Contents

1 The CCZ4 equations

We follow here the formulation presented in Ref. [1], although we will depart in places from their notation (which is prone to cause confusions). The starting point is the augmented version of the Einstein equations which we write in $\mathcal{D}$ dimensions with inclusion of a cosmological constant,

$$R_{\mu\nu} - \frac{2}{\mathcal{D}-2}\Lambda g_{\mu\nu} + \nabla_\mu Z_\nu + \nabla_\nu Z_\mu - c_1\left[n_\mu Z_\nu + n_\nu Z_\mu - (1+c_2)g_{\mu\nu}n_\sigma Z^\sigma\right] = 8\pi\left(T_{\mu\nu} - \frac{1}{\mathcal{D}-2}g_{\mu\nu}T\right). \tag{1.1}$$

Here, Z^μ is a vectorfield and n_μ is the timelike unit normal form we will later use for our space-time decomposition, i.e. we will adapt our coordinates such that surfaces of constant time will be level surfaces corresponding to the 1-form n_μ and the time coordinate is rescaled such that $n^\mu n_\mu = -1$.

Note that for $Z^\mu = 0$, Eq. (1.1) reduces to the Einstein equations, albeit in trace reversed form. In order to apply the Gauss-Codazzi-Mainardi equations, we will also need the Z4 equations without this trace reversal; in practice we can achieve that equivalently by simply computing the trace of (1.1) and taking whichever linear combinations of the 2 equations we find most appropriate. Contracting (1.1) with $g^{\mu\nu}$ gives by

$$R - \frac{2\mathcal{D}}{\mathcal{D}-2}\Lambda + 2\nabla^\mu Z_\mu - c_1\left[2n^\mu Z_\mu - (1+c_2)\mathcal{D}n_\sigma Z^\sigma\right] = -8\pi\frac{2}{\mathcal{D}-2}T\,. \tag{1.2}$$

Note that Alic *et al* [1] typically set $c_2 = 0$ and use $c_1 \approx 0.1\ M$ for BH spacetimes of mass M.

2 The $d + 1$ decomposition

We define the number of spacetime dimensions as $\mathcal{D}$ and the number of spatial dimensions as d, so that $\mathcal{D} = d+1$. We use Greek letters for spacetime indices and middle Latin letters for spatial indices.

Starting point of the spacetime split is a foliation of the spacetime $\mathcal{M}$ into spatial hypersurfaces Σ_t, defined as the set of points $t = \text{const}$, such that $\mathcal{M} = \cup_{t\in\mathbb{R}}\Sigma_t$. A coordinate system adapted to the foliation consists of $(t,\, x^i)$ where the x^i form a coordinate chart in each of the Σ_t. Evidently, the gradient $\mathbf{d}t$ is normal to the surfaces Σ_t and we rescale it to unit length by defining

$$\alpha := \frac{1}{\sqrt{-||\mathbf{d}t||^2}} \qquad \Rightarrow \qquad \boldsymbol{n} = -\alpha\mathbf{d}t \qquad \Rightarrow \qquad n_\mu = (-\alpha,\ 0)\,; \tag{2.1}$$

the reason for introducing the minus sign in the last two expressions will become clear shortly. Let us now assume that we have adapted coordinates and define the shift vector as

$$\boldsymbol{\beta} := \boldsymbol{\partial}_t - \alpha\boldsymbol{n} \qquad \Rightarrow \qquad \boldsymbol{n} = \frac{1}{\alpha}\left(\boldsymbol{\partial}_t - \boldsymbol{\beta}\right) \qquad \Rightarrow \qquad n^\mu = \left(\frac{1}{\alpha},\ -\frac{\beta^i}{\alpha}\right). \tag{2.2}$$

Here lies the reason for choosing the minus sign in Eq. (2.1): we want the vector n^μ to point towards the future direction viz. increasing t.

Using the relation $\langle \mathbf{d}t, \boldsymbol{\partial}_t\rangle = 1$, one straightforewardly finds

$$\langle \mathbf{d}t, \boldsymbol{\beta}\rangle = 0\,, \tag{2.3}$$

so the shift vector is tangent to Σ_t. Finally, we define the spatial metric or projector or first fundamental form as

$$\boldsymbol{\gamma} := \boldsymbol{g} + \boldsymbol{n}\otimes\boldsymbol{n} \qquad \Leftrightarrow \qquad \gamma_{\alpha\beta} := g_{\alpha\beta} + n_\alpha n_\beta\,. \tag{2.4}$$

In particular, the spatial components of the spatial metric are given by inserting the spatial basis vectors,

$$\gamma_{ij} := \boldsymbol{\gamma}(\boldsymbol{\partial}_i, \boldsymbol{\partial}_j)\,. \tag{2.5}$$

In the following, we will denote the projection or spatial metric either by $\gamma_{\alpha\beta}$ or $\perp_{\alpha\beta}$, depending on whether the focus is on its character as a metric or a projection operator.

The next step consists in expressing the components of the spacetime metric in terms of α, β^i and γ_{ij}. This is achieved by simply inserting the basis vectors,

$$\begin{aligned}
g_{00} &= \boldsymbol{g}(\boldsymbol{\partial}_t,\boldsymbol{\partial}_t) = \boldsymbol{g}(\boldsymbol{\beta}+\alpha\boldsymbol{n},\boldsymbol{\beta}+\alpha\boldsymbol{n}) = \boldsymbol{g}(\boldsymbol{\beta},\boldsymbol{\beta}) + \alpha^2\boldsymbol{g}(\boldsymbol{n},\boldsymbol{n}) = \gamma_{ij}\beta^i\beta^j - \alpha^2\,,\\
g_{0i} &= \boldsymbol{g}(\boldsymbol{\partial}_t,\boldsymbol{\partial}_i) = \boldsymbol{g}(\alpha\boldsymbol{n},\boldsymbol{\partial}_i) + \boldsymbol{g}(\boldsymbol{\beta},\boldsymbol{\partial}_i) = \beta_i\,,\\
g_{ij} &= \boldsymbol{g}(\boldsymbol{\partial}_i,\boldsymbol{\partial}_j) \stackrel{!}{=} \boldsymbol{\gamma}(\boldsymbol{\partial}_i,\boldsymbol{\partial}_j) = \gamma_{ij}\,,
\end{aligned} \tag{2.6}$$

where we have used the above result that $\boldsymbol{\beta}$ is a purely spatial vector, its inner product with $\mathbf{d}t$ and, hence, with $\boldsymbol{n}$ vanishes, and that the inner product of purely spatial vectors can be evaluated equivalently using either the spacetime metric $\boldsymbol{g}$ or the spatial metric $\boldsymbol{\gamma}$. In adapted coordinates, the spacetime metric therefore becomes

$$g_{\alpha\beta} = \begin{pmatrix} -\alpha^2 + \beta_m\beta^m & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \qquad \Leftrightarrow \qquad g^{\alpha\beta} = \begin{pmatrix} -\alpha^{-2} & \alpha^{-2}\beta^j \\ \alpha^{-2}\beta^i & \gamma^{ij} - \alpha^{-2}\beta^i\beta^j \end{pmatrix}, \tag{2.7}$$

where γ^{ij} is defined as the inverse of γ_{ij}.

In the CCZ4 system, we also need to consider the decomposition of the four vector Z^μ. This point is not very clearly spelled out in the literature I have read, but it is an important point to avoid confusion later on. We therefore define the time and spatial projections of the $Z4$ vector as

$$\boxed{\Theta := -n_\mu Z^\mu \quad \wedge \quad \Theta_\mu := \perp Z_\mu := \perp^\rho{}_\mu Z_\rho \qquad \Leftrightarrow \qquad Z^\mu = \Theta n^\mu + \Theta^\mu\,.} \tag{2.8}$$

By combining this expression with the $d+1$ split metric (2.7), we can find the components of Z^μ and Z_μ. We find

$$\Theta = -n_\mu Z^\mu = \alpha Z^0 \quad \Rightarrow \quad \boxed{Z^0 = \frac{\Theta}{\alpha}},$$

$$Z^\mu = \Theta n^\mu + \Theta^\mu \quad \Rightarrow \quad \begin{cases} Z^0 = \frac{1}{\alpha}\Theta + \Theta^0 \quad \Rightarrow \quad \boxed{\Theta^0 = 0}, \\[2ex] Z^i = -\frac{\beta^i}{\alpha}\Theta + \Theta^i \quad \Rightarrow \quad \Theta^i = Z^i + \frac{\beta^i}{\alpha}\Theta \quad \Rightarrow \quad \boxed{Z^i = \Theta^i - \frac{\beta^i}{\alpha}\Theta}, \end{cases}$$

$$Z_0 = g_{0\mu}Z^\mu = g_{00}Z^0 + g_{0i}Z^i = (-\alpha^2 + \beta_i\beta^i)\frac{\Theta}{\alpha} + \beta_i\left(\Theta^i - \frac{\beta^i}{\alpha}\right) \quad \Rightarrow \quad \boxed{Z_0 = -\alpha\Theta + \beta_i\Theta^i}$$

$$Z_i = g_{i\mu}Z^\mu = g_{i0}Z^0 + g_{ij}Z^j = \beta_i\frac{\Theta}{\alpha} + \gamma_{ij}\left(\Theta^i - \frac{\beta^i}{\alpha}\Theta\right) \quad \Rightarrow \quad \boxed{Z_i = \gamma_{ij}\Theta^i =: \Theta_i}\,. \tag{2.9}$$

We can summarize this as

$$Z_\mu = \left(-\alpha\Theta + \beta_j\Theta^j,\ \Theta_i\right) \qquad \Leftrightarrow \qquad Z^\mu = \left(\frac{\Theta}{\alpha},\ \Theta^i - \frac{\beta^i}{\alpha}\Theta\right)\,. \tag{2.10}$$

The components we will eventually be working with are Θ and Θ_i.

The foundation of decomposing the field equations are the Gauss-Codazzi-Mainardi equations which relate the projections of the Riemann tensor to the spatial Riemann and Ricci tensor. We will denote the latter by a caligraphic $\boldsymbol{\mathcal{R}}$, in order to distinguish them from the spacetime $\boldsymbol{R}$. First, however, we need to define the extrinsic curvature and the spatial covariant derivative. The extrinsic curvature or second fundamental form is defined by

$$K_{\alpha\beta} := -\perp\nabla_\alpha n_\beta = -\perp^\gamma{}_\alpha\perp^\delta{}_\beta\nabla_\gamma n_\delta\,. \tag{2.11}$$

The spatial covariant derivative of a tensor $T^\alpha{}_\beta$ is defined by projecting all free indices according to

$$D_\gamma T^\alpha{}_\beta := \perp^\lambda{}_\gamma\perp^\alpha{}_\mu\perp^\nu{}_\beta\nabla_\lambda T^\mu{}_\nu\,, \tag{2.12}$$

with obvious generalization to tensors of other rank.

With these definitions, the Gauss-Codazzi-Mainardi equations can be written as

$$\perp R^\alpha{}_{\beta\gamma\delta} = \mathcal{R}^\alpha{}_{\beta\gamma\delta} + K^\alpha{}_\gamma K_{\delta\beta} - K^\alpha{}_\delta K_{\gamma\beta}\,, \tag{2.13}$$

$$\perp R_{\alpha\beta} + \perp^\gamma{}_\alpha n^\delta\perp^\epsilon{}_\beta n^\rho R_{\gamma\delta\epsilon\rho} = \mathcal{R}_{\alpha\beta} + KK_{\alpha\beta} - K_{\alpha\gamma}K^\gamma{}_\beta\,, \tag{2.14}$$

$$R + 2R_{\gamma\delta}n^\gamma n^\delta = \mathcal{R} + K^2 - K_{\gamma\delta}K^{\gamma\delta}\,, \tag{2.15}$$

$$\perp^\delta{}_\alpha\perp^\epsilon{}_\beta\perp^\rho{}_\gamma n^\sigma R_{\delta\epsilon\rho\sigma} = -D_\alpha K_{\beta\gamma} + D_\beta K_{\alpha\gamma}\,, \tag{2.16}$$

$$\perp^\gamma{}_\beta R_{\gamma\delta}n^\delta = -D_\alpha K^\alpha{}_\beta + D_\beta K\,. \tag{2.17}$$

3 The ADM version of the Z4 equations

3.1 The equation for Θ

From Eq. (2.15), we see that we need to construct

$$2n^\mu n^\nu \times \text{Eq. (1.1)} + \text{Eq. (1.2)}\,, \tag{3.1}$$

in order to trade the spacetime Ricci tensor for its spatial counterpart. This gives us ($\rho \coloneqq T_{\mu\nu}n^\mu n^\nu$),

$$\begin{aligned}
&2n^\mu n^\nu R_{\mu\nu} + \frac{4}{\mathcal{D}-2}\Lambda + 4n^\mu n^\nu \nabla_\mu Z_\nu - 2c_1[-2n^\mu Z_\mu + (1+c_2)n_\sigma Z^\sigma] + R - \frac{2\mathcal{D}}{\mathcal{D}-2}\Lambda + 2\nabla^\mu Z_\mu \\
&-c_1[2n^\mu Z_\mu - (1+c_2)\mathcal{D} n_\sigma Z^\sigma] = 8\pi 2\rho + \frac{8\pi}{\mathcal{D}-2}2T - 8\pi T\frac{2}{\mathcal{D}-2}\,, \qquad (3.2)\\
\Rightarrow\quad &\mathcal{R} + K^2 - K_{\mu\nu}K^{\mu\nu} - 2\Lambda + 4n^\mu n^\nu \nabla_\mu Z_\nu + 2c_1 n^\mu Z_\mu + (\mathcal{D}-2)c_1(1+c_2)n_\sigma Z^\sigma + 2\nabla^\mu Z_\mu = 16\pi\rho\,.
\end{aligned}$$

A few expressions herein still need to be evaluated. First, we consider the term $\nabla^\mu Z_\mu$. Starting with the definition of the spatial covariant derivative we obtain

$$\begin{aligned}
&D_\mu Z_\nu = \perp^\alpha{}_\mu \perp^\beta{}_\nu \nabla_\alpha Z_\beta = (\delta^\alpha{}_\mu + n^\alpha n_\mu)(\delta^\beta{}_\nu + n^\beta n_\nu)\nabla_\alpha Z_\beta \\
\Rightarrow\quad &\gamma^{\mu\nu} D_\mu Z_\nu = (g^{\mu\nu} + n^\mu n^\nu) D_\mu Z_\nu = g^{\mu\nu} D_\mu Z_\nu = g^{\mu\nu}(\delta^\alpha{}_\mu + n^\alpha n_\mu)(\delta^\beta{}_\nu + n^\beta n_\nu)\nabla_\alpha Z_\beta \\
\Rightarrow\quad &\gamma^{\mu\nu} D_\mu Z_\nu = \nabla_\alpha Z_\beta (g^{\alpha\beta} + 2n^\alpha n^\beta - n^\alpha n^\beta) = \nabla_\alpha Z_\beta (g^{\alpha\beta} + n^\alpha n^\beta) \\
\Rightarrow\quad &g^{\alpha\beta}\nabla_\alpha Z_\beta = -n^\alpha n^\beta \nabla_\alpha Z_\beta + D^\mu Z_\mu\,. \qquad (3.3)
\end{aligned}$$

Inserting this into Eq. (3.2), we obtain

$$\mathcal{R} + K^2 - K_{\mu\nu}K^{\mu\nu} - 2\Lambda + 2n^\mu n^\nu \nabla_\mu Z_\nu + 2c_1 n^\mu Z_\mu + (\mathcal{D}-2)c_1(1+c_2)n_\sigma Z^\sigma + 2D^\mu Z_\mu = 16\pi\rho\,. \tag{3.4}$$

For the next evaluation, we recall that the extrinsic curvature satisfies

$$\begin{aligned}
&K_{\alpha\beta} = -\nabla_\alpha n_\beta - n_\alpha n^\gamma \nabla_\gamma n_\beta = -\nabla_\alpha n_\beta - n_\alpha \frac{\partial_\beta \alpha}{\alpha} \\
\Rightarrow\quad &-\nabla_\mu n_\nu = K_{\mu\nu} + n_\mu a_\nu\,, \quad \text{where } a_\nu \coloneqq n^\rho \nabla_\rho n_\nu = \frac{\partial_\nu \alpha}{\alpha}\,. \qquad (3.5)
\end{aligned}$$

Note that a^μ is spatial, i.e. $\langle \mathbf{d}t, \boldsymbol{a}\rangle = 0$; details of the derivations can be found in [3]. This result allows us to write

$$n^\mu n^\nu \nabla_\mu Z_\nu = n^\mu \nabla_\mu (n^\nu Z_\nu) - n^\mu Z_\nu \nabla_\mu n^\nu = -n^\mu \nabla_\mu \Theta + n^\mu Z^\nu \left(K_{\mu\nu} + n_\mu a_\nu\right)\,. \tag{3.6}$$

The second term on the right-hand-side vanishes, since $n^\mu K_{\mu\nu} = 0$ and we can insert the remainder into Eq. (3.4),

$$\mathcal{R} + K^2 - K_{\mu\nu}K^{\mu\nu} - 2\Lambda - 2n^\mu \nabla_\mu \Theta - 2Z^\nu a_\nu - 2c_1\Theta - (\mathcal{D}-2)c_1(1+c_2)\Theta + 2D^\mu Z_\mu = 16\pi\rho\,. \tag{3.7}$$

Next, we recall that n^μ is given by Eq. (2.2) which allows us to write

$$n^\mu \nabla_\mu \Theta = n^\mu \partial_\mu \Theta = \frac{1}{\alpha}\partial_t \Theta - \frac{\beta^m}{\alpha}\partial_m \Theta\,, \tag{3.8}$$

so that

$$\partial_t\Theta = \beta^m\partial_m\Theta + \frac{\alpha}{2}\left\{\mathcal{R} + K^2 - K_{\mu\nu}K^{\mu\nu} - 2\Lambda - 2Z_\nu a^\nu - c_1\Theta[\mathcal{D} + (\mathcal{D}-2)c_2] + 2D^\mu Z_\mu - 16\pi\rho\right\}. \tag{3.9}$$

There remains one subtle point: the expression $D^\mu Z_\mu$ needs to be decomposed into contributions from the spatial part of Z_μ and that part from the time component. Using Eq. (2.8), we find

$$\begin{aligned}
\gamma^{\mu\nu}D_\mu Z_\nu = g^{\mu\nu}D_\mu Z_\nu &= g^{\mu\nu}D_\mu(n_\nu\Theta + \Theta_\nu) = g^{\mu\nu}D_\mu\Theta_\nu + g^{\mu\nu}D_\mu(n_\nu\Theta) \\
&= \gamma^{\mu\nu}D_\mu\Theta_\nu + g^{\mu\nu}\perp^\alpha{}_\mu\perp^\beta{}_\nu\nabla_\alpha(n_\beta\Theta) = D^\mu\Theta_\mu + g^{\mu\nu}\perp^\alpha{}_\mu\underbrace{\perp^\beta{}_\nu[n_\beta}_{=0}\nabla_\alpha\Theta + \Theta\nabla_\alpha n_\beta] \\
&= D^\mu\Theta_\mu + g^{\mu\nu}\perp^\alpha{}_\mu\perp^\beta{}_\nu\Theta\nabla_\alpha n_\beta \qquad \Big|\; K_{\alpha\beta} = -\nabla_\alpha n_\beta - n_\alpha n^\mu\nabla_\mu n_\beta \\
&= D^\mu\Theta_\mu - g^{\mu\nu}\perp^\alpha{}_\mu\perp^\beta{}_\nu K_{\alpha\beta}\Theta \\
&= D^\mu\Theta_\mu - g^{\mu\nu}K_{\mu\nu}\Theta \\
&= D^\mu\Theta_\mu - K\Theta\,,
\end{aligned} \tag{3.10}$$

where in the last steps, we have used that the extrinsic curvature is a purely spatial quantity. With this substitution, Eq. (3.9) becomes, using $Z_\nu a^\nu = Z_i a^i = \gamma^{ij}Z_i a_j = \gamma^{ij}\Theta_i\partial_j\alpha/\alpha$,

$$\boxed{\begin{aligned}
\partial_t\Theta &= \beta^m\partial_m\Theta + \frac{\alpha}{2}\Big\{\mathcal{R} + K(K-2\Theta) - K_{mn}K^{mn} - 2\Lambda - 2\gamma^{ij}\Theta_i\frac{\partial_j\alpha}{\alpha} \\
&\qquad - c_1\Theta[\mathcal{D} + (\mathcal{D}-2)c_2] + 2D^m\Theta_m - 16\pi\rho\Big\}\,.
\end{aligned}} \tag{3.11}$$

3.2 The equation for Θ_i

The next equation is obtained by projecting Eq. (1.1) once onto the spatial and once onto the time direction. We will then use Eq. (2.17) to replace the spacetime Ricci tensor with spatial quantities. We obtain ($j_\alpha := -\perp^\mu{}_\alpha n^\nu T_{\mu\nu}$),

$$\begin{aligned}
&\perp^\mu{}_\alpha n^\nu \times \text{Eq. (1.1)} \\
\Rightarrow\quad &\perp^\mu{}_\alpha n^\nu R_{\mu\nu} - 0 + \perp^\mu{}_\alpha n^\nu\nabla_\mu Z_\nu + \perp^\mu{}_\alpha n^\nu\nabla_\nu Z_\mu - c_1[0 - \Theta_\alpha - (1+c_2)0] - 8\pi(T_{\mu\nu}\perp^\mu{}_\alpha n^\nu - 0) = 0 \\
\Rightarrow\quad &-D_\rho K^\rho{}_\alpha + D_\alpha K + \perp^\mu{}_\alpha n^\nu\nabla_\mu Z_\nu + \perp^\mu{}_\alpha n^\nu\nabla_\nu Z_\mu + c_1\Theta_\alpha + 8\pi j_\alpha = 0\,.
\end{aligned} \tag{3.12}$$

Here we have two terms that need further consideration. First,

$$\begin{aligned}
\perp^\mu{}_\alpha n^\nu\nabla_\mu Z_\nu &= \perp^\mu{}_\alpha\nabla_\mu(n^\nu Z_\nu) - \perp^\mu{}_\alpha Z_\nu\nabla_\mu n_\nu = -D_\alpha\Theta + \perp^\mu{}_\alpha Z^\nu(K_{\mu\nu} + n_\mu a_\nu) \\
&= -D_\alpha\Theta + K_{\alpha\nu}Z^\nu = -D_\alpha\Theta + K_{\alpha\nu}\Theta^\nu\,,
\end{aligned} \tag{3.13}$$

where we used in the very last step that $K_{\mu\nu}$ is spatial and hence its contraction with Z^μ and Θ^μ give the same result.

The second term is $\perp^\mu{}_\alpha n^\nu \nabla_\nu Z_\mu$ and requires more work. We first define the normal vector $\boldsymbol{m} = \alpha \boldsymbol{n} = \boldsymbol{\partial}_t - \boldsymbol{\beta}$, so that

$$\begin{aligned}
& m_\mu = \alpha n_\mu \qquad \wedge \qquad \nabla_\alpha n_\beta = -K_{\alpha\beta} - n_\alpha a_\beta \\
\Rightarrow \quad & \nabla_\alpha m_\beta = -\alpha K_{\alpha\beta} - \alpha n_\alpha a_\beta + n_\beta \nabla_\alpha \alpha \\
\Rightarrow \quad & \mathcal{L}_{\boldsymbol{m}} \Theta_\alpha = m^\mu \nabla_\mu \Theta_\alpha + \Theta_\mu \nabla_\alpha m^\mu = m^\mu \nabla_\mu \Theta_\alpha + \Theta_\mu \big(-\alpha K_\alpha{}^\mu - \alpha n_\alpha a^\mu + \underbrace{n^\mu \nabla_\alpha \alpha}_{\to 0} \big) ,
\end{aligned} \tag{3.14}$$

where the last term vanishes because Θ_μ is spatial. The Lie derivative along $\boldsymbol{m}$ will ultimately give us the time derivative of Θ_μ through the relation

$$\mathcal{L}_{\boldsymbol{m}} \Theta_i = \mathcal{L}_{\boldsymbol{\partial}_t} \Theta_i - \mathcal{L}_{\boldsymbol{\beta}} \Theta_i = \partial_t \Theta_i - \beta^m \partial_m \Theta_i - \Theta_m \partial_i \beta^m \,. \tag{3.15}$$

With these calculations in place, we can finally address the term

$$\begin{aligned}
\perp^\mu{}_\alpha n^\nu \nabla_\nu Z_\mu &= n^\nu \nabla_\nu (\perp^\mu{}_\alpha Z_\mu) - Z_\mu n^\nu \nabla_\nu \big(\delta^\mu{}_\alpha + n^\mu n_\alpha\big) = n^\nu \nabla_\nu \Theta_\alpha - Z_\mu n^\nu \big(n^\mu \nabla_\nu n_\alpha + n_\alpha \nabla_\nu n^\mu\big) \\
&= n^\nu \nabla_\nu \Theta_\alpha + Z_\mu n^\nu \big[n^\mu (K_{\nu\alpha} + n_\nu a_\alpha) + n_\alpha (K_\nu{}^\mu + n_\nu a^\mu)\big] \\
&= n^\nu \nabla_\nu \Theta_\alpha + \Theta a_\alpha - Z_\mu a^\mu n_\alpha = \frac{1}{\alpha} \mathcal{L}_{\boldsymbol{m}} \Theta_\alpha + \Theta_\mu K_\alpha{}^\mu + n_\alpha a^\mu \Theta_\mu + \Theta a_\alpha - Z_\mu a^\mu n_\alpha \\
&= \frac{1}{\alpha} \mathcal{L}_{\boldsymbol{m}} \Theta_\alpha + \Theta_\mu K_\alpha{}^\mu + \Theta a^\alpha \,,
\end{aligned} \tag{3.16}$$

where we have used Eq. (3.14) in the second last line. Finally, we plug (3.13) and (3.16) into (3.12) and solve for $\mathcal{L}_{\boldsymbol{m}}$,

$$\mathcal{L}_{\boldsymbol{m}} \Theta_\alpha = \alpha \Big(D_\rho K^\rho{}_\alpha - D_\alpha K - 2 K_{\alpha\mu} \Theta^\mu + D_\alpha \Theta - \Theta a_\alpha - c_1 \Theta_\alpha - 8\pi j_\alpha \Big) \,, \tag{3.17}$$

expand the Lie derivative according to Eq. (3.15) and recall Eq. (3.5) for a_i, which gives us

$$\boxed{\partial_t \Theta_i = \beta^m \partial_m \Theta_i + \Theta_m \partial_i \beta^m + \alpha \left(D_m K^m{}_i - D_i K - 2 K_{im} \Theta^m + D_i \Theta - \Theta \frac{\partial_i \alpha}{\alpha} - c_1 \Theta_i - 8\pi j_i \right) .} \tag{3.18}$$

3.3 The equation for K_{ij}

We finally consider the equations for the geometry. The defintion of the extrinsic curvature is unchanged from the case of pure GR, so that we obtain the standard equation

$$\boxed{\partial_t \gamma_{ij} = \beta^m \partial_m \gamma_{ij} + \gamma_{mj} \partial_i \beta^m + \gamma_{im} \partial_j \beta^m - 2\alpha K_{ij} \,.} \tag{3.19}$$

Besides the Gauss-Codazzi-Mainardi equations (2.13)-(2.17), we also need the final projection of the Riemann tensor which is given by [3]

$$\perp R_{\alpha\beta} = -\frac{1}{\alpha} \mathcal{L}_{\boldsymbol{m}} K_{\alpha\beta} - \frac{1}{\alpha} D_\alpha \partial_\beta \alpha + \mathcal{R}_{\alpha\beta} + K K_{\alpha\beta} - 2 K_{\alpha\mu} K^\mu{}_\beta \,. \tag{3.20}$$

We will also need the spatial projection of the energy momentum tensor

$$
\begin{aligned}
& S = \gamma^{\mu\nu} S_{\mu\nu} = \gamma^{\mu\nu} \perp^{\rho}{}_{\mu} \perp^{\sigma}{}_{\nu} T_{\rho\sigma} = \gamma^{\mu\nu} (\delta^{\rho}{}_{\mu} + n^{\rho} n_{\mu})(\delta^{\sigma}{}_{\nu} + n^{\sigma} n_{\nu}) T_{\rho\sigma} = \gamma^{\mu\nu} T_{\mu\nu} \\
\Rightarrow \quad & T = g^{\mu\nu} T_{\mu\nu} = (\gamma^{\mu\nu} - n^{\mu} n^{\nu}) T_{\mu\nu} = \gamma^{\mu\nu} T_{\mu\nu} - \rho = S - \rho \,. \qquad (3.21)
\end{aligned}
$$

Finally, we need the relation

$$
\begin{aligned}
\perp^{\mu}{}_{\alpha} \perp^{\nu}{}_{\beta} \nabla_{\mu} Z_{\nu} &= \perp^{\mu}{}_{\alpha} \perp^{\nu}{}_{\beta} \nabla_{\mu} (\Theta n_{\nu} + \Theta_{\nu}) = \perp^{\mu}{}_{\alpha} \perp^{\nu}{}_{\beta} \Theta \nabla_{\mu} n_{\nu} + D_{\alpha} \Theta_{\beta} \\
&= \perp^{\mu}{}_{\alpha} \perp^{\nu}{}_{\beta} \Theta (-K_{\mu\nu} - n_{\mu} a_{\nu}) + D_{\alpha} \Theta_{\beta} = -K_{\alpha\beta} \Theta + D_{\alpha} \Theta_{\beta} \,. \qquad (3.22)
\end{aligned}
$$

Let us then compute

$$
\begin{aligned}
& \perp^{\mu}{}_{\alpha} \perp^{\nu}{}_{\beta} \times \text{Eq. } (1.1) \\
\Rightarrow \quad & \perp R_{\alpha\beta} - \frac{2}{\mathcal{D}-2} \Lambda \gamma_{\alpha\beta} - K_{\alpha\beta} \Theta + D_{\alpha} \Theta_{\beta} + D_{\beta} \Theta_{\alpha} - K_{\alpha\beta} \Theta + c_1 (1 + c_2) \gamma_{\alpha\beta} (-\Theta) = 8\pi \left[S_{\alpha\beta} - \gamma_{\alpha\beta} \frac{S - \rho}{\mathcal{D}-2} \right] \\
\Rightarrow \quad & -\frac{1}{\alpha} \mathcal{L}_{\boldsymbol{m}} K_{\alpha\beta} - \frac{1}{\alpha} D_{\alpha} D_{\beta} \alpha + \mathcal{R}_{\alpha\beta} + K K_{\alpha\beta} - 2 K_{\alpha\mu} K^{\mu}{}_{\beta} - \frac{2}{\mathcal{D}-2} \Lambda \gamma_{\alpha\beta} - 2 K_{\alpha\beta} \Theta + D_{\alpha} \Theta_{\beta} + D_{\beta} \Theta_{\alpha} \\
& \qquad - c_1 (1 + c_2) \Theta \gamma_{\alpha\beta} - 8\pi \left[S_{\alpha\beta} - \frac{1}{\mathcal{D}-2} \gamma_{\alpha\beta} (S - \rho) \right] = 0 \,.
\end{aligned}
$$

Using the Lie derivative

$$
\mathcal{L}_{\boldsymbol{m}} K_{ij} = \mathcal{L}_{\boldsymbol{\partial}_t - \boldsymbol{\beta}} K_{ij} = \partial_t K_{ij} - \beta^m \partial_m K_{ij} - K_{mj} \partial_i \beta^m - K_{im} \partial_j \beta^m \,. \qquad (3.23)
$$

We obtain

$$
\boxed{
\begin{aligned}
\partial_t K_{ij} &= \beta^m \partial_m K_{ij} + K_{mj} \partial_i \beta^m + K_{im} \partial_j \beta^m - D_i \partial_j \alpha + \alpha \Big\{ \mathcal{R}_{ij} + K_{ij} (K - 2\Theta) - 2 K_{im} K^m{}_j \\
& \quad - \frac{2}{\mathcal{D}-2} \Lambda \gamma_{ij} + D_i \Theta_j + D_j \Theta_i - c_1 (1 + c_2) \gamma_{ij} \Theta - 8\pi \left[S_{ij} - \gamma_{ij} \frac{S - \rho}{\mathcal{D}-2} \right] \Big\} \,. \qquad (3.24)
\end{aligned}
}
$$

4 The BSSN like version of CCZ4

4.1 Preliminaries

The derivation of the BSSN equations makes frequent use of the conformal factor which is a measure for the determinant of the metric. We will quite often need important relations for the determinant of a matrix which we summarize here.

Let us assume that we have a matrix a_{ij} where the indices $i,\ j$ now simply run from 1 to n without making any requirements on n other than $n > 1$, $n \in \mathbb{N}_0$. First, we recall that the cofactor matrix C_{ij} is obtained by striking out row i and column j in a_{ij}, calculating the determinant of the resulting reduced matrix and finally by adjusting by the + or − sign obtained from the usual $\begin{matrix} + & - \\ - & + \end{matrix}$ pattern in the calculation of determinants. The determinant is then given by fixing a row, i.e. setting $i = i_0$ and computing

$$a := \det a_{ij} = \sum_{j=1}^{n} a_{i_0 j} C^{i_0 j} \,. \tag{4.1}$$

We could have chosen any row i_0 or, alternatively, have seleted a column j_0 and then performing the summation over i instead of j. Therefore, Eq. (4.1) is valid generically,

$$a = \sum_{j=1}^{n} a_{ij} C^{ij} = \sum_{i=1}^{n} a_{ij} C^{ij} \,. \tag{4.2}$$

Note that the cofactor matrix element C^{ij} by construction does not contain the original matrix element a_{ij} (as its row and column has been struck out). We can therefore differentiate Eq. (4.2) with respect to a_{ij} which gives us

$$\frac{\partial a}{\partial a_{ij}} = C^{ij} \,. \tag{4.3}$$

Next, we recall that the computation of the inverse of a matrix also uses the cofactor matrix. However, we need to use the adjunct of the cofactor matrix and also divide by the determinant of the original metric. More specifically, we have

$$(a^{-1})_{ij} = \frac{1}{a} C^{ji} \qquad \Rightarrow \qquad C^{ij} = a\,(a^{-1})_{ji} \,. \tag{4.4}$$

We can combine Eqs. (4.3) and (4.4) to obtain

$$\frac{\partial a}{\partial a_{ij}} = a\,(a^{-1})_{ji} \,. \tag{4.5}$$

The metric is symmetric, so that we are allowed to swap the indices on the right-hand side. Any metric, Riemannian or Lorentzian, therefore obeys

$$\frac{\partial g}{\partial g_{\alpha\beta}} = g\, g^{\alpha\beta} \,. \tag{4.6}$$

We find a similar relation for the inverse metric, but now a minus sign appears in the differentiation of its determinant:

$$\frac{\partial g^{-1}}{\partial g^{\alpha\beta}} = \frac{1}{g} g_{\alpha\beta} \qquad \Rightarrow \qquad \frac{\partial g}{\partial g^{\alpha\beta}} = -g\, g_{\alpha\beta} \,. \tag{4.7}$$

4.2 The BSSN variables

We begin our derivation of the BSSN like equations with the variables. We define

$$\begin{aligned}
\chi &:= \gamma^{-1/(\mathcal{D}-1)} \,, & & K := \gamma^{mn} K_{mn} \\
\tilde{\gamma}_{ij} &:= \chi \gamma_{ij} & \Leftrightarrow \quad & \tilde{\gamma}^{ij} = \frac{1}{\chi} \gamma^{ij} \\
\tilde{A}_{ij} &:= \chi \left(K_{ij} - \frac{1}{\mathcal{D}-1} \gamma_{ij} K \right) & \Leftrightarrow \quad & K_{ij} = \frac{1}{\chi} \left(\tilde{A}_{ij} + \frac{1}{\mathcal{D}-1} \tilde{\gamma}_{ij} K \right) \\
\tilde{\Gamma}^i &:= \tilde{\gamma}^{mn} \tilde{\Gamma}^i_{mn} \,, & & \tilde{\Gamma}^i_{mn} := \frac{1}{2} \tilde{\gamma}^{il} (\partial_m \tilde{\gamma}_{nl} + \partial_n \tilde{\gamma}_{lm} - \partial_l \tilde{\gamma}_{mn}) \,.
\end{aligned} \tag{4.8}$$

4.3 The χ equation

We first note that any derivative of χ is given by

$$\begin{aligned}\partial\chi &= \partial\gamma^{-1/(\mathcal{D}-1)} = \frac{-1}{\mathcal{D}-1}\gamma^{-1/(\mathcal{D}-1)-1}\partial\gamma = \frac{-1}{\mathcal{D}-1}\gamma^{-1/(\mathcal{D}-1)-1}\gamma\gamma^{ij}\partial\gamma_{ij} \\ &= \frac{-1}{\mathcal{D}-1}\underbrace{\gamma^{-1/(\mathcal{D}-1)}}_{=\chi}\gamma^{ij}\partial\gamma_{ij} = -\frac{1}{\mathcal{D}-1}\chi\gamma^{ij}\partial\gamma_{ij}\,. \end{aligned} \tag{4.9}$$

We therefore compute

$$\begin{aligned} &\frac{-1}{\mathcal{D}-1}\chi\gamma^{ij}\times \text{Eq. (3.19)} \\ \Rightarrow\quad &\partial_t\chi = \beta^m\partial_m\chi - \frac{1}{\mathcal{D}-1}\chi\left(\delta^i{}_m\partial_i\beta^m + \delta^j{}_m\partial_j\beta^m - 2\alpha K\right) \end{aligned}$$

$$\boxed{\Rightarrow\quad \partial_t\chi = \beta^m\partial_m\chi + \frac{2}{\mathcal{D}-1}\chi(\alpha K - \partial_m\beta^m)\,.} \tag{4.10}$$

4.4 The $\tilde{\gamma}_{ij}$ equation

Again, we note that any derivative of $\tilde{\gamma}_{ij}$ obeys

$$\partial\tilde{\gamma}_{ij} = \partial(\chi\gamma_{ij}) = \gamma_{ij}\partial\chi + \chi\partial\gamma_{ij}\,. \tag{4.11}$$

We therefore take

$$\begin{aligned} &\gamma_{ij}\times\text{Eq. (4.10)} + \chi\times\text{Eq. (3.19)} \\ \Rightarrow\quad &\partial_t\tilde{\gamma}_{ij} = \beta^m\partial_m\tilde{\gamma}_{ij} - \frac{2}{\mathcal{D}-1}\tilde{\gamma}_{ij}\partial_m\beta^m + \tilde{\gamma}_{mj}\partial_i\beta^m + \tilde{\gamma}_{im}\partial_j\beta^m - 2\alpha\underbrace{\left(\chi K_{ij} - \frac{1}{\mathcal{D}-1}\chi\gamma_{ij}K\right)}_{=\tilde{A}_{ij}} \end{aligned}$$

$$\boxed{\Rightarrow\quad \partial_t\tilde{\gamma}_{ij} = \beta^m\partial_m\tilde{\gamma}_{ij} + \tilde{\gamma}_{mj}\partial_i\beta^m + \tilde{\gamma}_{im}\partial_j\beta^m - \frac{2}{\mathcal{D}-1}\tilde{\gamma}_{ij}\partial_m\beta^m - 2\alpha\tilde{A}_{ij}\,.} \tag{4.12}$$

4.5 The K equation

Here we first notice that

$$\begin{aligned} &\gamma^{im}\gamma_{mn} = \delta^i{}_n \qquad\Rightarrow\qquad 0 = \partial(\gamma^{im}\gamma_{mn}) = \gamma_{mn}\partial\gamma^{im} + \gamma^{im}\partial\gamma_{mn}\ \Big|\ \times\gamma^{jn} \\ \Rightarrow\quad &\partial\gamma^{ij} = -\gamma^{im}\gamma^{jn}\partial\gamma_{mn}\,. \end{aligned} \tag{4.13}$$

With this result we obtain

$$\partial K = \partial(\gamma^{ij}K_{ij}) = \gamma^{ij}\partial K_{ij} - \gamma^{im}\gamma^{jn}K_{ij}\partial\gamma_{mn} = \gamma^{ij}\partial K_{ij} - K^{ij}\partial\gamma_{ij}\,, \tag{4.14}$$

and therefore take the linear combination

$$
\begin{aligned}
&\gamma^{ij} \times \text{Eq. (3.24)} - K^{ij} \times \text{Eq. (3.19)} \\
\Rightarrow\quad &\partial_t K = \beta^m \partial_m K + 2\alpha K_{ij}K^{ij} - \gamma^{mn} D_m D_n \alpha + \alpha \Big\{ \mathcal{R} + K(K - 2\Theta) - 2K_{nm}K^{mn} \\
&\qquad -2\frac{\mathcal{D}-1}{\mathcal{D}-2}\Lambda + 2\gamma^{mn} D_m \Theta_n - (\mathcal{D}-1)c_1(1+c_2)\Theta - 8\pi \left[\frac{-1}{\mathcal{D}-2} S + \frac{\mathcal{D}-1}{\mathcal{D}-2}\rho \right] \Big\}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow\quad \partial_t K &= \beta^m \partial_m K - D^m D_m \alpha + \alpha \Big\{ \mathcal{R} + K(K - 2\Theta) - 2\frac{\mathcal{D}-1}{\mathcal{D}-2}\Lambda + 2D^m \Theta_m \\
&\qquad -(\mathcal{D}-1)c_1(1+c_2)\Theta + \frac{8\pi}{\mathcal{D}-2}[S - (\mathcal{D}-1)\rho] \Big\} \qquad (4.15)
\end{aligned}
$$

4.6 The $\tilde{A}_{ij}$ equation

Our starting point is the definition of the traceless conformal extrinsic curvature $\tilde{A}_{ij}$ in Eq. (4.8). From this definition we find

$$
\begin{aligned}
\partial \tilde{A}_{ij} &= K_{ij}\partial\chi + \chi\partial K_{ij} - \frac{1}{\mathcal{D}-1}\left(\gamma_{ij}K\partial\chi + \chi K \partial\gamma_{ij} + \chi\gamma_{ij}\partial K\right) \\
&= A_{ij}\partial\chi + \chi\partial K_{ij} - \frac{1}{\mathcal{D}-1}\left(\chi K \partial\gamma_{ij} + \chi\gamma_{ij}\partial K\right) . \qquad (4.16)
\end{aligned}
$$

This motivates taking the linear combination

$$
A_{ij} \times \text{Eq. (4.10)} + \chi \times \text{Eq. (3.24)} - \frac{1}{\mathcal{D}-1}\chi K \times \text{Eq. (3.19)} - \frac{1}{\mathcal{D}-1}\chi\gamma_{ij} \times \text{Eq. (4.15)} \qquad (4.17)
$$

The resulting calculation is a bit lengthy, but all the cancelations and combinations of terms are rather straightforward. Getting all in place, this eventually leads to

$$
\begin{aligned}
\partial_t \tilde{A}_{ij} &= \beta^m \partial_m \tilde{A}_{ij} + \tilde{A}_{mj}\partial_i \beta^m + \tilde{A}_{im}\partial_j\beta^m - \frac{2}{\mathcal{D}-1}\tilde{A}_{ij}\partial_m\beta^m - 2\alpha\tilde{A}_{im}\tilde{A}^m{}_j + \alpha\tilde{A}_{ij}(K - 2\Theta) \\
&\quad + \chi\left[-D_i D_j \alpha + \alpha\left(\mathcal{R}_{ij} - 8\pi S_{ij} + D_i\Theta_j + D_j\Theta_i\right)\right]^{\text{TF}} \qquad (4.18)
\end{aligned}
$$

4.7 The $\tilde{\Gamma}^i$ equation

We first recall that for the determinant of a metric, Eq. (4.6) gives us

$$
\partial_\gamma g = g\, g^{\alpha\beta}\partial_\gamma g_{\alpha\beta} .
$$

The determinant of the conformal metric $\tilde{\gamma}_{ij} = 1$ by construction, so that

$$
\boxed{\partial_i\tilde{\gamma} = 0 \quad\Rightarrow\quad \tilde{\gamma}\,\tilde{\gamma}^{mn}\partial_i\tilde{\gamma}_{mn} = 0 \quad\Rightarrow\quad \tilde{\Gamma}^m_{im} = \frac{1}{2}\tilde{\gamma}^{mn}\partial_i\tilde{\gamma}_{mn} = 0} . \qquad (4.19)
$$

With this condition, we can write the conformal Christoffel symbols, which are defined in the standard way, as

$$\tilde{\Gamma}^i = \tilde{\gamma}^{mn}\tilde{\Gamma}^i_{mn} = \tilde{\gamma}^{mn}\frac{1}{2}\tilde{\gamma}^{ik}(\partial_m\tilde{\gamma}_{nk} + \partial_n\tilde{\gamma}_{km} - \partial_k\tilde{\gamma}_{mn}) = \tilde{\gamma}^{mn}\tilde{\gamma}^{ik}\partial_m\tilde{\gamma}_{nk}$$

$$\boxed{\tilde{\Gamma}^i = \tilde{\gamma}^{mn}\tilde{\gamma}^{ik}\partial_m\tilde{\gamma}_{nk}} \qquad \Leftrightarrow \qquad \boxed{\tilde{\gamma}_{ij}\tilde{\Gamma}^i = \tilde{\gamma}^{mn}\partial_m\tilde{\gamma}_{nj}}\,. \tag{4.20}$$

Also using

$$\partial\tilde{\gamma}^{mn} = -\tilde{\gamma}^{ml}\tilde{\gamma}^{nr}\partial\tilde{\gamma}_{lr}\,, \tag{4.21}$$

after a few lines of calculations, we find for arbitrary partial derivatives of $\tilde{\Gamma}^i$, that

$$\partial\tilde{\Gamma}^i = \tilde{\gamma}^{ik}\tilde{\gamma}^{mn}\left(\partial_m\partial\tilde{\gamma}_{nk} - \tilde{\gamma}^{lr}\partial_m\tilde{\gamma}_{lk}\,\partial\tilde{\gamma}_{nr} - \tilde{\gamma}^{lr}\partial_m\tilde{\gamma}_{nl}\,\partial\tilde{\gamma}_{kr}\right)\,. \tag{4.22}$$

This motivates deriving the evolution equation for $\tilde{\Gamma}^i$ by taking

$$\tilde{\gamma}^{ik}\tilde{\gamma}^{mn}\left\{\partial_m[\text{Eq. (4.12)}]_{nk} - \tilde{\gamma}^{lr}\partial_m\tilde{\gamma}_{lk}\times[\text{Eq. (4.12)}]_{nr} - \tilde{\gamma}^{lr}\partial_m\tilde{\gamma}_{nl}\times[\text{Eq. (4.12)}]_{kr}\right\}\,. \tag{4.23}$$

This computation is a bit longer but straightforward and, after cancelation of various terms, leads to

$$\begin{aligned}\partial_t\tilde{\Gamma}^i \;=\;& \beta^m\partial_m\tilde{\Gamma}^i + \tilde{\gamma}^{mn}\partial_m\partial_n\beta^i + \frac{\mathcal{D}-3}{\mathcal{D}-1}\tilde{\gamma}^{ik}\partial_k\partial_m\beta^m + \frac{2}{\mathcal{D}-1}\tilde{\Gamma}^i\partial_m\beta^m - 2\tilde{A}^{im}\partial_m\alpha - 2\alpha\tilde{\gamma}^{ik}\tilde{\gamma}^{mn}\partial_m\tilde{A}_{nk}\\ & -\tilde{\Gamma}^m\partial_m\beta^i + 2\alpha\left[\tilde{\gamma}^{ik}\tilde{A}^{mn}\partial_m\tilde{\gamma}_{nk} + \tilde{\Gamma}^l\tilde{A}^i{}_l\right]\,.\end{aligned} \tag{4.24}$$

In the BSSN system, we would next use the momentum constraint to modify this equation; in the CCZ4 system, the analogous procedure is to combine it with the evolution equation (3.17),

$$\partial_t\Theta_i = \beta^m\partial_m\Theta_i + \Theta_m\partial_i\beta^m + \alpha\left(D_mK^m{}_i - D_iK - 2K_{im}\Theta^m + D_i\Theta - \Theta a_i - c_1\Theta_i - 8\pi j_i\right)\,. \tag{4.25}$$

Our situation is complicated a bit by the fact that we really need the version of this equation for upstairs index i. Using our standard approach, we obtain this from

$$\partial\Theta^i = \partial(\gamma^{ij}\Theta_j) = -\Theta_j\gamma^{im}\gamma^{jn}\partial\gamma_{mn} + \gamma^{ij}\partial\Theta_j\,, \tag{4.26}$$

and therefore compute

$$-\Theta_j\gamma^{im}\gamma^{jn}\times[\text{Eq. (3.19)}]_{mn} + \gamma^{ij}\times[\text{Eq. (4.25)}]_j \tag{4.27}$$

This gives us

$$\partial_t\Theta^i = \beta^m\partial_m\Theta^i - \Theta^m\partial_m\beta^i + \alpha\left[D_mK^{mi} - D^iK + D^i\Theta - \Theta\frac{\partial^i\alpha}{\alpha} - c_1\Theta^i - 8\pi j^i\right]\,. \tag{4.28}$$

In order to combine this equation with (4.24), however, we still need to convert it to the conformal version. For this purpose, we first recall that

$$\Gamma^i_{jk} = \frac{1}{2}\gamma^{im}(\partial_j\gamma_{km} + \partial_k\gamma_{mj} - \partial_m\gamma_{jk})\,, \qquad \tilde{\Gamma}^i_{jk} = \frac{1}{2}\tilde{\gamma}^{im}(\partial_j\tilde{\gamma}_{km} + \partial_k\tilde{\gamma}_{mj} - \partial_m\tilde{\gamma}_{jk})\,, \qquad \tilde{\gamma}_{ij} = \chi\gamma_{ij}$$

$$\Rightarrow \qquad \Gamma^i_{jk} = \tilde{\Gamma}^i_{jk} - \frac{1}{2\chi}\left(\delta^i{}_k\partial_j\chi + \delta^i{}_j\partial_k\chi - \tilde{\gamma}_{jk}\tilde{\gamma}^{im}\partial_m\chi\right) . \tag{4.29}$$

We then find after a few lines of computation,

$$\begin{aligned}
D_jK_{mn} &= D_j\left(A_{mn} + \frac{1}{\mathcal{D}-1}\gamma_{mn}K\right) = D_jA_{mn} + \frac{1}{\mathcal{D}-1}\gamma_{mn}D_jK \qquad \Big| \quad \tilde{A}_{ij} = \chi A_{ij} \\
&= \partial_j\left(\frac{1}{\chi}\tilde{A}_{mn}\right) - \Gamma^l_{mj}A_{ln} - \Gamma^l_{nj}A_{ml} + \frac{1}{\mathcal{D}-1}\gamma_{mn}\partial_jK \\
&= -\frac{1}{\chi^2}\partial_j\chi\tilde{A}_{mn} + \frac{1}{\chi}\tilde{D}_j\tilde{A}_{mn} + \frac{1}{2\chi^2}\Big[\tilde{A}_{nj}\partial_m\chi + \tilde{A}_{mn}\partial_j\chi - \tilde{A}^k{}_n\tilde{\gamma}_{mj}\partial_k\chi + \tilde{A}_{mj}\partial_n\chi + \tilde{A}_{mn}\partial_j\chi \\
&\quad -\tilde{A}_m{}^k\tilde{\gamma}_{nj}\partial_k\chi\Big] + \frac{1}{\mathcal{D}-1}\gamma_{mn}\partial_jK \\
\Rightarrow D_mK^{mi} &= \gamma^{im}\gamma^{jn}D_jK_{mn} = \chi^2\tilde{\gamma}^{im}\tilde{\gamma}^{jn}D_jK_{mn} \\
\Rightarrow \ldots \Rightarrow D_mK^{mi} &= \chi\tilde{\gamma}^{im}\tilde{\gamma}^{jn}\tilde{D}_j\tilde{A}_{mn} - \frac{\mathcal{D}-1}{2}\tilde{A}^{ij}\partial_j\chi + \frac{1}{\mathcal{D}-1}\chi\tilde{\gamma}^{im}\partial_mK \,. \qquad (4.30)
\end{aligned}$$

We will need the covariant derivative $\tilde{D}_j\tilde{A}_{mn}$ in the form [using $\tilde{\gamma}^{mn}\partial_i\tilde{\gamma}_{mn} = 0$ from Eq. (4.19)]

$$\begin{aligned}
2\alpha\tilde{\gamma}^{im}\tilde{\gamma}^{jn}\tilde{D}_j\tilde{A}_{mn} &= 2\alpha\tilde{\gamma}^{ik}\tilde{\gamma}^{mn}\tilde{D}_m\tilde{A}_{nk} = 2\alpha\tilde{\gamma}^{ik}\tilde{\gamma}^{mn}(\partial_m\tilde{A}_{nk} - \tilde{\Gamma}^l_{nm}\tilde{A}_{lk} - \tilde{\Gamma}^l_{km}\tilde{A}_{nl}) \\
\Rightarrow \ldots \Rightarrow \quad 2\alpha\tilde{\gamma}^{im}\tilde{\gamma}^{jn}\tilde{D}_j\tilde{A}_{mn} - 2\alpha\tilde{\gamma}^{ik}\tilde{\gamma}^{mn}\partial_m\tilde{A}_{nk} &= -2\alpha\left[\tilde{A}^{ik}\tilde{\gamma}^{mn}\partial_m\tilde{\gamma}_{nk} + \frac{1}{2}\tilde{A}^{mn}\tilde{\gamma}^{ik}\partial_k\tilde{\gamma}_{mn}\right] .
\end{aligned} \tag{4.31}$$

Next, we can substitute Eq. (4.30) in Eq. (4.28) which gives us

$$\boxed{\partial_t\Theta^i = \beta^m\partial_m\Theta^i - \Theta^m\partial_m\beta^i + \alpha\left[\chi\tilde{\gamma}^{im}\tilde{\gamma}^{jn}\tilde{D}_j\tilde{A}_{mn} - \frac{\mathcal{D}-1}{2}\tilde{A}^{im}\partial_m\chi - \frac{\mathcal{D}-2}{\mathcal{D}-1}\partial^iK + \partial^i\Theta - c_1\Theta^i - 8\pi j^i\right] - \Theta\partial^i\alpha} . \tag{4.32}$$

Combining this equation with (4.10), (4.24) and using (4.31) enables us to determine the evolution of the variable

$$\hat{\Gamma}^i := \tilde{\Gamma}^i + 2\tilde{\gamma}^{ij}\Theta_j = \tilde{\Gamma}^i + \frac{2}{\chi}\Theta^i \,. \tag{4.33}$$

This gives us

$$\begin{aligned}
\partial_t\hat{\Gamma}^i &= \beta^m\partial_m\hat{\Gamma}^i + \tilde{\gamma}^{mn}\partial_m\partial_n\beta^i + \frac{\mathcal{D}-3}{\mathcal{D}-1}\tilde{\gamma}^{ik}\partial_k\partial_m\beta^m + \frac{2}{\mathcal{D}-1}\hat{\Gamma}^i\partial_m\beta^m - 2\tilde{A}^{im}\partial_m\alpha - 2\tilde{\gamma}^{im}\Theta\partial_m\alpha \\
&\quad -2\alpha\tilde{A}^{ik}\tilde{\gamma}^{mn}\partial_m\tilde{\gamma}_{nk} - \alpha\tilde{A}^{mn}\tilde{\gamma}^{ik}\partial_k\tilde{\gamma}_{mn} - \hat{\Gamma}^m\partial_m\beta^i + 2\alpha\tilde{\gamma}^{ik}\tilde{A}^{ml}\partial_m\tilde{\gamma}_{lk} + 2\alpha\tilde{\gamma}_{lm}\tilde{\Gamma}^m\tilde{A}^{il} - \frac{\mathcal{D}-1}{\chi}\alpha\tilde{A}^{im}\partial_m\chi \\
&\quad +2\alpha\tilde{\gamma}^{im}(\partial_m\Theta - \partial_mK) - 2\frac{\alpha}{\chi}c_1\Theta^i - 16\pi\alpha\tilde{\gamma}^{im}j_m - \frac{4}{\mathcal{D}-1}\frac{1}{\chi}\Theta^i\alpha K + \frac{2\alpha}{\mathcal{D}-1}\tilde{\gamma}^{im}\partial_mK \,.
\end{aligned} \tag{4.34}$$

One final simplification for the marked terms is obtained from (using $\tilde{\Gamma}^s = \tilde{\gamma}^{mn}\tilde{\gamma}^{sk}\partial_m\tilde{\gamma}_{nk}$)

$$\begin{aligned} &-2\alpha\tilde{A}^{ik}\tilde{\gamma}^{mn}\partial_m\tilde{\gamma}_{nk} - \alpha\tilde{A}^{mn}\tilde{\gamma}^{ik}\partial_k\tilde{\gamma}_{mn} + 2\alpha\tilde{\gamma}^{ik}\tilde{A}^{ml}\partial_m\tilde{\gamma}_{kl} + 2\alpha\tilde{\gamma}_{ls}\tilde{\Gamma}^s\tilde{A}^{il} \\ = \; & 2\alpha\tilde{A}^{mn}\frac{1}{2}\tilde{\gamma}^{ik}(2\partial_m\tilde{\gamma}_{nk} - \partial_k\tilde{\gamma}_{mn}) = 2\alpha\tilde{A}^{mn}\tilde{\Gamma}^i_{mn}\,, \end{aligned} \tag{4.35}$$

where the first and last term on the first row simply cancel by (4.20) and we renamed a few indices in the other terms. This enables us to rewrite (4.34) as

$$\begin{aligned} \partial_t\hat{\Gamma} \;=\; & \beta^m\partial_m\hat{\Gamma}^i + \tilde{\gamma}^{mn}\partial_m\partial_n\beta^i + \frac{\mathcal{D}-3}{\mathcal{D}-1}\tilde{\gamma}^{ik}\partial_k\partial_m\beta^m + \frac{2}{\mathcal{D}-1}\hat{\Gamma}^i\partial_m\beta^m - 2\partial_m\alpha(\tilde{A}^{im} + \Theta\tilde{\gamma}^{im}) - \hat{\Gamma}^m\partial_m\beta^i \\ & + 2\alpha\tilde{A}^{mn}\tilde{\Gamma}^i_{mn} - (\mathcal{D}-1)\alpha\tilde{A}^{im}\frac{\partial_m\chi}{\chi} + 2\alpha\tilde{\gamma}^{im}\left(\partial_m\Theta - \frac{\mathcal{D}-2}{\mathcal{D}-1}\partial_m K\right) - 2\frac{\alpha}{\chi}c_1\Theta^i - \frac{4}{\mathcal{D}-1}\frac{\alpha}{\chi}K\Theta^i \\ & - 16\pi\alpha\tilde{\gamma}^{im}j_m\,. \end{aligned} \tag{4.36}$$

This is not quite Eq. (19) of Ref. [1] since they employ the freedom to add some function of Θ and Θ^i to the right-hand side which merely adds 0 for Einstein's general relativity. More specifically, their Eq. (19) is obtained by adding to (4.36) the term

$$X^i_1 = 2(\kappa_3 - 1)\left(\frac{2}{\mathcal{D}-1}\frac{\Theta^i}{\chi}\partial_k\beta^k - \frac{\Theta^k}{\chi}\partial_k\beta^i\right)\,. \tag{4.37}$$

Together with terms we have absorbed in some places in $\hat{\Gamma}^i$ in Eq. (4.36), these terms lead to the overall term $2\kappa_3(\ldots)$ in Eq. (19) of [1]. It is also interesting to compare our Eq. (4.36) with the standard version used in the BSSN system as for example given in Eq. (64) of Ref. [2]. We indeed recover that equation if we set $\Theta = 0$, $\Theta^i = 0$ in our Eq. (4.36) which automatically implies $\hat{\Gamma}^i = \tilde{\Gamma}^i$. Interestingly, however, a modification also needs to be applied to the BSSN equation for $\tilde{\Gamma}^i$ in order to achieve long-term stable evolutions. In our own BSSN implementation, we follow [5] for this purpose and add to the right-hand side of the evolution of $\tilde{\Gamma}^i$ a term

$$X^i_2 = -\left(\sigma + \frac{2}{\mathcal{D}-1}\right)\left(\tilde{\Gamma}^i - \tilde{\gamma}^{mn}\tilde{\Gamma}^i_{mn}\right)\partial_k\beta^k\,. \tag{4.38}$$

Now we can combine the two additions (4.37) and (4.38), defining

$$\tilde{\sigma} := \frac{2(\kappa_3 - 1)}{\mathcal{D}-1} = -\left(\sigma + \frac{2}{\mathcal{D}-1}\right)\,. \tag{4.39}$$

This gives us after a little manipulation

$$X^i_1 + X^i_2 = \tilde{\sigma}\left(\hat{\Gamma}^i - \tilde{\gamma}^{mn}\tilde{\Gamma}^i_{mn}\right)\partial_k\beta^k - (\mathcal{D}-1)\tilde{\sigma}\frac{\Theta^k}{\chi}\partial_k\beta^i\,. \tag{4.40}$$

Alic et al. [1] note that they need $\kappa_3 \neq 1$ for their black-hole evolutions (in practice they use $\kappa_3 < 1$ which corresponds to $\tilde{\sigma} < 0$ and, I believe, is necessary for constraint damping). At this stage, we would therefore propose to evolve $\hat{\Gamma}^i$ to the following equation,

$$\begin{aligned}\partial_t\hat{\Gamma} &= \beta^m\partial_m\hat{\Gamma}^i + \tilde{\gamma}^{mn}\partial_m\partial_n\beta^i + \frac{\mathcal{D}-3}{\mathcal{D}-1}\tilde{\gamma}^{ik}\partial_k\partial_m\beta^m + \frac{2}{\mathcal{D}-1}\hat{\Gamma}^i\partial_m\beta^m - 2\partial_m\alpha(\tilde{A}^{im}+\Theta\tilde{\gamma}^{im}) - \hat{\Gamma}^m\partial_m\beta^i \\ &\quad +2\alpha\tilde{A}^{mn}\tilde{\Gamma}^i_{mn} - (\mathcal{D}-1)\alpha\tilde{A}^{im}\frac{\partial_m\chi}{\chi} + 2\alpha\tilde{\gamma}^{im}\left(\partial_m\Theta - \frac{\mathcal{D}-2}{\mathcal{D}-1}\partial_m K\right) - 2\frac{\alpha}{\chi}c_1\Theta^i - \frac{4}{\mathcal{D}-1}\frac{\alpha}{\chi}K\Theta^i \\ &\quad -16\pi\alpha\tilde{\gamma}^{im}j_m + \tilde{\sigma}\left(\hat{\Gamma}^i - \tilde{\gamma}^{mn}\tilde{\Gamma}^i_{mn}\right)\partial_k\beta^k - (\mathcal{D}-1)\tilde{\sigma}\frac{\Theta^k}{\chi}\partial_k\beta^i\,. \end{aligned} \tag{4.41}$$

In this form, the equation is still a bit misleading, since Θ^i is not an evolution variable, but instead is obtained from $\hat{\Gamma}^i$ according to

$$\Theta^i = \frac{\chi}{2}\left(\hat{\Gamma}^i - \tilde{\gamma}^{mn}\tilde{\Gamma}^i_{mn}\right)\,. \tag{4.42}$$

It is now a matter of taste to leave Eq. (4.41) as it is or to replace all terms of Θ^i in terms of (4.42) or, conversely, replace the conformal Christoffel symbols in terms of Θ^i. We follow the latter approach; the only term affected is the last constraint damping term (4.40) which shortens a bit,

$$\begin{aligned}\partial_t\hat{\Gamma}^i &= \beta^m\partial_m\hat{\Gamma}^i + \tilde{\gamma}^{mn}\partial_m\partial_n\beta^i + \frac{\mathcal{D}-3}{\mathcal{D}-1}\tilde{\gamma}^{ik}\partial_k\partial_m\beta^m + \frac{2}{\mathcal{D}-1}\hat{\Gamma}^i\partial_m\beta^m - 2\partial_m\alpha(\tilde{A}^{im}+\Theta\tilde{\gamma}^{im}) \\ &\quad -\hat{\Gamma}^m\partial_m\beta^i + 2\alpha\tilde{A}^{mn}\tilde{\Gamma}^i_{mn} - (\mathcal{D}-1)\alpha\tilde{A}^{im}\frac{\partial_m\chi}{\chi} + 2\alpha\tilde{\gamma}^{im}\left(\partial_m\Theta - \frac{\mathcal{D}-2}{\mathcal{D}-1}\partial_m K\right) - 2\frac{\alpha}{\chi}c_1\Theta^i \\ &\quad -\frac{4}{\mathcal{D}-1}\frac{\alpha}{\chi}K\Theta^i - 16\pi\alpha\tilde{\gamma}^{im}j_m + \tilde{\sigma}\frac{2}{\chi}\Theta^i\partial_k\beta^k - (\mathcal{D}-1)\tilde{\sigma}\frac{\Theta^k}{\chi}\partial_k\beta^i\,. \end{aligned} \tag{4.43}$$

We may have to explore a bit to see how exactly $\hat{\Gamma}^i$ is best evolved for stability and accuracy.

4.8 The equation for Θ

The evolution of Θ is governed by Eq. (3.11) and needs relatively minor changes for obtaining a conformal version. We obtain, using the relations

$$K^{mn}K_{mn} = \tilde{A}^{mn}\tilde{A}_{mn} + \frac{1}{\mathcal{D}-1}K^2\,, \qquad \gamma^{ij}\Theta_i\frac{\partial_j\alpha}{\alpha} = \Theta^j\frac{\partial_j\alpha}{\alpha}\,, \tag{4.44}$$

$$\begin{aligned}\Rightarrow\quad \partial_t\Theta &= \beta^m\partial_m\Theta + \frac{\alpha}{2}\left\{\mathcal{R} + 2D_m\Theta^m - \tilde{A}^{mn}\tilde{A}_{mn} + \frac{\mathcal{D}-2}{\mathcal{D}-1}K^2 - 2K\Theta - 2\Lambda\right. \\ &\quad \left. -2\Theta^m\frac{\partial_m\alpha}{\alpha} - c_1[\mathcal{D}+c_2(\mathcal{D}-2)]\Theta - 16\pi\rho\right\}\,. \end{aligned} \tag{4.45}$$

4.9 Auxiliary expressions and summary

In summary, we have so far derived a conformal version of the Z4 equations that govern the time evolution of χ, $\tilde{\gamma}_{ij}$, K, $\tilde{A}_{ij}$ and $\hat{\Gamma}^i$ according to Eqs. (4.10), (4.12), (4.15), (4.18), (4.43). There remain a few issues, however, that we still have to discuss. First, these equations still contain the variable Θ^i which we do *not* evolve explicitly, but rather derive from $\hat{\Gamma}^i$ and $\tilde{\gamma}_{ij}$ according to Eq. (4.42). This is fine as long as Θ^i appears in undifferentiated form, but terms like $D_i\Theta_j$ are not obvious to handle in this approach. A second issue is that we still need expressions for several auxiliary quantities, such as $D_iD_j\alpha$ or the Ricci tensor $\mathcal{R}_{ij}$. Finally, we will need in these expressions the covariant derivative of χ. Since χ is a tensor density, care needs to be taken when and when not to treat it as if it were a scalar. Let us handle this last issue first.

A tensor density of weight W is defined as an object that aquires under a coordinate transformation an extra factor of the Wth power of the Jacobian. E.g. $\mathcal{T}^\mu{}_\nu$ is a tensor density of weight W if it transforms under $x^\mu \to \bar{x}^\alpha$ according to

$$\mathcal{T}^\mu{}_\nu = \left(\det\left[\frac{\partial \bar{x}^\alpha}{\partial x^\mu}\right]\right)^W \frac{\partial x^\mu}{\partial \bar{x}^\alpha}\frac{\partial \bar{x}^\beta}{\partial x_\nu}\bar{\mathcal{T}}^\alpha{}_\beta\,. \tag{4.46}$$

A particular example is the determinant of the metric $g = \det g_{\mu\nu}$ whose transformation is easiest obtained by regarding the transformation

$$g_{\mu\nu} = \frac{\partial \bar{x}^\alpha}{\partial x^\mu}\bar{g}_{\alpha\beta}\frac{\partial \bar{x}^\beta}{\partial x^\nu} \tag{4.47}$$

as a matrix multiplication $g_{\mu\nu} = J^\alpha{}_\mu \bar{g}_{\alpha\beta} J^\beta{}_\nu$ with $J^\alpha{}_\mu := \partial\bar{x}^\alpha/\partial x^\mu$. The determinant is then simply the product of the determinants of the individual matrices, i.e.

$$g := \det g_{\mu\nu} = \left(\det\left[\frac{\partial \bar{x}^\alpha}{\partial x^\mu}\right]\right)^2 \bar{g}\,, \tag{4.48}$$

and by comparison with Eq. (4.46) we see that $(\det g_{\mu\nu})$ is a scalar density of weight $W = 2$.

This important result enables us to convert a tensor density of weight W into a tensor by dividing by $\sqrt{-g}^W$ (the minus sign applies for Lorentzian metrics which are our main concern here), i.e.

$$T^\mu{}_\nu := \sqrt{-g}^{-W}\mathcal{T}^\mu{}_\nu\,, \tag{4.49}$$

is a proper tensor. This allows us to determine the covariant derivative of a tensor density from the know expression for a tensor and using Leibniz rule for the additional factors of $\sqrt{-g}$.

First, we use Eq. (4.6) and $\Gamma^\mu_{\rho\mu} = (1/2)g^{\mu\nu}\partial_\rho g_{\mu\nu}$ to find

$$\partial_\rho\sqrt{-g}^W = \partial_\rho(-g)^{W/2} = -\frac{W}{2}(-g)^{W/2-1}\frac{\partial g}{\partial g_{\mu\nu}}\partial_\rho g_{\mu\nu} = -\frac{W}{2}(-g)^{W/2-1}g\,g^{\mu\nu}\partial_\rho g_{\mu\nu} = W\sqrt{-g}^W\Gamma^\mu_{\rho\mu}\,, \tag{4.50}$$

For the covariant derivative of our above $\mathcal{T}^\mu{}_\nu$ we then find

$$\begin{aligned}
&\nabla_\rho T^\mu{}_\nu = \partial_\rho T^\mu{}_\nu + \Gamma^\mu_{\sigma\rho}T^\sigma{}_\nu - \Gamma^\sigma_{\nu\rho}T^\mu{}_\sigma \qquad \Big| \quad \times\sqrt{-g}^W\\
\Rightarrow\quad &\sqrt{-g}^W\nabla_\rho T^\mu{}_\nu = \partial_\rho(\sqrt{-g}^W T^\mu{}_\nu) - T^\mu{}_\nu\partial_\rho(\sqrt{-g}^W) + \Gamma^\mu_{\sigma\rho}\mathcal{T}^\sigma{}_\nu - \Gamma^\sigma_{\nu\rho}\mathcal{T}^\mu_\sigma\\
\Rightarrow\quad &\nabla_\rho(T^\mu{}_\nu\sqrt{-g}^W) \overset{!}{=} \nabla_\rho\mathcal{T}^\mu{}_\nu = \partial_\rho\mathcal{T}^\mu{}_\nu + \Gamma^\mu_{\sigma\rho}\mathcal{T}^\sigma{}_\nu - \Gamma^\sigma_{\nu\rho}\mathcal{T}^\mu_\sigma - W\Gamma^\sigma_{\sigma\rho}\mathcal{T}^\mu{}_\nu\,,
\end{aligned} \tag{4.51}$$

where we pulled the $\sqrt{-g}^W$ into the covariant derivative because it is clearly a function of the metric components $g_{\mu\nu}$ and for the Levi-Civitia connection $\nabla_\rho g_{\mu\nu} = 0$ and, hence, $\nabla_\rho g = 0$. The derivation leading up to Eq. (4.51) applies in the same way to tensors of other rank, so the density nature implies adding a term $-W\Gamma^{\sigma}_{\sigma\rho}\mathcal{T}^{\alpha\ldots}{}_{\beta\ldots}$ for arbitrary tensor densities of weight W. It seems that this rule (4.51) is also used when the connection is not Levi-Civita, but I suspect that in such a case it may be part of the definition of the covariant derivative rather than a consequence of more fundamental principles; after all, for arbitrary connections, there is no relation between the metric and the connection.

But let us return to our original question, the covariant derivative of the variable χ. From the viewpoint of the $\mathcal{D}-1$ dimensional spatial submanifold, $\chi = \gamma^{-1/(\mathcal{D}-1)}$ is clearly a tensor density of weight $-2/(\mathcal{D}-1)$, so that

$$D_i\chi = \partial_i\chi + \frac{2}{\mathcal{D}-1}\Gamma^m_{mi} \neq \partial_i\chi\,. \tag{4.52}$$

Actually, a short calculation shows that, as expected,

$$\begin{aligned} D_i\chi &= \partial_i\chi + \frac{2}{\mathcal{D}-1}\left[\tilde{\Gamma}^m_{mi}\,\chi - \frac{\chi}{2\chi}(\delta^m{}_i\partial_m\chi + \delta^m{}_m\partial_i\chi - \tilde{\gamma}_{im}\tilde{\gamma}^{mn}\partial_n\chi)\right] \\ &= \partial_i\chi + \frac{2}{\mathcal{D}-1}\left[\tilde{\Gamma}^m_{im}\,\chi - \frac{\mathcal{D}-1}{2}\partial_i\chi\right] = 0\,. \end{aligned} \tag{4.53}$$

For the confirmal spatial metric, however, we know from Eq. (4.19) that $\det\tilde{\gamma}_{ij} = 1 = \text{const}$ and, hence, $\tilde{\Gamma}^m_{mi} = 0$ and therefore

$$\tilde{D}_i\chi = \partial_i\chi - \frac{2}{\mathcal{D}-1}\tilde{\Gamma}^m_{mi} = \partial_i\chi\,. \tag{4.54}$$

Intuitively, this is not too surprising: by fixing $\tilde{\gamma} = 1$, we have separated χ from the geometry of the conformal spatial submanifold on which it therefore lives akin to a scalar function. The upshot is that

$$\boxed{\tilde{D}_i\chi = \partial_i\chi\,, \qquad \tilde{D}_i\tilde{D}_j\chi = \partial_i\partial_j\chi - \tilde{\Gamma}^m_{ji}\partial_m\chi\,.} \tag{4.55}$$

Next, we consider the Ricci tensor which is a longer exercise. Starting with the definition of the Riemann tensor

$$\mathcal{R}^k{}_{lij} = \partial_i\Gamma^k_{lj} - \partial_j\Gamma^k_{li} + \Gamma^m_{lj}\Gamma^k_{mi} - \Gamma^m_{li}\Gamma^k_{mj}\,, \tag{4.56}$$

we find

$$\mathcal{R}_{ij} = \mathcal{R}^k{}_{ikj} = \partial_k\Gamma^k_{ij} - \partial_j\Gamma^k_{ik} + \Gamma^m_{ij}\Gamma^k_{mk} - \Gamma^m_{ik}\Gamma^k_{mj}\,. \tag{4.57}$$

Using the relation (4.29) between the spatial Christoffel symbols and their conformal counterparts, we directly see that $\mathcal{R}_{ij}$ splits into two contributions,

$$\mathcal{R}_{ij} = \tilde{\mathcal{R}}_{ij} + \mathcal{R}^{\chi}_{ij} \qquad \text{with} \qquad \tilde{R}_{ij} = \partial_k\tilde{\Gamma}^k_{ij} - \partial_j\tilde{\Gamma}^k_{ik} + \tilde{\Gamma}^m_{ij}\tilde{\Gamma}^k_{mk} - \tilde{\Gamma}^m_{ik}\tilde{\Gamma}^k_{mj}\,. \tag{4.58}$$

The term $\mathcal{R}^{\chi}_{ij}$ originates from the $-\frac{1}{2\chi}(\ldots)$ terms in Eq. (4.29) which appear here in differentiated form or in quadratic form. The resulting expression is a bit lengthy, but its bookkeeping is straightforward without using any tricks. After collecting everything, one finds

$$\begin{aligned} \mathcal{R}^{\chi}_{ij} &= \frac{1}{4\chi^2}\left[\partial_i\chi\,\partial_j\chi(-\mathcal{D}+3) + \tilde{\gamma}_{ij}\tilde{\gamma}^{mn}\partial_m\chi\,\partial_n\chi(-\mathcal{D}+1)\right] \\ &\quad + \frac{1}{2\chi}\left[\partial_j\partial_i\chi(\mathcal{D}-3) + \tilde{\gamma}_{ij}\tilde{\gamma}^{mn}\partial_m\partial_n\chi + \tilde{\gamma}_{ij}\,\partial_k\tilde{\gamma}^{kn}\,\partial_n\chi + \partial_k\tilde{\gamma}_{ij}\,\tilde{\gamma}^{kn}\partial_n\chi\right] \\ &\quad + \frac{1}{2\chi}\left[\tilde{\Gamma}^m_{ij}\partial_m\chi(3-\mathcal{D}) + \partial_n\chi\left(\tilde{\gamma}_{ij}\tilde{\gamma}^{mn}\tilde{\Gamma}^k_{mk} - \tilde{\Gamma}_{jim}\tilde{\gamma}^{mn} - \tilde{\Gamma}_{imj}\tilde{\gamma}^{mn}\right)\right]\,. \end{aligned} \tag{4.59}$$

This expression simplifies with the by now familiar $\tilde{\Gamma}^k_{ik} = 0$. For those terms appearing with a factor $\partial_n\chi$ in the last expression, we furthermore find

$$\tilde{\gamma}_{ij}\partial_k\tilde{\gamma}^{kn} + \partial_k\tilde{\gamma}_{ij}\,\tilde{\gamma}^{kn} - \frac{1}{2}\left(\partial_i\tilde{\gamma}_{mj} + \partial_m\tilde{\gamma}_{ji} - \partial_j\tilde{\gamma}_{im}\right)\tilde{\gamma}^{mn} - \frac{1}{2}\left(\partial_m\tilde{\gamma}_{ji} + \partial_j\tilde{\gamma}_{im} - \partial_i\tilde{\gamma}_{mj}\right)\tilde{\gamma}^{mn} = \tilde{\gamma}_{ij}\partial_k\tilde{\gamma}^{kn}\,. \tag{4.60}$$

Finally, we write the second covariant derivative of χ as

$$\begin{aligned}
\frac{1}{2\chi}\tilde{\gamma}_{ij}\tilde{\gamma}^{mn}\tilde{D}_m\tilde{D}_n\chi &= \frac{1}{2\chi}\tilde{\gamma}_{ij}\tilde{\gamma}^{mn}(\partial_m\partial_n\chi - \tilde{\Gamma}^l_{nm}\partial_l\chi) = \frac{1}{2\chi}\tilde{\gamma}_{ij}\tilde{\gamma}^{mn}\left[\partial_m\partial_n\chi - \frac{1}{2}\tilde{\gamma}^{lk}(\partial_n\tilde{\gamma}_{mk} + \partial_m\tilde{\gamma}_{kn} - \underbrace{\partial_k\tilde{\gamma}_{mn}}_{\to 0})\partial_l\chi\right] \\
&= \frac{1}{2\chi}\left[\tilde{\gamma}_{ij}\tilde{\gamma}^{mn}\partial_m\partial_n\chi - \tilde{\gamma}_{ij}\tilde{\gamma}^{kl}\tilde{\gamma}^{mn}\partial_k\tilde{\gamma}_{lm}\,\partial_n\chi\right]\,.
\end{aligned} \tag{4.61}$$

Applying these substitutions in Eq. (4.59), we obtain

$$\boxed{\mathcal{R}^\chi_{ij} = \frac{\mathcal{D}-3}{2\chi}\left[\tilde{D}_i\partial_j\chi - \frac{1}{2\chi}\partial_i\chi\,\partial_j\chi\right] + \frac{\tilde{\gamma}_{ij}}{2\chi}\left[\tilde{\gamma}^{mn}\tilde{D}_m\partial_n\chi - \frac{\mathcal{D}-1}{2\chi}\tilde{\gamma}^{mn}\partial_m\chi\,\partial_n\chi\right]} \tag{4.62}$$

The next expression is $\tilde{\mathcal{R}}_{ij}$ and this one is rather unpleasant to calculate because we have to convert combinations of derivatives of the metric into Christoffel symbols. Our task is mitigated by the fact that we know the result from the BSSN calculation, but we still have to be careful as one simple error screws up the entire matching of terms. Our starting point, the second relation in Eq. (4.58) looks innocent enough, especially after realizing that the second and third term on the right-hand side drop out due to $\tilde{\Gamma}^k_{ik} = 0$, so that

$$\tilde{\mathcal{R}}_{ij} = \partial_k\Gamma^k_{ij} - \tilde{\Gamma}^m_{ik}\tilde{\Gamma}^k_{mj}\,. \tag{4.63}$$

Let us first concentrate on those terms in $\tilde{\mathcal{R}}_{ij}$ that give rise to second derivatives of the metric. We first recall from Eq. (4.20) that

$$\tilde{\Gamma}^m = \tilde{\gamma}^{ms}\tilde{\gamma}^{kl}\partial_k\tilde{\gamma}_{ls}\,, \tag{4.64}$$

which allows us to write

$$\begin{aligned}
& \tilde{\gamma}_{mi}\partial_j\tilde{\Gamma}^m = \tilde{\gamma}_{mi}\partial_j(\tilde{\gamma}^{ms}\tilde{\gamma}^{kl}\partial_k\tilde{\gamma}_{ls}) \\
\Rightarrow\quad & \tilde{\gamma}_{mi}\partial_j\tilde{\Gamma}^m = \tilde{\gamma}_{mi}\tilde{\gamma}^{kl}\partial_j\tilde{\gamma}^{ms}\,\partial_k\tilde{\gamma}_{ls} + \partial_j\tilde{\gamma}^{kl}\,\partial_k\tilde{\gamma}_{li} + \tilde{\gamma}^{kl}\partial_j\partial_k\tilde{\gamma}_{li} \\
\Rightarrow\quad & \tilde{\gamma}^{km}\partial_k\partial_j\tilde{\gamma}_{mi} = \tilde{\gamma}_{mi}\partial_j\tilde{\Gamma}^m - \tilde{\gamma}_{mi}\tilde{\gamma}^{kl}\partial_j\tilde{\gamma}^{ms}\,\partial_k\tilde{\gamma}_{ls} - \partial_j\tilde{\gamma}^{kl}\,\partial_k\tilde{\gamma}_{li}\,.
\end{aligned} \tag{4.65}$$

Defining

$$\tilde{\Gamma}_{mij} := \tilde{\gamma}_{mn}\tilde{\Gamma}^n_{ij} \qquad \Leftrightarrow \qquad \tilde{\Gamma}_{mij} = \frac{1}{2}(\partial_i\tilde{\gamma}_{jm} + \partial_j\tilde{\gamma}_{mi} - \partial_m\tilde{\gamma}_{ij})\,, \tag{4.66}$$

We can use the previous result to express $\tilde{\mathcal{R}}_{ij}$ as

$$\begin{aligned}
\tilde{\mathcal{R}}_{ij} &= \partial_k\tilde{\gamma}^{km}\,\tilde{\Gamma}_{mij} + \frac{1}{2}\tilde{\gamma}^{km}(\partial_k\partial_i\tilde{\gamma}_{jm} + \partial_k\partial_j\tilde{\gamma}_{mi} - \partial_k\partial_m\tilde{\gamma}_{ij}) - \tilde{\Gamma}^m_{ik}\tilde{\Gamma}^k_{mj} \\
&= \partial_k\tilde{\gamma}^{km}\,\tilde{\Gamma}_{mij} - \frac{1}{2}\tilde{\gamma}^{km}\partial_k\partial_m\tilde{\gamma}_{ij} + \frac{1}{2}\tilde{\gamma}_{mi}\partial_j\tilde{\Gamma}^m - \frac{1}{2}\tilde{\gamma}_{mi}\tilde{\gamma}^{kl}\,\partial_j\tilde{\gamma}^{ms}\,\partial_k\tilde{\gamma}_{ls} - \frac{1}{2}\partial_j\tilde{\gamma}^{kl}\,\partial_k\tilde{\gamma}_{li} \\
&\quad + \frac{1}{2}\tilde{\gamma}_{mj}\,\partial_i\tilde{\Gamma}^m - \frac{1}{2}\tilde{\gamma}_{mj}\tilde{\gamma}^{kl}\,\partial_i\tilde{\gamma}^{ms}\,\partial_k\tilde{\gamma}_{ls} - \frac{1}{2}\partial_i\tilde{\gamma}^{kl}\,\partial_k\tilde{\gamma}_{lj} - \tilde{\gamma}^{mn}\tilde{\gamma}^{kl}\tilde{\Gamma}_{nik}\tilde{\Gamma}_{lmj}\,.
\end{aligned} \tag{4.67}$$

From (4.64), we find

$$-\tilde{\gamma}^{kl}\,\partial_i\tilde{\gamma}^{ms}\,\partial_k\tilde{\gamma}_{ls} = \tilde{\gamma}^{kl}\tilde{\gamma}^{mn}\tilde{\gamma}^{sr}\partial_i\tilde{\gamma}_{nr}\,\partial_k\tilde{\gamma}_{ls} = \tilde{\gamma}^{mn}\tilde{\Gamma}^r\,\partial_i\tilde{\gamma}_{nr}\,, \tag{4.68}$$

and, hence,

$$\begin{aligned}\tilde{\mathcal{R}}_{ij} \;=\; & -\frac{1}{2}\tilde{\gamma}^{km}\partial_k\partial_m\tilde{\gamma}_{ij} + \tilde{\gamma}_{m(i}\partial_{j)}\tilde{\Gamma}^m + \partial_k\tilde{\gamma}^{km}\tilde{\Gamma}_{mij} + \frac{1}{2}\tilde{\gamma}_{mi}\tilde{\gamma}^{mn}\tilde{\Gamma}^r\partial_j\tilde{\gamma}_{nr} - \frac{1}{2}\partial_j\tilde{\gamma}^{kl}\,\partial_k\tilde{\gamma}_{li} \\ & + \frac{1}{2}\tilde{\gamma}_{mj}\tilde{\gamma}^{mn}\,\tilde{\Gamma}^r\,\partial_i\tilde{\gamma}_{nr} - \frac{1}{2}\partial_i\tilde{\gamma}^{kl}\,\partial_k\tilde{\gamma}_{lj} - \tilde{\gamma}^{mn}\tilde{\gamma}^{kl}\tilde{\Gamma}_{nik}\tilde{\Gamma}_{lmj}\,. \end{aligned}\tag{4.69}$$

We further manipulate this expression by noticing that

$$\tilde{\Gamma}_{(ij)m} = \frac{1}{4}\left(-\partial_i\tilde{\gamma}_{jm}+\partial_j\tilde{\gamma}_{mi} + \partial_m\tilde{\gamma}_{ij}-\partial_j\tilde{\gamma}_{im}+\partial_i\tilde{\gamma}_{mj} + \partial_m\tilde{\gamma}_{ji}\right) = \frac{1}{2}\partial_m\tilde{\gamma}_{ij}\,, \tag{4.70}$$

$$\partial_k\tilde{\gamma}^{km}\tilde{\Gamma}_{mij} = -\tilde{\gamma}^{kr}\tilde{\gamma}^{ms}\partial_k\tilde{\gamma}_{rs}\,\frac{1}{2}(-\partial_m\tilde{\gamma}_{ij} + \partial_i\tilde{\gamma}_{jm} + \partial_j\tilde{\gamma}_{mi}) = -\frac{1}{2}\tilde{\Gamma}^m(-\partial_m\tilde{\gamma}_{ij} + \partial_i\tilde{\gamma}_{jm} + \partial_j\tilde{\gamma}_{mi})\,. \tag{4.71}$$

Plugging the latter relation into (4.69) cancels a few terms, leaving among the surviving terms one that equals the right-hand side of the former relation. That way we find

$$\tilde{\mathcal{R}}_{ij} = -\frac{1}{2}\tilde{\gamma}^{km}\partial_k\partial_m\tilde{\gamma}_{ij} + \tilde{\gamma}_{m(i}\partial_{j)}\tilde{\Gamma}^m + \tilde{\Gamma}^m\tilde{\Gamma}_{(ij)m} - \frac{1}{2}\partial_j\tilde{\gamma}^{kl}\,\partial_k\tilde{\gamma}_{li} - \frac{1}{2}\partial_i\tilde{\gamma}^{kl}\,\partial_k\tilde{\gamma}_{lj} - \tilde{\gamma}^{mn}\tilde{\gamma}^{kl}\tilde{\Gamma}_{nik}\tilde{\Gamma}_{lmj} \tag{4.72}$$

From Eq. (4.70) we find

$$\begin{aligned} & \tilde{\Gamma}_{lmj} + \tilde{\Gamma}_{mlj} = \partial_j\tilde{\gamma}_{lm} \\ \Rightarrow\quad & -\tilde{\gamma}^{mn}\tilde{\gamma}^{kl}\tilde{\Gamma}_{nik}\tilde{\Gamma}_{lmj} = \tilde{\gamma}^{mn}\tilde{\gamma}^{kl}\tilde{\Gamma}_{nik}(\tilde{\Gamma}_{mlj} - \partial_j\tilde{\gamma}_{lm}) = \tilde{\gamma}^{kl}\tilde{\Gamma}^m_{ik}\tilde{\Gamma}_{mjl} + \tilde{\Gamma}_{nik}\,\partial_j\tilde{\gamma}^{nk}\,, \end{aligned}\tag{4.73}$$

so that

$$\tilde{\mathcal{R}}_{ij} = -\frac{1}{2}\tilde{\gamma}^{km}\partial_k\partial_m\tilde{\gamma}_{ij} + \tilde{\gamma}_{m(i}\partial_{j)}\tilde{\Gamma}^m + \tilde{\Gamma}^m\tilde{\Gamma}_{(ij)m} + \tilde{\gamma}^{kl}\tilde{\Gamma}^m_{ik}\tilde{\Gamma}_{mjl} + \tilde{\Gamma}_{nik}\,\partial_j\tilde{\gamma}^{nk} - \frac{1}{2}\partial_j\tilde{\gamma}^{kl}\,\partial_k\tilde{\gamma}_{li} - \frac{1}{2}\partial_i\tilde{\gamma}^{kl}\,\partial_k\tilde{\gamma}_{lj}\,. \tag{4.74}$$

Finally, we find after a couple of lines of combining and canceling terms that

$$2\tilde{\gamma}^{mn}\tilde{\Gamma}^k_{m(i}\tilde{\Gamma}_{j)kn} = \tilde{\Gamma}_{nik}\,\partial_j\tilde{\gamma}^{nk} - \frac{1}{2}\partial_j\tilde{\gamma}^{kl}\,\partial_k\tilde{\gamma}_{li} - \frac{1}{2}\partial_i\tilde{\gamma}^{kl}\,\partial_k\tilde{\gamma}_{lj}\,. \tag{4.75}$$

Plugging this into the previous result gives us the final expression

$$\boxed{\tilde{\mathcal{R}}_{ij} = -\frac{1}{2}\tilde{\gamma}^{km}\partial_k\partial_m\tilde{\gamma}_{ij} + \tilde{\gamma}_{m(i}\partial_{j)}\tilde{\Gamma}^m + \tilde{\Gamma}^m\tilde{\Gamma}_{(ij)m} + \tilde{\gamma}^{mn}\left[2\tilde{\Gamma}^k_{m(i}\tilde{\Gamma}_{j)kn} + \tilde{\Gamma}^k_{im}\tilde{\Gamma}_{kjn}\right]\,.} \tag{4.76}$$

The next item on our todo list is to consider the derivatives of Θ_i that appear in the evolution equations (4.45) and (4.18). Note, however, that these derivatives appear exclusively in the form

$$\mathcal{R}_{ij} + D_i\Theta_j + D_j\Theta_i\,, \tag{4.77}$$

or a contraction thereof with $\tilde{\gamma}^{ij}$. This allows us to trade these derivatives for replacing $\tilde{\Gamma}^i$ with $\hat{\Gamma}^i$ in a few places. More specifically, recalling

$$\Theta_j = \gamma_{jn}\Theta^n = \frac{1}{\chi}\tilde{\gamma}_{jn}\Theta^n\,, \qquad \hat{\Gamma}^m = \tilde{\Gamma}^m + \frac{2}{\chi}\Theta^m\,, \qquad \tilde{\gamma}_{ij} = \chi\gamma_{ij}\,,$$

we find

$$\begin{aligned}
D_i\Theta_j &= \partial_i\Theta_j - \Gamma^m_{ji}\Theta_m = \partial_i\left(\frac{1}{\chi}\tilde{\gamma}_{jn}\Theta^n\right) - \Gamma^m_{ji}\Theta_m = \frac{1}{\chi}\Theta^n\partial_i\tilde{\gamma}_{jn} + \tilde{\gamma}_{jn}\partial_i\left(\frac{\Theta^n}{\chi}\right) - \Gamma^m_{ji}\Theta_m \\
&= \frac{1}{\chi}\Theta^n\partial_i\tilde{\gamma}_{nj} + \tilde{\gamma}_{nj}\partial_i\left(\frac{\Theta^n}{\chi}\right) - \tilde{\Gamma}^m_{ji}\Theta_m + \frac{1}{2\chi}\left(\delta^m{}_j\partial_i\chi + \delta^m{}_i\partial_j\chi - \tilde{\gamma}_{ji}\tilde{\gamma}^{mn}\partial_n\chi\right)\Theta_m \\
&= \ldots \\
&= \underset{\sim\sim\sim}{\frac{1}{\chi}\Theta^n\partial_i\tilde{\gamma}_{jn}} + \frac{1}{2}\tilde{\gamma}_{jm}\partial_i(\hat{\Gamma}^m - \tilde{\Gamma}^m) \underset{\sim\sim\sim}{- \frac{1}{2\chi}\Theta^n\partial_j\tilde{\gamma}_{in}} \underset{\sim\sim\sim}{- \frac{1}{2\chi}\Theta^n\partial_i\tilde{\gamma}_{nj}} + \frac{1}{2\chi}\Theta^n\partial_n\tilde{\gamma}_{ij} + \frac{1}{2\chi^2}\tilde{\gamma}_{jn}\Theta^n\partial_i\chi \\
&\quad + \frac{1}{2\chi^2}\tilde{\gamma}_{in}\Theta^n\partial_j\chi - \frac{1}{2\chi^2}\tilde{\gamma}_{ji}\Theta^n\partial_n\chi\,.
\end{aligned} \tag{4.78}$$

Symmetrizing over ij eliminates all the marked terms and combines some of the other ones, so that we find

$$D_i\Theta_j + D_j\Theta_i = \tilde{\gamma}_{m(j}\partial_{i)}(\hat{\Gamma}^m - \tilde{\Gamma}^m) + \mathcal{R}^Z_{ij}\,, \tag{4.79}$$

where we have defined

$$\boxed{\mathcal{R}^Z_{ij} = \frac{\Theta^n}{\chi^2}\left[\chi\partial_n\tilde{\gamma}_{ij} + \tilde{\gamma}_{jn}\partial_i\chi + \tilde{\gamma}_{in}\partial_j\chi - \tilde{\gamma}_{ij}\partial_n\chi\right].} \tag{4.80}$$

With Eq. (4.79) and the expressions (4.62), (4.76) and (4.80) for the Ricci tensor we are now able to write

$$\mathcal{R}_{ij} + 2D_{(i}\Theta_{j)} = \tilde{\mathcal{R}}_{ij} + \mathcal{R}^\chi_{ij} + \tilde{\gamma}_{m(i}\partial_{j)}(\hat{\Gamma}^m - \tilde{\Gamma}^m) + \mathcal{R}^Z_{ij}\,. \tag{4.81}$$

This motivates replacing the derivative $\partial\tilde{\Gamma}^m$ in Eq. (4.76) with $\partial\hat{\Gamma}^m$, so that we obtain the equivalent set of equations

$$\boxed{\begin{aligned}
&\mathcal{R}_{ij} + 2D_{(i}\Theta_{j)} = \hat{\mathcal{R}}_{ij} + \mathcal{R}^\chi_{ij} + \mathcal{R}^Z_{ij}\,, \\
&\hat{\mathcal{R}}_{ij} = -\frac{1}{2}\tilde{\gamma}^{km}\partial_k\partial_m\tilde{\gamma}_{ij} + \tilde{\gamma}_{m(i}\partial_{j)}\hat{\Gamma}^m + \tilde{\Gamma}^m\tilde{\Gamma}_{(ij)m} + \tilde{\gamma}^{mn}\left[2\tilde{\Gamma}^k_{m(i}\tilde{\Gamma}_{j)kn} + \tilde{\Gamma}^k_{im}\tilde{\Gamma}_{kjn}\right], \\
&\mathcal{R}^\chi_{ij} = \frac{\mathcal{D}-3}{2\chi}\left[\tilde{D}_i\partial_j\chi - \frac{1}{2\chi}\partial_i\chi\,\partial_j\chi\right] + \frac{\tilde{\gamma}_{ij}}{2\chi}\left[\tilde{\gamma}^{mn}\tilde{D}_m\partial_n\chi - \frac{\mathcal{D}-1}{2\chi}\tilde{\gamma}^{mn}\partial_m\chi\,\partial_n\chi\right], \\
&\mathcal{R}^Z_{ij} = \frac{\Theta^n}{\chi^2}\left[\chi\partial_n\tilde{\gamma}_{ij} + \tilde{\gamma}_{jn}\partial_i\chi + \tilde{\gamma}_{in}\partial_j\chi - \tilde{\gamma}_{ij}\partial_n\chi\right].
\end{aligned}} \tag{4.82}$$

The latter two equations are (4.62) and (4.80) with no changes and have only been reproduced here in order to have all equations to compute the Ricci tensor in one place.

For the second covariant derivative of the lapse function we find directly with (4.29) that

$$D_i D_j \alpha = \tilde{D}_i \partial_j \alpha + \frac{1}{\chi} \partial_{(i} \chi \partial_{j)} \alpha - \frac{1}{2\chi} \tilde{\gamma}_{ij} \tilde{\gamma}^{mn} \partial_m \chi \partial_n \alpha \,. \tag{4.83}$$

In summary, we evolve the variables χ, $\tilde{\gamma}_{ij}$, K, $\tilde{A}_{ij}$, $\hat{\Gamma}^i$, Θ according to equations (4.10), (4.12), (4.15), (4.43), (4.45) using Eqs. (4.82) to compute the extended Ricci tensor $\mathcal{R}_{ij} + 2D_{(i}\Theta_{j)}$ and (4.83) to compute $D_i D_j \alpha$. The variable Θ^i then only appears in undifferentiated form in the source terms of the evolution equations and is obtained from (4.42).

The ADM constraints in conformal form are now given by setting $\Theta = 0$, $\Theta^i = 0$ in the evolution equations (4.45), (4.32) for Θ and Θ^i. This leads to the familiar BSSN version of the Hamiltonian and momentum constraints

$$\mathcal{H} \coloneqq \mathcal{R} - \tilde{A}^{mn} \tilde{A}_{mn} + \frac{\mathcal{D}-2}{\mathcal{D}-1} K^2 - 2\Lambda - 16\pi\rho = 0 \,, \tag{4.84}$$

$$\mathcal{M}_i \coloneqq \tilde{\gamma}^{mn} \tilde{D}_m \tilde{A}_{ni} - \frac{\mathcal{D}-1}{2\chi} \tilde{A}^m{}_i \partial_m \chi - \frac{\mathcal{D}-2}{\mathcal{D}-1} \partial_i K - 8\pi j_i = 0 \,. \tag{4.85}$$

5 The electromagnetic decomposition of the Weyl tensor

In this section, we will derive in more detail the decomposition of the Weyl tensor into its electric and magnetic part. This derivation is surprisingly involved and we will need to discuss some more generic properties of 2-forms. We follow to some extent in this section the discussion in Stephani's book [4], but note a potentially dangerous typo in Eq. (32.5) therein: The quantity on the left-hand-side should be ${}^{\sim}C^{\sim}_{arsq}$, i.e. without the asterisk.

5.1 2-forms

A 2-form is simply defined as an asymmetric rank $\binom{0}{2}$ tensor,

$$\omega_{\alpha\beta} = -\omega_{\beta\alpha}\,. \tag{5.1}$$

We always assume that we have a metric and therefore will not distinguish between upstairs and downstairs tensors. We may therefore write our 2-form also as $\omega_\alpha{}^\beta = -\omega^\beta{}_\alpha$ or in any other combination of indices.

We now define the dual of a 2-form, which we denote either by a tilde or an operator D,

$$\text{Dual:} \qquad D(\omega)_{\alpha\beta} = \tilde{\omega}_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta}{}^{\mu\nu}\omega_{\mu\nu}\,. \tag{5.2}$$

A 2-form is defined to be self-dual if it satisfies the following relation,

$$\text{Self-dual:} \qquad \omega_{\alpha\beta} = \mathrm{i}\,\tilde{\omega}_{\alpha\beta} = \mathrm{i}\,D(\omega)_{\alpha\beta} = \frac{\mathrm{i}}{2}\epsilon_{\alpha\beta}{}^{\mu\nu}\omega_{\mu\nu}\,. \tag{5.3}$$

Now consider a timelike unit normal vector n^α with $n_\mu n^\mu = -1$. The entire following discussion would also work for spatial unit fields, but signs might be changing in many places, and our practical consideration exclusively concerns timelike unit fields, so that we will focus on this case. Given this field, we can define the projection of our 2-form onto this field,

$$\text{Time projection:} \qquad \varpi_\alpha = n^\mu\omega_{\mu\alpha}\,. \tag{5.4}$$

Our final definition is the "identity operator",

$$\text{Identity operator:} \qquad I_{\alpha\beta\mu\nu} = -(g_{\alpha\mu}g_{\beta\nu} - g_{\beta\mu}g_{\alpha\nu} + \mathrm{i}\epsilon_{\alpha\beta\mu\nu}) \tag{5.5}$$

Note that there are myriads of ways to shift signs in these definitions without altering the mathematics. Stephani, for example, does not have the minus sign in the identity operator but instead introduces it in the definition of the self-dual. These variations in the conventions can be confusing, but are not problematic once one has fixed a convention. We shall use the convention of Eqs. (5.2)-(5.5) and forget about any other possibilties.

The key feature of a self-dual 2-form is that it is completely determined by its time projection. More specifically,

$$\omega_{\alpha\beta} = I_{\alpha\beta\mu\nu}n^\mu\varpi^\nu = I_{\alpha\beta\mu\nu}n^\mu n^\rho\omega_\rho{}^\nu\,. \tag{5.6}$$

We proof this as follows,

$$
\begin{aligned}
I_{\alpha\beta\mu\nu} n^\mu \varpi^\nu &= -(g_{\alpha\mu} g_{\beta\nu} - g_{\beta\mu} g_{\alpha\nu} + \mathrm{i}\epsilon_{\alpha\beta\mu\nu}) n^\mu \varpi^\nu \\
&= -n_\alpha n^\rho \omega_{\rho\beta} + n_\beta n^\rho \omega_{\rho\alpha} - \mathrm{i}\epsilon_{\alpha\beta\mu}{}^{\nu} n^\mu n^\rho \omega_{\rho\nu} \qquad \Big| \; \omega_{\rho\nu} \overset{!}{=} \mathrm{i}\tilde{\omega}_{\rho\nu} = \frac{\mathrm{i}}{2}\epsilon_{\rho\nu}{}^{\gamma\delta}\omega_{\gamma\delta} \\
&= -n_\alpha n^\mu \omega_{\mu\beta} + n_\beta n^\mu \omega_{\mu\alpha} + \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}\epsilon^{\rho\nu\gamma\delta} n^\mu n_\rho \omega_{\gamma\delta} \\
&= -n_\alpha n^\mu \omega_{\mu\beta} + n_\beta n^\mu \omega_{\mu\alpha} + \frac{1}{2}\Big(- \delta_\alpha{}^\gamma \delta_\beta{}^\delta \delta_\mu{}^\rho - \delta_\alpha{}^\delta \delta_\beta{}^\rho \delta_\mu{}^\gamma - \delta_\alpha{}^\rho \delta_\beta{}^\gamma \delta_\mu{}^\delta \\
&\qquad\qquad + \delta_\alpha{}^\gamma \delta_\beta{}^\rho \delta_\mu{}^\delta + \delta_\alpha{}^\delta \delta_\beta{}^\gamma \delta_\mu{}^\rho + \delta_\alpha{}^\rho \delta_\beta{}^\delta \delta_\mu{}^\gamma \Big)\; n^\mu n_\rho \omega_{\gamma\delta} \\
&= -n_\alpha n^\mu \omega_{\mu\beta} + n_\beta n^\mu \omega_{\mu\alpha} + \Big(- n^\rho n_\rho \omega_{\alpha\beta} - n^\gamma n_\beta \omega_{\gamma\alpha} - n^\delta n_\alpha \omega_{\beta\delta} \Big) \\
&= \omega_{\alpha\beta}\,, \qquad (5.7)
\end{aligned}
$$

where the marked terms cancel in the last step.

Note that we have also used that the product of two Levi-Civita tensors can be written entirely in terms of Kronecker deltas. The precise equation depends on the number of indices that are contracted and are given by Eqs. (6.17)-(6.21) in Stephani [4]. It is not a bad idea to have an electronic version of these very useful relations which we provide here. For typing convenience, we switch on this one occasion from Greek to Latin indices and obtain,

$$
\begin{aligned}
\epsilon_{abcd}\epsilon^{efmn} &= -\delta^e_a\delta^f_b\delta^m_c\delta^n_d - \delta^e_a\delta^m_b\delta^n_c\delta^f_d - \delta^e_a\delta^n_b\delta^f_c\delta^m_d + \delta^e_a\delta^f_b\delta^n_c\delta^m_d + \delta^e_a\delta^m_b\delta^f_c\delta^n_d + \delta^e_a\delta^n_b\delta^m_c\delta^f_d \\
&\quad +\delta^f_a\delta^m_b\delta^n_c\delta^e_d + \delta^f_a\delta^n_b\delta^e_c\delta^m_d + \delta^f_a\delta^e_b\delta^m_c\delta^n_d - \delta^f_a\delta^m_b\delta^e_c\delta^n_d - \delta^f_a\delta^n_b\delta^m_c\delta^e_d - \delta^f_a\delta^e_b\delta^n_c\delta^m_d \\
&\quad -\delta^m_a\delta^n_b\delta^e_c\delta^f_d - \delta^m_a\delta^e_b\delta^f_c\delta^n_d - \delta^m_a\delta^f_b\delta^n_c\delta^e_d + \delta^m_a\delta^n_b\delta^f_c\delta^e_d + \delta^m_a\delta^e_b\delta^n_c\delta^f_d + \delta^m_a\delta^f_b\delta^e_c\delta^n_d \\
&\quad +\delta^n_a\delta^e_b\delta^f_c\delta^m_d + \delta^n_a\delta^f_b\delta^m_c\delta^e_d + \delta^n_a\delta^m_b\delta^e_c\delta^f_d - \delta^n_a\delta^e_b\delta^m_c\delta^f_d - \delta^n_a\delta^f_b\delta^e_c\delta^m_d - \delta^n_a\delta^m_b\delta^n_c\delta^e_d\,, \qquad (5.8) \\
\epsilon_{abcd}\epsilon^{efmd} &= -\delta^e_a\delta^f_b\delta^m_c - \delta^f_a\delta^m_b\delta^e_c - \delta^m_a\delta^e_b\delta^f_c + \delta^e_a\delta^m_b\delta^f_c + \delta^f_a\delta^e_b\delta^m_c + \delta^m_a\delta^f_b\delta^e_c\,, \qquad (5.9) \\
\epsilon_{abcd}\epsilon^{efcd} &= 2\big(- \delta^e_a\delta^f_b + \delta^f_a\delta^e_b\big)\,, \qquad (5.10) \\
\epsilon_{abcd}\epsilon^{ebcd} &= -6\delta^e_a \qquad (5.11) \\
\epsilon_{abcd}\epsilon^{abcd} &= -24\,. \qquad (5.12)
\end{aligned}
$$

Note that there are two traps in the evaluations of these expressions. First, we obtain a ubiquitous minus sign from the Lorentzian signature of our metric. If we define $\epsilon^{0123} = +1$ in Minkowski spacetime, as is often done, then $\epsilon_{0123} = g_{00}\,g_{11}\,g_{22}\,g_{33}\,\epsilon^{0123} = -1$. We likewise pick up a minus sign in general Lorentzian spacetimes. The second trap arises in the construction of Eqs. (5.8)-(5.12). Note that we always reduce the more complex case to a sequence of the next simpler case. For instance, the six terms in (5.9) are obtained as follows: Fix the lower indices to be the same in all terms and use for this purpose the order given in the downstairs ϵ. In the very first term, write the upstairs indices in the order given by the upstairs ϵ. Then we permute the upstairs indices cyclically. This gives us the first three terms. The other three are obtained by swapping the last two upstairs indices. We thus get 3 pairs of terms and each term has the structure of the right-hand side of Eq. (5.10).

So far so good. The danger is the parity of the cyclic permutations. The cyclic permutation of an *odd* number of elements is *even*, but the cyclic permutation of an *even* number of elements is *odd*! So we can safely cycle the 3 upstairs indices in Eq. (5.9), but we pick up a minus sign each time we do this in Eq. (5.8).

Let us now return to self-dual 2-forms. We have seen in Eq. (5.6) that knowledge of the time projection $n^\mu \omega_{\mu\alpha}$ of a self-dual 2-form is enough to know the entire $\omega_{\alpha\beta}$. In the next step, we will see how we can systematically construct for any 2-form a self-dualized version. Let $F_{\alpha\beta} = -F_{\beta\alpha}$ and we find its self-dualized version from,

$$\text{Self-dualization:} \qquad \overset{\star}{F}_{\alpha\beta} = F_{\alpha\beta} + \mathrm{i}\,\tilde{F}_{\alpha\beta} = F_{\alpha\beta} + \frac{\mathrm{i}}{2}\epsilon_{\alpha\beta}{}^{\rho\sigma} F_{\rho\sigma}\,. \tag{5.13}$$

We proof this as follows,

$$\begin{aligned}
\mathrm{i}\,\widetilde{\overset{\star}{F}}_{\mu\nu} &= \frac{\mathrm{i}}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}\overset{\star}{F}_{\rho\sigma} = \frac{\mathrm{i}}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}\left(F_{\rho\sigma} + \frac{\mathrm{i}}{2}\epsilon_{\rho\sigma}{}^{\gamma\delta}F_{\gamma\delta}\right) \\
&= \frac{\mathrm{i}}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma} - \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\rho\sigma\gamma\delta}F_{\gamma\delta} = \frac{\mathrm{i}}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma} - \frac{1}{4}2\big(-\delta_\mu{}^\gamma\delta_\nu{}^\delta + \delta_\nu{}^\gamma\delta_\mu{}^\delta\big)F_{\gamma\delta} \\
&= \frac{\mathrm{i}}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma} - \frac{1}{2}(-F_{\mu\nu} + F_{\nu\mu}) = \frac{\mathrm{i}}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma} + F_{\mu\nu} \\
&= \overset{\star}{F}_{\mu\nu}\,,
\end{aligned} \tag{5.14}$$

where in the second line we have used Eq. (5.10).

There remains one important feature of the dual of a 2-form that we shall need later on. The dual operator is equal to minus its inverse or, equivalently, applying the dual operator twice returns minus the original 2-form,

$$D^{-1}(\omega)_{\alpha\beta} = -D(\omega)_{\alpha\beta} = -\tilde{\omega}_{\alpha\beta} \qquad \Leftrightarrow \qquad D\big(D(\omega)\big)_{\alpha\beta} = \tilde{\tilde{\omega}}_{\alpha\beta} = -\omega_{\alpha\beta}\,. \tag{5.15}$$

The existence of this inverse follows from

$$\begin{aligned}
-D\big(D(\omega)\big)_{\alpha\beta} &= -\frac{1}{4}\epsilon_{\alpha\beta}{}^{\mu\nu}\epsilon_{\mu\nu}{}^{\rho\sigma}\omega_{\rho\sigma} = -\frac{1}{4}\epsilon_{\alpha\beta\mu\nu}\epsilon^{\mu\nu\rho\sigma}\omega_{\rho\sigma} = -\frac{1}{4}\epsilon_{\alpha\beta\mu\nu}\epsilon^{\rho\sigma\mu\nu}\omega_{\rho\sigma} \\
&= -\frac{1}{4}2\left(-\delta_\alpha{}^\rho\delta_\beta{}^\sigma + \delta_\alpha{}^\sigma\delta_\beta{}^\rho\right)\omega_{\rho\sigma} = -\frac{1}{2}(-\omega_{\alpha\beta} + \omega_{\beta\alpha}) \\
&= \omega_{\alpha\beta} \\
\Rightarrow \sigma_{\alpha\beta} &= -D(\omega)_{\alpha\beta} \quad \text{satisfies} \quad D(\sigma)_{\alpha\beta} = D\big(-D(\omega)\big) = -D\big(D(\omega)\big)_{\alpha\beta} = \omega_{\alpha\beta}\,.
\end{aligned} \tag{5.16}$$

The inverse is also unique. Let us assume, we have an inverse $\sigma_{\alpha\beta}$. Then by definition

$$\begin{aligned} & \tilde{\sigma}_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta}{}^{\mu\nu}\sigma_{\mu\nu} = \omega_{\alpha\beta} && \Big| \times \frac{1}{2}\epsilon_{\rho\sigma}{}^{\alpha\beta} \\ \Rightarrow\ & \frac{1}{4}\epsilon_{\rho\sigma}{}^{\alpha\beta}\epsilon_{\alpha\beta}{}^{\mu\nu}\sigma_{\mu\nu} = \frac{1}{2}\epsilon_{\rho\sigma}{}^{\alpha\beta}\omega_{\alpha\beta} \\ \Rightarrow\ & \frac{1}{4}2\left(-\delta_\rho{}^\mu\delta_\sigma{}^\nu + \delta_\rho{}^\nu\delta_\sigma{}^\mu\right)\sigma_{\mu\nu} = \tilde{\omega}_{\rho\sigma} \\ \Rightarrow\ & \frac{1}{2}(-\sigma_{\rho\sigma} + \sigma_{\sigma\rho}) = -\sigma_{\rho\sigma} = \omega_{\rho\sigma}\,. \end{aligned} \tag{5.17}$$

So any inverse must be minus the original 2-form. In index free notation, we can write this as

$$D\big(D(\omega)\big) = -\omega \qquad \Leftrightarrow \qquad D^{-1}(\omega) = -D(\omega)\,. \tag{5.18}$$

5.2 The Weyl tensor

The Weyl tensor on a d-dimensional manifold with metric $g_{\alpha\beta}$ is defined by

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - \frac{1}{d-2}\big(g_{\alpha\gamma}R_{\beta\delta} + g_{\beta\delta}R_{\alpha\gamma} - g_{\alpha\delta}R_{\beta\gamma} - g_{\beta\gamma}R_{\alpha\delta}\big) + \frac{1}{(d-1)(d-2)}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma})R\,, \tag{5.19}$$

where $R_{\alpha\beta\gamma\delta}$, $R_{\alpha\beta}$ and R are the Riemann tensor, Ricci tensor and Ricci scalar, respectively.

The Weyl tensor has the following properties (assuming we use the Levi-Civita connection on our manifold),

$$C_{\alpha\beta\gamma\delta} = -C_{\beta\alpha\gamma\delta} = -C_{\alpha\beta\delta\gamma}\,, \tag{5.20}$$

$$C_{\alpha\beta\gamma\delta} = C_{\gamma\delta\alpha\beta}\,, \tag{5.21}$$

$$C_{\alpha\beta\gamma\delta} + C_{\alpha\gamma\delta\beta} + C_{\alpha\delta\beta\gamma} = 0 \qquad \Leftrightarrow \qquad C_{\alpha[\beta\gamma\delta]} = 0\,, \tag{5.22}$$

$$C^\mu{}_{\beta\mu\delta} = 0\,. \tag{5.23}$$

The former three are the symmetries of the Riemann tensor which the Weyl tensor inherits. The final condition shows that the Weyl tensor is traceless; note that by virtue of the other symmetries, any contraction of the Weyl tensor vanishes.

The Riemann tensor is known to have $(d^2(d-1)^2/12$ independent components and the Ricci tensor has $d(d+1)/2$ independent coponents. This leaves

$$\frac{d^2(d^2-1)}{12} - \frac{d(d+1)}{2} = \frac{d^4-d^2-6d^2-6d}{12} = \frac{d(d^3-7d-6)}{12} = \frac{d(d+1)(d+2)(d-3)}{12} \tag{5.24}$$

components for the Weyl tensor. In $d = 4$, Ricci and Weyl tensor have 10 free components each, totaling the Riemann tensor's 20 independent components.

From Eq. (5.20), we see that the Weyl tensor is antisymmetric in its first and in its second index pair. We can therefore treat the Weyl tensor as a $\binom{0}{2}$ tensor valued 2-form and we have two options to do so, in its first or its second index pair. In either case, we have the entire machinery of Sec. 5.1 on 2-forms available. The only tricky thing is that we have to specify which index pair we are considering as the 2-form indices. For this purpose, we will introduce the following notation for the dual of the Weyl tensor with respect to its first or second index pair,

$$D_1(C)_{\alpha\beta\gamma\delta} = {}^{\sim}C_{\alpha\beta\gamma\delta} = \frac{1}{2}\epsilon_{\alpha\beta}{}^{\mu\nu}C_{\mu\nu\gamma\delta}\,, \tag{5.25}$$

$$D_2(C)_{\alpha\beta\gamma\delta} = C^{\sim}_{\alpha\beta\gamma\delta} = \frac{1}{2}\epsilon_{\gamma\delta}{}^{\rho\sigma}C_{\alpha\beta\rho\sigma}\,. \tag{5.26}$$

The notation with the left and right tilde is used by Stephani [4], the D_1 and D_2 operators are our own invention that will become very convenient later on.

The central goal of this section is to express the Weyl tensor in terms of its electric and magnetic part, defined by

$$E_{\alpha\beta} = C_{\alpha\mu\beta\nu}n^{\mu}n^{\nu}\,, \tag{5.27}$$

$$B_{\alpha\beta} = D_1(C)_{\alpha\mu\beta\nu}n^{\mu}n^{\nu} = {}^{\sim}C_{\alpha\mu\beta\nu}n^{\mu}n^{\nu} = \frac{1}{2}\epsilon_{\alpha\mu}{}^{\rho\sigma}C_{\rho\sigma\beta\nu}n^{\mu}n^{\nu}\,. \tag{5.28}$$

Note that we have defined the magnetic part in terms of the dual of the Weyl tensor taken over its *first* index pair. As it turns out, we can equivalently define it with regard to the second index pair. But that is one of the results we wish to show and we will therefore rigorously stick to the definition given in Eq. (5.28). We will also show that both $E_{\alpha\beta}$ and $B_{\alpha\beta}$ are symmetric. This is obvious for the electric part but not for the magnetic part.

With the definitions (5.27), (5.28), the Weyl tensor can be written as

$$C_{\alpha\beta\gamma\delta} = l_{\alpha\gamma}E_{\beta\delta} - l_{\alpha\delta}E_{\beta\gamma} + l_{\beta\delta}E_{\alpha\gamma} - l_{\beta\gamma}E_{\alpha\delta} - n_{\gamma}B_{\delta\mu}\epsilon^{\mu}{}_{\alpha\beta} + n_{\delta}B_{\gamma\mu}\epsilon^{\mu}{}_{\alpha\beta} - n_{\alpha}B_{\beta\mu}\epsilon^{\mu}{}_{\gamma\delta} + n_{\beta}B_{\alpha\mu}\epsilon^{\mu}{}_{\gamma\delta}\,,$$

$$\text{where}\qquad l_{\mu\nu} = \gamma_{\mu\nu} + n_{\mu}n_{\nu} = g_{\mu\nu} + 2n_{\mu}n_{\nu}\,, \qquad \epsilon_{\beta\gamma\delta} = \epsilon_{\mu\beta\gamma\delta}n^{\mu}\,. \tag{5.29}$$

Proving this relation is the central goal of this section.

5.3 The duals of the Weyl tensor

We will now derive Eq. (5.29). This is a complex calculation that proceeds in several steps. First, we will show that the self-dualization of the Weyl tensor according to Eq. (5.13) will give the same result irrespective of whether we compute it using the first or the second index pair.

For this purpose, we need to derive the second equality in Stephani's [4] Eq. (32.5). Because of the avalanche of indices, we again switch to Latin rather than Greek indices. Using Eq. (5.8) for the product of the Levi-Civita tensors in the third line of the following calculation, we find,

$$
\begin{aligned}
&\frac{1}{4}\epsilon_{ar}{}^{ik}\epsilon_{sq}{}^{mn}C_{ikmn} = \frac{1}{4}g^{ic}g^{kd}g_{se}g_{qf}\epsilon_{arcd}\epsilon^{efmn}C_{ikmn} \\
= \quad &\frac{1}{4}g^{ic}g^{kd}g_{se}g_{qf}\Big\{ \\
&-\delta_a^e\delta_r^f\delta_c^m\delta_d^n - \delta_a^e\delta_r^m\delta_c^n\delta_d^f - \delta_a^e\delta_r^n\delta_c^f\delta_d^m + \delta_a^e\delta_r^f\delta_c^n\delta_d^m + \delta_a^e\delta_r^m\delta_c^f\delta_d^n + \delta_a^e\delta_r^n\delta_c^m\delta_d^f \\
&+\delta_a^f\delta_r^m\delta_c^n\delta_d^e + \delta_a^f\delta_r^n\delta_c^e\delta_d^m + \delta_a^f\delta_r^e\delta_c^m\delta_d^n - \delta_a^f\delta_r^m\delta_c^e\delta_d^n - \delta_a^f\delta_r^n\delta_c^m\delta_d^e - \delta_a^f\delta_r^e\delta_c^n\delta_d^m \\
&-\delta_a^m\delta_r^n\delta_c^e\delta_d^f - \delta_a^m\delta_r^e\delta_c^f\delta_d^n - \delta_a^m\delta_r^f\delta_c^n\delta_d^e + \delta_a^m\delta_r^n\delta_c^f\delta_d^e + \delta_a^m\delta_r^e\delta_c^n\delta_d^f + \delta_a^m\delta_r^f\delta_c^e\delta_d^n \\
&+\delta_a^n\delta_r^e\delta_c^f\delta_d^m + \delta_a^n\delta_r^f\delta_c^m\delta_d^e + \delta_a^n\delta_r^m\delta_c^e\delta_d^f - \delta_a^n\delta_r^e\delta_c^m\delta_d^f - \delta_a^n\delta_r^f\delta_c^e\delta_d^m - \delta_a^n\delta_r^m\delta_c^f\delta_d^e\Big\} C_{ikmn} \\
= \quad &\frac{1}{4}\Big\{ - g^{im}g^{kn}g_{sa}g_{qr} - g^{in}g^{kf}g_{sa}g_{qf}\delta_r^m - g^{if}g^{km}g_{sa}g_{qf}\delta_r^n + g^{in}g^{km}g_{sa}g_{qr} + g^{if}g^{kn}g_{sa}g_{qf}\delta_r^m + g^{im}g^{kf}g_{sa}g_{qf}\delta_r^n \\
&+g^{in}g^{ke}g_{se}g_{qa}\delta_r^m + g^{ie}g^{km}g_{se}g_{qa}\delta_r^n + g^{im}g^{kn}g_{sr}g_{qa} - g^{ie}g^{kn}g_{se}g_{qa}\delta_r^m - g^{im}g^{ke}g_{se}g_{qa}\delta_r^n - g^{in}g^{km}g_{sr}g_{qa} \\
&-g^{ie}g^{kf}g_{se}g_{qf}\delta_a^m\delta_r^n - g^{if}g^{kn}g_{sr}g_{qf}\delta_a^m - g^{in}g^{ke}g_{se}g_{qr}\delta_a^m + g^{if}g^{ke}g_{se}g_{qf}\delta_a^m\delta_r^n + g^{in}g^{kf}g_{sr}g_{qf}\delta_a^m + g^{ie}g^{kn}g_{se}g_{qr}\delta_a^m \\
&+g^{if}g^{km}g_{sr}g_{qf}\delta_a^n + g^{im}g^{ke}g_{se}g_{qr}\delta_a^n + g^{ie}g^{kf}g_{se}g_{qf}\delta_a^n\delta_r^m - g^{im}g^{kf}g_{sr}g_{qf}\delta_a^n - g^{ie}g^{km}g_{se}g_{qr}\delta_a^n - g^{if}g^{ke}g_{se}g_{qf}\delta_a^n\delta_r^m \\
&\Big\}C_{ikmn} \\
= \quad &\frac{1}{4}\big(- C_{sqar} + C_{qsar} + C_{sqra} - C_{qsra}\big) = -C_{arsq}\,. \qquad (5.30)
\end{aligned}
$$

Here we have marked in the third step by a wavy underline all those metric terms that take a trace of the Weyl tensor and hence vanish, and by a dotted underline the four terms that remain after this elimination. The final step follows directly from applying the symmetries (5.20) and (5.21) of the Weyl tensor. Relabeling the indices and recalling our notation of the D_1 and D_2 operators from Eqs. (5.25), (5.26), we can write this important result as

$$
\boxed{D_1\big(D_2(C)\big)_{\alpha\beta\gamma\delta} = \epsilon_{\alpha\beta}{}^{\mu\nu}D_2(C)_{\mu\nu\gamma\delta} = \epsilon_{\alpha\beta}{}^{\mu\nu}\epsilon_{\gamma\delta}{}^{\rho\sigma}C_{\mu\nu\rho\sigma} \stackrel{!}{=} D_2\big(D_1(C)\big)_{\alpha\beta\gamma\delta} = -C_{\alpha\beta\gamma\delta}\,, \quad (5.31)}
$$

or, in short,

$$
\boxed{D_1\big(D_2(C)\big) = D_2\big(D_1(C)\big) = -C\,. \qquad (5.32)}
$$

As a side result, we also see that the operators D_1 and D_2 commute when acting on the Weyl tensor.

Next, we construct the self-dualized version of the Weyl tensor according to Eq. (5.13). Again, we have two options, self-dualizing on the first or on the second index pair. Let us temporarily denote the self-dual by caligraphic letters rather than the $\star$ symbol. For want of a better idea, we call the two versions,

$$\mathcal{C}_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + \frac{\mathrm{i}}{2}\epsilon_{\alpha\beta}{}^{\mu\nu}C_{\mu\nu\gamma\delta} \qquad \Leftrightarrow \qquad \mathcal{C} = C + \mathrm{i}\, D_1(C)\,, \tag{5.33}$$

$$\mathcal{B}_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + \frac{\mathrm{i}}{2}\epsilon_{\gamma\delta}{}^{\rho\sigma}C_{\alpha\beta\rho\sigma} \qquad \Leftrightarrow \qquad \mathcal{B} = C + \mathrm{i}\, D_2(C)\,, \tag{5.34}$$

recalling once again the definition of the two dual operators in Eqs. (5.25), (5.26). We will now show that these two self-dualized tensors are actually the same. In order to show this, we recall the properties of the inverse dual from Eq. (5.18) and note that this relation applies individually to D_1 and D_2 operating on the Weyl tensor. Combining this with Eq. (5.32), we find (note that we can use the operator D_2 on $\mathcal{C}$ since $\mathcal{C}_{\alpha\beta\gamma\delta} = -\mathcal{C}_{\alpha\beta\delta\gamma}$),

$$-\mathcal{C} \overset{(5.18)}{=} D_2D_2\mathcal{C} \overset{(5.33)}{=} D_2D_2C + \mathrm{i}D_2\big(D_2(D_1C)\big) \overset{(5.32)}{=} D_2D_2C + \mathrm{i}D_2(-C) \overset{(5.18)}{=} -C - \mathrm{i}D_2(-C) \overset{(5.34)}{=} -\mathcal{B}\,. \tag{5.35}$$

We therefore have only one self-dualized version of the Weyl tensor and can revert to our notation using the $\star$ symbol, so that

$$\begin{aligned} \overset{\star}{C}_{\alpha\beta\gamma\delta} &= \mathcal{C}_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + \mathrm{i}\epsilon_{\alpha\beta}{}^{\mu\nu}C_{\mu\nu\gamma\delta} && (5.36)\\ &= \mathcal{B}_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + \mathrm{i}\epsilon_{\gamma\delta}{}^{\rho\sigma}C_{\alpha\beta\rho\sigma}\,. && (5.37)\end{aligned}$$

Combining this relation with Eqs.(5.33) and (5.34), we immediately conclude that $\mathrm{i}\,D_1(C) = \mathcal{C} - C \overset{!}{=} \mathcal{B} - C = \mathrm{i}D_2(C)$ and, hence, the symmetry of the magnetic part,

$$B_{\alpha\beta} = D_1(C)_{\alpha\mu\beta\nu}n^\mu n^\nu = D_2(C)_{\alpha\mu\beta\nu}n^\mu n^\nu = \frac{1}{2}\epsilon_{\beta\nu}{}^{\rho\sigma}C_{\alpha\mu\rho\sigma}n^\mu n^\nu = \frac{1}{2}\epsilon_{\beta\nu}{}^{\rho\sigma}C_{\rho\sigma\alpha\mu}n^\nu n^\mu = B_{\beta\alpha}\,. \tag{5.38}$$

As an important consequence of the result (5.36), (5.37), we can use the reconstruction of a 2-form in terms of its time projection according to Eq. (5.6) independently on both index pairs of the self-dualized Weyl tensor, because by Eq. (5.36) it is self-dual with respect to the first index pair and by (5.37) it is self-dual with respect to the second index pair. We therefore have analogous to Eq. (5.6),

$$\overset{\star}{C}_{\alpha\beta\gamma\delta} = I_{\alpha\beta\mu}{}^{\nu}n^\mu n^\rho \overset{\star}{C}_{\rho\nu\gamma\delta} \qquad \wedge \qquad \overset{\star}{C}_{\alpha\beta\gamma\delta} = I_{\gamma\delta\mu}{}^{\nu}n^\mu n^\rho \overset{\star}{C}_{\alpha\beta\rho\nu} \tag{5.39}$$

The key step in the derivation of Eq. (5.29) consists in applying both these identities successively to the self-dualized Weyl tensor. This gives us

$$\begin{aligned} \overset{\star}{C}_{\alpha\beta\gamma\delta} &= I_{\alpha\beta\mu}{}^{\nu}n^\mu n^\kappa \overset{\star}{C}_{\kappa\nu\gamma\delta} = I_{\alpha\beta\mu}{}^{\nu}n^\mu n^\kappa I_{\gamma\delta\rho}{}^{\sigma}n^\rho n^\lambda \overset{\star}{C}_{\kappa\nu\lambda\sigma}\\ &= I_{\alpha\beta\mu}{}^{\nu}I_{\gamma\delta\rho}{}^{\sigma}n^\mu n^\kappa n^\rho n^\lambda\left(C_{\kappa\nu\lambda\sigma} + \frac{\mathrm{i}}{2}\epsilon_{\kappa\nu}{}^{\epsilon\tau}C_{\epsilon\tau\lambda\sigma}\right)\\ &= I_{\alpha\beta\mu}{}^{\nu}I_{\gamma\delta\rho}{}^{\sigma}n^\mu n^\rho\left(C_{\nu\kappa\sigma\lambda}n^\kappa n^\lambda + \frac{\mathrm{i}}{2}\epsilon_{\nu\kappa}{}^{\epsilon\tau}C_{\epsilon\tau\sigma\lambda}n^\kappa n^\lambda\right)\\ &\overset{(5.27),(5.28)}{=} I_{\alpha\beta\mu}{}^{\nu}I_{\gamma\delta\rho}{}^{\sigma}n^\mu n^\rho\left(E_{\nu\sigma} + \mathrm{i}\,B_{\nu\sigma}\right)\,. \end{aligned} \tag{5.40}$$

Note that here, as well in the folllowing steps, we repeatedly trade minus signs for swapping indices in the Weyl and the Levi-Civita tensors, preferably swapping two index pairs at a time with no overall sign change.

The remainder consists in expanding the identitiy operator according to Eq. (5.5). This will include contractions of the 4-dimensional Levi-Civita tensor and the timelike normal n^μ and we define the 3-dimensional Levi-Civita tensor as

$$\boxed{\epsilon_{\beta\gamma\delta} := \epsilon_{\mu\beta\gamma\delta} n^\mu \,. \qquad (5.41)}$$

Now there is another subtle trap. In fact, given the number of traps we are facing, this entire calculation feels like an Indiana Jones quest through a tomb in the land of abstract manifolds. The challenge is that we need to carefully distinguish between four-dimensional spacetime and three-dimensional spatial indices. Naively, one might be tempted to conclude from Eq. (5.8) that the following were an equality,

$$\epsilon_{ijm}\epsilon^{kln} \neq \delta^k_i\delta^l_j\delta^n_m + \delta^l_i\delta^n_j\delta^k_m + \delta^n_i\delta^k_j\delta^l_m - \delta^k_i\delta^n_j\delta^l_m - \delta^l_i\delta^k_j\delta^n_m - \delta^n_i\delta^l_j\delta^k_m. \qquad (5.42)$$

But as a tensorial equation this is not an equality. Indeed, a straightforward calculation applying Eq. (5.8) to the definition (5.41) gives, using $n_a n^a = -1$,

$$\begin{aligned}
\epsilon_{ijm}\epsilon^{kln} &= \epsilon_{aijm}\epsilon^{bkln}n^a n_b \\
&= \delta^k_i\delta^l_j\delta^n_m + \delta^l_i\delta^n_j\delta^k_m + \delta^n_i\delta^k_j\delta^l_m - \delta^k_i\delta^n_j\delta^l_m - \delta^l_i\delta^k_j\delta^n_m - \delta^n_i\delta^l_j\delta^k_m \\
&\quad + n^k n_m\delta^l_i\delta^n_j + n^k n_j\delta^n_i\delta^l_m + n^k n_i\delta^l_j\delta^n_m - n^k n_j\delta^l_i\delta^n_m - n^k n_m\delta^n_i\delta^l_j - n^k n_i\delta^n_j\delta^l_m \\
&\quad - n^l n_j\delta^n_i\delta^k_m - n^l n_i\delta^k_j\delta^n_m - n^l n_m\delta^k_i\delta^n_j + n^l n_m\delta^n_i\delta^k_j + n^l n_i\delta^n_j\delta^k_m + n^l n_j\delta^k_i\delta^n_m \\
&\quad + n^n n_i\delta^k_j\delta^l_m + n^n n_m\delta^k_i\delta^l_j + n^n n_j\delta^l_i\delta^k_m - n^n n_i\delta^l_j\delta^k_m - n^n n_j\delta^k_i\delta^l_m - n^n n_m\delta^l_i\delta^k_j \,. \qquad (5.43)
\end{aligned}$$

The additional terms involving the timelike normal n_i vanish if we are using adapted three-dimensional coordinates in which case we would recover the equality version of (5.42). Thus proceeding with (5.42) in our calculation of the Weyl tensor, however, would require special care when raising or lowering indices; more specifically, we would have to use the spatial 3-metric $\gamma_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta$ in several places. But this is tricky and potentially confusing. There is a cleaner and more efficient alternative.

For this purpose, we recall the projection operator

$$\perp^b_a = \perp^b{}_a := \delta^b{}_a + n^b n_a \,. \qquad (5.44)$$

We immediately conclude that

$$\begin{aligned}
\epsilon_{ijm} &= \epsilon_{aijm}n^a \overset{!}{=} \perp^c_i\perp^d_j\perp^e_m\epsilon_{acde}n^a \,, \\
\epsilon^{kln} &= \epsilon^{bkln}n_b \overset{!}{=} \perp^k_f\perp^l_g\perp^n_h\epsilon^{bfgh}n_b \,, \qquad (5.45)
\end{aligned}$$

since $n^c\epsilon_{abcde}n^a = 0$ and likewise for any other double contraction of the Levi-Civita tensor with n.

We can therefore multiply Eq. (5.43) with six projection operators, one for each free index, and find

$$\begin{aligned}
\epsilon_{ijm}\epsilon^{kln} &= \perp_i^c\perp_j^d\perp_m^e\perp_f^k\perp_g^l\perp_h^n n^a n_b \epsilon_{acde}\epsilon^{bfgh} \\
&= \perp_i^c\perp_j^d\perp_m^e\perp_f^k\perp_g^l\perp_h^n n^a \\
&\quad \left[\delta_c^f\delta_d^g\delta_e^h + \delta_c^g\delta_d^h\delta_e^f + \delta_c^h\delta_d^f\delta_e^g - \delta_c^f\delta_d^h\delta_e^g - \delta_c^g\delta_d^f\delta_e^h - \delta_c^h\delta_d^g\delta_e^f + n^f(\ldots) - n^g(\ldots) + n^h(\ldots)\right] \\
&= \perp_i^k\perp_j^l\perp_m^n + \perp_i^l\perp_j^n\perp_m^k + \perp_i^n\perp_j^k\perp_m^l - \perp_i^k\perp_j^n\perp_m^l - \perp_i^l\perp_j^k\perp_m^n - \perp_i^n\perp_j^l\perp_m^k\,. \qquad (5.46)
\end{aligned}$$

Of couse, for three-dimensional coordinates adapted to the space-time split, we have $\perp_j^i = {}^{(3)}\delta_j^i$ and we recover the equality version of (5.42). But in Eq. (5.46) we now have a coordinate independent, geometric version of this relation and will proceed accordingly. Note, incidentally, that the overall minus sign, which we encountered in the spacetime case of (5.8), has vanished in the spatial case (5.46) by virtue of $n_a n^a = -1$.

Continuing with Eq. (5.40), we thus obtain

$$\begin{aligned}
\overset{\star}{C}_{\alpha\beta\gamma\delta} &= \left(g_{\alpha\mu}\delta_\beta{}^\nu - g_{\beta\mu}\delta_\alpha{}^\nu + \mathrm{i}\epsilon_{\alpha\beta\mu}{}^\nu\right)\left(g_{\gamma\rho}\delta_\delta{}^\sigma - g_{\delta\rho}\delta_\gamma{}^\sigma + \mathrm{i}\epsilon_{\gamma\delta\rho}{}^\sigma\right)n^\mu n^\rho(E_{\nu\sigma} + \mathrm{i}\,B_{\nu\sigma}) \\
&= n_\alpha n_\gamma(E_{\beta\delta} + \mathrm{i}B_{\beta\delta}) - n_\alpha n_\delta(E_{\beta\gamma} + \mathrm{i}B_{\beta\delta}) + \mathrm{i}n_\alpha\epsilon_{\gamma\delta\rho}{}^\sigma n^\rho(E_{\beta\sigma} + \mathrm{i}B_{\beta\sigma}) \\
&\quad -n_\beta n_\gamma(E_{\alpha\delta} + \mathrm{i}B_{\alpha\delta}) + n_\beta n_\delta(E_{\alpha\gamma} + \mathrm{i}B_{\alpha\gamma}) - \mathrm{i}\,n_\beta\epsilon_{\gamma\delta\rho}{}^\sigma n^\rho(E_{\alpha\sigma} + \mathrm{i}B_{\alpha\sigma}) \\
&\quad +\mathrm{i}\,\epsilon_{\alpha\beta\mu}{}^\nu n^\mu n_\gamma(E_{\nu\delta} + \mathrm{i}B_{\nu\delta}) - \mathrm{i}\epsilon_{\alpha\beta\mu}{}^\nu n^\mu n_\delta(E_{\nu\gamma} + \mathrm{i}B_{\nu\gamma}) - \epsilon_{\alpha\beta\mu}{}^\nu\epsilon_{\gamma\delta\rho}{}^\sigma n^\mu n^\rho(E_{\nu\sigma} + \mathrm{i}B_{\nu\sigma}) \\
\Rightarrow C_{\alpha\beta\gamma\delta} &\overset{!}{=} \mathrm{Re}\left[\overset{\star}{C}_{\alpha\beta\gamma\delta}\right] \\
&= n_\alpha n_\gamma E_{\beta\delta} - n_\alpha n_\delta E_{\beta\gamma} - n_\beta n_\gamma E_{\alpha\delta} + n_\beta n_\delta E_{\alpha\gamma} - \epsilon_{\alpha\beta}{}^\nu\epsilon_{\gamma\delta}{}^\sigma E_{\nu\sigma} \\
&\quad -n_\alpha\epsilon_{\gamma\delta}{}^\sigma B_{\beta\sigma} + n_\beta\epsilon_{\gamma\delta}{}^\sigma B_{\alpha\sigma} - n_\gamma\epsilon_{\alpha\beta}{}^\nu B_{\nu\delta} + n_\delta\epsilon_{\alpha\beta}{}^\nu B_{\nu\gamma}\,. \qquad (5.47)
\end{aligned}$$

Using Eq. (5.46), the fifth term on the final right-hand side becomes (writing $\perp_{\alpha\beta} = \gamma_{\alpha\beta}$ for the 3-metric),

$$\begin{aligned}
-\epsilon_{\alpha\beta}{}^\nu\epsilon_{\gamma\delta}{}^\sigma E_{\nu\sigma} &= -\perp_{\alpha\gamma}\perp_{\beta\delta}\underbrace{\perp^{\nu\sigma}E_{\nu\sigma}}_{=0} - \perp_{\alpha\delta}\perp_\beta{}^\sigma\perp^\nu{}_\gamma E_{\nu\sigma} - \perp_\alpha{}^\sigma\perp_{\beta\gamma}\perp^\nu{}_\delta E_{\nu\sigma} \\
&\quad +\perp_{\alpha\gamma}\perp_\beta{}^\sigma\perp^\nu{}_\delta E_{\nu\sigma} + \perp_{\alpha\delta}\perp_{\beta\gamma}\perp^{\nu\sigma}E_{\nu\sigma} + \perp_\alpha{}^\sigma\perp_{\beta\delta}\perp^\nu{}_\gamma E_{\nu\sigma} \\
&= -\gamma_{\alpha\delta}E_{\beta\gamma} - \gamma_{\beta\gamma}E_{\delta\alpha} + \gamma_{\alpha\gamma}E_{\delta\beta} + \gamma_{\beta\delta}E_{\gamma\alpha} \\
\Rightarrow C_{\alpha\beta\gamma\delta} &= E_{\beta\delta}(\gamma_{\alpha\gamma} + n_\alpha n_\gamma) - E_{\beta\gamma}(\gamma_{\alpha\delta} + n_\alpha n_\delta) - E_{\alpha\delta}(\gamma_{\beta\gamma} + n_\beta n_\gamma) + E_{\alpha\gamma}(\gamma_{\beta\delta} + n_\beta n_\delta) \\
&\quad -n_\gamma B_{\delta\nu}\epsilon^\nu{}_{\alpha\beta} + n_\delta B_{\gamma\nu}\epsilon^\nu{}_{\alpha\beta} - n_\alpha B_{\beta\sigma}\epsilon^\sigma{}_{\gamma\delta} + n_\beta B_{\alpha\sigma}\epsilon^\sigma{}_{\gamma\delta}\,, \qquad (5.48)
\end{aligned}$$

where we have used that $E_{\alpha\beta}$ and $B_{\alpha\beta}$ are symmetric and purely spatial, so that $\perp_\alpha^\mu E_{\mu\beta} = E_{\alpha\beta}$ and $\gamma^{\mu\nu}E_{\mu\nu} = g^{\mu\nu}E_{\mu\nu}$, and that the Weyl tensor is traceless, whence $g^{\mu\nu}E_{\mu\nu} = C^\mu{}_{\alpha\mu\beta}n^\alpha n^\beta = 0$. Equation (5.48) indeed gives us the decomposition (5.29) that we wished to derive in the first place. We're done.

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