_u\partial_w T_{xx} = 0\,, \\
\partial_w\partial_w T_{xx} &= \frac{\partial_z T_{xx}}{z}\,.
\end{aligned}
\tag{4.3}
$$

[US: Check the $\partial_w\partial_w T_{zz}$ term. We added a $-$ sign in front of the 2 here. See if hand written notes agree.] With all the expressions in place, we can now rewrite them in a form more convenient for calculating the modifications of the 3+1 dimensional BSSN equations. For this purpose we introduce the following notation: late latin indices i, j, k, ... cover the x, y and z directions as usual, i.e. run from 1 to 3 whereas early latin indices a, b, c, ... cover the additional coordinates w_1, w_2, ..., i.e. run from 4 to $D-1$. Furthermore, we use the indices z and w to denote the quasi-radial coordinate of the computational domain and a Cartesian coordinate off the hyperplane. It turns out that all components off the hyperplane are the same and w denotes the corresponding expressions. The z and w indices are therefore fixed and are *not* summed over repeated appearance as is done for other indices according to the Einstein summation convention. In this notation, we can reformulate Eqs. (4.1)-(4.3) as

$$\partial_a \Psi = 0\,, \tag{4.4}$$
$$\partial_i \partial_a \Psi = 0\,, \tag{4.5}$$
$$\partial_a \partial_b \Psi = \delta_{ab} \frac{\partial_z \Psi}{z}\,, \tag{4.6}$$

$$V^a = 0\,, \tag{4.7}$$
$$\partial_i V^a = 0\,, \tag{4.8}$$
$$\partial_a V^b = \delta^b{}_a \frac{V^z}{z}\,, \tag{4.9}$$
$$\partial_i \partial_a V^b = \delta^b{}_a \left(\frac{\partial_i V^z}{z} - \delta_{iz} \frac{V^z}{z^2} \right)\,, \tag{4.10}$$
$$\partial_a \partial_b V^c = 0\,, \tag{4.11}$$
$$\partial_a V^i = 0\,, \tag{4.12}$$
$$\partial_a \partial_b V^i = \delta_{ab} \left(\frac{\partial_z V^i}{z} - \delta^i_z \frac{V^z}{z^2} \right)\,, \tag{4.13}$$

$$T_{ab} = \delta_{ab} T_{ww}\,, \tag{4.14}$$
$$\partial_a T_{bc} = 0\,, \tag{4.15}$$
$$\partial_a \partial_b T_{cd} = (\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) \frac{T_{zz} - T_{ww}}{z^2} + \delta_{ab}\delta_{cd} \frac{\partial_z T_{ww}}{z}\,, \tag{4.16}$$
$$\partial_i \partial_a T_{bc} = 0\,, \tag{4.17}$$
$$T_{ia} = 0\,, \tag{4.18}$$
$$\partial_a T_{ib} = \delta_{ab} \frac{T_{iz} - \delta_{iz} T_{ww}}{z}\,, \tag{4.19}$$
$$\partial_a \partial_b T_{ic} = 0\,, \tag{4.20}$$
$$\partial_i \partial_a T_{jb} = \delta_{ab} \left(\frac{\partial_i T_{jz} - \delta_{jz} \partial_i T_{ww}}{z} - \delta_{iz} \frac{T_{jz} - \delta_{jz} T_{ww}}{z^2} \right)\,, \tag{4.21}$$
$$\partial_a T_{ij} = 0\,, \tag{4.22}$$
$$\partial_a \partial_b T_{ij} = \delta_{ab} \left(\frac{\partial_z T_{ij}}{z} - \frac{\delta_{iz} T_{jz} + \delta_{jz} T_{iz} - 2\delta_{iz}\delta_{jz} T_{ww}}{z^2} \right)\,, \tag{4.23}$$
$$\partial_i \partial_a T_{jk} = 0\,. \tag{4.24}$$

5 The BSSN equations

The next step in our derivation is to use the relations (4.1)-(4.3) in the D dimensional BSSN equations to trade derivatives off the xyz hyperplane for derivatives inside that plane. We first recall the BSSN equations which govern the time evolution of the variables defined in Eq. (2.5)

$$\begin{aligned}
\partial_t\phi &= \beta^M\partial_M\phi + \frac{1}{2(D-1)}(\partial_M\beta^M - \alpha K)\,, && (5.1)\\
\partial_t\tilde{\gamma}_{IJ} &= \beta^M\partial_M\tilde{\gamma}_{IJ} + 2\tilde{\gamma}_{M(I}\partial_{J)}\beta^M - \frac{2}{D-1}\tilde{\gamma}_{IJ}\partial_M\beta^M - 2\alpha\tilde{A}_{IJ}\,, && (5.2)\\
\partial_t K &= \beta^M\partial_M K - e^{-4\phi}\tilde{\gamma}^{MN}D_M D_N\alpha + \alpha\tilde{A}^{MN}\tilde{A}_{MN} + \frac{1}{D-1}\alpha K^2\\
&\quad + \frac{8\pi}{D-2}\alpha[S + (D-3)\rho] - \frac{2}{D-2}\alpha\Lambda\,, && (5.3)\\
\partial_t\tilde{A}_{IJ} &= \beta^M\partial_M\tilde{A}_{IJ} + 2\tilde{A}_{M(I}\partial_{J)}\beta^M - \frac{2}{D-1}\tilde{A}_{IJ}\partial_M\beta^M + \alpha K\tilde{A}_{IJ} - 2\alpha\tilde{A}_{IM}\tilde{A}^M{}_J\\
&\quad + e^{-4\phi}\left(\alpha\mathcal{R}_{IJ} - D_I D_J\alpha - 8\pi\alpha S_{IJ}\right)^{\rm TF}\,, && (5.4)\\
\partial_t\tilde{\Gamma}^I &= \beta^M\partial_M\tilde{\Gamma}^I + \frac{2}{D-1}\tilde{\Gamma}^I\partial_M\beta^M - \tilde{\Gamma}^M\partial_M\beta^I + \tilde{\gamma}^{MN}\partial_M\partial_N\beta^I + \frac{D-3}{D-1}\tilde{\gamma}^{IM}\partial_M\partial_N\beta^N - \sigma\partial_M\beta^M\mathcal{G}^I\\
&\quad + 2\tilde{A}^{IM}\left[2(D-1)\alpha\partial_M\phi - \partial_M\alpha\right] + 2\alpha\tilde{\Gamma}^I_{MN}\tilde{A}^{MN} - 2\frac{D-2}{D-1}\alpha\tilde{\gamma}^{IM}\partial_M K - 16\pi\frac{\alpha}{\chi}j^I\,. && (5.5)
\end{aligned}$$

Here the superscript "TF" denotes the tracefree part and we further use

$$\begin{aligned}
\mathcal{G}^I &= \tilde{\Gamma}^I - \tilde{\gamma}^{MN}\tilde{\Gamma}^I_{MN}\,, && (5.6)\\
\tilde{\Gamma}^I_{JK} &= \frac{1}{2}\tilde{\gamma}^{IM}(\partial_J\tilde{\gamma}_{KM} + \partial_K\tilde{\gamma}_{MJ} - \partial_M\tilde{\gamma}_{IJ})\,, && (5.7)\\
\Gamma^I_{JK} &= \tilde{\Gamma}^I_{JK} + 2\left(\delta^I{}_K\partial_J\phi + \delta^I{}_J\partial_K\phi - \tilde{\gamma}_{JK}\tilde{\gamma}^{IM}\partial_M\phi\right)\,, && (5.8)\\
\tilde{D}_I\tilde{D}_J\phi &= \partial_I\partial_J\phi - \tilde{\Gamma}^M_{JI}\partial_M\phi\,, && (5.9)\\
\mathcal{R}_{IJ} &= \tilde{\mathcal{R}}_{IJ} + \mathcal{R}^\phi_{IJ}\,, && (5.10)\\
\mathcal{R}^\phi_{IJ} &= 2(3-D)\tilde{D}_I\tilde{D}_J\phi - 2\tilde{\gamma}_{IJ}\tilde{\gamma}^{MN}\tilde{D}_M\tilde{D}_N\phi + 4(D-3)(\partial_I\phi\,\partial_J\phi - \tilde{\gamma}_{IJ}\tilde{\gamma}^{MN}\partial_M\phi\partial_N\phi)\,, && (5.11)\\
\tilde{\mathcal{R}}_{IJ} &= -\frac{1}{2}\tilde{\gamma}^{MN}\partial_M\partial_N\tilde{\gamma}_{IJ} + \tilde{\gamma}_{M(I}\partial_{J)}\tilde{\Gamma}^M + \tilde{\Gamma}^M\tilde{\Gamma}_{(IJ)M} + \tilde{\gamma}^{MN}\left[2\tilde{\Gamma}^K_{M(I}\tilde{\Gamma}_{J)KN} + \tilde{\Gamma}^K_{IM}\tilde{\Gamma}_{KJN}\right]\,, && (5.12)\\
D_I D_J\alpha &= \tilde{D}_I\tilde{D}_J\alpha - 2(\partial_I\phi\,\partial_J\alpha + \partial_J\phi\,\partial_I\alpha) + 2\tilde{\gamma}_{IJ}\tilde{\gamma}^{MN}\partial_M\phi\,\partial_N\alpha\,. && (5.13)
\end{aligned}$$

6 The BSSN equations with $SO(D-3)$ isometry

We will now replace ϕ by the variable $\chi = e^{-4\phi}$ and also write the symmetry components of the ADM variables

$$\gamma_{ab} = \delta_{ab}\gamma_{ww}\,, \quad K_{ab} = \delta_{ab}K_{ww}\,. \qquad (6.1)$$

The BSSN variables are now defined by

$$\begin{aligned}
\phi = \tfrac{1}{4(D-1)}\ln\gamma, \quad \gamma = \det\gamma_{IJ} = \gamma_{ww}^{n}\det\gamma_{ij} \qquad & K = \gamma^{MN}K_{MN} = \gamma^{mn}K_{mn} + n\gamma^{ww}K_{ww}, \\
\tilde{\gamma}_{ij} = \chi\gamma_{ij}\,, \quad \tilde{\gamma}_{ww} = \chi\gamma_{ww} \qquad & \Leftrightarrow \quad \tilde{\gamma}^{ij} = \frac{1}{\chi}\gamma^{ij}\,, \quad \tilde{\gamma}^{ww} = \frac{1}{\chi}\gamma^{ww}\,, \\
\tilde{A}_{ij} = \chi\left(K_{ij} - \tfrac{1}{D-1}\gamma_{ij}K\right) \qquad & \Leftrightarrow \quad K_{ij} = \frac{1}{\chi}\left(\tilde{A}_{ij} + \frac{1}{D-1}\tilde{\gamma}_{ij}K\right)\,, \\
\tilde{A}_{ww} = \chi\left(K_{ww} - \tfrac{1}{D-1}\gamma_{ww}K\right) \qquad & \Leftrightarrow \quad K_{ww} = \frac{1}{\chi}\left(\tilde{A}_{ww} + \frac{1}{D-1}\tilde{\gamma}_{ww}K\right)\,, \\
\tilde{\Gamma}^{i} = \tilde{\gamma}^{MN}\tilde{\Gamma}^{i}_{MN} = \tilde{\gamma}^{mn}\tilde{\Gamma}^{i}_{mn} + n\tilde{\gamma}^{ww}\tilde{\Gamma}^{i}_{ww}\,, &
\end{aligned} \tag{6.2}$$

where

$$\tilde{\Gamma}^{i}_{ww} = -\frac{1}{2}\tilde{\gamma}^{im}\partial_m\tilde{\gamma}_{ww} + \frac{\delta^{i}{}_{z} - \tilde{\gamma}^{zi}\tilde{\gamma}_{ww}}{z}\,. \tag{6.3}$$

The BSSN equations (5.1)-(5.5) now become

$$\partial_t\chi = \beta^m\partial_m\chi + \frac{2}{D-1}\chi\left(\alpha K - \partial_m\beta^m - n\frac{\beta^z}{z}\right)\,, \tag{6.4}$$

$$\partial_t\tilde{\gamma}_{ij} = \beta^m\partial_m\tilde{\gamma}_{ij} + 2\tilde{\gamma}_{m(i}\partial_{j)}\beta^m - \frac{2}{D-1}\tilde{\gamma}_{ij}\left(\partial_m\beta^m + n\frac{\beta^z}{z}\right) - 2\alpha\tilde{A}_{ij}\,, \tag{6.5}$$

$$\partial_t\tilde{\gamma}_{ww} = \beta^m\partial_m\tilde{\gamma}_{ww} - \frac{2}{D-1}\tilde{\gamma}_{ww}\left(\partial_m\beta^m - 3\frac{\beta^z}{z}\right) - 2\alpha\tilde{A}_{ww}\,, \tag{6.6}$$

$$\partial_t K = \beta^m\partial_m K - \chi\tilde{\gamma}^{mn}D_mD_n\alpha + \alpha\tilde{A}^{mn}\tilde{A}_{mn} + \frac{1}{D-1}\alpha K^2 + n\tilde{\gamma}^{ww}\left(\alpha\frac{\tilde{A}^2_{ww}}{\tilde{\gamma}_{ww}} - \chi D_wD_w\alpha\right) \tag{6.7}$$

$$\begin{aligned}
\partial_t\tilde{A}_{ij} = {}& \beta^m\partial_m\tilde{A}_{ij} + 2\tilde{A}_{m(i}\partial_{j)}\beta^m - \frac{2}{D-1}\tilde{A}_{ij}\left(\partial_m\beta^m + n\frac{\beta^z}{z}\right) + \alpha K\tilde{A}_{ij} - 2\alpha\tilde{A}_{im}\tilde{A}^{m}{}_{j} \\
& + \chi\left[\alpha\mathcal{R}_{ij} - D_iD_j\alpha\right]^{\mathrm{TF}}\,,
\end{aligned} \tag{6.8}$$

$$\begin{aligned}
\partial_t\tilde{A}_{ww} = {}& \beta^m\partial_m\tilde{A}_{ww} - \frac{2}{D-1}\tilde{A}_{ww}\left(\partial_m\beta^m - 3\frac{\beta^z}{z}\right) + \alpha\tilde{A}_{ww}(K - 2\tilde{\gamma}^{ww}\tilde{A}_{ww}) \\
& + \chi\left[\alpha\mathcal{R}_{ww} - D_wD_w\alpha\right]^{\mathrm{TF}}\,,
\end{aligned} \tag{6.9}$$

$$\begin{aligned}
\partial_t\tilde{\Gamma}^{i} = {}& \beta^m\partial_m\tilde{\Gamma}^{i} + \frac{2}{D-1}\tilde{\Gamma}^{i}\left(\partial_m\beta^m + n\frac{\beta^z}{z}\right) + \tilde{\gamma}^{mn}\partial_m\partial_n\beta^i + n\tilde{\gamma}^{ww}\left(\frac{\partial_z\beta^i}{z} - \delta^{i}{}_{z}\frac{\beta^z}{z^2}\right) \\
& - \tilde{\Gamma}^{m}\partial_m\beta^i + \frac{D-3}{D-1}\tilde{\gamma}^{im}\partial_m\partial_n\beta^n + \frac{D-3}{D-1}n\left(\tilde{\gamma}^{im}\frac{\partial_m\beta^z}{z} - \tilde{\gamma}^{iz}\frac{\beta^z}{z^2}\right) - 2\frac{D-2}{D-1}\alpha\tilde{\gamma}^{im}\partial_m K \\
& - \tilde{A}^{im}\left[(D-1)\alpha\frac{\partial_m\chi}{\chi} + 2\partial_m\alpha\right] + 2\alpha\left(\tilde{\Gamma}^{i}_{mn}\tilde{A}^{mn} + n\tilde{\Gamma}^{i}_{ww}\tilde{A}^{ww}\right) - \sigma\partial_M\beta^M\mathcal{G}^i\,,
\end{aligned} \tag{6.10}$$

In these equations we need the auxiliary quantities

$$\tilde{\Gamma}_{ijk} = \frac{1}{2}(\partial_j \tilde{\gamma}_{ki} + \partial_k \tilde{\gamma}_{ij} - \partial_i \tilde{\gamma}_{jk})\,, \tag{6.11}$$

$$\tilde{\Gamma}_{ajk} = \tilde{\Gamma}_{ibk} = \tilde{\Gamma}_{abc} = 0\,, \tag{6.12}$$

$$\tilde{\Gamma}_{abk} = \frac{1}{2}\delta_{ab}\partial_k \tilde{\gamma}_{ww}\,, \tag{6.13}$$

$$\tilde{\Gamma}_{iab} = \frac{1}{2}\delta_{ab}\left(2\frac{\tilde{\gamma}_{iz} - \delta_{iz}\tilde{\gamma}_{ww}}{z} - \partial_i \tilde{\gamma}_{ww}\right)\,, \tag{6.14}$$

$$\tilde{\Gamma}^i_{jk} = \frac{1}{2}\tilde{\gamma}^{im}\left(\partial_j \tilde{\gamma}_{km} + \partial_k \tilde{\gamma}_{mj} - \partial_m \tilde{\gamma}_{jk}\right)\,, \tag{6.15}$$

$$\tilde{\Gamma}^a_{jk} = \tilde{\Gamma}^i_{ak} = \tilde{\Gamma}^a_{bc} = 0\,, \tag{6.16}$$

$$\tilde{\Gamma}^a_{bk} = \frac{1}{2}\delta^a{}_b \tilde{\gamma}^{ww}\partial_k \tilde{\gamma}_{ww}\,, \tag{6.17}$$

$$\tilde{\Gamma}^i_{ab} = \delta_{ab}\tilde{\Gamma}^i_{ww}\,, \tag{6.18}$$

and

$$\begin{aligned}
\partial_M \beta^M \mathcal{G}^i &= \left(\partial_m \beta^m + n\frac{\beta^z}{z}\right)\left(\tilde{\Gamma}^i - \tilde{\gamma}^{mn}\tilde{\Gamma}^i_{mn} - n\tilde{\gamma}^{ww}\tilde{\Gamma}^i_{ww}\right), && (6.19)\\
\tilde{D}_i\tilde{D}_j\alpha &= \partial_i\partial_j\alpha - \tilde{\Gamma}^m_{ji}\partial_m\alpha\,, && (6.20)\\
\tilde{D}_a\tilde{D}_b\alpha &= \delta_{ab}\left(\frac{\partial_z\alpha}{z} - \tilde{\Gamma}^m_{ww}\partial_m\alpha\right), && (6.21)\\
D_iD_j\alpha &= \tilde{D}_i\tilde{D}_j\alpha + \frac{1}{2\chi}(\partial_i\chi\ \partial_j\alpha + \partial_j\chi\ \partial_i\alpha) - \frac{1}{2\chi}\tilde{\gamma}_{ij}\tilde{\gamma}^{mn}\partial_m\chi\ \partial_n\alpha\,, && (6.22)\\
D_wD_w\alpha &= \left(\frac{1}{2}\tilde{\gamma}^{mn}\partial_n\tilde{\gamma}_{ww} + \frac{\tilde{\gamma}^{zm}}{z}\tilde{\gamma}_{ww}\right)\partial_m\alpha - \frac{1}{2\chi}\tilde{\gamma}_{ww}\tilde{\gamma}^{mn}\partial_m\chi\ \partial_n\alpha\,, && (6.23)\\
\tilde{D}^M\tilde{D}_M\alpha &= \tilde{\gamma}^{mn}\tilde{D}_m\tilde{D}_n\alpha + n\left(\frac{1}{2}\tilde{\gamma}^{ww}\tilde{\gamma}^{mn}\partial_n\tilde{\gamma}_{ww} + \frac{\tilde{\gamma}^{zm}}{z}\right)\partial_m\alpha\,, && (6.24)\\
\frac{1}{\chi}D^MD_M\alpha &= \tilde{\gamma}^{mn}\tilde{D}_m\tilde{D}_n\alpha - \frac{D-3}{2\chi}\tilde{\gamma}^{mn}\partial_m\chi\ \partial_n\alpha + n\left(\frac{1}{2}\tilde{\gamma}^{ww}\tilde{\gamma}^{mn}\partial_n\tilde{\gamma}_{ww} + \frac{\tilde{\gamma}^{zm}}{z}\right)\partial_m\alpha\,, && (6.25)\\
[D_iD_j\alpha]^{\mathrm{TF}} &= D_iD_j\alpha - \frac{1}{D-1}\tilde{\gamma}_{ij}\left(\tilde{\gamma}^{mn}D_mD_n\alpha + n\tilde{\gamma}^{ww}D_wD_w\alpha\right), && (6.26)\\
[D_aD_b\alpha]^{\mathrm{TF}} &= \delta_{ab}\frac{1}{D-1}\left(3D_wD_w\alpha - \tilde{\gamma}_{ww}\tilde{\gamma}^{mn}D_mD_n\alpha\right), && (6.27)
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}_{ij} &= \mathcal{R}^\phi_{ij} + \tilde{\mathcal{R}}_{ij}\,, && (6.28)\\
\mathcal{R}_{ww} &= \mathcal{R}^\phi_{ww} + \tilde{\mathcal{R}}_{ww}\,, && (6.29)\\
\mathcal{R}^\phi_{ij} &= \frac{1}{2\chi}\tilde{\gamma}_{ij}\left[\tilde{\gamma}^{mn}\tilde{D}_m\tilde{D}_n\chi + n\left(\frac{1}{2}\tilde{\gamma}^{ww}\tilde{\gamma}^{mn}\partial_n\tilde{\gamma}_{ww} + \frac{\tilde{\gamma}^{mz}}{z}\right)\partial_m\chi - \frac{D-1}{2\chi}\tilde{\gamma}^{mn}\partial_m\chi\ \partial_n\chi\right]\\
&\quad + \frac{D-3}{2\chi}\left(\tilde{D}_i\tilde{D}_j\chi - \frac{1}{2\chi}\partial_i\chi\ \partial_j\chi\right) && (6.30)\\
\mathcal{R}^\phi_{ww} &= \frac{\tilde{\gamma}_{ww}}{2\chi}\left[\tilde{\gamma}^{mn}\tilde{D}_m\tilde{D}_n\chi + (2D-7)\left(\frac{1}{2}\tilde{\gamma}^{ww}\tilde{\gamma}^{mn}\partial_n\tilde{\gamma}_{ww} + \frac{\tilde{\gamma}^{mz}}{z}\right)\partial_m\chi - \frac{D-1}{2\chi}\tilde{\gamma}^{mn}\partial_m\chi\ \partial_n\chi\right], && (6.31)\\
\tilde{\mathcal{R}}_{ij} &= +n\tilde{\gamma}^{ww}\left[-\frac{1}{2}\frac{\partial_z\tilde{\gamma}_{ij}}{z} + \frac{\delta_{z(i}\tilde{\gamma}_{j)z} - \delta_{iz}\delta_{jz}\tilde{\gamma}_{ww}}{z^2} + \frac{\tilde{\gamma}^{ww}\tilde{\gamma}_{z(j} - \delta_{z(j}}{z}\partial_{i)}\tilde{\gamma}_{ww} - \frac{1}{4}\tilde{\gamma}^{ww}\partial_i\tilde{\gamma}_{ww}\ \partial_j\tilde{\gamma}_{ww}\right]\\
&\quad -\frac{1}{2}\tilde{\gamma}^{mn}\partial_m\partial_n\tilde{\gamma}_{ij} + \tilde{\gamma}_{m(i}\partial_{j)}\tilde{\Gamma}^m + \tilde{\Gamma}^m\tilde{\Gamma}_{(ij)m} + \tilde{\gamma}^{mn}\left[2\tilde{\Gamma}^k_{m(i}\tilde{\Gamma}_{j)kn} + \tilde{\Gamma}^k_{im}\tilde{\Gamma}_{kjn}\right], && (6.32)\\
\tilde{\mathcal{R}}_{ww} &= -\frac{1}{2}\tilde{\gamma}^{mn}\partial_m\partial_n\tilde{\gamma}_{ww} + \frac{1}{2}\tilde{\gamma}^{ww}\tilde{\gamma}^{mn}\partial_m\tilde{\gamma}_{ww}\ \partial_n\tilde{\gamma}_{ww} - \frac{n}{2}\tilde{\gamma}^{ww}\frac{\partial_z\tilde{\gamma}_{ww}}{z} + \tilde{\gamma}_{ww}\frac{\tilde{\Gamma}^z}{z}\\
&\quad +\frac{1}{2}\tilde{\Gamma}^m\partial_m\tilde{\gamma}_{ww} + \frac{\tilde{\gamma}^{zz}\tilde{\gamma}_{ww} - 1}{z^2}\,, && (6.33)\\
[\mathcal{R}_{ij}]^{\mathrm{TF}} &= \mathcal{R}_{ij} - \frac{1}{D-1}\tilde{\gamma}_{ij}\tilde{\gamma}^{mn}\mathcal{R}_{mn} - \frac{n}{D-1}\tilde{\gamma}_{ij}\tilde{\gamma}^{ww}\mathcal{R}_{ww}\,, && (6.34)\\
[\mathcal{R}_{ww}]^{\mathrm{TF}} &= \frac{1}{D-1}\left(3\mathcal{R}_{ww} - \tilde{\gamma}_{ww}\tilde{\gamma}^{mn}\mathcal{R}_{mn}\right). && (6.35)
\end{aligned}$$

A key ingredient in the BSSN formulation is the subtraction of the trace of the extrinsic curvature $\tilde{A}_{IJ} \to \tilde{A}_{IJ} - \tilde{\gamma}_{IJ}\tilde{A}/(D-1)$ which is now given by

$$\begin{aligned}
\mathrm{tr}\tilde{A}_{ij} &= \tilde{A} = \tilde{\gamma}^{mn}\tilde{A}_{mn} + n\tilde{\gamma}^{ww}\tilde{A}_{ww}\,, && (6.36)\\
\tilde{A}_{ij} &\to \tilde{A}_{ij} - \frac{1}{D-1}\tilde{\gamma}_{ij}\tilde{A}\,, && (6.37)\\
\tilde{A}_{ww} &\to \tilde{A}_{ww} - \frac{1}{D-1}\tilde{\gamma}_{ww}\tilde{A}\,. && (6.38)
\end{aligned}$$

We further note that the traceless extrinsic curvature with upper indices is obtained from

$$\begin{aligned}
\tilde{A}^{ij} &= \tilde{\gamma}^{iM}\tilde{\gamma}^{jN}\tilde{A}_{MN} = \tilde{\gamma}^{im}\tilde{\gamma}^{jn}\tilde{A}_{mn}\,, && (6.39)\\
\tilde{A}^{i}{}_{j} &= \tilde{\gamma}^{iM}\tilde{A}_{Mj} = \tilde{\gamma}^{im}\tilde{A}_{mj}\,, && (6.40)\\
\tilde{A}^{ab} &= \delta^{ab}(\tilde{\gamma}^{ww})^2\tilde{A}_{ww}\,, && (6.41)\\
\tilde{A}^{a}{}_{i} &= \tilde{A}^{i}{}_{a} = 0\,, && (6.42)\\
\tilde{A}^{a}{}_{b} &= \delta^{a}{}_{b}\tilde{\gamma}^{ww}\tilde{A}_{ww}\,, && (6.43)
\end{aligned}$$

The *moving puncture* gauge conditions remain almost unchanged from their four-dimensional version and are given by

$$\begin{aligned}
\partial_t\alpha &= \beta^M\partial_M\alpha - 2\alpha K = \beta^m\partial_m\alpha - 2\alpha K\,, && (6.44)\\
\partial_t\beta^i &= \beta^M\partial_M\beta^i + \frac{3}{4}\tilde{\Gamma}^i - \eta\beta^i = \beta^m\partial_m\beta^i + \frac{3}{4}\tilde{\Gamma}^i - \eta\beta^i\,, && (6.45)
\end{aligned}$$

so that the only change arises from the extra terms in the definition of $\tilde{\Gamma}^i$ in Eq. (6.2). Note that all vector components off the xyz hyperplane vanish by symmetry and we therefore do not need to evolve β^w.

We still have to adress the Hamiltonian and momentum constraints. In BSSN they are given by

$$\begin{aligned}
\mathcal{H} &= \mathcal{R} + \frac{D-2}{D-1}K^2 - \tilde{A}^{MN}\tilde{A}_{MN} - 16\pi\rho - 2\Lambda = 0\,, && (6.46)\\
\mathcal{M}_I &= \tilde{\gamma}^{MN}\tilde{D}_M\tilde{A}_{NI} - \frac{D-2}{D-1}\partial_I K + 2(D-1)\tilde{A}^M{}_I\partial_M\phi - 8\pi j_I = 0\,, && (6.47)
\end{aligned}$$

Their version with $SO(D-3)$ isometry is given by

$$\begin{aligned}
\mathcal{H} &= \chi\tilde{\gamma}^{mn}\mathcal{R}_{mn} - \tilde{A}^{mn}\tilde{A}_{mn} + \frac{D-2}{D-1}K^2 + n\left(\chi\tilde{\gamma}^{ww}\mathcal{R}_{ww} - \frac{\tilde{A}^2_{ww}}{\tilde{\gamma}^2_{ww}}\right) = 0\,, && (6.48)\\
\mathcal{M}_i &= \tilde{\gamma}^{mn}\partial_m\tilde{A}_{ni} - \tilde{\Gamma}^m\tilde{A}_{mi} - \tilde{\gamma}^{ml}\tilde{\Gamma}^n_{im}\tilde{A}_{nl} - \frac{D-2}{D-1}\partial_i K - \frac{D-1}{2\chi}\tilde{A}^m{}_i\partial_m\chi \\
&\quad + n\tilde{\gamma}^{ww}\left(\frac{\tilde{A}_{iz} - \delta_{iz}\tilde{A}_{ww}}{z} - \tilde{\Gamma}^m_{ww}\tilde{A}_{mi} - \frac{1}{2}\tilde{\gamma}^{ww}\tilde{A}_{ww}\partial_i\tilde{\gamma}_{ww}\right)\,. && (6.49)
\end{aligned}$$

7 Regularization

The equations of the previous section contain various expressions with divisions by z which implies division by zero in the xy plane. All these expressions can be regularized which is the topic of this section.

We start with several preliminary results that will be needed throughout the following calculations. The conformal spatial metric is given by

$$\tilde{\gamma}_{IJ} = \left(\begin{array}{ccc|ccc} \tilde{\gamma}_{xx} & \tilde{\gamma}_{xy} & \tilde{\gamma}_{xz} & 0 & \cdots & 0 \\ \tilde{\gamma}_{yx} & \tilde{\gamma}_{yy} & \tilde{\gamma}_{yz} & 0 & \cdots & 0 \\ \tilde{\gamma}_{zx} & \tilde{\gamma}_{zy} & \tilde{\gamma}_{zz} & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & \tilde{\gamma}_{ww} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \tilde{\gamma}_{ww} \end{array}\right) \quad \overset{z\to 0}{\longrightarrow} \quad \left(\begin{array}{ccc|ccc} \tilde{\gamma}_{xx} & \tilde{\gamma}_{xy} & 0 & 0 & \cdots & 0 \\ \tilde{\gamma}_{yx} & \tilde{\gamma}_{yy} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \tilde{\gamma}_{zz} & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & \tilde{\gamma}_{ww} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \tilde{\gamma}_{ww} \end{array}\right). \tag{7.1}$$

For the xyz subset of this matrix, we will need the inverse metric which is calculated from the cofactor matrix C_{ij} which is obtained by striking out row i and column j in $\tilde{\gamma}_{IJ}$, calculating the determinant of the resulting reduced matrix and finally adjusting by the $+$ or $-$ sign obtained from the usual $\begin{array}{cc} + & - \\ - & + \end{array}$ pattern in the calculation of determinants. We thus obtain

$$\begin{array}{lll} C_{xx} = \tilde{\gamma}^n_{ww}(\tilde{\gamma}_{yy}\tilde{\gamma}_{zz} - \tilde{\gamma}^2_{yz})\,, & C_{xy} = -\tilde{\gamma}^n_{ww}(\tilde{\gamma}_{yx}\tilde{\gamma}_{zz} - \tilde{\gamma}_{zx}\tilde{\gamma}_{yz})\,, & C_{xz} = \tilde{\gamma}^n_{ww}(\tilde{\gamma}_{yx}\tilde{\gamma}_{zy} - \tilde{\gamma}_{zx}\tilde{\gamma}_{yy})\,, \\ \ldots & C_{yy} = \tilde{\gamma}^n_{ww}(\tilde{\gamma}_{xx}\tilde{\gamma}_{zz} - \tilde{\gamma}^2_{zx})\,, & C_{yz} = -\tilde{\gamma}^n_{ww}(\tilde{\gamma}_{xx}\tilde{\gamma}_{zy} - \tilde{\gamma}_{zx}\tilde{\gamma}_{xy})\,, \\ \ldots & \ldots & C_{zz} = \tilde{\gamma}^n_{ww}(\tilde{\gamma}_{xx}\tilde{\gamma}_{yy} - \tilde{\gamma}^2_{yx})\,, \end{array} \tag{7.2}$$

where the dots denote symmetric terms. The inverse metric is derived from the cofactor matrix by taking the adjunct and dividing by the determinant. By construction, the determinant (easiest obtained using the "diagonal" method) satisfies

$$\det\tilde{\gamma}_{IJ} = \tilde{\gamma}^n_{ww}\left(\tilde{\gamma}_{xx}\tilde{\gamma}_{yy}\tilde{\gamma}_{zz} + 2\tilde{\gamma}_{xy}\tilde{\gamma}_{xz}\tilde{\gamma}_{yz} - \tilde{\gamma}_{xx}\tilde{\gamma}^2_{yz} - \tilde{\gamma}_{yy}\tilde{\gamma}^2_{xz} - \tilde{\gamma}_{zz}\tilde{\gamma}^2_{xy}\right) \overset{!}{=} 1\,, \tag{7.3}$$

$$\overset{z\to 0}{\longrightarrow} \quad \tilde{\gamma}^n_{ww}\tilde{\gamma}_{zz}\left(\tilde{\gamma}_{xx}\tilde{\gamma}_{yy} - \tilde{\gamma}^2_{xy}\right) \overset{!}{=} 1\,. \tag{7.4}$$

Note, however, the n factors of $\tilde{\gamma}_{ww}$; in general, the determinant of the xyz subset of the conformal metric $\det\tilde{\gamma}_{ij} \neq 1$. The inverse metric is then given by $\tilde{\gamma}^{IJ} = C_{JI}/\det\tilde{\gamma}_{IJ} = C_{JI}$:

$$\begin{array}{lll} \tilde{\gamma}^{xx} = \tilde{\gamma}^n_{ww}(\tilde{\gamma}_{yy}\tilde{\gamma}_{zz} - \tilde{\gamma}^2_{yz})\,, & \tilde{\gamma}^{xy} = \tilde{\gamma}^n_{ww}(\tilde{\gamma}_{yz}\tilde{\gamma}_{xz} - \tilde{\gamma}_{xy}\tilde{\gamma}_{zz})\,, & \tilde{\gamma}^{xz} = \tilde{\gamma}^n_{ww}(\tilde{\gamma}_{xy}\tilde{\gamma}_{yz} - \tilde{\gamma}_{xz}\tilde{\gamma}_{yy})\,, \\ \ldots & \tilde{\gamma}^{yy} = \tilde{\gamma}^n_{ww}(\tilde{\gamma}_{xx}\tilde{\gamma}_{zz} - \tilde{\gamma}^2_{xz})\,, & \tilde{\gamma}^{yz} = \tilde{\gamma}^n_{ww}(\tilde{\gamma}_{xy}\tilde{\gamma}_{xz} - \tilde{\gamma}_{xx}\tilde{\gamma}_{yz})\,, \\ \ldots & \ldots & \tilde{\gamma}^{zz} = \tilde{\gamma}^n_{ww}(\tilde{\gamma}_{xx}\tilde{\gamma}_{yy} - \tilde{\gamma}^2_{xy}) \overset{z\to 0}{\longrightarrow} \frac{1}{\tilde{\gamma}_{zz}}\,, \end{array} \tag{7.5}$$

We further note that across $z = 0$, an expansion of scalars yields only even orders in z and likewise vector and tensor components exclusively in the x or y directions. Vector components in the z direction and tensor components involving exactly one z are odd across $z = 0$ and only contain odd orders of z in a series expansion. Tensor components T_{zz}, on the other hand are again even in z. We shall use these properties frequently when we trade divisions by z for derivatives.

One further ingredient we shall need is the absence of a conical singularity at $z = 0$. Consider for this purpose the coordinate transformation given by Eq. (1.4) which changes (z, w) to polar coordinates

(ρ, φ). Restricting attention to these two coordinates, we have

$$\gamma_{\rho\rho} = \frac{z^2}{\rho^2}\gamma_{zz} + 2\frac{zw}{\rho^2}\gamma_{zw} + \frac{w^2}{\rho^2}\gamma_{ww}\,, \tag{7.6}$$

$$\gamma_{\varphi\varphi} = w^2\gamma_{zz} - 2wz\gamma_{zm} + z^2\gamma_{ww}\,, \tag{7.7}$$

$$ds^2 = \gamma_{\rho\rho}d\rho^2 + \gamma_{\varphi\varphi}d\varphi^2 d\varphi^2\,. \tag{7.8}$$

We require the proper circumference to be related to the proper radius by a factor of 2π which implies $\gamma_{\varphi\varphi} = \rho^2\gamma_{\rho\rho}$. Substituting the above expressions for the polar metric components and letting $z \to 0$ implies

$$\gamma_{zz} - \gamma_{ww} = \mathcal{O}(z^2)\,. \tag{7.9}$$

This condition is valid for arbitrary values of w and by smoothness therefore also aplies in the limit $w \to 0$. It furthermore directly transfers to the conformal metric, so that $\tilde{\gamma}_{ww} - \tilde{\gamma}_{zz} = \mathcal{O}(z^2)$. Finally, for $z \to 0$, the inverse metric components are simply $\tilde{\gamma}^{zz} = 1/\tilde{\gamma}_{zz}$ and $\tilde{\gamma}^{ww} = 1/\tilde{\gamma}_{ww}$, so that we also have $\tilde{\gamma}^{ww} - \tilde{\gamma}^{zz} = \mathcal{O}(z^2)$. Let us now discuss all terms appearing in Eqs. (6.3)-(6.49).

1) We expand $\beta^z = b_1 z + b_3 z^3 + \ldots$ and conclude

$$\frac{\beta^z}{z} = \partial_z\beta^z\,. \tag{7.10}$$

Likewise, this applies to all vectorial quantities such as $\tilde{\Gamma}^z$.

2) We use the above expression (7.5) for the inverse metric to first replace inverse metric components by their covariant counterparts and then apply the same trading of divisions by z for z-derivatives as above for β^z. This gives us

$$\frac{\delta^i{}_z - \tilde{\gamma}^{zi}\tilde{\gamma}_{ww}}{z} = \begin{cases} \tilde{\gamma}^{n+1}_{ww}(\tilde{\gamma}_{yy}\partial_z\tilde{\gamma}_{xz} - \tilde{\gamma}_{xy}\partial_z\tilde{\gamma}_{yz}) & \text{if } i = x \\ \tilde{\gamma}^{n+1}_{ww}(\tilde{\gamma}_{xx}\partial_z\tilde{\gamma}_{yz} - \tilde{\gamma}_{xy}\partial_z\tilde{\gamma}_{xz}) & \text{if } i = y \\ 0 & \text{if } i = z \end{cases} \tag{7.11}$$

3) Expanding $\beta^x = b_0 + b_2 z^2 + \ldots$ and $\beta^z = b_1 z + b_3 z^3 + \ldots$, we trade two divisions for a second derivative in

$$\frac{\partial_z\beta^i}{z} - \delta^i{}_z\frac{\beta^z}{z^2} = \begin{cases} \partial_z\partial_z\beta^A & \text{if } i = A \in \{x,\ y\} \\ 0 & \text{if } i = z \end{cases} \tag{7.12}$$

4) We rewrite the term

$$\frac{\tilde{\gamma}^{im}\partial_m\beta^z}{z} - \tilde{\gamma}^{iz}\frac{\beta^z}{z^2} = \tilde{\gamma}^{im}\left(\frac{\partial_m\beta^z}{z} - \delta^z{}_m\frac{\beta^z}{z^2}\right)\,, \tag{7.13}$$

and use the standard expansion of β^i which leads to

$$\frac{\partial_m\beta^z}{z} - \delta^z{}_m\frac{\beta^z}{z^2} = \begin{cases} \partial_A\partial_z\beta^z & \text{if } i = A \\ 0 & \text{if } i = z \end{cases} \tag{7.14}$$

5) The derivative of a scalar behaves like a vector component, so that

$$\frac{\partial_z \alpha}{z} = \partial_z \partial_z \alpha \,. \tag{7.15}$$

6) By using expression (7.5) for the inverse metric and then trading derivatives, we can rewrite

$$\frac{\tilde{\gamma}^{zm}}{z}\partial_m \alpha = \tilde{\gamma}^n_{ww}\left[\left(\tilde{\gamma}_{xy}\partial_z\tilde{\gamma}_{yz} - \tilde{\gamma}_{yy}\partial_z\tilde{\gamma}_{xz}\right)\partial_x\alpha + \left(\tilde{\gamma}_{xy}\partial_z\tilde{\gamma}_{xz} - \tilde{\gamma}_{xx}\partial_z\tilde{\gamma}_{yz}\right)\partial_y\alpha\right] + \tilde{\gamma}^{zz}\partial_z\partial_z\alpha \,. \tag{7.16}$$

7) We consider the condition $\tilde{\gamma}_{zz} - \tilde{\gamma}_{ww} = \mathcal{O}(z^2)$ and take the difference of the evolution equations (6.5) and (6.6) for $\tilde{\gamma}_{zz}$ and $\tilde{\gamma}_{ww}$. In fact, almost all terms trivially reduce to expressions containing $\tilde{\gamma}_{zz} - \tilde{\gamma}_{ww}$. The only exception is

$$2\tilde{\gamma}_{mz}\partial_z\beta^m - \frac{2n}{D-1}\tilde{\gamma}_{zz}\frac{\beta^z}{z} - \frac{2}{D-1}\tilde{\gamma}_{ww}3\frac{\beta^z}{z} = \mathcal{O}(z^2) + \tilde{\gamma}_{zz}\frac{\beta^z}{z}\underbrace{\left(2 - \frac{2n}{D-1}\right)}_{=\frac{6}{D-1}} - \frac{6}{D-1}\tilde{\gamma}_{ww}\frac{\beta^z}{z}\,, \tag{7.17}$$

where the factorization of $\tilde{\gamma}_{zz} - \tilde{\gamma}_{ww}$ requires a small calculation. The difference of Eqs. (6.5) and (6.6) therefore results in

$$\tilde{A}_{zz} - \tilde{A}_{ww} = \Big[\ldots\Big](\tilde{\gamma}_{zz} - \tilde{\gamma}_{ww}) + \mathcal{O}(z^2) = \mathcal{O}(z^2)\,, \tag{7.18}$$

where the dots denote a collection of terms and operators that ubiquitously preserve the z^2 behaviour inherited by $\tilde{\gamma}_{zz} - \tilde{\gamma}_{ww}$.
With Eq. (7.18), we can regularize

$$\frac{\tilde{A}_{iz} - \delta_{iz}\tilde{A}_{ww}}{z} = \begin{cases} \partial_z \tilde{A}_{Az} & \text{if } i = A \in \{x, y\} \\ 0 & \text{if } i = z \end{cases} \tag{7.19}$$

8) Using again $\tilde{\gamma}_{zz} - \tilde{A}_{ww} = \mathcal{O}(z^2)$ and trading a division by z for a z derivative, we find

$$-\frac{1}{2}\frac{\partial_z\tilde{\gamma}_{ij}}{z} + \frac{\delta_{z(i}\tilde{\gamma}_{j)z} - \delta_{iz}\delta_{jz}\tilde{\gamma}_{ww}}{z^2} = \begin{cases} -\frac{1}{2}\partial_z\partial_z\tilde{\gamma}_{AB} & \text{if } i = A,\ j = B \\ 0 & \text{if } i = A,\ j = z \text{ or } i = z,\ j = B \\ -\frac{1}{2}\partial_z\partial_z\tilde{\gamma}_{ww} & \text{if } i = j = z \end{cases} \tag{7.20}$$

9) Using $\tilde{\gamma}^{ww}\tilde{\gamma}_{zz} - 1 = \tilde{\gamma}^{ww}(\tilde{\gamma}_{zz} - \tilde{\gamma}_{ww}) = \tilde{\gamma}^{ww}\mathcal{O}(z^2)$ and $\tilde{\gamma}_{zA}/z = \partial_z\tilde{\gamma}_{zA}$, we can rewrite

$$\frac{\tilde{\gamma}^{ww}\tilde{\gamma}_{z(j} - \delta_{z(j}}{z}\partial_{i)}\tilde{\gamma}_{ww} = \begin{cases} \tilde{\gamma}^{ww}\partial_z\tilde{\gamma}_{z(B}\partial_{A)}\tilde{\gamma}_{ww} & \text{if } i = A,\ j = B \\ 0 & \text{if } i = A,\ j = z \text{ or } i = z,\ j = B \\ 0 & \text{if } i = z,\ j = z \end{cases} \tag{7.21}$$

10) The expression $\tilde{\gamma}^{ww}(\tilde{\gamma}_{zz} - 2\tilde{\gamma}_{ww} + \tilde{\gamma}^{zz}\tilde{\gamma}^2_{ww})/z^2$ is a bit dangerous because the two factors of z in the denominator do not allow us to simply use the conical singularity condition $\tilde{\gamma}_{zz} - \tilde{\gamma}_{ww} = \mathcal{O}(z^2)$ to simply eliminate terms. Instead, the division by z^2 will leave a finite contribution and we need to be very careful when applying simplifications. By trading derivatives for divisions in the usual way and using $\tilde{\gamma}_{ww} = 1/\tilde{\gamma}^{ww}$, which is valid exactly at any z, we arrive at

$$\tilde{\gamma}^{ww}\frac{\tilde{\gamma}_{zz} - 2\tilde{\gamma}_{ww} + \tilde{\gamma}^{zz}\tilde{\gamma}^2_{ww}}{z^2} = \frac{1}{2}\left(\tilde{\gamma}^{ww}\partial_z\partial_z\tilde{\gamma}_{zz} + \tilde{\gamma}_{ww}\partial_z\partial_z\tilde{\gamma}^{zz}\right) . \tag{7.22}$$

We can then directly substitute for the derivative of the inverse metric using

$$\begin{aligned}\partial_z\partial_z\tilde{\gamma}^{zz} &= n\tilde{\gamma}^{n-1}_{ww}\underbrace{\left(\tilde{\gamma}_{xx}\tilde{\gamma}_{yy} - \tilde{\gamma}^2_{yz}\right)}_{=\frac{1}{\tilde{\gamma}^n_{ww}\tilde{\gamma}_{zz}}}\partial_z\partial_z\tilde{\gamma}_{ww} + \tilde{\gamma}^n_{ww}\left[\tilde{\gamma}_{yy}\partial_z\partial_z\tilde{\gamma}_{xx} + \tilde{\gamma}_{xx}\partial_z\partial_z\tilde{\gamma}_{yy} - 2\tilde{\gamma}_{xy}\partial_z\partial_z\tilde{\gamma}_{xy}\right] \\ &= n\tilde{\gamma}^{ww}\tilde{\gamma}^{zz}\partial_z\partial_z\tilde{\gamma}_{ww} + \tilde{\gamma}^n_{ww}\left[\tilde{\gamma}_{yy}\partial_z\partial_z\tilde{\gamma}_{xx} + \tilde{\gamma}_{xx}\partial_z\partial_z\tilde{\gamma}_{yy} - 2\tilde{\gamma}_{xy}\partial_z\partial_z\tilde{\gamma}_{xy}\right] .\end{aligned} \tag{7.23}$$

By using $\det\tilde{\gamma}_{IJ} = 1$, taking the z derivative thereof and also using the $z \to 0$ limit of the determinant (7.4), a more lengthy calculation gives the simpler expression

$$\tilde{\gamma}^{ww}\frac{\tilde{\gamma}_{zz} - 2\tilde{\gamma}_{ww} + \tilde{\gamma}^{zz}\tilde{\gamma}^2_{ww}}{z^2} = -\tilde{\gamma}^n_{ww}\left[2\tilde{\gamma}_{xy}\partial_z\tilde{\gamma}_{xz}\ \partial_z\tilde{\gamma}_{yz} - \tilde{\gamma}_{xx}(\partial_z\tilde{\gamma}_{yz})^2 - \tilde{\gamma}_{yy}(\partial_z\tilde{\gamma}_{xz})^2\right] . \tag{7.24}$$

Attempts at further simplifying this expression have not yet succeeded. Maybe, this is just as good as it gets. I'd probably still feel a little more comfortable with Eq. (7.24) because we take first z derivatives of quantities that manifestly vanish at $z = 0$, whereas in (7.23) we take second z derivatives of functions which do not vanish at $z = 0$.

11) By trading two divisions for a second derivative, we finally get

$$\frac{\tilde{\gamma}_{zz} - \tilde{\gamma}_{ww}}{z^2} = \frac{1}{2}(\partial_z\partial_z\tilde{\gamma}_{zz} - \partial_z\partial_z\tilde{\gamma}_{ww}) . \tag{7.25}$$

12) It seems we should have simplified the terms in Eq. (6.33) and therefore combine the regularization steps from 10 and 11 into

$$\frac{\tilde{\gamma}^{zz}\tilde{\gamma}_{ww} - 1}{z^2} = -\tilde{\gamma}^n_{ww}\left[2\tilde{\gamma}_{xy}\partial_z\tilde{\gamma}_{xz}\ \partial_z\tilde{\gamma}_{yz} - \tilde{\gamma}_{xx}(\partial_z\tilde{\gamma}_{yz})^2 - \tilde{\gamma}_{yy}(\partial_z\tilde{\gamma}_{xz})^2\right] + \frac{1}{2}\left(\frac{\partial_z\partial_z\tilde{\gamma}_{ww}}{\tilde{\gamma}_{ww}} - \frac{\partial_z\partial_z\tilde{\gamma}_{zz}}{\tilde{\gamma}_{zz}}\right) . \tag{7.26}$$

8 Some analytic data

Here we summarize some analytic solutions to the Einstein equations or the initial data equations that are helpful in testing the equations of the previous section as well as the code. A reference I have found very helpful in this regard is Kunstatter, Maeda & Taves [7].

1) Tangherlini in isotropic coordinates

The puncture equivalent of a Schwarzschild black-hole in higher dimensions has a line element

$$ds^2 = -\left(\frac{4r^{D-3} - \mu^{D-3}}{4r^{D-3} + \mu^{D-3}}\right)dt^2 + \left(1 + \frac{\mu^{D-3}}{4r^{D-3}}\right)^{4/(D-3)}\left(dx^2 + dy^2 + dz^2 + \sum_a dw^2_a\right) . \tag{8.1}$$

This implies for the ADM variables

$$\gamma_{ij} = \psi^{4/(D-3)} , \quad \psi = 1 + \frac{\mu^{D-3}}{4r^{D-3}} , \quad \beta^I = 0 , \quad K_{ij} = 0 . \tag{8.2}$$

2) Brill-Lindquist data in D dimensions

Brill Lindquist data are not a complete solution of the Einstein equations but merely solve the constraints at the moment of time symmetry. They are very helpful data to start dynamic time evolutions. They are given by

$$\gamma_{ij} = \psi^{4/(D-3)}\delta_{ij}\,, \quad \psi = 1+\frac{\mu_A^{D-3}}{4[(x-x_A)^2+y^2+z^2]^{(D-3)/2}}+\frac{\mu_B^{D-3}}{4[(x-x_B)^2+y^2+z^2]^{(D-3)/2}}\,, \quad K_{ij}=0\,. \tag{8.3}$$

Here, two black holes are assumed to be located at $x = x_A$ and $x = x_B$, respectively. The solution generalizes straightforwardly to > 2 holes.

3) Tangherlini in Kerr-Schild coordinates

A non-rotating black hole in Kerr-Schild coordinates is particularly helpful to check the equations and also the code implementation because it contains far more non-trivial terms than the conformally flat solutions. Tangherlini in Kerr-Schild coordinates is given by

$$ds^2 = -\left(1+\frac{\mu}{r^{D-3}}\right)dt^2 + 2\frac{\mu}{r^{D-3}}dt\,dr + \left(1+\frac{\mu}{r^{D-3}}\right)dr^2 + r^2 d\Omega^2\,, \tag{8.4}$$

where

$$d\Omega^2 = h_{ab}\,d\phi^a\,d\phi^b\,, \quad a,\,b = 1,\,\ldots,\,D-1\,. \tag{8.5}$$

For the transformation to Cartesian coordinates, we use

$$\begin{aligned}
x_1 &= r\cos\phi_1\,, & r &= \sqrt{x_1^2+\ldots x_n^2}\,,\\
x_2 &= r\sin\phi_1\,\cos\phi_2\,, & \phi_1 &= \mathrm{arccot}\frac{x_1}{\sqrt{x_n^2+\ldots+x_2^2}}\,,\\
x^3 &= r\sin\phi_1\,\sin\phi_2\,\cos\phi_3\,, & \phi_2 &= \mathrm{arccot}\frac{x_2}{\sqrt{x_n^2+\ldots+x_3^2}}\,,\\
&\;\,\vdots & &\;\,\vdots\\
x^{n-1} &= r\sin\phi_1\,\sin\phi_2\,\ldots\,\sin\phi_{n-1}\,, & \phi_{n-2} &= \mathrm{arccot}\frac{x_{n-2}}{\sqrt{x_n^2+x_{n-1}^2}}\,,\\
x^n &= r\sin\phi_1\,\sin\phi_2\,\ldots\,\cos\phi_{n-1}\,, & \phi_{n-1} &= \mathrm{arccot}\frac{x_{n-1}}{x_n}\left(=2\mathrm{arccot}\frac{\sqrt{x_n^2+x_{n-1}^2}+x_{n-1}}{x_n}\right),
\end{aligned} \tag{8.6}$$

where $n = D-1$ and the last expression for ϕ_{n-1} in parentheses is strictly speaking more accurate for $x_{n-1} < 0$ where it eliminates a jump by 2π. For most purposes, the simpler, first expression should work fine, though. Using

$$g_{\alpha\beta} = \frac{\partial x^{\bar{\mu}}}{\partial x^\alpha}\frac{\partial x^{\bar{\nu}}}{\partial x^\beta}g_{\bar{\mu}\bar{\nu}}\,, \tag{8.7}$$

we obtain

$$ds^2 = -\left(1-\frac{\mu}{r^{D-3}}\right)dt^2 + 2\frac{\mu}{r^{D-3}}\frac{x^i}{r}dt\,dx^i + \left(\delta_{ij}+\frac{\mu}{r^{D-3}}\frac{x_i}{r}\frac{x_j}{r}\right)dx^i\,dx^j\,, \tag{8.8}$$

and for the ADM variables

$$
\begin{aligned}
\gamma_{ij} &= \delta_{ij} + \frac{\mu}{r^{D-3}}\frac{x_i x_j}{r^2}\,, \quad \beta_i = \frac{\mu x_i}{r^{D-3} r}\,, \quad \beta^i = \frac{\mu}{r^{D-3}+\mu}\frac{x^i}{r}\,, \quad \alpha = \sqrt{\frac{r^{D-3}}{r^{D-3}+\mu}}\,,\\
K_{ij} &= \frac{2\mu(r^{10}\mu + r^{7+D})\delta_{ij} - \mu x_i x_j\left[(D-3)r^{11-D}\mu^2 + (3D-7)\mu r^8 + (2D-4)r^{5+D}\right]}{2r^5(r^D+\mu r^3)\sqrt{r^{2D}+\mu r^{D+3}}}\,,\\
\gamma_{ww} &= 1\,, \quad K_{ww} = \frac{\mu r^2}{\sqrt{r^{2D}+\mu r^{3+D}}}\,. \qquad (8.9)
\end{aligned}
$$

For the special case $D = 7$, we also get the following results

$$
\det\gamma_{ij} = 1 + \frac{\mu}{r^4}\,, \quad K_{ij} = \frac{\mu}{r\sqrt{r^4(r^4+\mu)}}\left[\delta_{ij} - \frac{x_i x_j(5r^4+2\mu)}{r^6}\right]\,, \quad K = \frac{\mu(r^4+3\mu)}{r^3(r^4+\mu)^{3/2}}\,. \qquad (8.10)
$$

The remaining ADM and BSSN variables can then be calculated, e.g. inside `Maple`, from their standard definitions.

9 A modified version using rescaled variables

In Sec. 7 we have seen how we can regularize terms arising in the BSSN equations which contain factors of z or z^2 in the denominator. Here we will explore a different version of the higher dimensional equations which employs a set of variables rescaled with factors of z and results in a set of equations where the only irregular terms that appear are of the type ***5)*** in Sec. 7. More specifically, the only terms containing factors of z in the denominator are of the form $\partial_z f/z$ where f is either a (possibly rescaled) BSSN variable or a z derivative thereof and is straightforwardly regularized as

$$
\frac{\partial_z f}{z} = \partial_z\partial_z f\,, \qquad (9.1)
$$

as $z \to 0$.
For this purpose, we use the following notation. Early, lower case Latin indices a, b, ... now denote the x and y directions; we used them previously for the w components, but this will not be needed anymore, so that they are free for this different use.
The set of rescaled variables we shall be using is given by

$$
\begin{aligned}
\hat\beta^z &:= \frac{\beta^z}{z}\,, \quad \hat\gamma_{xz} := \frac{\tilde\gamma_{xz}}{z}\,, \quad \hat\gamma_{yz} := \frac{\tilde\gamma_{yz}}{z}\,, \quad h_{ww} := \frac{\tilde\gamma_{ww} - \tilde\gamma_{zz}}{z^2}\,,\\
\hat\Gamma^z &:= \frac{\tilde\Gamma^z}{z}\,, \quad \hat A_{xz} := \frac{\tilde A_{xz}}{z}\,, \quad \hat A_{yz} := \frac{\tilde A_{yz}}{z}\,, \quad H_{ww} := \frac{\tilde A_{ww} - \tilde A_{zz}}{z^2}\,. \qquad (9.2)
\end{aligned}
$$

Likewise, we consider a rescaled version of the ADM variables, where we use the same $\hat\beta^z$ as well as

$$
\begin{aligned}
\check\gamma_{xz} &:= \frac{\gamma_{xz}}{z}\,, \quad \check\gamma_{yz} := \frac{\gamma_{yz}}{z}\,, \quad \xi_{ww} := \frac{\gamma_{ww} - \gamma_{zz}}{z^2}\,,\\
\check K_{xz} &:= \frac{K_{xz}}{z}\,, \quad \check K_{yz} := \frac{K_{yz}}{z}\,, \quad \Xi_{ww} := \frac{K_{ww} - K_{zz}}{z^2}\,. \qquad (9.3)
\end{aligned}
$$

For the conversion from ADM to BSSN variables, we need the determinant of the physical metric which is given by

$$
\gamma = \det\gamma_{IJ} = \gamma_{ww}^n\left(\gamma_{xx}\gamma_{yy}\gamma_{zz} + 2z^2\gamma_{xy}\check\gamma_{xz}\check\gamma_{yz} - z^2\gamma_{xx}\check\gamma_{yz}^2 - z^2\gamma_{yy}\check\gamma_{xz}^2 - \gamma_{zz}\gamma_{xy}^2\right)\,, \qquad (9.4)
$$

where $\gamma_{ww} = \gamma_{zz} + z^2 \xi_{ww}$. We can calculate the inverse physical metric from

$$\begin{aligned}
&\gamma^{xx} = \gamma^n_{ww}(\gamma_{yy}\gamma_{zz} - z^2\check{\gamma}^2_{yz})\,, \quad && \gamma^{xy} = \gamma^n_{ww}(z^2\check{\gamma}_{yz}\check{\gamma}_{xz} - \gamma_{xy}\gamma_{zz})\,, \quad && \check{\gamma}^{xz} = \gamma^n_{ww}(\gamma_{xy}\check{\gamma}_{yz} - \check{\gamma}_{xz}\gamma_{yy})\,,\\
&\ldots && \gamma^{yy} = \gamma^n_{ww}(\gamma_{xx}\gamma_{zz} - z^2\check{\gamma}^2_{xz})\,, && \check{\gamma}^{yz} = \gamma^n_{ww}(\gamma_{xy}\check{\gamma}_{xz} - \gamma_{xx}\check{\gamma}_{yz})\,,\\
&\ldots && \ldots && \gamma^{zz} = \gamma^n_{ww}(\gamma_{xx}\gamma_{yy} - \gamma^2_{xy}) \xrightarrow{z\to 0} \tfrac{1}{\gamma_{zz}}\,,
\end{aligned} \tag{9.5}$$

The rescaled BSSN variables are then obtained from

$$\begin{gathered}
\chi = \gamma^{\frac{-1}{D-1}}, \quad \gamma = \det \gamma_{IJ} = \gamma^n_{ww} \det \gamma_{ij}\,,\\
K = \gamma^{MN}K_{MN} = \gamma^{cd}K_{cd} + 2z^2\check{\gamma}^{cz}\check{K}_{cz} + \gamma^{zz}K_{zz} + n\gamma^{ww}K_{ww}\,,\\
\tilde{\gamma}_{ab} = \chi\gamma_{ab}\,, \quad \hat{\gamma}_{az} = \chi\check{\gamma}_{az}\,, \quad \tilde{\gamma}_{zz} = \chi\gamma_{zz}\,,\\
\tilde{\gamma}_{ww} = \chi\gamma_{ww}\,, \quad h_{ww} = \chi\xi_{ww}\,,\\
\tilde{\gamma}^{ab} = \tfrac{1}{\chi}\gamma^{ab}, \quad \hat{\gamma}^{az} = \tfrac{1}{\chi}\check{\gamma}^{az}\,, \quad \tilde{\gamma}^{zz} = \tfrac{1}{\chi}\gamma^{zz}\,,\\
\tilde{\gamma}^{ww} = \tfrac{1}{\chi}\gamma^{ww} = \tfrac{1}{\tilde{\gamma}_{zz}+z^2h_{ww}}\,,\\
\tilde{A}_{ab} = \chi\left(K_{ab} - \tfrac{1}{D-1}\gamma_{ab}K\right)\,, \quad \hat{A}_{az} = \chi\left(\hat{K}_{az} - \tfrac{1}{D-1}\check{\gamma}_{az}K\right)\,, \quad \tilde{A}_{zz} = \chi\left(K_{zz} - \tfrac{1}{D-1}\gamma_{zz}K\right)\,,\\
\tilde{A}_{ww} = \chi\left(K_{ww} - \tfrac{1}{D-1}\gamma_{ww}K\right)\,, \quad H_{ww} = \chi\left(\check{K}_{ww} - \tfrac{1}{D-1}\check{\gamma}_{ww}K\right)\,,\\
K_{ab} = \tfrac{1}{\chi}\left(\tilde{A}_{ab} + \tfrac{1}{D-1}\tilde{\gamma}_{ab}K\right)\,, \quad \check{K}_{az} = \tfrac{1}{\chi}\left(\hat{A}_{az} + \tfrac{1}{D-1}\hat{\gamma}_{az}K\right)\,, \quad K_{zz} = \tfrac{1}{\chi}\left(\tilde{A}_{zz} + \tfrac{1}{D-1}\tilde{\gamma}_{zz}K\right)\,,\\
K_{ww} = \tfrac{1}{\chi}\left(\tilde{A}_{ww} + \tfrac{1}{D-1}\tilde{\gamma}_{ww}K\right)\,,\\
\tilde{\Gamma}^i = \tilde{\gamma}^{MN}\tilde{\Gamma}^i_{MN} = \tilde{\gamma}^{mn}\tilde{\Gamma}^i_{mn} + n\tilde{\gamma}^{ww}\tilde{\Gamma}^i_{ww}\,,
\end{gathered} \tag{9.6}$$

We will leave the BSSN Γ^i variable for now and first discuss in more detail the BSSN metric. The determinant (7.3) of the BSSN metric is given by

$$\det \tilde{\gamma}_{IJ} = \tilde{\gamma}^n_{ww}\left(\tilde{\gamma}_{xx}\tilde{\gamma}_{yy}\tilde{\gamma}_{zz} + 2z^2\tilde{\gamma}_{xy}\hat{\gamma}_{xz}\hat{\gamma}_{yz} - z^2\tilde{\gamma}_{xx}\hat{\gamma}^2_{yz} - z^2\tilde{\gamma}_{yy}\hat{\gamma}^2_{xz} - \tilde{\gamma}_{zz}\tilde{\gamma}^2_{xy}\right) \overset{!}{=} 1\,, \tag{9.7}$$

where $\tilde{\gamma}_{ww} = \tilde{\gamma}_{zz} + z^2 + h_{ww}$. Equation (7.5) for the inverse metric now becomes

$$\begin{aligned}
&\tilde{\gamma}^{xx} = \tilde{\gamma}^n_{ww}(\tilde{\gamma}_{yy}\tilde{\gamma}_{zz} - z^2\hat{\gamma}^2_{yz})\,, \quad && \tilde{\gamma}^{xy} = \tilde{\gamma}^n_{ww}(z^2\hat{\gamma}_{yz}\hat{\gamma}_{xz} - \tilde{\gamma}_{xy}\tilde{\gamma}_{zz})\,, \quad && \hat{\gamma}^{xz} = \tilde{\gamma}^n_{ww}(\tilde{\gamma}_{xy}\hat{\gamma}_{yz} - \hat{\gamma}_{xz}\tilde{\gamma}_{yy})\,,\\
&\ldots && \tilde{\gamma}^{yy} = \tilde{\gamma}^n_{ww}(\tilde{\gamma}_{xx}\tilde{\gamma}_{zz} - z^2\hat{\gamma}^2_{xz})\,, && \hat{\gamma}^{yz} = \tilde{\gamma}^n_{ww}(\tilde{\gamma}_{xy}\hat{\gamma}_{xz} - \tilde{\gamma}_{xx}\hat{\gamma}_{yz})\,,\\
&\ldots && \ldots && \tilde{\gamma}^{zz} = \tilde{\gamma}^n_{ww}(\tilde{\gamma}_{xx}\tilde{\gamma}_{yy} - \tilde{\gamma}^2_{xy}) \xrightarrow{z\to 0} \tfrac{1}{\tilde{\gamma}_{zz}}\,,
\end{aligned} \tag{9.8}$$

where the rescaled components of the inverse metric are given in analogy to their covariant counterparts

$$\hat{\gamma}^{xz} := \frac{\tilde{\gamma}^{xz}}{z}\,, \quad \hat{\gamma}^{yz} := \frac{\tilde{\gamma}^{yz}}{z}\,. \tag{9.9}$$

Next we calculate the Christoffel symbols. We start with the components of the Christoffel symbols of the first kind in the computational domain

$$\begin{aligned}
\tilde{\Gamma}_{abc} &= \frac{1}{2}\left(\partial_b\tilde{\gamma}_{ca} + \partial_c\tilde{\gamma}_{ab} - \partial_a\tilde{\gamma}_{bc}\right)\,, \\
\hat{\Gamma}_{zbc} &= \frac{1}{2}\left(\partial_b\hat{\gamma}_{cz} + \partial_c\hat{\gamma}_{zb} - \frac{\partial_z\tilde{\gamma}_{bc}}{z}\right)\,, \\
\hat{\Gamma}_{abz} &= \frac{1}{2}\left(\partial_b\hat{\gamma}_{za} + \frac{\partial_z\gamma_{ab}}{z} - \partial_a\hat{\gamma}_{bz}\right)\,, \\
\tilde{\Gamma}_{zbz} &= \frac{1}{2}\partial_b\tilde{\gamma}_{zz}\,, \\
\tilde{\Gamma}_{azz} &= \hat{\gamma}_{za} + z\partial_z\hat{\gamma}_{za} - \frac{1}{2}\tilde{\gamma}_{zz}\,, \\
\hat{\Gamma}_{zzz} &= \frac{1}{2}\frac{\partial_z\tilde{\gamma}_{zz}}{z}\,.
\end{aligned} \tag{9.10}$$

Note that for all components with a 'hat' we have "$\tilde{\Gamma} = z\hat{\Gamma}$".

10 The BSSN equations with $SO(D-1)$ isometry

We now consider the modified Cartoon version of the BSSN equations for the case of spherical symmetry. In this case, we only have one effective spatial direction which we choose to be the z direction. This coordinate is a radius and hence $z \geq 0$. All other spatial directions are now contained in the $D-2$ variables we tended to call w or denoted by early Latin indices a, b, ... Note that the constant n is now given by

$$n = D - 2\,, \tag{10.1}$$

instead of $n = D - 4$. As before, we write the symmetry components of the ADM variables

$$\gamma_{ab} = \delta_{ab}\gamma_{ww}\,, \quad K_{ab} = \delta_{ab}K_{ww}\,. \tag{10.2}$$

The BSSN variables are now defined by

$$\begin{aligned}
&\chi = \gamma^{\frac{-1}{D-1}}, \quad \gamma = \det\gamma_{IJ} = \gamma_{ww}^{n}\gamma_{zz} && K = \gamma^{MN}K_{MN} = \gamma^{mn}K_{mn} + n\gamma^{ww}K_{ww}, \\
&\tilde{\gamma}_{zz} = \chi\gamma_{zz}\,, \quad \tilde{\gamma}_{ww} = \chi\gamma_{ww} && \Leftrightarrow \quad \tilde{\gamma}^{zz} = \frac{1}{\chi}\gamma^{zz}\,, \quad \tilde{\gamma}^{ww} = \frac{1}{\chi}\gamma^{ww}\,, \\
&\tilde{A}_{zz} = \chi\left(K_{zz} - \tfrac{1}{D-1}\gamma_{zz}K\right) && \Leftrightarrow \quad K_{zz} = \frac{1}{\chi}\left(\tilde{A}_{zz} + \frac{1}{D-1}\tilde{\gamma}_{zz}K\right)\,, \\
&\tilde{A}_{ww} = \chi\left(K_{ww} - \tfrac{1}{D-1}\gamma_{ww}K\right) && \Leftrightarrow \quad K_{ww} = \frac{1}{\chi}\left(\tilde{A}_{ww} + \frac{1}{D-1}\tilde{\gamma}_{ww}K\right)\,, \\
&\tilde{\Gamma}^{z} = \tilde{\gamma}^{MN}\tilde{\Gamma}^{i}_{MN} = \tilde{\gamma}^{zz}\tilde{\Gamma}^{z}_{zz} + n\tilde{\gamma}^{ww}\tilde{\Gamma}^{z}_{ww}\,,
\end{aligned} \tag{10.3}$$

where

$$\tilde{\Gamma}^{z}_{ww} = -\frac{1}{2}\tilde{\gamma}^{zz}\partial_z\tilde{\gamma}_{ww} + \frac{1 - \tilde{\gamma}^{zz}\tilde{\gamma}_{ww}}{z}\,. \tag{10.4}$$

The BSSN equations (5.1)-(5.5) now become

$$
\begin{aligned}
\partial_t \chi &= \beta^z \partial_z \chi + \frac{2}{D-1}\chi\left(\alpha K - \partial_z\beta^z - n\frac{\beta^z}{z}\right), && (10.5)\\
\partial_t \tilde{\gamma}_{zz} &= \beta^z \partial_z \tilde{\gamma}_{zz} + 2\tilde{\gamma}_{zz}\partial_z\beta^z - \frac{2}{D-1}\tilde{\gamma}_{zz}\left(\partial_z\beta^z + n\frac{\beta^z}{z}\right) - 2\alpha\tilde{A}_{zz}, && (10.6)\\
\partial_t \tilde{\gamma}_{ww} &= \beta^z \partial_z \tilde{\gamma}_{ww} - \frac{2}{D-1}\tilde{\gamma}_{ww}\left(\partial_z\beta^z - \frac{\beta^z}{z}\right) - 2\alpha\tilde{A}_{ww}, && (10.7)\\
\partial_t K &= \beta^z\partial_z K - \chi\tilde{\gamma}^{zz}D_zD_z\alpha + \alpha\tilde{A}^{zz}\tilde{A}_{zz} + \frac{1}{D-1}\alpha K^2 + n\tilde{\gamma}^{ww}\left(\alpha\frac{\tilde{A}^2_{ww}}{\tilde{\gamma}_{ww}} - \chi D_wD_w\alpha\right), && (10.8)\\
\partial_t \tilde{A}_{zz} &= \beta^z\partial_z\tilde{A}_{zz} + 2\tilde{A}_{zz}\partial_z\beta^z - \frac{2}{D-1}\tilde{A}_{zz}\left(\partial_z\beta^z + n\frac{\beta^z}{z}\right) + \alpha K\tilde{A}_{zz} - 2\alpha\tilde{A}_{zz}\tilde{A}^z{}_z \\
&\quad +\chi\left[\alpha\mathcal{R}_{zz} - D_zD_z\alpha\right]^{\mathrm{TF}}, && (10.9)\\
\partial_t \tilde{A}_{ww} &= \beta^z\partial_z\tilde{A}_{ww} - \frac{2}{D-1}\tilde{A}_{ww}\left(\partial_z\beta^z - \frac{\beta^z}{z}\right) + \alpha\tilde{A}_{ww}(K - 2\tilde{\gamma}^{ww}\tilde{A}_{ww}) \\
&\quad +\chi\left[\alpha\mathcal{R}_{ww} - D_wD_w\alpha\right]^{\mathrm{TF}}, && (10.10)\\
\partial_t\tilde{\Gamma}^z &= \beta^z\partial_z\tilde{\Gamma}^z + \frac{2}{D-1}\tilde{\Gamma}^z\left(\partial_z\beta^z + n\frac{\beta^z}{z}\right) + \tilde{\gamma}^{zz}\partial_z\partial_z\beta^z + n\tilde{\gamma}^{ww}\left(\frac{\partial_z\beta^z}{z} - \frac{\beta^z}{z^2}\right) \\
&\quad -\tilde{\Gamma}^z\partial_z\beta^z + \frac{D-3}{D-1}\tilde{\gamma}^{zz}\partial_z\partial_z\beta^z + \frac{D-3}{D-1}n\tilde{\gamma}^{zz}\left(\frac{\partial_z\beta^z}{z} - \frac{\beta^z}{z^2}\right) - 2\frac{D-2}{D-1}\alpha\tilde{\gamma}^{zz}\partial_z K \\
&\quad -\tilde{A}^{zz}\left[(D-1)\alpha\frac{\partial_z\chi}{\chi} + 2\partial_z\alpha\right] + 2\alpha\left(\tilde{\Gamma}^z_{zz}\tilde{A}^{zz} + n\tilde{\Gamma}^z_{ww}\tilde{A}^{ww}\right) - \sigma\partial_M\beta^M\mathcal{G}^z, && (10.11)
\end{aligned}
$$

In these equations we need the auxiliary quantities

$$
\begin{aligned}
\tilde{\Gamma}_{zzz} &= \frac{1}{2}\partial_z\tilde{\gamma}_{zz}, && (10.12)\\
\tilde{\Gamma}_{azz} &= \tilde{\Gamma}_{zbz} = \tilde{\Gamma}_{zzz} = 0, && (10.13)\\
\tilde{\Gamma}_{abz} &= \frac{1}{2}\delta_{ab}\partial_z\tilde{\gamma}_{ww} = \delta^a{}_b\tilde{\Gamma}_{wwz}, && (10.14)\\
\tilde{\Gamma}_{zab} &= \frac{1}{2}\delta_{ab}\left(2\frac{\tilde{\gamma}_{zz} - \tilde{\gamma}_{ww}}{z} - \partial_z\tilde{\gamma}_{ww}\right), && (10.15)\\
\tilde{\Gamma}^z_{zz} &= \frac{1}{2}\tilde{\gamma}^{zz}\partial_z\tilde{\gamma}_{zz}, && (10.16)\\
\tilde{\Gamma}^a_{zz} &= \tilde{\Gamma}^z_{az} = \tilde{\Gamma}^a_{bc} = 0, && (10.17)\\
\tilde{\Gamma}^a_{bz} &= \frac{1}{2}\delta^a{}_b\tilde{\gamma}^{ww}\partial_z\tilde{\gamma}_{ww} = \delta^a{}_b\tilde{\Gamma}^w_{wz}, && (10.18)\\
\tilde{\Gamma}^z_{ab} &= \delta_{ab}\tilde{\Gamma}^z_{ww}, && (10.19)
\end{aligned}
$$

and

$$\begin{aligned}
\partial_M \beta^M \mathcal{G}^z &= \left(\partial_z \beta^z + n\frac{\beta^z}{z}\right)\left(\tilde{\Gamma}^z - \tilde{\gamma}^{zz}\tilde{\Gamma}^z_{zz} - n\tilde{\gamma}^{ww}\tilde{\Gamma}^z_{ww}\right), && (10.20)\\
\tilde{D}_z\tilde{D}_z\alpha &= \partial_z\partial_z\alpha - \tilde{\Gamma}^z_{zz}\partial_z\alpha\,, && (10.21)\\
\tilde{D}_a\tilde{D}_b\alpha &= \delta_{ab}\tilde{D}_w\tilde{D}_w\alpha = \delta_{ab}\left(\frac{\partial_z\alpha}{z} - \tilde{\Gamma}^z_{ww}\partial_z\alpha\right) = \delta_{ab}\left(\frac{1}{2}\tilde{\gamma}^{zz}\partial_z\tilde{\gamma}_{ww} + \frac{\tilde{\gamma}^{zz}\tilde{\gamma}_{ww}}{z}\right)\partial_z\alpha\,, && (10.22)\\
\tilde{D}_z\tilde{D}_z\alpha &= \partial_z\partial_z\alpha - \tilde{\Gamma}^z_{zz}\partial_z\alpha\,, && (10.23)\\
D_zD_z\alpha &= \tilde{D}_z\tilde{D}_z\alpha + \frac{1}{2\chi}\partial_z\chi\;\partial_z\alpha\,, && (10.24)\\
D_wD_w\alpha &= \tilde{D}_w\tilde{D}_w\partial_m\alpha - \frac{1}{2\chi}\tilde{\gamma}_{ww}\tilde{\gamma}^{zz}\partial_z\chi\;\partial_z\alpha\,, && (10.25)\\
[D_zD_z\alpha]^{\mathrm{TF}} &= \frac{D-2}{D-1}\left(D_zD_z\alpha - \tilde{\gamma}_{zz}\tilde{\gamma}^{ww}D_wD_w\alpha\right), && (10.26)\\
[D_aD_b\alpha]^{\mathrm{TF}} &= \delta_{ab}\frac{1}{D-1}\left(3D_wD_w\alpha - \tilde{\gamma}_{ww}\tilde{\gamma}^{mn}D_mD_n\alpha\right), && (10.27)
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}_{zz} &= \mathcal{R}^\chi_{zz} + \tilde{\mathcal{R}}_{zz}\,, && (10.28)\\
\mathcal{R}_{ww} &= \mathcal{R}^\chi_{ww} + \tilde{\mathcal{R}}_{ww}\,, && (10.29)\\
\mathcal{R}^\chi_{zz} &= \frac{1}{2\chi}\left[(D-2)\tilde{D}_z\tilde{D}_z\chi + n\left(\frac{1}{2}\tilde{\gamma}^{ww}\partial_z\tilde{\gamma}_{ww} + \frac{1}{z}\right)\partial_z\chi - \frac{D-2}{\chi}\partial_z\chi\;\partial_z\chi\right]\\
\mathcal{R}^\chi_{ww} &= \frac{\tilde{\gamma}_{ww}\tilde{\gamma}^{zz}}{2\chi}\left[\tilde{D}_z\tilde{D}_z\chi + (2D-5)\left(\frac{1}{2}\tilde{\gamma}^{ww}\partial_z\tilde{\gamma}_{ww} + \frac{1}{z}\right)\partial_z\chi - \frac{D-1}{2\chi}\partial_z\chi\;\partial_z\chi\right], \\
& && (10.30)\\
\tilde{\mathcal{R}}_{zz} &= +n\tilde{\gamma}^{ww}\left[-\frac{1}{2}\frac{\partial_z\tilde{\gamma}_{zz}}{z} + \frac{\tilde{\gamma}_{zz}-\tilde{\gamma}_{ww}}{z^2} + \frac{\tilde{\gamma}^{ww}\tilde{\gamma}_{zz}-1}{z}\partial_z\tilde{\gamma}_{ww} - \frac{1}{4}\tilde{\gamma}^{ww}\partial_z\tilde{\gamma}_{ww}\;\partial_z\tilde{\gamma}_{ww}\right]\\
&\quad -\frac{1}{2}\tilde{\gamma}^{zz}\partial_z\partial_z\tilde{\gamma}_{zz} + \tilde{\gamma}_{zz}\partial_z\tilde{\Gamma}^z + \tilde{\Gamma}^z\tilde{\Gamma}_{zzz} + 3\tilde{\gamma}^{zz}\tilde{\Gamma}^z_{zz}\tilde{\Gamma}_{zzz}\,, && (10.31)\\
\tilde{\mathcal{R}}_{ww} &= -\frac{1}{2}\tilde{\gamma}^{zz}\partial_z\partial_z\tilde{\gamma}_{ww} + \frac{1}{2}\tilde{\gamma}^{ww}\tilde{\gamma}^{zz}\partial_z\tilde{\gamma}_{ww}\;\partial_z\tilde{\gamma}_{ww} - \frac{n}{2}\tilde{\gamma}^{ww}\frac{\partial_z\tilde{\gamma}_{ww}}{z} + \tilde{\gamma}_{ww}\frac{\tilde{\Gamma}^z}{z}\\
&\quad +\frac{1}{2}\tilde{\Gamma}^z\partial_z\tilde{\gamma}_{ww} + \frac{\tilde{\gamma}^{zz}\tilde{\gamma}_{ww}-1}{z^2}\,, && (10.32)\\
[\mathcal{R}_{zz}]^{\mathrm{TF}} &= \frac{D-2}{D-1}\mathcal{R}_{zz} - \frac{n}{D-1}\tilde{\gamma}_{zz}\tilde{\gamma}^{ww}\mathcal{R}_{ww}\,, && (10.33)\\
[\mathcal{R}_{ww}]^{\mathrm{TF}} &= \frac{1}{D-1}\left(\mathcal{R}_{ww} - \tilde{\gamma}_{ww}\tilde{\gamma}^{zz}\mathcal{R}_{zz}\right). && (10.34)
\end{aligned}$$

A key ingredient in the BSSN formulation is the subtraction of the trace of the extrinsic curvature $\tilde{A}_{IJ} \to \tilde{A}_{IJ} - \tilde{\gamma}_{IJ}\tilde{A}/(D-1)$ which is now given by

$$\begin{aligned}
\tilde{A} = \mathrm{tr}\tilde{A}_{IJ} &= \tilde{A} = \tilde{\gamma}^{zz}\tilde{A}_{zz} + n\tilde{\gamma}^{ww}\tilde{A}_{ww}\,, && (10.35)\\
\tilde{A}_{zz} &\to \tilde{A}_{zz} - \frac{1}{D-1}\tilde{\gamma}_{zz}\tilde{A}\,, && (10.36)\\
\tilde{A}_{ww} &\to \tilde{A}_{ww} - \frac{1}{D-1}\tilde{\gamma}_{ww}\tilde{A}\,. && (10.37)
\end{aligned}$$

We further note that the traceless extrinsic curvature with upper indices is obtained from

$$\begin{aligned}
\tilde{A}^{zz} &= \tilde{\gamma}^{zM}\tilde{\gamma}^{zN}\tilde{A}_{MN} = \tilde{\gamma}^{zz}\tilde{\gamma}^{zz}\tilde{A}_{zz}\,, && (10.38)\\
\tilde{A}^{z}{}_{z} &= \tilde{\gamma}^{zM}\tilde{A}_{Mz} = \tilde{\gamma}^{zz}\tilde{A}_{zz}\,, && (10.39)\\
\tilde{A}^{ab} &= \delta^{ab}(\tilde{\gamma}^{ww})^2\tilde{A}_{ww}\,, && (10.40)\\
\tilde{A}^{a}{}_{z} &= \tilde{A}^{z}{}_{a} = 0\,, && (10.41)\\
\tilde{A}^{a}{}_{b} &= \delta^{a}{}_{b}\tilde{\gamma}^{ww}\tilde{A}_{ww}\,, && (10.42)
\end{aligned}$$

The *moving puncture* gauge conditions remain almost unchanged from their four-dimensional version and are given by

$$\begin{aligned}
\partial_t\alpha &= \beta^M\partial_M\alpha - 2\alpha K = \beta^z\partial_z\alpha - 2\alpha K\,, && (10.43)\\
\partial_t\beta^z &= \beta^M\partial_M\beta^z + \frac{3}{4}\tilde{\Gamma}^z - \eta\beta^z = \beta^z\partial_z\beta^z + \frac{3}{4}\tilde{\Gamma}^z - \eta\beta^z\,, && (10.44)
\end{aligned}$$

so that the only change arises from the extra terms in the definition of $\tilde{\Gamma}^z$ in Eq. (10.3). Note that all vector components off the z axis vanish by symmetry and we therefore do not need to evolve β^w.
We still have to adress the Hamiltonian and momentum constraints. In BSSN they are given by

$$\begin{aligned}
\mathcal{H} &= \mathcal{R} + \frac{D-2}{D-1}K^2 - \tilde{A}^{MN}\tilde{A}_{MN} - 16\pi\rho - 2\Lambda = 0\,, && (10.45)\\
\mathcal{M}_I &= \tilde{\gamma}^{MN}\tilde{D}_M\tilde{A}_{NI} - \frac{D-2}{D-1}\partial_I K + 2(D-1)\tilde{A}^{M}{}_{I}\partial_M\phi - 8\pi j_I = 0\,, && (10.46)
\end{aligned}$$

Their version with $SO(D-1)$ isometry is given by

$$\begin{aligned}
\mathcal{H} &= \chi\tilde{\gamma}^{zz}\mathcal{R}_{zz} - \tilde{A}^{zz}\tilde{A}_{zz} + \frac{D-2}{D-1}K^2 + n\left(\chi\tilde{\gamma}^{ww}\mathcal{R}_{ww} - \frac{\tilde{A}^2_{ww}}{\tilde{\gamma}^2_{ww}}\right) = 0\,, && (10.47)\\
\mathcal{M}_z &= -\frac{D-2}{D-1}\partial_z K + n\tilde{\gamma}^{ww}\left(\frac{\tilde{A}_{zz}-\tilde{A}_{ww}}{z} - \frac{1}{2}\tilde{\gamma}^{ww}\tilde{A}_{ww}\,\partial_z\tilde{\gamma}_{ww}\right) - n\tilde{\gamma}^{ww}\tilde{\Gamma}^z_{ww}\tilde{A}_{zz}\\
&\quad + \tilde{\gamma}^{zz}\left(\tilde{D}\tilde{A}_{zz} - \frac{D-1}{2\chi}\tilde{A}_{zz}\,\partial_z\chi\right)\,. && (10.48)
\end{aligned}$$

We also note the following relation that might be helpful

$$\tilde{D}_w\tilde{D}_w = \tilde{\gamma}^{zz}\left(\tilde{\gamma}_{ww}\frac{\partial_z\chi}{z} + \frac{1}{2}\partial_z\tilde{\gamma}_{ww}\,\partial_z\chi\right)\,. \qquad (10.49)$$

11 Regularization

The equations of the previous section contain various expressions with divisions by z which implies division by zero in the xy plane. All these expressions can be regularized which is the topic of this section.
We start with several preliminary results that will be needed throughout the following calculations. The conformal spatial metric is given by

$$\tilde{\gamma}_{IJ} = \left(\begin{array}{c|ccc}
\tilde{\gamma}_{zz} & 0 & \cdots & 0\\
\hline
0 & \tilde{\gamma}_{ww} & \cdots & 0\\
\vdots & \vdots & \ddots & \vdots\\
0 & 0 & \cdots & \tilde{\gamma}_{ww}
\end{array}\right)\,. \qquad (11.1)$$

The determinant is given by

$$\det \tilde{\gamma}_{IJ} = \tilde{\gamma}_{zz}\tilde{\gamma}_{ww}^{n} \quad \stackrel{!}{=} \quad 1\,, \tag{11.2}$$
$$\tag{11.3}$$

where the BSSN definition implies that the determinant is unity. Inversion of the metric is thus trivial and we obtain

$$\tilde{\gamma}^{zz} = \frac{1}{\tilde{\gamma}_{zz}}\,, \qquad \tilde{\gamma}^{ww} = \frac{1}{\tilde{\gamma}_{ww}}\,, \tag{11.4}$$

which we already used in the derivation of the specific BSSN equations for $SO(D-1)$ isometry. One further ingredient we shall need is the absence of a conical singularity at $z=0$. Consider for this purpose the coordinate transformation given by Eq. (1.4) which changes (z,w) to polar coordinates (ρ,φ). Restricting attention to these two coordinates, we have

$$\gamma_{\rho\rho} = \frac{z^2}{\rho^2}\gamma_{zz} + 2\frac{zw}{\rho^2}\gamma_{zw} + \frac{w^2}{\rho^2}\gamma_{ww}\,, \tag{11.5}$$
$$\gamma_{\varphi\varphi} = w^2\gamma_{zz} - 2wz\gamma_{zm} + z^2\gamma_{ww}\,, \tag{11.6}$$
$$ds^2 = \gamma_{\rho\rho}d\rho^2 + \gamma_{\varphi\varphi}d\varphi^2 d\varphi^2\,. \tag{11.7}$$

We require the proper circumference to be related to the proper radius by a factor of 2π which implies $\gamma_{\varphi\varphi} = \rho^2\gamma_{\rho\rho}$. Substituting the above expressions for the polar metric components and letting $z \to 0$ implies

$$\gamma_{zz} - \gamma_{ww} = \mathcal{O}(z^2)\,. \tag{11.8}$$

This condition is valid for arbitrary values of w and by smoothness therefore also aplies in the limit $w \to 0$. It furthermore directly transfers to the conformal metric, so that $\tilde{\gamma}_{ww} - \tilde{\gamma}_{zz} = \mathcal{O}(z^2)$. Finally, since the inverse metric components are simply $\tilde{\gamma}^{zz} = 1/\tilde{\gamma}_{zz}$ and $\tilde{\gamma}^{ww} = 1/\tilde{\gamma}_{ww}$, we also have $\tilde{\gamma}^{ww} - \tilde{\gamma}^{zz} = \mathcal{O}(z^2)$. Let us now discuss all terms appearing in Eqs. (6.3)-(6.49).

1) We expand $\beta^z = b_1 z + b_3 z^3 + \ldots$ and conclude

$$\frac{\beta^z}{z} = \partial_z\beta^z\,. \tag{11.9}$$

Likewise, this applies to all vectorial quantities such as $\tilde{\Gamma}^z$.

2) We use the above expression (11.4) for the inverse metric to factor out $\tilde{\gamma}^{zz}$. Then

$$\frac{1-\tilde{\gamma}^{zz}\tilde{\gamma}_{ww}}{z} = \tilde{\gamma}^{zz}\frac{\tilde{\gamma}_{zz}-\tilde{\gamma}_{ww}}{z} = \frac{\mathcal{O}(z^2)}{z} = 0\,. \tag{11.10}$$

3) Expanding $\beta^z = b_1 z + b_3 z^3 + \ldots$, we trade two divisions for a second derivative in

$$\frac{\partial_z\beta^i}{z} - \frac{\beta^z}{z^2} = 0\,. \tag{11.11}$$

4) The derivative of a scalar behaves like a vector component, so that

$$\frac{\partial_z\alpha}{z} = \partial_z\partial_z\alpha\,. \tag{11.12}$$

5) We consider the condition $\tilde{\gamma}_{zz} - \tilde{\gamma}_{ww} = \mathcal{O}(z^2)$ and take the difference of the evolution equations (10.6) and (10.7) for $\tilde{\gamma}_{zz}$ and $\tilde{\gamma}_{ww}$. After a brief calculation, all terms reduce to expressions containing $\tilde{\gamma}_{zz} - \tilde{\gamma}_{ww}$. We thus obtain

$$\tilde{A}_{zz} - \tilde{A}_{ww} = \Big[\dots\Big](\tilde{\gamma}_{zz} - \tilde{\gamma}_{ww}) = \mathcal{O}(z^2)\,, \tag{11.13}$$

where the dots denote a collection of terms and operators that ubiquitously preserve the z^2 behaviour inherited by $\tilde{\gamma}_{zz} - \tilde{\gamma}_{ww}$. ***6)*** Using again $\tilde{\gamma}_{zz} - \tilde{A}_{ww} = \mathcal{O}(z^2)$ and trading a division by z for a z derivative, we find

$$-\frac{1}{2}\frac{\partial_z \tilde{\gamma}_{zz}}{z} + \frac{\tilde{\gamma}_{zz} - \tilde{\gamma}_{ww}}{z^2} = -\frac{1}{2}\partial_z\partial_z\tilde{\gamma}_{ww}\,, \tag{11.14}$$

7) By trading two divisions for a second derivative, we finally get

$$\frac{\tilde{\gamma}_{zz} - \tilde{\gamma}_{ww}}{z^2} = \frac{1}{2}(\partial_z\partial_z\tilde{\gamma}_{zz} - \partial_z\partial_z\tilde{\gamma}_{ww})\,. \tag{11.15}$$

12 Some analytic data

Here we summarize some analytic solutions to the Einstein equations or the initial data equations that are helpful in testing the equations of the previous section as well as the code. A reference I have found very helpful in this regard is Kunstatter, Maeda & Taves [7].

1) Tangherlini in isotropic coordinates
The puncture equivalent of a Schwarzschild black-hole in higher dimensions has a line element

$$ds^2 = -\left(\frac{4r^{D-3} - \mu^{D-3}}{4r^{D-3} + \mu^{D-3}}\right)dt^2 + \left(1 + \frac{\mu^{D-3}}{4r^{D-3}}\right)^{4/(D-3)}\left(dx^2 + dy^2 + dz^2 + \sum_a dw_a^2\right)\,. \tag{12.1}$$

This implies for the ADM variables

$$\gamma_{zz} = \psi^{4/(D-3)}\,, \quad \psi = 1 + \frac{\mu^{D-3}}{4r^{D-3}}\,, \quad \beta^I = 0\,, \quad K_{zz} = 0\,. \tag{12.2}$$

2) Tangherlini in Kerr-Schild coordinates
A non-rotating black hole in Kerr-Schild coordinates is particularly helpful to check the equations and also the code implementation because it contains far more non-trivial terms than the conformally flat solutions. Tangherlini in Kerr-Schild coordinates is given by

$$ds^2 = -\left(1 + \frac{\mu}{r^{D-3}}\right)dt^2 + 2\frac{\mu}{r^{D-3}}dt\,dr + \left(1 + \frac{\mu}{r^{D-3}}\right)dr^2 + r^2 d\Omega^2\,, \tag{12.3}$$

where

$$d\Omega^2 = h_{ab}\,d\phi^a\,d\phi^b\,, \quad a, b = 1, \dots, D-1\,. \tag{12.4}$$

For the transformation to Cartesian coordinates, we use

$$
\begin{aligned}
x_1 &= r\cos\phi_1\,, & r &= \sqrt{x_1^2+\ldots x_n^2}\,,\\
x_2 &= r\sin\phi_1\,\cos\phi_2\,, & \phi_1 &= \mathrm{arccot}\frac{x_1}{\sqrt{x_n^2+\ldots+x_2^2}}\,,\\
x^3 &= r\sin\phi_1\,\sin\phi_2\,\cos\phi_3\,, & \phi_2 &= \mathrm{arccot}\frac{x_2}{\sqrt{x_n^2+\ldots+x_3^2}}\,,\\
&\vdots & &\vdots\\
x^{n-1} &= r\sin\phi_1\,\sin\phi_2\,\ldots\,\sin\phi_{n-1}\,, & \phi_{n-2} &= \mathrm{arccot}\frac{x_{n-2}}{\sqrt{x_n^2+x_{n-1}^2}}\,,\\
x^n &= r\sin\phi_1\,\sin\phi_2\,\ldots\,\cos\phi_{n-1}\,, & \phi_{n-1} &= \mathrm{arccot}\frac{x_{n-1}}{x_n}\left(=2\mathrm{arccot}\frac{\sqrt{x_n^2+x_{n-1}^2}+x_{n-1}}{x_n}\right)
\end{aligned}
\tag{12.5}
$$

where $n=D-1$ and the last expression for ϕ_{n-1} in parentheses is strictly speaking more accurate for $x_{n-1}<0$ where it eliminates a jump by 2π. For most purposes, the simpler, first expression should work fine, though. Using

$$
g_{\alpha\beta}=\frac{\partial x^{\bar\mu}}{\partial x^\alpha}\frac{\partial x^{\bar\nu}}{\partial x^\beta}g_{\bar\mu\bar\nu}\,,
\tag{12.6}
$$

we obtain

$$
ds^2=-\left(1-\frac{\mu}{r^{D-3}}\right)dt^2+2\frac{\mu}{r^{D-3}}\frac{x^i}{r}dt\,dx^i+\left(\delta_{ij}+\frac{\mu}{r^{D-3}}\frac{x_i}{r}\frac{x_j}{r}\right)dx^i\,dx^j\,,
\tag{12.7}
$$

and for the ADM variables

$$
\begin{aligned}
&\gamma_{ij}=\delta_{ij}+\frac{\mu}{r^{D-3}}\frac{x_ix_j}{r^2}\,,\quad \beta_i=\frac{\mu x_i}{r^{D-3}r}\,,\quad \beta^i=\frac{\mu}{r^{D-3}+\mu}\frac{x^i}{r}\,,\quad \alpha=\sqrt{\frac{r^{D-3}}{r^{D-3}+\mu}}\,,\\
&K_{ij}=\frac{2\mu(r^{10}\mu+r^{7+D})\delta_{ij}-\mu x_ix_j\left[(D-3)r^{11-D}\mu^2+(3D-7)\mu r^8+(2D-4)r^{5+D}\right]}{2r^5(r^D+\mu r^3)\sqrt{r^{2D}+\mu r^{D+3}}}\,,\\
&\gamma_{ww}=1\,,\quad K_{ww}=\frac{\mu r^2}{\sqrt{r^{2D}+\mu r^{3+D}}}\,.
\end{aligned}
\tag{12.8}
$$

For the special case $D=7$, we also get the following results

$$
\det\gamma_{ij}=1+\frac{\mu}{r^4}\,,\quad K_{ij}=\frac{\mu}{r\sqrt{r^4(r^4+\mu)}}\left[\delta_{ij}-\frac{x_ix_j(5r^4+2\mu)}{r^6}\right]\,,\quad K=\frac{\mu(r^4+3\mu)}{r^3(r^4+\mu)^{3/2}}\,.
\tag{12.9}
$$

For the case where we only have the z directions, the above more general result becomes

$$
\begin{aligned}
\gamma_{zz} &= 1 + \frac{\mu}{z^{D-3}}\,, \\
\beta^z &= \frac{\mu}{z^{D-3}+\mu}\,, \\
\alpha &= \sqrt{\frac{z^{D-3}}{z^{D-3}+\mu}}\,, \\
K_{zz} &= \frac{2\mu(z^{10}\mu + z^{7+D}) - \mu z^2\left[(D-3)z^{11-D}\mu^2 + (3D-7)\mu z^8 + (2D-4)z^{5+D}\right]}{2z^5(z^D+\mu z^3)\sqrt{z^{2D}+\mu z^{D+3}}}\,, \\
\gamma_{ww} &= 1\,, \\
K_{ww} &= \frac{\mu z^2}{\sqrt{z^{2D}+\mu z^{3+D}}}\,, \\
\det\gamma &= \gamma_{zz}\,.
\end{aligned}
\tag{12.10}
$$

The remaining ADM and BSSN variables can then be calculated, e.g. inside `Maple`, from their standard definitions.

13 The ADM equations in modified cartoon form

Here we present the ADM equations in their modified-cartoon version which we use for verifying the analytic solution for the black ring. We recall that late Latin indices $i, j, \ldots$ run from 1 to 3 and w denotes the components off the computational domain; w is a label and not a running index. The ADM equations are given by

$$\partial_t\gamma_{ij} = \beta^m\partial_m\gamma_{ij} + \gamma^{mj}\partial_i\beta^m + \gamma^{im}\partial_j\beta^m - 2\alpha K_{ij}\,, \tag{13.1}$$

$$\partial\gamma_{ww} = \beta^m\partial_m\gamma_{ww} + 2\gamma_{ww}\frac{\beta^z}{z} - 2\alpha K_{ww}\,, s \tag{13.2}$$

$$\partial_t K_{ij} = \beta^m\partial_m K_{ij} + K_{mj}\partial_i\beta^m + K_{im}\partial_j\beta^m - D_iD_j\alpha + \alpha\left[\mathcal{R}_{ij} + KK_{ij} - 2K_{im}K^m{}_j\right]\,, \tag{13.3}$$

$$\partial_t K_{ww} = \beta^m\partial_m K_{ww} + 2K_{ww}\frac{\beta^z}{z} - D_wD_w\alpha_+\left[\mathcal{R}_{ww} + KK_{ww} - 2\gamma^{ww}K_{ww}^2\right]\,. \tag{13.4}$$

Note that the trace of the extrinsic curvature has extra terms arising from the additional spacetime dimensions

$$K = K^m{}_m + (D-4)\gamma^{ww}K_{ww}\,. \tag{13.5}$$

We furthermore have the following auxiliary variables in the ADM equations

$$D_iD_j\alpha = \partial_i\partial_j\alpha - \Gamma^m_{ji}\partial_m\alpha\,, \tag{13.6}$$

$$D_wD_w\alpha = \frac{\partial_z\alpha}{z} - \Gamma^m_{ww}\partial_m\alpha\,, \tag{13.7}$$

$$
\begin{aligned}
\mathcal{R}_{ij} = {} & (D-4)\gamma^{ww}\left[-\frac{1}{2}\frac{\partial_z\gamma_{ij}}{z} + \frac{\delta_{z(i}\gamma_{j)z} - \delta_{iz}\delta_{jz}\gamma_{ww}}{z^2} + \frac{\gamma^{ww}\gamma_{z(i} - \delta_{z(i}}{z}\delta_{j)}\gamma_{ww} - \frac{1}{4}\gamma^{ww}\partial_i\gamma_{ww}\,\partial_j\gamma_{ww}\right] \\
& -\frac{1}{2}\gamma^{mn}\partial_m\partial_n\gamma_{ij} + \gamma_{m(i}\partial_{j)}\Gamma^m + \Gamma^m\Gamma_{(ij)m} + \gamma^{mn}\left[2\Gamma^k_{m(i}\Gamma_{j)nk} + \Gamma^k_{im}\Gamma_{kjn}\right]\,,
\end{aligned}
\tag{13.8}
$$

$$\begin{aligned}
\mathcal{R}_{ww} &= -\frac{1}{2}\gamma^{mn}\partial_m\partial_n\gamma_{ww} + \frac{1}{2}\gamma^{ww}\gamma^{mn}\partial_m\gamma_{ww}\,\partial_n\gamma_{ww} - \frac{D-4}{2}\gamma^{ww}\frac{\partial_z\gamma_{ww}}{z} \\
&\quad + \gamma_{ww}\frac{\Gamma^z}{z} + \frac{1}{2}\Gamma^m\partial_m\gamma_{ww} + \frac{\gamma^{zz}\gamma_{ww}-1}{z^2}\,, \qquad (13.9) \\
\Gamma^i &= \gamma^{mn}\Gamma^i_{mn} + (D-4)\gamma^{ww}\Gamma^i_{ww}\,, \qquad (13.10) \\
\Gamma^i_{ww} &= -\frac{1}{2}\gamma^{im}\partial_m\gamma_{ww} + \frac{\delta^{iz}-\gamma^{iz}\gamma_{ww}}{z}\,, \qquad (13.11) \\
\gamma^{ww} &= \frac{1}{\gamma_{ww}}\,. \qquad (13.12)
\end{aligned}$$

14 Wave extraction using the generalized Newman-Penrose scalars

A method to extract the energy radiated in gravitational waves in higher-dimensional spacetimes generalizing the Newman-Penrose scalars has been developed in [5, 6]. Here we describe the implementation in the framework of our modified cartoon formalism.
We need to distinguish sensitively between different index ranges and employ the following conventions.

- Early upper case Latin indices $A,\ B,\ \ldots$ run from 0 to $D-1$, i.e. cover all spacetime indices.
- Late upper case Latin indices $I,\ J,\ \ldots$ cover all spatial indices $1,\ \ldots,\ D-1$.
- Greek indices $\alpha,\ \eta,\ \ldots$ run from 0 to 3 and cover the entire computational domain.
- Late lower case Latin indices $i,\ j,\ldots$ cover the spatial part of the computational domain and run from 1 to 3.
- Early lower case Latin indices $a,\ b,\ \ldots$ cover the spatial compactified directions $4,\ \ldots,\ D-1$.
- Cartesian coordinates are given by $t,\ x,\ y,\ z,\ w_a$ and we shall at times use w or u to denote specific off-domain coordinates.
- Spherical coordinates are denoted by $t,\ r,\ \phi_2,\ \phi_3,\ \ldots,\ \phi_{D-1}$. We use early lower-case Latin characters with a caret to denote the angular indices, i.e. $\hat{a},\ \hat{b},\ \ldots = 2,\ \ldots,\ D-1$. Note that our index range for $\phi_{\hat{a}}$ is shifted by 1 relative to Will's notes.

We need to distinguish between the spacetime Riemann tensor denoted by R_{ABCD} and its spatial counterpart $\mathcal{R}_{IJKM}$. Their respective Ricci tensors and scalar are obtained by contraction with the spacetime metric g^{AB} or the spatial $(D-1)$ metric γ^{IJ}. A good deal of what we have derived for the BSSN equations remains valid here, but note that we can not use the simplification of a unit determinant here, i.e. $\det\gamma_{IJ} \neq 1$ in general.

14.1 The spacetime Riemann tensor

We use the convention where the Riemann tensor is defined by

$$R^A{}_{BCD} = \partial_C\Gamma^A_{BD} - \partial_D\Gamma^A_{BC} + \Gamma^F_{BD}\Gamma^A_{FC} - \Gamma^F_{BC}\Gamma^A_{FD}\,. \qquad (14.1)$$

The components of the spacetime Riemann tensor are given in terms of its spatial analog and the extrinsic curvature by the Gauss equation

$$\perp R_{ABCD} = \mathcal{R}_{ABCD} + K_{AC}K_{BD} - K_{AD}K_{CB}\,. \tag{14.2}$$

Here $\perp$ denotes the set of projection operators $\perp^A{}_E$ required to project every free index of the argument, so for instance

$$\perp(T_{ABC}v^B) \equiv \perp^E{}_A \perp^F{}_C T_{EBF}v^B\,. \tag{14.3}$$

We shall also need the time components of the spacetime Riemann tensor, although it will turn out convenient to instead use the corresponding contractions with the unit timelike normal n_A which appear naturally in the $(D-1)+1$ split of the Einstein equations. The Codazzi equation gives us

$$\perp(R_{ABCD}n^B) = -D_C K_{AD} + D_D K_{AC}\,. \tag{14.4}$$

Note that our notation is a bit dodgy here, since the 0 index does not denote the time component but the contraction with n.

The final contraction we need is that containing twice the timelike unit normal which is given by the contracted Gauss equation

$$\perp(R_{ABCD}n^B n^D) = -\perp R_{AC} + \mathcal{R}_{AC} + KK_{AC} - K_{AE}K^E{}_C\,, \tag{14.5}$$

where $K \equiv \gamma^{AB}K_{AB}$ is the trace of the extrinsic curvature. We have written all these equations using spacetime coordinates but will in practice always work in coordinates adapted to the $(D-1)+1$ split. We can therefore replace spacetime with spatial indices and rewrite the preceding equations in the form

$$R_{IJKL} \;=\; \mathcal{R}_{IJKL} + K_{IK}K_{JL} - K_{IL}K_{JK}\,, \tag{14.6}$$

$$R_{I0KL} \;\equiv\; \perp^A{}_I \perp^C{}_K \perp^D{}_L R_{ABCD}n^B = -D_K K_{IL} + D_L K_{IK}\,. \tag{14.7}$$

$$R_{I0K0} \;\equiv\; \perp^A{}_I \perp^C{}_K R_{ABCD}n^B n^D = -\perp R_{IK} + \mathcal{R}_{IK} + KK_{IK} - K_{IM}K^M{}_K\,. \tag{14.8}$$

The notation of using a 0 index for a contraction with n is a bit fishy but will turn out convenient throughout this discussion. We furthermore note that gravitational waves will in practice always be extracted in vacuum where the spacetime Ricci tensor $R_{AB} = 0$, so that also $-\perp R_{IK} = 0$ and we will from now on ignore this term on the right-hand side of Eq. (14.8).

14.2 The spatial Riemann tensor

Using the spatial analog of the definition of the Riemann tensor (14.1) and rearranging terms, a slightly lengthy but straightforward calculation gives the spatial Riemann tensor in terms of the metric derivatives and Christoffel symbols as

$$\mathcal{R}_{IJKL} = \frac{1}{2}\big(\partial_K\partial_J\gamma_{IL} + \partial_L\partial_I\gamma_{JK} - \partial_K\partial_I\gamma_{JL} - \partial_L\partial_J\gamma_{IK}\big) - \gamma_{MN}\Gamma^M_{JL}\Gamma^N_{IK} + \gamma_{MN}\Gamma^M_{JK}\Gamma^N_{IL}\,. \tag{14.9}$$

Before returning to the spacetime Riemann tensor, we stay with its spatial counterpart for a little longer and consider how the components look like when considering the two subsectors of the index range $I=(i,a)$. After eliminating all those (typically mixed) components that vanish by $SO(D-3)$ symmetry, we arrive at the following expressions.

$$\mathcal{R}_{ijkl} = \frac{1}{2}\big(\partial_l\partial_i\gamma_{jk}+\partial_k\partial_j\gamma_{il}-\partial_k\partial_i\gamma_{jl}-\partial_l\partial_j\gamma_{ik}\big)-\gamma_{mn}\Gamma^n_{ik}\Gamma^m_{jl}+\gamma_{mn}\Gamma^n_{il}\Gamma^m_{jk}\,, \tag{14.10}$$

$$\mathcal{R}_{ajkl} = 0\,, \tag{14.11}$$

$$\mathcal{R}_{ijkd} = 0\,, \tag{14.12}$$

$$\mathcal{R}_{ibkd} = \delta_{bd}\mathcal{R}_{iwkw}\,, \tag{14.13}$$

$$\begin{aligned}\mathcal{R}_{iwkw} \equiv\ & \frac{\partial_{(i}\gamma_{k)z}-\delta_{z(k}\partial_{i)}\gamma_{ww}}{z}-\delta_{z(i}\frac{\gamma_{k)z}-\delta_{k)z}\gamma_{ww}}{z^2}-\frac{1}{2}\partial_k\partial_i\gamma_{ww}-\gamma_{mn}\Gamma^n_{ik}\Gamma^m_{ww}\\ & -\frac{1}{2}\frac{\partial_z\gamma_{ik}}{z}+\frac{\delta_{z(i}\gamma_{k)z}-\delta_{iz}\delta_{kz}\gamma_{ww}}{z^2}+\frac{1}{4}\gamma^{ww}(\partial_i\gamma_{ww})\,\partial_k\gamma_{ww}\,,\end{aligned} \tag{14.14}$$

$$\Gamma^m_{ww} \equiv -\frac{1}{2}\gamma^{ml}\partial_l\gamma_{ww}+\frac{\delta^m{}_z-\gamma^{mz}\gamma_{ww}}{z}\,, \tag{14.15}$$

$$\mathcal{R}_{abcl} = 0\,, \tag{14.16}$$

$$\mathcal{R}_{abcd} = (\delta_{ac}\delta_{bd}-\delta_{bc}\delta_{ad})\mathcal{R}_{wuwu}\,, \tag{14.17}$$

$$\mathcal{R}_{wuwu} \equiv -\frac{1}{4}\gamma^{mn}(\partial_m\gamma_{ww})\partial_n\gamma_{ww}-\gamma_{ww}\frac{\gamma^{zm}}{z}\partial_m\gamma_{ww}+\frac{\gamma_{ww}-\gamma^{zz}\gamma^2_{ww}}{z^2}\,. \tag{14.18}$$

Every other combination of indices for the spatial Riemann tensor can be constructed directly from those given through the symmetry properties.
We shall also need the components of the spatial Ricci tensor which are obtained straightforwardly from the previous expressions by contracting over the first and third index. This gives

$$\mathcal{R}_{ij} = \gamma^{mn}\mathcal{R}_{minj}+(D-4)\gamma^{ww}\mathcal{R}_{iwjw}\,, \tag{14.19}$$

$$\mathcal{R}_{ab} = \delta_{ab}\mathcal{R}_{ww}\,, \tag{14.20}$$

$$\mathcal{R}_{ww} \equiv \gamma^{mn}\mathcal{R}_{mwnw}+(D-5)\gamma^{ww}\mathcal{R}_{wuwu}\,. \tag{14.21}$$

Note that the components $\mathcal{R}_{iwjw}$ and $\mathcal{R}_{mwnw}$ are given in Eq. (14.14).

14.3 Components of the spacetime Riemann tensor

We now return to the spacetime components of the Riemann tensor given in Eqs. (14.6)-(14.8). In $SO(D-3)$ symmetry in the modified Cartoon formalism, we distinguishing between the (x,y,z) and the w^a components and obtain for the index range $I = (i, a)$

$$R_{ijkl} = \mathcal{R}_{ijkl} + K_{ik}K_{jl} - K_{il}K_{jk}\,, \tag{14.22}$$

$$R_{ibkd} = \mathcal{R}_{ibkd} + K_{ik}K_{bd} - K_{id}K_{bk} = \delta_{bd}R_{iwkw}\,, \tag{14.23}$$

$$R_{iwkw} \equiv \mathcal{R}_{iwkw} + K_{ik}K_{ww}\,, \tag{14.24}$$

$$R_{abcd} = \mathcal{R}_{abcd} + K_{ac}K_{bd} - K_{ad}K_{bc} = (\delta_{ac}\delta_{bd} - \delta_{bc}\delta_{ad})\big(\mathcal{R}_{wuwu} + K_{ww}^2\big)\,, \tag{14.25}$$

$$R_{ajkl} = R_{abcl} = 0\,, \tag{14.26}$$

$$R_{i0kl} = D_l K_{ik} - D_k K_{il}\,, \tag{14.27}$$

$$R_{a0ck} = \delta_{ac}R_{w0wk}\,, \tag{14.28}$$

$$R_{w0wk} \equiv \partial_k K_{ww} - \frac{1}{2}\gamma^{ww}K_{ww}\partial_k\gamma_{ww} - \frac{K_{kz} - \delta_{kz}K_{ww}}{z} + \Gamma^m_{ww}K_{mk}\,, \tag{14.29}$$

$$R_{a0cd} = R_{i0kd} = R_{a0kl} = 0\,, \tag{14.30}$$

$$R_{i0j0} = \mathcal{R}_{ij} + KK_{ij} - K_{im}K^m{}_j\,, \tag{14.31}$$

$$K = \gamma^{mn}K_{mn} + (D-4)\gamma^{ww}K_{ww}\,, \tag{14.32}$$

$$R_{a0b0} = \delta_{ab}R_{w0w0}\,, \tag{14.33}$$

$$R_{w0w0} \equiv \mathcal{R}_{ww} + (K - \gamma^{ww}K_{ww})K_{ww}\,, \tag{14.34}$$

$$R_{a0i0} = 0\,. \tag{14.35}$$

14.4 Regularization

In the expressions of the previous subsections, we have various terms which need to be regularized at $z = 0$. Many of these terms are already included in the list of terms discussed in Sec. 7 and we shall refer to the corresponding terms as ***reg??)***. Some, however, are new terms and will be discussed here in more detail. We continue the count with number***13*** here as we left Sec. 7 at number ***12***. We distinguish the indices z and (x, y) where for the latter we shall only write x; substitution $x \to y$ on both sides of the respective equations leaves equalities unchanged.
13) First 2 terms in Eq. (14.14),

$$\text{reg13}_{ik} = \frac{1}{2}\left(\frac{\partial_i\gamma_{kz} - \delta_{zk}\partial_i\gamma_{ww}}{z} - \delta_{zi}\frac{\gamma_{kz} - \delta_{kz}\gamma_{ww}}{z^2} + \frac{\partial_k\gamma_{iz} - \delta_{zi}\partial_k\gamma_{ww}}{z} - \delta_{zk}\frac{\gamma_{iz} - \delta_{iz}\gamma_{ww}}{z^2}\right)\,. \tag{14.36}$$

Case 1: $i = z$, $k = x$. By expanding $\gamma_{xz} = \gamma_1 z + \gamma_3 z^3 + \ldots$ and $\gamma_{zz} - \gamma_{ww} = \mathcal{O}(z^2)$, we directly obtain

$$\text{reg13}_{zx} = \frac{1}{2}\left\{\frac{\partial_z\gamma_{xz}}{z} - \frac{\gamma_{xz}}{z^2} + \frac{\partial_x\gamma_{zz} - \partial_x\gamma_{ww}}{z}\right\} \to 0 + 0\,, \tag{14.37}$$

and the entire expression vanishes.
Case 2: $i = x$, $k = z$. Since the entire expression is symmetric in i, k, this reduces to case 1 and yields $\mathrm{reg13}_{xz} = 0$ as well.
Case 3: $i = x$, $k = y$. All the Kronecker deltas vanish and are left with

$$\mathrm{reg13}_{xy} = \frac{1}{2}\left(\frac{\partial_x \gamma_{yz}}{z} + \frac{\partial_y \gamma_{xz}}{z}\right) \to \frac{1}{2}\left(\partial_x \partial_z \gamma_{yz} + \partial_y \partial_z \gamma_{xz}\right) . \tag{14.38}$$

This result remains valid if x and y coincide, i.e.

$$\mathrm{reg13}_{xx} \quad \to \quad \partial_x \partial_z \gamma_{xz} , \tag{14.39}$$

$$\mathrm{reg13}_{yy} \quad \to \quad \partial_y \partial_z \gamma_{yz} . \tag{14.40}$$

Case 4: $i = z$, $k = z$. We can now combine terms and then trade once division by z for a derivative ∂_z and use reg11 for the other terms and combine the result into

$$\mathrm{reg13}_{zz} = \frac{\partial_z \gamma_{zz} - \partial_z \gamma_{ww}}{z} - \frac{\gamma_{zz} - \gamma_{ww}}{z^2} \to \frac{1}{2}\partial_z \partial_z (\gamma_{zz} - \gamma_{ww}) . \tag{14.41}$$

Next, we consider the 5th and 6th term in Eq. (14.14)

$$\frac{1}{2}\left(-\frac{\partial_z \gamma_{ik}}{z} + \frac{\delta_{zi}\gamma_{kz} + \delta_{zk}\gamma_{iz} - 2\delta_{iz}\delta_{kz}\gamma_{ww}}{z^2}\right) , \tag{14.42}$$

and this is exactly reg08 of Sec. 7. We already encountered the final term in Eq. (14.15) as reg02 in Sec. 7, but on that occasion used the unit determinant $\det \tilde{\gamma}_{IJ} = 1$ to modify the result. The generic version, valid for any value of the determinant is given by

$$\mathrm{reg02}' = \frac{\delta^m{}_z - \gamma^{zm}\gamma_{ww}}{z} = \begin{cases} \frac{\gamma_{yy}\partial_z\gamma_{xz} - \gamma_{xy}\partial_z\gamma_{yz}}{\gamma_{xx}\gamma_{yy} - \gamma_{xy}^2} & \text{if } m = x \\ \frac{\gamma_{xx}\partial_z\gamma_{yz} - \gamma_{xy}\partial_z\gamma_{xz}}{\gamma_{xx}\gamma_{yy} - \gamma_{xy}^2} & \text{if } m = y \\ 0 & \text{if } m = z \end{cases} \tag{14.43}$$

where we also used that $\gamma_{ww} = \gamma_{zz} + \mathcal{O}(z^2)$ in order to cancel $\gamma_{ww}/\gamma_{zz} \to 1$.
Next, we consider the regularization of terms on the right-hand side of Eq. (14.18). We call the first of these reg14 and obtain

$$\begin{aligned} \mathrm{reg14} &= -\gamma_{ww}\frac{\gamma^{zm}}{z}\partial_m \gamma_{ww} = -\gamma_{ww}\left(\frac{\gamma^{zx}}{z}\partial_z \gamma_{ww} + \frac{\gamma^{zy}}{z}\partial_y \gamma_{ww} + \gamma^{zz}\frac{\partial_z \gamma_{ww}}{z}\right) \\ &\to \frac{\gamma_{yy}\partial_z\gamma_{xz} - \gamma_{xy}\partial_z\gamma_{yz}}{\gamma_{xx}\gamma_{yy} - \gamma_{xy}^2}\partial_x \gamma_{ww} + \frac{\gamma_{xx}\partial_z\gamma_{yz} - \gamma_{xy}\partial_z\gamma_{xz}}{\gamma_{xx}\gamma_{yy} - \gamma_{xy}^2}\partial_y \gamma_{ww} - \partial_z\partial_z\gamma_{ww} . \end{aligned} \tag{14.44}$$

The second term to be regularized in Eq. (14.18) is

$$\begin{aligned} \mathrm{reg15} &= \frac{\gamma_{ww} - \gamma^{zz}\gamma_{ww}^2}{z^2} = \gamma_{ww}\gamma^{zz}\frac{\frac{1}{\gamma^{zz}} - \gamma_{ww}}{z^2} \\ &\to \frac{1}{2}(\partial_z\partial_z\gamma_{zz} - \partial_z\partial_z\gamma_{ww}) + \frac{2\gamma_{xy}(\partial_z\gamma_{xz})\partial_z\gamma_{yz} - \gamma_{xx}(\partial_z\gamma_{yz})^2 - \gamma_{yy}(\partial_z\gamma_{xz})^2}{\gamma_{xx}\gamma_{yy} - \gamma_{xy}^2} . \end{aligned} \tag{14.45}$$

The expressions for the spacetime Riemann tensor contain only one case for regularization which is the last but one term in Eq. (14.29). This is identical to reg07 in 7, though with $\tilde{A}_{\alpha\beta}$ replaced by $K_{\alpha\beta}$ and α and β stand for any index in Eq. (7.19), i.e.

$$\frac{K_{iz} - \delta_{iz} K_{ww}}{z} = \begin{cases} \partial_z K_{Xz} & \text{if } i = X \in \{x, y\} \\ 0 & \text{if } i = z \end{cases} \tag{14.46}$$

The derivation of this relation proceeds along the same lines as that of Eq. (7.19) with the only exception that we take the difference of the ADM equations (13.1) and (13.2) instead of their BSSN counter parts.

14.5 The normal frame

The normal frame we need for the calculation of $\Omega_{\hat{a}\hat{b}}$ consists of D unit vectors: (i) The ingoing null vector which we denote by k^A because we would like to reserve the letter n for the timelike unit normal vector on the spatial hypersurfaces. (ii) The outgoing null vector l^A which does not appear explicitly in the wave scalars we wish to calculate. (iii) $D-2$ vectors pointing in the angular directions $\phi_{\hat{a}}$; we will call these vectors $m^A_{(\hat{a})}$, where the parentheses in the subscript indicate that this is not a component label but merely an index distinguishing the $D-2$ different vectors $m_{(\hat{a})}$. Recall the index range for $\hat{a}$ is 2, ..., $D-1$, so that we have the vectors

$$m_{(2)}, \ldots, m_{(D-1)}, \tag{14.47}$$

and, thus,

$$\Omega_{\hat{a}\hat{b}} = R_{ABCD} k^A m^B_{(\hat{a})} k^C m^D_{(\hat{b})} . \tag{14.48}$$

For the construction of these vectors, we start by quoting Will's relation between angular and Cartesian coordinates (note that we shift the index of the angles ϕ by 1 relative to Will's notes)

$$\begin{aligned}
(w_{-2} =)\ x &= r\cos\phi_2 , \\
(w_{-1} =)\ y &= r\sin\phi_2 \cos\phi_3 , \\
(w_0 =)\ z &= r\sin\phi_2 \sin\phi_3 \cos\phi_4 , \\
w_1 &= r\sin\phi_2 \sin\phi_3 \sin\phi_4 \cos\phi_5 , \\
w_2 &= r\sin\phi_2 \sin\phi_3 \sin\phi_4 \sin\phi_5 \cos\phi_6 , \\
\vdots\ &\quad \vdots \\
w_{D-6} &= r\sin\phi_2 \ldots \sin\phi_{D-3} \cos\phi_{D-2} , \\
w_{D-5} &= r\sin\phi_2 \ldots \sin\phi_{D-3} \sin\phi_{D-2} \cos\phi_{D-1} , \\
w_{D-4} &= r\sin\phi_2 \ldots \sin\phi_{D-3} \sin\phi_{D-2} \sin\phi_{D-1} .
\end{aligned} \tag{14.49}$$

The angular vectors for the null frame are then obtained in Cartesian coordinates through chainrule according to

$$\frac{\partial}{\partial \phi_{\hat a}} = \frac{\partial x^I}{\partial \phi_{\hat a}} \frac{\partial}{\partial x^I}\,, \tag{14.50}$$

where $x^I = (x,\, y,\, z,\, w_1,\, \ldots,\, w_{D-4})$. We can ignore time components here, because by construction, all spatial vectors have a zero time component and no mixing arises from the Gram-Schmidt orthogonalization procedure of the spatial vectors.
Applying Eq. (14.50), we obtain the following explicit expressions for the angular vectors $\tilde m_{(\hat a)}$, $\hat a = 2,\, \ldots,\, D-1$,

$$\underbrace{\begin{pmatrix} -\sum_{i=1}^{D-2} w^2_{D-3-i} \\ w_{-2}w_{-1} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ w_{-2}w_{D-4} \end{pmatrix}}_{=\tilde m_{(2)}}, \ldots, \underbrace{\begin{pmatrix} \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} (\hat a - 2)\times \\ -\sum_{i}^{D-\hat a} w^2_{D-3-i\text{‘}} \\ w_{\hat a-4}w_{\hat a-3} \\ \vdots \\ w_{\hat a-4}w_{D-5} \\ w_{\hat a-4}w_{D-4} \end{pmatrix}}_{=\tilde m_{(\hat a)}}, \ldots, \underbrace{\begin{pmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ -(w^2_{D-5}+w^2_{D-4}) \\ w_{D-6}w_{D-5} \\ w_{D-6}w_{D-4} \end{pmatrix}}_{=\tilde m_{(D-2)}}, \underbrace{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ -w_{D-4}(w_{D-4}) \\ w_{D-5}(w_{D-4}) \end{pmatrix}}_{=\tilde m_{(D-1)}}. \tag{14.51}$$

Note that we differ here in the normalization of the last vector $m_{(D-1)}$ from Will's notes such that all vectors have the same normalization (or, rather, lack thereof). This is also the reason we introduced the *tilde* on the $\tilde m$; these vectors are not unit even in a Minkowski background. In this notation, however, we can write all vectors generically in a form giving unit length for a Minkowski background

$$m_{(\hat a)} = \frac{1}{\sqrt{\left(\sum_{i=1}^{D-\hat a} w^2_{D-3-i}\right)\left(\sum_{i=1}^{D-\hat a+1} w^2_{D-3-i}\right)}} \begin{pmatrix} \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} (\hat a-2)\times \\ -\sum_{i=1}^{D-\hat a} w^2_{D-3-i} \\ \left.\begin{matrix} w_{\hat a-4}w_{\hat a-3} \\ \vdots \\ w_{\hat a-4}w_{D-5} \\ w_{\hat a-4}w_{D-4} \end{matrix}\right\} (D-\hat a)\times \end{pmatrix}, \qquad \hat a = 2,\, \ldots,\, D-1\,. \tag{14.52}$$

Now we are facing an apparent conundrum, however. In the modified cartoon approach, we will ultimately be working on a computational domain where $w_1 = \ldots = w_{D-4} = 0$, so that all components in the $m_{(\hat a)}$ vanish and then are divided by zero through normalization. How do we identify which components survive in the unit normal vectors?
For that purpose it is necessary to return to spherical coordinates and it turns out convenient to begin

with a number of definitions for achieving a short-hand notation. We therefore set

$$\rho_1^2 \equiv x^2 + y^2 + z^2 + w_1^2 + \ldots + w_{D-4}^2 = r^2 \,, \tag{14.53}$$

$$\rho_2^2 \equiv y^2 + z^2 + w_1^2 + \ldots + w_{D-4}^2 \,, \tag{14.54}$$

$$\rho_3^2 \equiv z^2 + w_1^2 + \ldots + w_{D-4}^2 \,, \tag{14.55}$$

$$\rho_4^2 \equiv w_1^2 + \ldots + w_{D-4}^2 \,, \tag{14.56}$$

$$\vdots$$

$$\rho_{D-2}^2 \equiv w_{D-5}^2 + w_{D-4}^2 \,, \tag{14.57}$$

$$\rho_{D-1}^2 \equiv w_{D-4}^2 \,. \tag{14.58}$$

Note that the m vectors in Eq. (14.52) can now be written in the alternative form

$$m_{(\hat{a})} = \frac{1}{\rho_{\hat{a}}\,\rho_{\hat{a}-1}} \begin{pmatrix} \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} (\hat{a}-2)\times \\ -\rho_{\hat{a}}^2 \\ \left.\begin{matrix} w_{\hat{a}-4} w_{\hat{a}-3} \\ \vdots \\ w_{\hat{a}-4} w_{D-5} \\ w_{\hat{a}-4} w_{D-4} \end{matrix}\right\} (D-\hat{a})\times \end{pmatrix} \,, \qquad \hat{a} = 2, \ldots, D-1 \,. \tag{14.59}$$

With the definitions of the $\rho_1, \ldots, \rho_{D-1}$,

$$\begin{aligned} \sin\phi_2 &= \frac{\rho_2}{r} \,, & \cos\phi_2 &= \frac{x}{r} \,, \\ \sin\phi_3 &= \frac{\rho_3}{\rho_2} \,, & \cos\phi_3 &= \frac{y}{\rho_2} \,, \\ \sin\phi_4 &= \frac{\rho_4}{\rho_3} \,, & \cos\phi_4 &= \frac{z}{\rho_3} \,, \\ \sin\phi_5 &= \frac{\rho_5}{\rho_4} \,, & \cos\phi_5 &= \frac{w_1}{\rho_4} \,, \\ &\vdots & &\vdots \\ \sin\phi_{D-2} &= \frac{\rho_{D-2}}{\rho_{D-3}} \,, & \cos\phi_{D-2} &= \frac{w_{D-6}}{\rho_{D-3}} \,, \\ \sin\phi_{D-1} &= \frac{\rho_{D-1}}{\rho_{D-2}} = \frac{w_{D-4}}{\rho_{D-2}} \,, & \cos\phi_{D-1} &= \frac{w_{D-5}}{\rho_{D-2}} \,. \end{aligned} \tag{14.60}$$

For notation, it turns out convenient, to also identify

$$\text{“}\cos\phi_D = \frac{w_{D-4}}{\rho_{D-1}} = 1\text{”}\,. \tag{14.61}$$

Equation (14.60) can then be summarized in the short form

$$\sin\phi_{\hat{a}} = \frac{\rho_{\hat{a}}}{\rho_{\hat{a}-1}}\,, \qquad \cos\phi_{\hat{a}} = \frac{w_{\hat{a}-4}}{\rho_{\hat{a}-1}}\,, \qquad \hat{a} = 2,\,\ldots,\,D-1\,, \tag{14.62}$$

and we can write the vectors $m_{(\hat{a})}$ as

$$m_{(\hat{a})} = \begin{pmatrix} 0 \\ \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\}(\hat{a}-2)\times \\ -\sin\phi_{\hat{a}} \\ \left.\begin{matrix} \cos\phi_{\hat{a}}\,\cos\phi_{\hat{a}+1} \\ \cos\phi_{\hat{a}}\,\sin\phi_{\hat{a}+1}\,\cos\phi_{\hat{a}+2} \\ \vdots \\ \cos\phi_{\hat{a}}\,\sin\phi_{\hat{a}+1}\,\ldots\,\sin\phi_{D-2}\,\cos\phi_{D-1} \\ \cos\phi_{\hat{a}}\,\sin\phi_{\hat{a}+1}\,\ldots\,\sin\phi_{D-2}\,\sin\phi_{D-1} \end{matrix}\right\}(D-\hat{a})\times \end{pmatrix} = \begin{pmatrix} 0 \\ \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\}(\hat{a}-2)\times \\ -\sin\phi_{\hat{a}} \\ \vdots \\ \cos\phi_{\hat{a}}\left(\prod_{j=\hat{a}+1}^{\hat{a}+n-1}\sin\phi_j\right)\cos\phi_{\hat{a}+n} \\ \vdots \end{pmatrix}\,, \tag{14.63}$$

where $n = 1,\,\ldots,\,D-\hat{a}$, $\hat{a} = 2,\,\ldots,\,D-1$, and the bottom block in the last vector has $D-\hat{a}$ entries. Furthermore, we define $\Pi_{j=\hat{a}+1}^{\hat{a}}\sin\phi_j \equiv 1$, i.e. the neutral element of multiplication. Note that we also introduced in the above vectors for completeness the vanishing time component.

With these expressions for the vectors $m_{\hat{a}}$, we are finally in the position to construct the starting point for a unit normal basis. Let us first consider the fact that on the computational domain,

$$\rho_4^2 = w_1^2 + \ldots + w_{D-4}^2 = 0\,, \tag{14.64}$$

because all $w^a = 0$. Expressing the Cartesian w^a in terms of spherical coordinates according to Eqs. (14.49), this implies

$$r^2\sin^2\phi_2\,\sin^2\phi_3\,\sin^2\phi_4 = 0\,. \tag{14.65}$$

On the computational domain, the angles ϕ_2 and ϕ_3 are arbitrary (in $D=4$, we would call them θ and ϕ), so that for points in the computational domain we have

$$\sin\phi_4 = 0\,. \tag{14.66}$$

This implies $\phi_4 = 0$ or $\phi_4 = \pi$ and, thus, $\cos\phi_4 = \pm 1$. We have the freedom to choose our first spatial basis vector to correspond to the case $\cos\phi_4 = +1$ and obtain

$$m_{(2)} \;=\; \big(0\;|\;-\sin\phi_2,\;\cos\phi_2\cos\phi_3,\;\cos\phi_2\sin\phi_3,\;\underbrace{0,\;\ldots,\;0}_{(D-4)\times}\big)\,. \tag{14.67}$$

$$m_{(3)} \;=\; \big(0\;|\;0,\;-\sin\phi_3,\;\cos\phi_3,\;\underbrace{0,\;\ldots,\;0}_{(D-4)\times}\big)\,. \tag{14.68}$$

For the remaining vectors, we still have rotational freedom in the angles $\phi_5, \ldots, \phi_{D-1}$. For any values of these angles, we still satisfy $w_1 = \ldots = w_{D-4} = 0$. All we need to ensure, however, is orthonormality of the system of vectors $m_{\hat{a}}$, and this is most conveniently achieved by setting

$$\phi_5 = \ldots = \phi_{D-1} = 0\,, \tag{14.69}$$

so that our system of vectors is completed through

$$\begin{aligned}
m_{(4)} &= \big(0 \mid 0,\ 0,\ 0,\ 1,\ \underbrace{0,\ \ldots,\ 0}_{(D-5)\times}\big)\,,\\
m_{(5)} &= \big(0 \mid 0,\ 0,\ 0,\ 0,\ 1,\ \underbrace{0,\ \ldots,\ 0}_{(D-6)\times}\big)\,,\\
&\ \,\vdots \qquad \vdots\\
m_{(D-1)} &= \big(0 \mid \underbrace{0,\ \ldots,\ 0}_{(D-2)\times},\ 1\big)\,,
\end{aligned} \tag{14.70}$$

or, in short notation,

$$m^A_{(a)} = \delta^A{}_a\,, \qquad \text{for } a = 4,\ \ldots,\ D-1\,. \tag{14.71}$$

These vectors $m_{(a)}$ still need to be orthonormalized, but this becomes straightforward once we realize that $m_{(2)}$ and $m_{(3)}$ on the one side and $m_{(4)}, \ldots, m_{(D-1)}$ do not mix; the former have x, y, z components only and the latter exclusively have components in the $w_1, \ldots, w_{D-4}$ directions. We therefore apply Gram-Schmidt orthonormalization to the tetrad k^A, l^A, $m^A_{(2)}$, $m^A_{(3)}$ in the standard manner known from four-dimensional GR whereas the remaining vectors are orthonormalized trivially according to

$$m^A_{(a)} = \frac{1}{\sqrt{\gamma_{ww}}}\delta^A{}_a\,, \qquad \text{for } a = 4,\ \ldots,\ D-1\,. \tag{14.72}$$

Let us finally turn our attention to the non-trivial vectors k^A and $m^A_{(i)}$. First, we note that these vectors have, by construction, only non-zero components in the first four indices, so that we write them as k^α, $m^\alpha_{(i)}$, $i = 1, 2, 3$. Let us first consider the non-normalized starting expressions which we denote with a tilde

$$\tilde{m}_{(1)} = (0,\ x,\ y,\ z)\,, \tag{14.73}$$

$$\tilde{m}_{(2)} = (0,\ -y^2 - z^2,\ xy,\ xz)\,, \tag{14.74}$$

$$\tilde{m}_{(3)} = (0,\ 0,\ -z,\ y)\,. \tag{14.75}$$

We orthonormalize these vectors via Gram-Schmidt orthonormalization

$$m^{\alpha}_{(1)} = \frac{\tilde{m}^{\alpha}_{(1)}}{\sqrt{\gamma_{kl}\tilde{m}^{k}_{(1)}\tilde{m}^{l}_{(1)}}}\,, \tag{14.76}$$

$$\bar{m}^{\alpha}_{(2)} = \tilde{m}^{\alpha}_{(2)} - \left[\gamma_{kl}\tilde{m}^{k}_{(2)}m^{l}_{(1)}\right] m^{\alpha}_{(1)}\,, \tag{14.77}$$

$$m^{\alpha}_{(2)} = \frac{\bar{m}^{\alpha}_{(2)}}{\sqrt{\gamma_{kl}\bar{m}^{k}_{(2)}\bar{m}^{l}_{(2)}}}\,, \tag{14.78}$$

$$\bar{m}^{\alpha}_{(3)} = \tilde{m}^{\alpha}_{(3)} - \left[\gamma_{kl}\tilde{m}^{k}_{(3)}m^{l}_{(1)}\right] m^{\alpha}_{(1)} - \left[\gamma_{kl}\tilde{m}^{k}_{(3)}m^{l}_{(2)}\right] m^{\alpha}_{(2)}\,, \tag{14.79}$$

$$m^{\alpha}_{(3)} = \frac{\bar{m}^{\alpha}_{(3)}}{\sqrt{\gamma_{kl}\bar{m}^{k}_{(3)}\bar{m}^{l}_{(3)}}}\,, \tag{14.80}$$

noting that the time component of all these vectors stays zero throughout the entire procedure which allows us to calculate the norms using the spatial metric γ_{kl} instead of the spacetime metric $g_{\mu\nu}$. Of course, the components $4\ldots(D-1)$ also remain zero throughout. Note furthermore that

$$n_{\alpha} = g_{\alpha\mu}n^{\mu} = (g_{0\mu}n^{\mu},\ g_{i\mu}n^{\mu}) = (-\alpha,\ 0,\ 0,\ 0)\,, \tag{14.81}$$

so that

$$g_{\mu\nu}n^{\mu}m^{\nu}_{(i)} = n_{\mu}m^{\mu}_{(i)} = 0\,. \tag{14.82}$$

Finally, the ingoing null vector is given by

$$k^{\alpha} = \frac{1}{\sqrt{2}}\left(\frac{1}{\alpha},\ -\frac{\beta^{i}}{\alpha} - m^{i}_{(1)}\right)\,, \tag{14.83}$$

which differs from Bondi's ingoing null vector by a factor $\sqrt{2}$: $k^{\alpha}_{\text{Bondi}} = k^{\alpha}/\sqrt{2}$.
In summary, we have now an ingoing null vector k given by Eq. (14.83) and $D-2$ angular vectors $m_{\hat{a}}$ which we organize as

$$\{m_{(\hat{a})}\} = \{m_{(2)},\ m_{(3)},\ m_{(a)}\}\,, \qquad a = 4,\ \ldots,\ D-1\,, \tag{14.84}$$

where $m_{(2)}$ and $m_{(3)}$ are given by Eqs. (14.73)-(14.80) and the $m_{(a)}$ by Eq. (14.72).

14.6 The projections $\Omega_{\hat{a}\hat{b}}$

We are now left with the contraction of the Riemann (i.e. Weyl) tensor with the null-frame vectors to obtain the components of Ω,

$$\Omega_{\hat{a}\hat{b}} = R_{ABCD}k^{A}m^{B}_{(\hat{a})}k^{C}m^{D}_{(\hat{b})}\,. \tag{14.85}$$

These fall into three categories. (i) both indices are inside the computational domain, i.e. $\hat{a} = \hat{\imath}$, $\hat{b} = \hat{\jmath}$ where $\hat{\imath}$, $\hat{\jmath} = 2, 3$; (ii) one index is inside and the other off the domain, i.e. $\hat{a} = \hat{\imath}$, $\hat{b} = b$ which also covers the reverse case due to symmetry of $\Omega_{\hat{a}\hat{b}}$; (iii) both indices are off domain $\hat{a} = a$, $\hat{b} = b$. We obtain

$$\Omega_{\hat{i}\hat{j}} = \frac{1}{2}\left\{R_{0k0l}m^k_{(\hat{i})}m^l_{(\hat{j})} - R_{mk0l}m^m_{(1)}m^k_{(\hat{i})}m^l_{(\hat{j})} - R_{0kml}m^k_{(\hat{i})}m^m_{(1)}m^l_{(\hat{j})} + R_{mknl}m^m_{(1)}m^k_{(\hat{i})}m^n_{(1)}m^l_{(\hat{j})}\right\}, \tag{14.86}$$

$$\Omega_{\hat{i}b} = \frac{1}{2\sqrt{\gamma_{ww}}}\left\{R_{0k0b}m^k_{(\hat{i})} - R_{mk0b}m^m_{(1)}m^k_{(\hat{i})} - R_{0kmb}m^k_{(\hat{i})}m^m_{(1)} + R_{mknb}m^m_{(1)}m^k_{(\hat{i})}m^n_{(1)}\right\} = 0\,, \tag{14.87}$$

$$\Omega_{ab} = \frac{1}{2\gamma_{ww}}\delta_{ab}\Omega_{ww}\,, \tag{14.88}$$

$$\Omega_{ww} \equiv \frac{1}{2\gamma_{ww}}\left(R_{w0w0} - R_{w0wk}m^k_{(1)} - R_{w0wl}m^l_{(1)} + R_{wkwl}m^k_{(1)}m^l_{(1)}\right). \tag{14.89}$$

14.7 The Kretschmann scalar

The Kretschmann scalar is defined as

$$C = R^{ABCD}R_{ABCD} = g^{AE}g^{BF}g^{CG}g^{DH}R_{ABCD}R_{EFGH}\,, \tag{14.90}$$

and we will now derive its expression in the modified cartoon formalism of the $(D-1)+1$ decomposition. As we will see, its calculation is closely related to that of the wave signal and we will obtain our final expression in terms of the components (14.22)-(14.35) of the Riemann tensor as derived above for the wave extraction. First, we recall

$$\gamma^{AB} = g^{AB} + n^A n^B \qquad \Leftrightarrow \qquad g^{AB} = \gamma^{AB} - n^A n^B\,, \tag{14.91}$$

which we insert in the expression for the Kretschmann scalar to sort between spatial and time components. By using the symmetry properties of the Riemann tensor, we can cancel or combine a large number of terms and obtain after a little while,

$$C = \underbrace{\gamma^{AE}\gamma^{BF}\gamma^{CG}\gamma^{DH}R_{ABCD}R_{EFGH}}_{=:C_1}\underbrace{-4\gamma^{CG}\gamma^{BF}\gamma^{DH}R_{B0CD}R_{F0GH}}_{=:C_2}+\underbrace{4\gamma^{BF}\gamma^{DH}R_{B0D0}R_{F0H0}}_{=:C_3}, \tag{14.92}$$

where an index '0' denotes contraction with n; e.g. $R_{A0CD} = R_{ABCD}n^B$. For the calculation of the individual C_i, we switch to spatial indices I, J, ... and decompose into on-domain and off-domain directions. We find,

$$\begin{aligned} C_1 &= \gamma^{IM}\gamma^{JN}\gamma^{KP}\gamma^{LQ}R_{IJKL}R_{MNPQ} \\ &= \gamma^{im}\gamma^{jn}\gamma^{kp}\gamma^{lq}R_{ijkl}R_{mnpq} + (\gamma^{ww})^4\delta^{ae}\delta^{bf}\delta^{cg}\delta^{dh}R_{abcd}R_{efgh} \\ &\quad +2\gamma^{im}\gamma^{jn}\gamma^{cg}\gamma^{dh}\underbrace{R_{ijcd}}_{=0}R_{mngh} + 4\gamma^{im}\gamma^{bf}\gamma^{kp}\gamma^{dh}R_{ibkd}R_{mfph}\,, \end{aligned} \tag{14.93}$$

where we used that R_{ijcd} has to vanish since it would inevitably have to be antisymmetric in cd due to the symmetry of the Riemann tensor and symmetric in cd because of the symmetry in the extra dimensions. We can now use Eqs. (14.22)-(14.35) and expand the various Kronecker δ symbols recalling that $\delta_a{}^a = D - 4$, to obtain

$$C_1 = \gamma^{im}\gamma^{jn}\gamma^{kp}\gamma^{lq}R_{ijkl}R_{mnpq} + 4(D-4)\gamma^{im}\gamma^{kp}(\gamma^{ww})^2 R_{iwkw}R_{mwpw} + 2(D-4)(D-5)(\gamma^{ww})^4 (R_{wuwu})^2 \,. \tag{14.94}$$

The other contributions to the Kretschmann scalar are computed accordingly which gives us

$$C_2 = -4\gamma^{jm}\gamma^{il}\gamma^{kn}R_{i0jk}R_{l0mn} - 8(\gamma^{ww})^2(D-4)\gamma^{jm}R_{w0wj}R_{w0wm} \,, \tag{14.95}$$

$$C_3 = 4\gamma^{ik}\gamma^{jl}R_{i0j0}R_{k0l0} + 4(D-4)(\gamma^{ww})^2(R_{w0w0})^2 \,. \tag{14.96}$$

For completeness, we also list here the computation of the Kretschmann scalar in $D = 4$ dimension. Again, this calculation makes good use of quantities we already have computed for the wave extraction; cf. App. C in [9].

For the 3+1 dimensional case, we assume for our calculation that we are in vacuum, so that the Riemann and Weyl tensor are identical. The electric and magnetic part of the Weyl tensor are defined as

$$\mathcal{E}_{\alpha\beta} = \perp^{\mu}{}_{\alpha}\perp^{\nu}{}_{\beta}C_{\mu\rho\nu\sigma}n^{\rho}n^{\sigma} \,, \tag{14.97}$$

$$\mathcal{B}_{\alpha\beta} = \perp^{\mu}{}_{\alpha}\perp^{\nu}{}_{\beta}\,{}^{*}C_{\mu\rho\nu\sigma}n^{\rho}n^{\sigma} = \frac{1}{2}\perp^{\mu}{}_{\alpha}\perp^{\nu}{}_{\beta}C_{\mu\rho\eta\tau}\epsilon^{\eta\tau}{}_{\nu\sigma}n^{\rho}n^{\sigma} \,, \tag{14.98}$$

where

$${}^{*}C_{\mu\rho\nu\sigma} = \frac{1}{2}C_{\mu\rho\kappa\lambda}\epsilon^{\kappa\lambda}{}_{\nu\sigma} \,. \tag{14.99}$$

Strictly speaking, we wouldn't need the spatial projection operators on the left-hand side of Eqs. (14.98), (14.97), since any further multiplications with the unit normal n would vanish anyway by virtue of the antisymmetry of the Weyl tensor. We still write the $\perp$ operators to emphasize that the electric and magnetic parts fof the Weyl tensor are manifestly spatial tensors.

Using the Gauss-Codazzi equations in 3+1 dimensions, it is straightforward to show that

$$\mathcal{B}_{\alpha\beta} = \epsilon_{\beta}{}^{\mu\nu}D_{\mu}K_{\alpha\nu} \,, \tag{14.100}$$

$$\mathcal{E}_{\alpha\beta} = \mathcal{R}_{\alpha\beta} + KK_{\alpha\beta} - K^{\mu}{}_{\beta}K_{\mu\alpha} \,, \tag{14.101}$$

where the 3-dimensional antisymmetric Levi-Civita tensor is related to its spacetime counterpart by

$$\epsilon_{\alpha\beta\gamma} = \epsilon_{\mu\alpha\beta\gamma}n^{\mu} \,. \tag{14.102}$$

Note that by this defintion, the spatial Levi-Civita tensor is indeed purely spatial, i.e.

$$\epsilon_{\alpha\beta\gamma}n^{\alpha} = 0 \,. \tag{14.103}$$

We recall the definition of the projector and also define

$$\perp^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + n^{\mu}n_{\nu} \,, \qquad l^{\mu}{}_{\nu} = \perp^{\mu}{}_{\nu} + n^{\mu}n_{\nu} = \delta^{\mu}{}_{\nu} + 2n^{\mu}n_{\nu} \,. \tag{14.104}$$

With this definition, one can reconstruct the Weyl tensor according to [4]

$$C_{\mu\nu\lambda\rho} = l_{\mu\lambda}\mathcal{E}_{\rho\nu} - l_{\mu\rho}\mathcal{E}_{\lambda\nu} - l_{\nu\lambda}\mathcal{E}_{\rho\mu} + l_{\nu\rho}\mathcal{E}_{\lambda\mu} - n_{\lambda}\mathcal{B}_{\rho\tau}\epsilon^{\tau}{}_{\mu\nu} + n_{\rho}\mathcal{B}_{\lambda\tau}\epsilon^{\tau}{}_{\mu\nu} - n_{\mu}\mathcal{B}_{\nu\tau}\epsilon^{\tau}{}_{\lambda\rho} + n_{\nu}\mathcal{B}_{\mu\tau}\epsilon^{\tau}{}_{\lambda\rho} \,. \tag{14.105}$$

With our definitions, we have the following useful relations,

$$
\begin{aligned}
l^{\mu\nu} l_{\mu\nu} &= 4\,, \\
\perp^{\mu\nu} \perp_{\mu\nu} &= 3\,, \\
l^{\mu\nu} l_{\mu\rho} &= \delta^{\nu}{}_{\rho}\,, \\
\perp^{\mu\nu} \perp_{\mu\rho} &= \perp^{\nu}{}_{\rho}\,, \\
l_{\nu\lambda} \mathcal{E}^{\rho\nu} &= \mathcal{E}^{\rho}{}_{\lambda}\,, \\
\mathcal{E}^{\mu}{}_{\mu} &= g_{\alpha\beta} C^{\alpha}{}_{\rho}{}^{\beta}{}_{\sigma} n^{\rho} n^{\sigma} = 0\,,
\end{aligned}
\tag{14.106}
$$

since the Weyl tensor is traceless. The remainder of the calculation consists in expanding $C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma}$ using the left-hand side of Eq. (14.105) and collect the non-zero terms; most terms vanish due to some contraction of the timelike normal with a spatial tensor component. It is convenient to split the total sum into 8 terms taking in each term the first factor of the Weyl tensor and multiply it with the entire Weyl tensor as the second factor. This gives, for instance,

$$
\begin{aligned}
K_1 &:= l^{\mu\nu} \mathcal{E}^{\rho\nu} C_{\mu\nu\rho\sigma} \\
&= 4\mathcal{E}^{\rho\nu}\mathcal{E}_{\rho\nu} - \delta^{\lambda}{}_{\rho}\mathcal{E}^{\rho\nu}\mathcal{E}_{\lambda\nu} - \delta^{\mu}{}_{\nu}\mathcal{E}^{\rho\nu}\mathcal{E}_{\rho\mu} + 0 + n^{\mu}\mathcal{E}^{\rho\nu}\mathcal{B}_{\rho\tau}\epsilon^{\tau}{}_{\mu\nu} + 0 + n^{\lambda}\mathcal{E}^{\rho\nu}\mathcal{B}_{\nu\tau}\epsilon^{\tau}{}_{\lambda\rho} + 0 \\
&= 2\mathcal{E}^{\rho\nu}\mathcal{E}_{\rho\nu}\,.
\end{aligned}
\tag{14.107}
$$

$$
\begin{aligned}
K_5 &:= -n^{\lambda}\mathcal{B}^{\rho}{}_{\tau}\epsilon^{\tau\mu\nu} C_{\mu\nu\rho\sigma} \\
&= n_{\mu}\mathcal{B}^{\rho}{}_{\tau}\epsilon^{\tau\mu\mu}\mathcal{E}_{\rho\nu} + 0 - n_{\nu}\mathcal{B}^{\rho}{}_{\tau}\epsilon^{\tau\mu\nu}\mathcal{E}_{\rho\mu} - 0 \\
&\quad -\mathcal{B}^{\rho}{}_{\sigma}\epsilon^{\sigma\mu\nu}\mathcal{B}_{\rho\tau}\epsilon^{\tau}{}_{\mu\nu} - 0 + n^{\lambda}n_{\mu}\mathcal{B}^{\rho}{}_{\sigma}\mathcal{B}_{\nu\tau}\epsilon^{\sigma\mu\nu}\epsilon^{\tau}{}_{\lambda\rho} - n^{\lambda}n_{\nu}\mathcal{B}^{\rho}{}_{\sigma}\epsilon^{\sigma\mu\nu}\mathcal{B}_{\mu\tau}\epsilon^{\tau}{}_{\lambda\rho} \\
&= -\mathcal{B}^{\rho}{}_{\sigma}\epsilon^{\sigma\mu\nu}\mathcal{B}_{\rho\tau}\epsilon^{\tau}{}_{\mu\nu} = -2\delta^{\sigma\tau}\mathcal{B}^{\rho}{}_{\sigma}\mathcal{B}_{\rho\tau} = -2\mathcal{B}^{\rho\tau}\mathcal{B}_{\rho\tau}\,,
\end{aligned}
\tag{14.108}
$$

where we have used

$$
\epsilon^{\sigma\mu\nu}\epsilon^{\tau}{}_{\mu\nu} = 2\delta^{\sigma\tau}\,. \tag{14.109}
$$

The terms K_2 to K_4 turn out to be the same as K_1 and terms K_6 to K_8 give the same result as K_5 which leaves us with the final expression for the Kretschmann scalar,

$$
C = 8(\mathcal{E}^{\rho\nu}\mathcal{E}_{\rho\nu} - \mathcal{B}^{\rho\nu}\mathcal{B}_{\rho\nu})\,. \tag{14.110}
$$

Finally, we consider the interpretation of the Kretschmann scalar in terms of super Planckian curvature. This has been studied in Ref. [8] but their presentation of the normalization for a Planck mass BH is rather cryptic, although we eventually agree with it.
Let us start by setting $\hbar = c = 1$ which means that the Compton wavelength of an object of mass m is now given by

$$
\lambda\!\!\!^{-} = \frac{\hbar}{mc} = \frac{1}{m}\,, \tag{14.111}
$$

i.e. we now measure a particle's mass in terms of its inverse Compton wavelength. We keep, however, the gravitational constant G in our equations for now. For $c = 1$, the gravitational constant represents a conversion between length and mass. This point may appear a bit confusing since we now relate mass to inverse length and to length at the same time. This is not quite the case, however. G is a conversion factor to translate mass into length, but it does not imply that the two are the same. Say, we decide to measure length in meters, then the value of G defines which units we measure mass in. By setting $\hbar$ and c to unity, we really equate the (inverse) Compton wavelength and the mass of an object. Let us bear in mind this subtle difference.

Quantum gravity becomes important when the Schwarzschild radius of an object becomes comparable to its Compton wavelength and, following [8] we shall use this relation to define the Planck mass. Up to a factor of unity, this agrees with other definitions. Consider a Schwarzschild-Tangherlini BH in areal radius and polar slicing

$$ds^2 = -\left(1 - \frac{\mu}{r_S^{D-3}}\right) dt^2 + \left(1 - \frac{\mu}{r_S^{D-3}}\right)^{-1} dr^2 + r^2 d\Omega_{D-2}^2\,, \tag{14.112}$$

where $d\Omega_{D-2}^2$ is the metric on the unit $D-2$ sphere,

$$d\Omega_{D-2}^2 = d\phi_1^2 + \sin^2\phi_1\, d\phi_2^2 + \sin^2\phi_1\, \sin^2\phi_2\, d\phi_3^2 + \sin^2\phi_1\, \sin^2\phi_2\, \sin^2\phi_3\, d\phi_4^2 + \ldots \tag{14.113}$$

A straightforward calulcation shows that the Kretschmann scalar C of this spacetime is given by

$$\begin{aligned} C &= \frac{12\mu^2}{r^6} && \text{in } D = 4\,, \\ C &= \frac{72\mu^2}{r^8} && \text{in } D = 5\,, \\ C &= \frac{240\mu^2}{r^{10}} && \text{in } D = 6\,, \\ C &= \frac{600\mu^2}{r^{12}} && \text{in } D = 7\,, \\ C &= \frac{1260\mu^2}{r^{14}} && \text{in } D = 8\,. \end{aligned} \tag{14.114}$$

For $D = 4$, we recover the more common $C = 48M^2/r^6$ by identifying $\mu = 2M$.
There is some free convention about choosing the constant in front of the matter terms in the Einstein equations. Following [3], we use

$$G_{\alpha\beta} = 8\pi G T_{\alpha\beta}\,, \tag{14.115}$$

for all values of D. Note that this differs from [8]. At first glance, it may appear surprising that this factor plays a role in the interpretation of vacuum BH spacetimes where the right-hand side of the Einstein equations vanishes and the factor $8\pi G$ disappears with the matter terms. The answer is that the interpretation of the BH mass is based on the Newtonian limit at large r and the computation of the weak-field limit actually uses the energy momentum tensor, blissfully unaware whether a potential source term arises from a vacuum BH source or a self respecting star composed, e.g., of a perfect fluid. In either case, the Newtonian potential is associated with the total mass through the spacetime metric's far field limit

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)dr^2 + r^2 d\Omega_{D-2}\,. \tag{14.116}$$

In $D = 4$ this leads to the well known $\Phi = GM/r$ and confirms the factor $8\pi G$ in the Einstein equations. In higher D, one needs to adjust the geometric factor 4π in the Poisson equation $\vec{\nabla}\Phi = 4\pi\rho$ to account for the surface area of the higher dimensional n sphere, but the principle remains that the mass M is inferred by relating the far-field limit to a Newtonian law $\ddot{x}^k = -\partial_k\Phi$ with $\Phi = GM/r^{D-3}$. For our choice of writing the Einstein equations as (14.115) for all D, the mass of a BH with parameter μ is [3]

$$M = \frac{(D-2)\Omega_{D-2}\,\mu}{16\pi G}\,, \tag{14.117}$$

where Ω_{D-2} denotes the surface area of the $D-2$ sphere. For reference, we list some values here,

$$\begin{aligned} \Omega_0 &= 2\,, & \Omega_1 &= 2\pi\,, \\ \Omega_2 &= 4\pi\,, & \Omega_3 &= 2\pi^2\,, \\ \Omega_4 &= \tfrac{8}{3}\pi^2\,, & \Omega_5 &= \pi^3\,, \\ \Omega_6 &= \tfrac{16}{15}\pi^3\,, & \Omega_7 &= \frac{1}{3}\pi^4\,. \end{aligned} \tag{14.118}$$

We now have a relation between the Schwarzschild radius of a BH and the BH's mass,

$$r_S^{D-3} \stackrel{!}{=} \mu = \frac{16\pi G M}{(D-2)\Omega_{D-2}}\,. \tag{14.119}$$

The Planck regime is defined when r_S and the Compton wavelength of the BH are the same, i.e. when $r_S = 1/M$ which we substitute in the last equation to obtain

$$M^{D-2} = \frac{(D-2)\Omega_{D-2}}{16\pi G}\,. \tag{14.120}$$

This gives us a Planck mass for any D and we again list some values for reference,

$$\begin{aligned} D=4: &\qquad M_p^2 = \frac{1}{2G}\,, \\ D=5: &\qquad M_p^3 = \frac{3\pi}{8G}\,, \\ D=6: &\qquad M_p^4 = \frac{2\pi}{3G}\,, \\ D=7: &\qquad M_p^5 = \frac{5\pi^2}{16G}\,, \\ D=8: &\qquad M_p^6 = \frac{6\pi^2}{15G}\,. \end{aligned} \tag{14.121}$$

In particular, this agrees with Okawa *et al* [8] for the value $D=5$ which they consider. They write the Einstein equations in the form

$$G_{ab} = 3\pi^2 E_p^{-3} T_{ab}\,, \tag{14.122}$$

which corresponds to ours if $E_p^{-3} = 8G/(3\pi)$, as should indeed be the case according to our list above. There now remains the task of normalizing the Kretschmann scalar. For this purpose, Okawa *et al* consider its value on the horizon of a Tangherlini BH. In $D=5$, this gives, using (14.119) and the $D=5$ entry in (14.121),

$$C = \frac{72}{r_S^4} = \frac{72}{\mu^2} = \frac{72}{M^2}\left(\frac{3\pi}{8G}\right)^2 = \frac{72}{M^2} M_p^6\,. \tag{14.123}$$

We are particularly interested in the Kretschmann scalar on the horizon of a Tangherlini BH with $M = M_p$ which becomes

$$C_p = 72\, M_p^4 = 72\left(\frac{3\pi}{8G}\right)^{4/3}\,. \tag{14.124}$$

Okawa *et al* then normalize their Kretschmann scalar according to

$$\mathcal{K} := \frac{\sqrt{R^{abcd}R_{abcd}}}{6\sqrt{2}\,E_p^2} = \frac{\sqrt{R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}}}{6\sqrt{2}\left(\frac{3\pi}{8G}\right)^{2/3}}\,, \tag{14.125}$$

where the first expression uses their notation and the second ours. Whether to use the square root here or not is a convention we may test in analysing the data. So we may laternatively consider

$$\mathcal{K}^2 = \frac{1}{72}\left(\frac{8G}{3\pi}\right)^{4/3} R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}\,. \tag{14.126}$$

There remains one subtle question, namely what is the value of G in our code? As mentioned above, the only place where G enters our physics is in determining the units of the BH mass relative to the length units and our units are chosen according to $G = 1$. In $D = 4$, for instance, this choice implies that we measure the BH mass in terms of (half of) its Schwarzschild radius and me might more accurately call the “BH mass” the “BH length”.
In $D = 6$, the corresponding calculation gives

$$\begin{aligned} C_p &= \frac{180\pi}{G} \\ \Rightarrow\quad \mathcal{K}^2 &= \frac{G}{180\pi} R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}\,. \end{aligned} \tag{14.127}$$

14.8 The Chern-Pontryagin scalar

In our computation of the Kretschmann scalar above, we were a bit sloppy in one regard. Strictly speaking, the Kretschmann scalar (and its cousins that we shall be discussing shortly) are defined as contractions of the Riemann tensor rather than the Weyl tensor. Since we work in vacuum most of the time, the latter are the same, but it does not hurt to give their generalization here. In doing so, we follow the notation and results of Cherubini *et al* [2] who denote the Kretschmann, Chern-Pontryagin and Euler scalars, respectively, by

$$K_1 = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\,, \tag{14.128}$$

$$K_2 = {}^{\sim}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\,, \tag{14.129}$$

$$K_3 = {}^{\sim}R^{\sim}_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\,, \tag{14.130}$$

where the left and right dual of the Riemann tensor are defined by

$${}^{\sim}R_{\alpha\beta\gamma\delta} = \frac{1}{2}\epsilon_{\alpha\beta}{}^{\mu\nu}R_{\mu\nu\gamma\delta}\,, \tag{14.131}$$

$$R^{\sim}_{\alpha\beta\gamma\delta} = \frac{1}{2}\epsilon_{\gamma\delta}{}^{\rho\sigma}R_{\alpha\beta\rho\sigma}\,. \tag{14.132}$$

For completeness, we recall here the relation between the Riemann and the Weyl tensor in D and 4 dimensions,

$$
\begin{aligned}
C_{ABCD} &= R_{ABCD} + \frac{1}{D-2}\left(g_{AD}R_{BC} - g_{AC}R_{BD} + g_{BC}R_{AD} - g_{BD}R_{AC}\right), && (14.133)\\
& + \frac{1}{(D-1)(D-2)}(g_{AC}g_{BD} - g_{AD}g_{BC})R\,, && (14.134)\\
C_{\alpha\beta\gamma\delta} &= R_{\alpha\beta\gamma\delta} + \frac{1}{2}\left(g_{\alpha\delta}R_{\beta\gamma} - g_{\alpha\gamma}R_{\beta\delta} + g_{\beta\gamma}R_{\alpha\delta} - g_{\beta\delta}R_{\alpha\gamma}\right) + \frac{1}{6}\left(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\right)R && (14.135)
\end{aligned}
$$

From now on we shall stay in $D = 4$ dimensions in this subsection. With the definition of the Weyl tensor, we can construct its analogs of the scalars (14.128)-(14.130),

$$
\begin{aligned}
I_1 &= C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}\,, && (14.136)\\
I_2 &= {}^\sim C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}\,. && (14.137)
\end{aligned}
$$

The analog of the Euler scalar becomes trivial in this case because ${}^\sim C^\sim_{\alpha\beta\gamma\delta} = -C_{\alpha\beta\gamma\delta}$, so we need not consider it any further here. We note on the side that the self-duality $[{}^\sim \boldsymbol{C}^\sim] = -\boldsymbol{C}$ of the Weyl-tensor is not obvious but follows from a non-trivial calculation we have summarized in the companion document `ccz4.pdf`.

Using the tracelessness of the Weyl tensor, one can show that $K_2 = I_2$ while the other scalars are related by[1]

$$
\begin{aligned}
K_1 &= I_1 + 2R_{\alpha\beta}R^{\alpha\beta} - \frac{1}{3}R^2\,, && (14.138)\\
K_3 &= -I_1 + 2R_{\alpha\beta}R^{\alpha\beta} - \frac{2}{3}R^2\,. && (14.139)
\end{aligned}
$$

In this new notation, the quantity we have calculated as the Kretschmann scalar $\mathcal{C}$ in Eq. (14.110) is only correct in vacuum and should be called I_1 in our new notation,

$$
K_1 \overset{\text{vacuum}}{=} I_1 = \mathcal{C} = 8(\mathcal{E}^{\rho\nu}\mathcal{E}_{\rho\nu} - \mathcal{B}^{\rho\nu}\mathcal{B}_{\rho\nu})\,. \tag{14.140}
$$

Our goal in this subsection is to derive the corresponding equation for $K_2 = I_2$ which, fortunately, even holds without the assumption of vacuum.

We start this calculation by summarizing some general relations that will be required later on in our compuation of K_2. First, we need the composition of the Weyl tensor in terms of its electric and magnetic parts,

$$
\begin{aligned}
E_{\alpha\beta} &= C_{\alpha\mu\beta\nu}n^\mu n^\nu\,, && (14.141)\\
B_{\alpha\beta} &= D_1(C)_{\alpha\mu\beta\nu}n^\mu n^\nu \quad = \quad {}^\sim C_{\alpha\mu\beta\nu}n^\mu n^\nu \quad = \quad \frac{1}{2}\epsilon_{\alpha\mu}{}^{\rho\sigma}C_{\rho\sigma\beta\nu}n^\mu n^\nu\,, && (14.142)
\end{aligned}
$$

which is given by

[1] I have not checked these relations yet, but taken them from Ref. [2].

$$C_{\alpha\beta\gamma\delta} = l_{\alpha\gamma}E_{\beta\delta} - l_{\alpha\delta}E_{\beta\gamma} + l_{\beta\delta}E_{\alpha\gamma} - l_{\beta\gamma}E_{\alpha\delta} - n_\gamma B_{\delta\mu}\epsilon^\mu{}_{\alpha\beta} + n_\delta B_{\gamma\mu}\epsilon^\mu{}_{\alpha\beta} - n_\alpha B_{\beta\mu}\epsilon^\mu{}_{\gamma\delta} + n_\beta B_{\alpha\mu}\epsilon^\mu{}_{\gamma\delta}\,,$$
$$\text{where} \qquad l_{\mu\nu} = \gamma_{\mu\nu} + n_\mu n_\nu = g_{\mu\nu} + 2n_\mu n_\nu\,, \qquad \epsilon_{\beta\gamma\delta} = \epsilon_{\mu\beta\gamma\delta}n^\mu\,. \tag{14.143}$$

It is this last equation that turns out a Marathon des Sables to proof as we have done in the already mentioned `ccz4.pdf` document.
Something else we need are the contractions of the Levi-Civita tensor. Borrowing again from `ccz4.pdf`, these are given in 4 dimensions by (switching to Latin indices for typing convenience),

$$\begin{aligned}
\epsilon_{abcd}\epsilon^{efmn} &= -\delta^e_a\delta^f_b\delta^m_c\delta^n_d - \delta^e_a\delta^m_b\delta^n_c\delta^f_d - \delta^e_a\delta^n_b\delta^f_c\delta^m_d + \delta^e_a\delta^f_b\delta^n_c\delta^m_d + \delta^e_a\delta^m_b\delta^f_c\delta^n_d + \delta^e_a\delta^n_b\delta^m_c\delta^f_d \\
&\quad +\delta^f_a\delta^m_b\delta^n_c\delta^e_d + \delta^f_a\delta^n_b\delta^e_c\delta^m_d + \delta^f_a\delta^e_b\delta^m_c\delta^n_d - \delta^f_a\delta^m_b\delta^e_c\delta^n_d - \delta^f_a\delta^n_b\delta^m_c\delta^e_d - \delta^f_a\delta^e_b\delta^n_c\delta^m_d \\
&\quad -\delta^m_a\delta^n_b\delta^e_c\delta^f_d - \delta^m_a\delta^e_b\delta^f_c\delta^n_d - \delta^m_a\delta^f_b\delta^n_c\delta^e_d + \delta^m_a\delta^n_b\delta^f_c\delta^e_d + \delta^m_a\delta^e_b\delta^n_c\delta^f_d + \delta^m_a\delta^f_b\delta^e_c\delta^n_d \\
&\quad +\delta^n_a\delta^e_b\delta^f_c\delta^m_d + \delta^n_a\delta^f_b\delta^m_c\delta^e_d + \delta^n_a\delta^m_b\delta^e_c\delta^f_d - \delta^n_a\delta^e_b\delta^m_c\delta^f_d - \delta^n_a\delta^f_b\delta^e_c\delta^m_d - \delta^n_a\delta^m_b\delta^n_c\delta^e_d\,, && (14.144) \\
\epsilon_{abcd}\epsilon^{efmd} &= -\delta^e_a\delta^f_b\delta^m_c - \delta^f_a\delta^m_b\delta^e_c - \delta^m_a\delta^e_b\delta^f_c + \delta^e_a\delta^m_b\delta^f_c + \delta^f_a\delta^e_b\delta^m_c + \delta^m_a\delta^f_b\delta^e_c\,, && (14.145) \\
\epsilon_{abcd}\epsilon^{efcd} &= 2\big(- \delta^e_a\delta^f_b + \delta^f_a\delta^e_b\big)\,, && (14.146) \\
\epsilon_{abcd}\epsilon^{ebcd} &= -6\delta^e_a && (14.147) \\
\epsilon_{abcd}\epsilon^{abcd} &= -24\,. && (14.148)
\end{aligned}$$

We can derive analogous expressions for the spatial (3-dimensional) Levi-Civita tensor defined by Eq. (14.102),

$$\epsilon_{\alpha\beta\gamma} = n^\mu\epsilon_{\mu\alpha\beta\gamma}\,;$$

note that our convention is to contract over the first index of $\boldsymbol{\epsilon}$. By using this definition and replacing contractions of the 4-dimensional $\boldsymbol{\epsilon}$ in terms of Eqs. (14.144)-(14.148), one finds after some tedious collection of terms that

$$\begin{aligned}
\epsilon_{\alpha\beta\gamma}\epsilon^{\lambda\mu\nu} &= \perp^\lambda{}_\alpha\perp^\mu{}_\beta\perp^\nu{}_\gamma + \perp^\mu{}_\alpha\perp^\nu{}_\beta\perp^\lambda{}_\gamma + \perp^\nu{}_\alpha\perp^\lambda{}_\beta\perp^\mu{}_\gamma \\
&\quad -\perp^\lambda{}_\alpha\perp^\nu{}_\beta\perp^\mu{}_\gamma - \perp^\mu{}_\alpha\perp^\lambda{}_\beta\perp^\nu{}_\gamma - \perp^\nu{}_\alpha\perp^\mu{}_\beta\perp^\lambda{}_\gamma && (14.149) \\
\epsilon_{\alpha\beta\rho}\epsilon^{\lambda\mu\rho} &= \perp^\lambda{}_\alpha\perp^\mu{}_\beta - \perp^\mu{}_\alpha\perp^\lambda{}_\beta && (14.150) \\
\epsilon_{\alpha\rho\sigma}\epsilon^{\lambda\rho\sigma} &= 2\perp^\lambda{}_\alpha\,, && (14.151) \\
\epsilon_{\rho\sigma\tau}\epsilon^{\rho\sigma\tau} &= 6\,. && (14.152)
\end{aligned}$$

Note that we have the projectors or 3-metrics $\perp$ here in place of the $\boldsymbol{\delta}$ in Eqs. (14.144)-(14.148). This is due to the fact that we are still working in full spacetime here. In the case of a genuinely 3-dimensional manifold, we can replace $\perp^\alpha{}_\beta$ with $\delta^i{}_j$ where i, j, $\ldots = 1, 2, 3$, but in our full spacetime picture, using $\delta^\alpha{}_\beta$ in Eqs. (14.149)-(14.151) would be manifestly wrong!

We complete this preliminary discussion with a collection of various useful relations involving the different tensors encountered in our calculations. Symmetry, tracelessness and perpendicularity of these objects can be summarized in the form

$$\mathcal{E}_{\alpha\beta} = \mathcal{E}_{\beta\alpha}\,, \qquad \mathcal{B}_{\alpha\beta} = \mathcal{B}_{\beta\alpha}\,, \tag{14.153}$$

$$\mathcal{E}^{\mu}{}_{\mu} = \mathcal{B}^{\mu}{}_{\mu} = 0\,, \qquad l^{\mu}{}_{\nu} := \perp^{\mu}{}_{\nu} + n^{\mu}n_{\nu} = l_{\nu}{}^{\mu}\,, \tag{14.154}$$

$$l^{\mu\nu}l_{\mu\nu} = 4\,, \qquad l^{\mu\rho}l_{\rho\nu} = \delta^{\mu}{}_{\nu} \tag{14.155}$$

$$l_{\nu\rho}\mathcal{E}^{\mu\rho} = \mathcal{E}^{\mu}{}_{\nu}\,, \qquad l^{\alpha\mu}\epsilon^{\tau}{}_{\mu\beta} = \epsilon^{\tau\alpha}{}_{\beta}\,, \tag{14.156}$$

$$\perp_{\alpha\beta}l^{\alpha\beta} = 3\,, \qquad l_{\alpha\mu}\perp^{\mu\beta} = \perp^{\beta}{}_{\alpha}\,, \tag{14.157}$$

$$\epsilon_{\alpha\beta\gamma\delta} = -\epsilon_{\beta\gamma\delta\alpha}\,, \qquad \epsilon_{\alpha\beta\gamma} = \epsilon_{\beta\gamma\alpha}\,, \tag{14.158}$$

$$l^{\mu\nu}n_{\mu} = -n^{\nu}\,. \tag{14.159}$$

Finally, we can compute $K_2 = I_2$,

$$K_2 = I_2 = \frac{1}{2}\epsilon_{\alpha\beta}{}^{\mu\nu}C_{\mu\nu\gamma\delta}C^{\alpha\beta\gamma\delta} = \frac{1}{2}\epsilon_{\alpha\beta}{}^{\mu\nu}C^{\alpha\beta\gamma\delta}C_{\mu\nu\gamma\delta} \tag{14.160}$$

We now expand the Weyl tensor according to Eq. (14.143),

$$C^{\alpha\beta\gamma\delta} = l^{\alpha\gamma}\mathcal{E}^{\delta\beta} - l^{\alpha\delta}\mathcal{E}^{\gamma\beta} - l^{\gamma\beta}\mathcal{E}^{\alpha\delta} + l^{\delta\beta}\mathcal{E}^{\alpha\gamma} - n^{\gamma}\mathcal{B}^{\delta\tau}\epsilon_{\tau}{}^{\alpha\beta} + n^{\delta}\mathcal{B}^{\gamma\tau}\epsilon_{\tau}{}^{\alpha\beta} - n^{\alpha}\mathcal{B}^{\beta\tau}\epsilon_{\tau}{}^{\gamma\delta} + n^{\beta}\mathcal{B}^{\alpha\tau}\epsilon_{\tau}{}^{\gamma\delta}\,,$$

and likewise for $C_{\mu\nu\gamma\delta}$. We then sort the resulting 64 terms into groups of 8 where we take the individual constributions from $C^{\alpha\beta\gamma\delta}$ and multiply each with the entire $C_{\mu\nu\gamma\delta}$. We show the calculation for the first of these terms in detail,

$$\begin{aligned}
\mathrm{term}_1 \;=\;& \frac{1}{2}\epsilon_{\alpha\beta}{}^{\mu\nu}l^{\alpha\gamma}\mathcal{E}^{\delta\beta}\times \\
& (l_{\mu\gamma}\mathcal{E}_{\delta\nu} - l_{\mu\delta}\mathcal{E}_{\gamma\nu} - l_{\nu\gamma}\mathcal{E}_{\delta\mu} + l_{\nu\delta}\mathcal{E}_{\gamma\mu} - n_{\gamma}\mathcal{B}_{\delta\tau}\epsilon^{\tau}{}_{\mu\nu} + n_{\delta}\mathcal{B}_{\gamma\tau}\epsilon^{\tau}{}_{\mu\nu} - n_{\mu}\mathcal{B}_{\nu\tau}\epsilon^{\tau}{}_{\gamma\delta} + n_{\nu}\mathcal{B}_{\mu\tau}\epsilon^{\tau}{}_{\gamma\delta}) \\
=\;& \frac{1}{2}\underbrace{\epsilon_{\alpha\beta}{}^{\mu\nu}\delta^{\alpha}{}_{\mu}}_{=0}\mathcal{E}^{\delta\beta}\mathcal{E}_{\delta\nu} - \frac{1}{2}\underbrace{\epsilon_{\alpha\beta}{}^{\mu\nu}\mathcal{E}^{\alpha}{}_{\nu}}_{=0}\mathcal{E}^{\beta}{}_{\mu} - \frac{1}{2}\underbrace{\epsilon_{\alpha\beta}{}^{\mu\nu}\delta^{\alpha}{}_{\nu}}_{=0}\mathcal{E}^{\delta\beta}\mathcal{E}_{\delta\mu} + \frac{1}{2}\underbrace{\epsilon_{\alpha\beta}{}^{\mu\nu}\mathcal{E}^{\beta}{}_{\nu}}_{=0}\mathcal{E}^{\alpha}{}_{\nu} \\
& + \frac{1}{2}\epsilon_{\alpha\beta}{}^{\mu\nu}n^{\alpha}\mathcal{E}^{\delta\beta}\mathcal{B}_{\delta\tau}\epsilon^{\tau}{}_{\mu\nu} + 0 - \frac{1}{2}n_{\mu}\epsilon_{\alpha\beta}{}^{\mu\nu}\mathcal{B}_{\nu\tau}\mathcal{E}^{\delta\beta}\epsilon^{\tau\alpha}{}_{\delta} + \frac{1}{2}n_{\nu}\epsilon_{\alpha\beta}{}^{\mu\nu}\mathcal{B}_{\mu\tau}\mathcal{E}^{\delta\beta}\epsilon^{\tau\alpha}{}_{\delta} \\
=\;& \frac{1}{2}\epsilon_{\beta\mu\nu}\epsilon^{\tau\mu\nu}\mathcal{E}^{\delta\beta}\mathcal{B}_{\delta\tau} + \frac{1}{2}\epsilon_{\beta\alpha\nu}\epsilon^{\tau\alpha\delta}\mathcal{B}^{\nu}{}_{\tau}\mathcal{E}_{\delta}{}^{\beta} - \frac{1}{2}\epsilon_{\beta\mu\alpha}\epsilon^{\tau\alpha\delta}\mathcal{B}^{\mu}{}_{\tau}\mathcal{E}_{\delta}{}^{\beta} \\
=\;& \perp^{\tau}{}_{\beta}\mathcal{E}^{\delta\beta}\mathcal{B}_{\delta\tau} + \frac{1}{2}\epsilon_{\beta\nu\alpha}\epsilon^{\tau\delta\alpha}\mathcal{B}^{\nu}{}_{\tau}\mathcal{E}_{\delta}{}^{\beta} + \frac{1}{2}\epsilon_{\beta\mu\alpha}\epsilon^{\tau\delta\alpha}\mathcal{B}^{\mu}{}_{\tau}\mathcal{E}_{\delta}{}^{\beta} \\
=\;& \mathcal{E}^{\delta\tau}\mathcal{B}_{\delta\tau} + \frac{1}{2}\left(\perp^{\tau}{}_{\beta}\perp^{\delta}{}_{\nu} - \perp^{\tau}{}_{\nu}\perp^{\delta}{}_{\beta}\right)\mathcal{B}^{\nu}{}_{\tau}\mathcal{E}_{\delta}{}^{\beta} + \frac{1}{2}\left(\perp^{\tau}{}_{\beta}\perp^{\delta}{}_{\mu} - \perp^{\tau}{}_{\mu}\perp^{\delta}{}_{\beta}\right)\mathcal{B}^{\mu}{}_{\tau}\mathcal{E}_{\delta}{}^{\beta} \\
=\;& \mathcal{E}^{\delta\tau}\mathcal{B}_{\delta\tau} + \frac{1}{2}\mathcal{B}^{\delta}{}_{\beta}\mathcal{E}_{\delta}{}^{\beta} - 0 + \frac{1}{2}\mathcal{B}^{\delta}{}_{\beta}\mathcal{E}_{\delta}{}^{\beta} - 0 \;=\; 2\mathcal{E}^{\delta\tau}\mathcal{B}_{\delta\tau}\,.
\end{aligned} \tag{14.161}$$

Quite amazingly all the other 7 terms give exactly the same contribution and our final result, merged with our earlier result (14.110), becomes

$$\boxed{\begin{aligned} I_1 &= 8(\mathcal{E}^{\mu\nu}\mathcal{E}_{\mu\nu} - \mathcal{B}^{\mu\nu}\mathcal{B}_{\mu\nu})\,, \\ K_2 = I_2 &= 16\mathcal{E}^{\mu\nu}\mathcal{B}_{\mu\nu}\,. \end{aligned}}$$

For the Schwarzschild metric, one can compute the Chern-Pontryagin using, for example, Maple and GRTensor and obtains $K_2 = 0$. For a non-trivial test case, we need to relax some symmetry and switch to the Kerr metric given in Boyer-Lindquist coordinates by

$$\begin{aligned} \mathrm{d}s^2 &= -\left(1-\frac{2Mr}{\rho^2}\right)\mathrm{d}t^2 + \frac{\rho^2}{a^2-2Mr+r^2}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 + \sin^2\theta\left(r^2+a^2+\frac{2Mra^2\sin^2\theta}{\rho^2}\right)\mathrm{d}\phi^2 \\ &\quad -4\frac{Mar\sin^2\theta}{\rho^2}\mathrm{d}t\,\mathrm{d}\phi \\ &= -\frac{r^2-2Mr+a^2}{\rho^2}\left(\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi\right)^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)\mathrm{d}\phi - a\mathrm{d}t\right]^2 + \frac{\rho^2}{r^2-2Mr+a^2}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 \\ &\quad \text{with} \quad \rho^2 = r^2 + a^2\cos^2\theta\,. \end{aligned} \tag{14.162}$$

The Chern-Pontryagin scalar for this metric is

$$K_2 = \frac{288r\cos\theta(a^2\cos^2\theta - 3r^2)\,a\sin\theta\left(a^2\cos^2\theta - \frac{r^2}{3}\right)M^2}{\sqrt{\sin^2\theta(r^2+a^2\cos^2\theta)^2}\,(r^2+a^2\cos^2\theta)^5}\,. \tag{14.163}$$

15 Some font tests

$$\mathfrak{r} \quad \mathfrak{r} \quad \mathfrak{r} \quad \mathfrak{r} \quad \mathscr{R} \tag{15.1}$$

16 Boosting initial data

Purpose of this section is to write down the expressions for a boosted Tangherlini solution. We will first deal with a single BH and then consider superposing two such solutions.

16.1 Notation

We keep the index notation used so far:

$$\begin{array}{ll} A,\, B,\, \ldots = 0,\, \ldots,\, D-1 & \text{Full spacetime} \\ I,\, J,\, \ldots = 1,\, \ldots,\, D-1 & \text{All spatial dimensions} \\ \alpha,\, \beta, \ldots = 0.\, \ldots,\, 3 & \text{Reduced spacetime} \\ i,\, j,\, \ldots = 1,\, 2,\, 3 & \text{Reduced spatial dimensions} \\ a,\, b,\, \ldots = 4,\, \ldots,\, D-1 & \text{Off-domain spatial directions} \end{array} \tag{16.1}$$

We also need to distinguish between indices in the rest frame and the boosted frame. We put a tilde on the latter, both on the indices and the object. For example, the spacetime metric in the boosted frame is written as $\tilde{g}_{\tilde{A}\tilde{B}}$.

16.2 Lorentz transformations

We will allow for arbitrary boost velocities in the three spatial dimensions that constitute our computational domain, but will set the velocity to zero in the off-domain directions. Probably, we can also assume the z component of the velocity to vanish, but we leave it unconstrained which is best in line with all the notation we have used so far. We thus consider two frames, $\mathcal{O}$ where the black hole is at rest. The coordinates in this frame are x^A. We wish to describe this spacetime from the point of view of an observer $\tilde{\mathcal{O}}$ moving with velocity v^i relative to $\mathcal{O}$. In this frame $\tilde{x}^{\tilde{A}}$, the black hole is thus moving with velocity $-v^i$. Note that indices of the velocity are raised and lowered with the Euclidean metric: $v_i = \delta_{im} v^m$.
Tensors are transformed from $\mathcal{O}$ to $\tilde{\mathcal{O}}$ with the Lorentz transformation matrices

$$\Lambda^{\tilde{A}}{}_M = \left(\begin{array}{c|c} \gamma & -\gamma v_J \\ \hline -\gamma v^I & \delta^I{}_J + (\gamma-1)\frac{v^I v_J}{\vec{v}^2} \end{array}\right) \qquad \Leftrightarrow \qquad \Lambda^M{}_{\tilde{A}} = \left(\begin{array}{c|c} \gamma & \gamma v_J \\ \hline \gamma v^I & \delta^I{}_J + (\gamma-1)\frac{v^I v_J}{\vec{v}^2} \end{array}\right) \tag{16.2}$$

In our case, the Lorentz transformation can be written with a slightly simplified version,

$$\Lambda^{\tilde{A}}{}_M = \left(\begin{array}{c|c|c} \gamma & -\gamma v_j & 0 \\ \hline -\gamma v^i & \delta^i{}_j + (\gamma-1)\frac{v^i v_j}{\vec{v}^2} & 0 \\ \hline 0 & 0 & \delta^a{}_b \end{array}\right) \qquad \Leftrightarrow \qquad \Lambda^M{}_{\tilde{A}} = \left(\begin{array}{c|c|c} \gamma & \gamma v_j & 0 \\ \hline \gamma v^i & \delta^i{}_j + (\gamma-1)\frac{v^i v_j}{\vec{v}^2} & 0 \\ \hline 0 & 0 & \delta^a{}_b \end{array}\right), \tag{16.3}$$

which more clearly shows that most of our operations will happen on the computational domain.

Step 1, Coordinates: We assume as input the coordinates, in Cartesian form, in the boosted frame. These are the coordinates that are used in the time evolution and they are *not* the coordinates in which we have the analytic solution. They are denoted by

$$\tilde{X}^{\tilde{A}} = \left(\tilde{t},\, \tilde{x}^{\tilde{i}},\, \tilde{w}^{\tilde{a}}\right) = \left(\tilde{t},\, \tilde{x},\, \tilde{y},\, \tilde{z},\, \tilde{w}^{\tilde{a}}\right) . \tag{16.4}$$

The first step consists in transforming these back to the non-boosted frame where we know the analytic form of the metric. These rest frame coordinates are

$$X^A = \Lambda^A{}_{\tilde{B}} \tilde{X}^{\tilde{B}} = (t,\, x^i,\, w^a) = (t,\, x,\, y,\, z,\, w^a)\,. \tag{16.5}$$

Note that derivatives with respect to the boosted and the non-boosted coordinates are related by analogous transformations.

Step 2, Inverse metric and derivatives: We now assume that the metric is given as g_{AB} in the non-boosted coordinate system X^A. We write the metric in block structure analogous to (16.3) as

$$g_{AB} = \left(\begin{array}{c|c|c} g_{00} & g_{0j} & g_{0b} \\ \hline g_{i0} & g_{ij} & g_{ib} \\ \hline g_{a0} & g_{aj} & g_{ab} \end{array}\right) = \left(\begin{array}{c|c|c} g_{00} & g_{0j} & 0 \\ \hline g_{i0} & g_{ij} & 0 \\ \hline 0 & 0 & g_{ww}\,\delta_{ab} \end{array}\right) , \tag{16.6}$$

where we again used the $SO(D-3)$ symmetry to simplify the expressions in the last equality.
In our specific case, this will be the Tangherlini metric for a single non-rotating black hole given in isotropic Cartesian coordinates by

$$ds^2 = \left(\frac{4R^{D-3} - \mu}{4R^{D-3} + \mu}\right) + \left(1 + \frac{\mu}{4R^{D-3}}\right)^{\frac{4}{D-3}} \delta_{MN} dx^M\, dx^N\,,$$

$$R = \sqrt{\delta_{MN}(x^M - x_0^M)(x^N - x_0^N)}\,. \tag{16.7}$$

Using the conformal factor

$$\psi = 1 + \frac{\mu}{4R^{D-3}}\,, \tag{16.8}$$

we can write the components of the blocks of Eq. (16.6) as

17 The ADM mass

The ADM mass of a higher-dimensional spacetime can be computed from Eq. (134) of Ref. [1],

$$M_{\rm ADM} = \frac{1}{16\pi G} \oint \delta^{MN}(\partial_N \gamma_{MK} - \partial_K \gamma_{MN}) \hat{r}^K\, dS\,, \tag{17.1}$$

where $\hat{r}^K$ is the unit radial vector and dS denotes the surface element of the $D-2$ sphere over which we are integrating.
In the case of $SO(D-3)$ isometry, we can reduce this integral to an integral on the 3-dimensional hypersurface of our computational domain or, to be more specific, a 2-dimensional surface integral on the hypersurface. Let us first consider the integrand. On the computation domain, we clearly have

$\hat{r}^a = 0$, since all vector components in the extra dimensions vanish. The summation over K therefore trivially reduces to that over the index $k = 1, 2, 3$. The integrand $E_{\rm den}$ thus becomes [using Eq. (4.19 in the third equality],

$$\begin{aligned} E_{\rm den} &= \delta^{MN}(\partial_N \gamma_{MK} - \partial_K \gamma_{MN})\hat{r}^K \\ &= \delta^{mn}(\partial_n \gamma_{mk} - \partial_k \gamma_{mn})\hat{r}^k + \underbrace{\delta^{mb}}_{=0}(\partial_b \gamma_{mk} - \partial_k \gamma_{mb})\hat{r}^k \\ &\quad + \underbrace{\delta^{an}}_{=0}(\partial_n \gamma_{ak} - \partial_k \gamma_{an})\hat{r}^k + \delta^{ab}(\partial_b \gamma_{ak} - \partial_k \gamma_{ab})\hat{r}^k \\ &= (\partial_m \gamma_{mk} - \partial_k \gamma_{mn})\hat{r}^k + \delta^{ab}\delta_{ab}\frac{\gamma_{kz} - \delta_{kz}\gamma_{ww}}{z}\hat{r}^k - \delta^{ab}\delta_{ab}\hat{r}^k \partial_k \gamma_{ww} \\ &= (\partial_m \gamma_{mk} - \partial_k \gamma_{mn})\hat{r}^k + (D-4)\frac{\gamma_{kz} - \delta_{kz}\gamma_{ww}}{z}\hat{r}^k - (D-4)\hat{r}^k \partial_k \gamma_{ww}\,. \end{aligned} \tag{17.2}$$

With $\hat{r}^k = (\frac{x}{r},\ \frac{y}{r},\ \frac{z}{r})$, this is exactly the relation we use in the code.
For the linear momentum, we likewise obtain from the general formulat

$$P_I = \frac{1}{8\pi G}\oint (K_{MI} - \delta_{MI}K)\hat{r}^M\, dS\,, \tag{17.3}$$

that

$$\begin{aligned} P_{x,\rm den} &= K_{mx}\hat{r}^m - K\frac{x}{r}\,, \\ P_{y,\rm den} &= K_{my}\hat{r}^m - K\frac{y}{r}\,. \end{aligned} \tag{17.4}$$

Finally, Eq. (136) of Ref. [1] gives us the ADM analog for the angular momentum as

$$J_I = \frac{1}{8\pi G}\oint (K_{JK} - K\gamma_{JK})\xi^J_{(I)}\hat{r}^K, dS\,, \tag{17.5}$$

where $\xi^J_{(I)}$ is the Killing vector associated with the asymtptotic rotational symmetry. In our case, we only have one rotational plane, the xy plane and the corresponding Killing vector is given by

$$\xi^J_{(xy)} = -y\partial_x + x\partial_y\,. \tag{17.6}$$

The angular momentum density function is then given by

$$J_{(xy),\rm den} = -xK\hat{r}^y + yK\hat{r}^x + xK_{my}\hat{r}^m - yK_{mx}\hat{r}^m = xK_{my}\hat{r}^m - yK_{mx}\hat{r}^m\,. \tag{17.7}$$

The integration over the sphere differs from that in 3 spatial dimensions, however, as now other powers of $\sin\theta$ and $\sin\phi$ appear in the integral. In order to see that, let us recall the relation between

Cartesian and spherical coordinates for n spatial dimensions,

$$
\begin{aligned}
x_1 &= r\cos\theta_1\,, \\
x_2 &= r\sin\theta_1\ \cos\theta_2\,, && (17.8)\\
x_3 &= r\sin\theta_1\ \sin\theta_2\ \cos\theta_3\,, && (17.9)\\
&\vdots && (17.10)\\
x_{n-2} &= r\sin\theta_1\ \sin\theta_2\ \sin\theta_3\ldots\sin\theta_{n-3}\ \cos\theta_{n-2}\,, && (17.11)\\
x_{n-1} &= r\sin\theta_1\ \sin\theta_2\ \sin\theta_3\ldots\sin\theta_{n-3}\ \sin\theta_{n-2}\ \cos\theta_{n-1}\,, && (17.12)\\
x_n &= r\sin\theta_1\ \sin\theta_2\ \sin\theta_3\ldots\sin\theta_{n-3}\ \sin\theta_{n-2}\ \sin\theta_{n-1}\,, && (17.13)
\end{aligned}
$$

Here, $\theta_{n-1}\in[0,2\pi)$ and all other $\theta_i\in[0,\pi]$. A tedious but straightforward calculation gives us the Jacobian

$$\frac{\partial(x_1,\,x_2,\,\ldots,\,x_n)}{\partial(r,\,\theta_1,\,\ldots,\,\theta_{n-1})} = r^{n-1}\sin^{n-2}\theta_1\ \sin^{n-3}\theta_2\ldots\sin^2\theta_{n-3}\ \sin\theta_{n-2}\,. \qquad (17.14)$$

The volume element on the $n-1$ sphere is then given by

$$dS = r^{n-1}\sin^{n-2}\theta_1\ \sin^{n-3}\theta_2\ldots\sin^2\theta_{n-3}\ \sin\theta_{n-2}d\theta_1\times\ldots\times d\theta_{n-1}\,. \qquad (17.15)$$

For $n=3$, for example, we obtain the familiar

$$dS = r^2\sin\theta_1\,, \qquad (17.16)$$

though note that our notation in Eq. (17.13) would correspond to the usual Cartesian coordinates $x_1=z,\ x_2=x,\ x_3=y$. In $n=4$ spatial dimensions, we have

$$
\begin{aligned}
x_1 &= r\cos\theta_1\,, && (17.17)\\
x_2 &= r\sin\theta_1\ \cos\theta_2\,, && (17.18)\\
x_3 &= r\sin\theta_1\ \sin\theta_2\ \cos\theta_3\,, && (17.19)\\
x_4 &= r\sin\theta_1\ \sin\theta_2\ \sin\theta_3\,, && (17.20)\\
\\
dS &= r^2\sin^2\theta_1\ \sin\theta_2\,. && (17.21)
\end{aligned}
$$

Note that we again have the unorthodox identification $x_1=z,\ x_2=x,\ x=y$ inside our 3-dimensional computational domain. This is of no consequence, however, as long as we are consistently labeling coordinates inside our computational domain. In fact, our code corresponds to the identification $x_3=z$, since we use z as our quasi-radial coordinate and it is x_3 that appears as a multiplier of all our $x_4,\,\ldots,\,x_n$. The extra dimensions manifest themselves in the volume element we need to use in the integration of Eq. (17.1), however. More specifically, we have $D-4$ extra dimensions, all of which with rotational symmetry, so that the integrand does not depend on the extra angles θ_3, θ_4 etc. We can therefore write the integral as

$$
\begin{aligned}
M_{\rm ADM} &= \frac{1}{16\pi G}\int E_{\rm den}r^{D-2}\sin^{D-3}\theta_1\ \sin^{D-4}\theta_2\ \ldots\ \sin^2\theta_{D-4}\ \sin\theta_{D-3}\ d\theta_1\times\ \ldots\ \times\ d\theta_{D-2} && (17.22)\\
&= \frac{r^{D-2}}{16\pi G}\int E_{\rm den}\sin^{D-3}\theta_1\ \sin^{D-4}\theta_2\ d\theta_1\ d\theta_2\times A_{D-4}\,, && (17.23)
\end{aligned}
$$

where A_{D-4} denotest the area of the $D-4$ sphere,

$$A_{D-4} = 2\pi \frac{D-3}{2\Gamma\left(\frac{D-3}{2}\right)}\,. \tag{17.24}$$

With the additional identification $\theta_1 = \theta$, $\theta_2 = \phi$, we recover exactly the expression used in the code. For the linear and angular momentum, we obtain likewise,

$$P_{x,\mathrm{ADM}} = \frac{A_{D-4} r^{D-2}}{8\pi G} \int\int P_{x,\mathrm{den}} \sin^{D-4}\phi d\phi \sin^{D-3}\theta\, d\theta\,, \tag{17.25}$$

$$P_{y,\mathrm{ADM}} = \frac{A_{D-4} r^{D-2}}{8\pi G} \int\int P_{y,\mathrm{den}} \sin^{D-4}\phi d\phi \sin^{D-3}\theta\, d\theta\,, \tag{17.26}$$

$$J_{xy,\mathrm{ADM}} = \frac{A_{D-4} r^{D-2}}{8\pi G} \int\int J_{xy,\mathrm{den}} \sin^{D-4}\phi d\phi \sin^{D-3}\theta\, d\theta\,. \tag{17.27}$$

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