Contents

1 Introduction

In this document, we consider scalar-field boson stars in the framework of scalar-tensor gravity, which here means Brans-Dicke theory [4] with a generalized coupling function between the scalar and tensor sectors of the theory. Eventually, we will focus on Damour-Esposito-Farèse [7] gravity which includes a linear (Brans-Dicke) plus a quadratic term in the coupling function, but for now, we leave this coupling function generic. We will end up with lots of variables we called some variant of the Greek `phi`. The Greeks have given us three symbols for this letter, but that won't be enough, so that we replace one by ψ. More specifically, we have the following variables,

$$\phi \quad = \quad \text{gravitational scalar field in the Jordan frame}\,, \tag{1.1}$$

$$\varphi \quad = \quad \text{gravitational scalar field in the Einstein frame}\,, \tag{1.2}$$

$$\Theta \quad = \quad \text{gravitational scalar field in the Brans-Dicke formalism}\,, \tag{1.3}$$

$$\Phi \quad = \quad \text{logarithm of the lapse function}\,, \tag{1.4}$$

$$\psi \quad = \quad \text{complex scalar field of the boson star}\,. \tag{1.5}$$

At this point, we assume that there is no interaction between the two scalar fields except gravitational through the Einstein field equation.

As in the core collapse work, we will denote quantities in the Einstein frame by an overbar as in $\bar{g}_{\mu\nu}$ whereas variables in the physical Jordan frame are written as plain letters as in $g_{\mu\nu}$. Since the overbar is now engaged, we denote the complex conjugate by a star as in ψ^*.

2 The action and covariant equations

The action for scalar-tensor gravity in the Einstein frame is given, for example, by Eq. (3) of Damour & Esposito-Farèse [7]. The only modification of their action is that we allow for a potential function W of the gravitational scalar field φ, so that we start with the action

$$\boxed{S_E = \frac{1}{16\pi G}\int \mathrm{d}x^4 \sqrt{-\bar{g}}\left\{\bar{R} - 2\bar{g}^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi - 4W(\varphi)\right\} + S_m[\psi_m, F(\varphi)^{-1}\bar{g}_{\mu\nu}]\,,} \tag{2.1}$$

where ψ_m denotes the (ordinary) matter fields, $F(\varphi)$ is the coupling function[1] and $\bar{R}$ the Ricci scalar associated with the Einstein metric $\bar{g}_{\mu\nu}$. The physical or Jordan metric is related to the Einstein metric by

$$g_{\mu\nu} = F^{-1}\bar{g}_{\mu\nu} \qquad \Leftrightarrow \qquad g^{\mu\nu} = F\bar{g}^{\mu\nu}\,. \tag{2.2}$$

We see from Eq. (2.1) that the matter sources couple to the physical metric $a^2\bar{g}_{\mu\nu}$.

The physical or Jordan frame is obtained by changing the scalar field variable according to

$$\frac{\partial\varphi}{\partial\phi} = \sqrt{\frac{3}{4}\frac{F_\phi^2}{F^2} + \frac{4\pi}{F}} \qquad \Leftrightarrow \qquad \frac{\partial\phi}{\partial\varphi} = \sqrt{\frac{4F^2 - 3F_{,\varphi}^2}{16\pi F}}\,. \tag{2.3}$$

[1] In the literature one also finds the conformal factor $a(\varphi)$ related to our F by $F(\varphi) = a(\varphi)^{-2}$. Following [16], we got used to F and shall stick to it in these notes.

Taking into account some boundary terms which do not affect the variation, these redefinitions of variables lead to the action in the physical or Jordan-Fierz frame,

$$S_J = \int \mathrm{d}x^4 \sqrt{-g} \left\{ \frac{F(\phi)}{16\pi G} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - U(\phi) \right\} + S_m(\psi_m, g_{\mu\nu}) \,, \tag{2.4}$$

where R denotes the Ricci scalar associated with the physical metric and the potential functions are related by

$$U(\phi) = \frac{F^2}{4\pi} W(\varphi) \,. \tag{2.5}$$

The two frames provide us with two alternative ways to formulate the field equations of the geometrical and matter fields. In the Einstein frame, these are given in covariant form by

$$\begin{aligned}
\bar{G}_{\alpha\beta} &= 2\partial_\alpha \varphi \, \partial_\beta \varphi - \bar{g}_{\alpha\beta} \bar{g}^{\mu\nu} \partial_\mu \varphi \, \partial_\nu \varphi + 8\pi \bar{T}_{\alpha\beta} - 2W(\varphi) \bar{g}_{\alpha\beta} \,, \\
\bar{\nabla}^\mu \bar{\nabla}_\mu \varphi &= 2\pi \frac{F_{,\varphi}}{F} \bar{T} + W_{,\varphi} \,, \\
\bar{\nabla}_\mu \bar{T}^{\mu\alpha} &= -\frac{1}{2} \frac{F_{,\varphi}}{F} \bar{T} \bar{g}^{\alpha\mu} \bar{\nabla}_\mu \varphi \,.
\end{aligned} \tag{2.6}$$

Here, the energy momentum tensor $\bar{T}^{\alpha\beta}$ is defined as

$$\bar{T}^{\alpha\beta} = \frac{2}{\sqrt{-\bar{g}}} \frac{\delta S_m}{\delta \bar{g}_{\alpha\beta}} = \frac{1}{F^3} T^{\alpha\beta} \quad \Rightarrow \quad \bar{T}^\alpha{}_\beta = \frac{1}{F^2} T^\alpha{}_\beta \quad \Rightarrow \quad \bar{T}_{\alpha\beta} = \frac{1}{F} T_{\alpha\beta} \,, \tag{2.7}$$

since the indices of $\bar{T}^{\alpha\beta}$ and $T^{\alpha\beta}$ are lowered with $\bar{g}_{\mu\nu}$ and $g_{\mu\nu}$, respectively.

In the physical frame, the field equations are obtained by varying the action (2.4) with respect to $g_{\alpha\beta}$, ϕ and ψ_m which results in

$$\begin{aligned}
G_{\alpha\beta} &= \frac{8\pi}{F} (T^F_{\alpha\beta} + T^\phi_{\alpha\beta} + T_{\alpha\beta}) \,, \qquad \nabla^\mu T_{\mu\alpha} = 0 \,, \\
T^F_{\alpha\beta} &= \frac{1}{8\pi} (\nabla_\alpha \nabla_\beta F - g_{\alpha\beta} \nabla^\mu \nabla_\mu F) \,, \\
T^\phi_{\alpha\beta} &= \partial_\alpha \phi \, \partial_\beta \phi - g_{\alpha\beta} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi + U(\phi) \right] \,, \\
\nabla^\mu \nabla_\mu \phi &= -\frac{1}{16\pi} F_{,\phi} R + U_{,\phi} \,.
\end{aligned} \tag{2.8}$$

The two sets of equations (2.6) and (2.8) are in principle equivalent, but some subtlety arises from the possible non-uniqueness of the redefinition of the scalar field (2.3) which, in turn, is a consequence of the square root which allows for multiple signs and may also have a negative argument. This has resulted in numerous discussions in the literature about the validity and meaning of the different frames [8]. We will not dive into this discussion but work with the respective frames as long as they give us plausible results.

We finally note one other common version of the action that is also given in the physical frame but promotes the conformal factor itself to the status of the scalar field, i.e. defines[2]

$$\Theta := F(\phi)\,. \tag{2.9}$$

This is the form used by [4] and we will therefore call it the Brans-Dicke frame (though noone else does for all I know). The action can then be written as

$$S_{BD} = \frac{1}{16\pi G}\int \mathrm{d}x^4\sqrt{-g}\left\{\Theta R - \frac{\omega(\Theta)}{\Theta}g^{\mu\nu}\partial_\mu\Theta\,\partial_\nu\Theta + 2\Theta\lambda(\Theta)\right\} + S_m[\psi_m, g_{\mu\nu}]\,, \tag{2.10}$$

where

$$\omega = \frac{8\pi\Theta}{F_{,\phi}^2}\,, \qquad \lambda = -\frac{8\pi U(\phi)}{\Theta}\,. \tag{2.11}$$

Brans & Dicke [4] considered the special case $\omega = \omega_{\rm BD} = \text{const}$ in their work which leads to

$$F(\phi) = \frac{2\pi}{\omega_{\rm BD}}(\phi - \phi_0)^2\,, \tag{2.12}$$

as one readily checks by insertion. We will likely not use the Brans-Dicke variables, but some of the literature does, so it's good to have it here as a reference.

3 The equations for Boson-Star matter

In the physical frame, a minimally coupled complex scalar field's dynamics are governed by the matter action[3]

$$S_m = \int \mathrm{d}x^4\sqrt{-g}\left\{-\frac{1}{2}\left[g^{\mu\nu}\nabla_\mu\psi^*\,\nabla_\nu\psi + V(\psi)\right]\,.\right\} \tag{3.1}$$

Compared with a boson star in general relativity, the only difference is therefore that the Einstein Hilbert part of the action, $\frac{1}{16\pi}R$ is replaced with the augmented term within the braces in Eq. (2.4).

Our goal in this section is to derive the field equations (2.6) and (2.8) for the special case of a complex scalar field. We start with the evolution equation for the scalar field itself which is most easily obtained first in the Jordan frame. Varying the action (2.4) with respect to the scalar field ψ, we see that this proceeds exactly as in general relativity. The only term depending on ψ is S_m and

[2] The scalar field in this Brans-Dicke formulation is more commonly denoted by Φ, but we already use this letter for the logarithmic lapse function, so resort to Θ to avoid confusion.

[3] We have inherited the actions for the two scalar fields from previous work and, unfortunately, not used quite the same conventions: For the boson-star field, the potential function appears with a factor $1/2$ that we have not used for $U(\phi)$ in the Jordan action (2.4) while we use some factor of π difference for the potential $W(\varphi)$ in the Einstein action (2.1). For now, we keep our notation as is and consider at a later stage whether we want to change anything.

we obtain

$$\begin{aligned}
\delta S_m &= S_m[\psi+\delta\psi, g_{\mu\nu}] - S_m[\psi, g_{\mu\nu}] \\
&= -\frac{1}{2}\int \mathrm{d}x^4\sqrt{-g}\,\{g^{\rho\sigma}\partial_\rho(\psi^*+\delta\psi^*)\partial_\sigma(\psi+\delta\psi) + V(\psi+\delta\psi) - g^{\rho\sigma}\partial_\rho\psi^*\partial_\sigma\psi - V(\psi)\} \\
&= -\frac{1}{2}\int \mathrm{d}x^4\sqrt{-g}\,\left\{g^{\rho\sigma}\partial_\rho\delta\psi^*\,\partial_\sigma\psi + g^{\rho\sigma}\partial_\rho\psi^*\,\partial_\sigma\delta\psi + V'(\psi)\delta\psi\right\} \\
&= -\frac{1}{2}\int \mathrm{d}x^4\sqrt{-g}\big\{\underbrace{g^{\rho\sigma}\nabla_\rho(\delta\psi^*\,\nabla_\sigma\psi)}_{\to 0} - g^{\rho\sigma}\delta\psi^*\nabla_\rho\nabla_\sigma\psi + \underbrace{g^{\rho\sigma}\nabla_\sigma(\delta\psi\partial_\rho\psi^*)}_{\to 0} \\
&\quad - g^{\rho\sigma}\delta\psi\,\nabla_\sigma\nabla_\rho\psi^* + V'\delta\psi\big\} \\
&= \frac{1}{2}\int \mathrm{d}x^4\sqrt{-g}\,\left\{g^{\rho\sigma}\delta\psi^*\nabla_\rho\nabla_\sigma\psi + g^{\rho\sigma}\delta\psi\nabla_\sigma\nabla_\rho\psi^* - V'\delta\psi\right\} \qquad (3.2)
\end{aligned}$$

Next, we define real and imaginary part of the scalar field by $\psi = \psi_R + \mathrm{i}\psi_I$ and find

$$\delta\psi^*\nabla_\rho\nabla_\sigma\psi = \delta\psi_R\nabla_\rho\nabla_\sigma\psi_R + \delta\psi_I\nabla_\rho\nabla_\sigma\psi_I + \mathrm{i}[-\delta\psi_I\nabla_\rho\nabla_\sigma\psi_R + \delta\psi_R\nabla_\rho\nabla_\sigma\psi_I]\,,$$

$$\delta\psi\nabla_\rho\nabla_\sigma\psi^* = \delta\psi_R\nabla_\rho\nabla_\sigma\psi_R + \delta\psi_I\nabla_\rho\nabla_\sigma\psi_I + \mathrm{i}[\delta\psi_I\nabla_\rho\nabla_\sigma\psi_R - \delta\psi_R\nabla_\rho\nabla_\sigma\psi_I]\,.$$

$$\Rightarrow\quad \delta\psi^* g^{\rho\sigma}\nabla_\rho\nabla_\sigma\psi + \delta\psi g^{\rho\sigma}\nabla_\rho\nabla_\sigma\psi^* = 2\delta\psi_R g^{\rho\sigma}\nabla_\rho\nabla_\sigma\psi_R + 2\delta\psi_I g^{\rho\sigma}\nabla_\rho\nabla_\sigma\psi_I$$

$$\Rightarrow\quad \delta S_m = \int \mathrm{d}x^4\sqrt{-g}\left\{\delta\psi_R g^{\rho\sigma}\nabla_\rho\nabla_\sigma\psi_R + \delta\psi_I g^{\rho\sigma}\nabla_\rho\nabla_\sigma\psi_I - \frac{1}{2}\frac{\partial V}{\partial\psi_R}\delta\psi_R - \frac{1}{2}\frac{\partial V}{\partial\psi_I}\delta\psi_I\right\}\,. \qquad (3.3)$$

Now, the variations $\delta\psi_R$ and $\delta\psi_I$ are independent, so that δS_m vanishes if the equations

$$g^{\rho\sigma}\nabla_\rho\nabla_\sigma\psi_R - \frac{1}{2}\frac{\partial V}{\partial\psi_R} = 0\,,\qquad g^{\rho\sigma}\nabla_\rho\nabla_\sigma\psi_I - \frac{1}{2}\frac{\partial V}{\partial\psi_I} = 0\,, \qquad (3.4)$$

hold. The linear combination of these two equations gives us the Klein-Gordon equation for a complex scalar field in the physical frame,

$$\nabla^\mu\nabla_\mu\psi = \frac{1}{2}\left(\frac{\partial V}{\partial\psi_R} + \mathrm{i}\,\frac{\partial V}{\partial\psi_I}\right)\,. \qquad (3.5)$$

In practice, we will consider potentials that only depend on $|\psi|^2$, so that

$$\frac{\partial V}{\partial\psi_R} = \frac{\partial V}{\partial|\psi|^2}\frac{\partial|\psi|^2}{\partial\psi_R} = V_{,|\psi|^2}2\psi_R \qquad \wedge \qquad \frac{\partial V}{\partial\psi_I} = \frac{\partial V}{\partial|\psi|^2}2\psi_I\,, \qquad (3.6)$$

so that

$$\boxed{\nabla^\mu\nabla_\mu\psi = V_{,|\psi|^2}\psi =: V'\,\psi\,. \qquad (3.7)}$$

We can convert this equation into the Einstein frame by replacing the physical metric in terms of the conformal metric according to Eq. (2.2). This modifies the wave operator on the right-hand side,

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\psi = F\bar{g}^{\mu\nu}(\partial_\mu\partial_\nu\psi - \Gamma^\rho_{\mu\nu}\partial_\rho\psi)\,. \tag{3.8}$$

For the Christoffel symbols we find

$$\begin{aligned}
\Gamma^\rho_{\mu\nu} &= \frac{1}{2}g^{\rho\sigma}(-\partial_\sigma g_{\mu\nu} + \partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu}) \\
&= \frac{1}{2}F\bar{g}^{\rho\sigma}\left[-\partial_\sigma(\bar{g}_{\mu\nu}/F) + \partial_\mu(\bar{g}_{\nu\sigma}/F) + \partial_\nu(\bar{g}_{\sigma\mu}/F)\right] \\
&= \bar{\Gamma}^\rho_{\mu\nu} + \frac{1}{2}F^{-1}\bar{g}^{\rho\sigma}\left[\bar{g}_{\mu\nu}\partial_\sigma F - \bar{g}_{\nu\sigma}\partial_\mu F - \bar{g}_{\sigma\mu}\partial_\nu F\right] \\
&= \bar{\Gamma}^\rho_{\mu\nu} + \frac{1}{2}F^{-1}\left[\bar{g}^{\rho\sigma}\bar{g}_{\mu\nu}\partial_\sigma F - \delta^\rho{}_\nu\partial_\mu F - \delta^\rho{}_\mu\partial_\nu F\right]\,,
\end{aligned} \tag{3.9}$$

so that

$$\nabla^\mu\nabla_\mu\psi = F\bar{g}^{\mu\nu}\bar{\nabla}_\mu\bar{\nabla}_\nu\psi - \frac{1}{2}\left[4\bar{g}^{\rho\sigma}\partial_\sigma F - 2\bar{g}^{\rho\mu}\partial_\mu F\right]\partial_\rho\psi = F\bar{\nabla}^\mu\bar{\nabla}_\mu\psi - \bar{g}^{\rho\mu}\partial_\mu F\partial_\rho\psi\,. \tag{3.10}$$

Combining this result with Eq. (3.7), we obtain the Klein-Gordon equation in the Einstein frame,

$$\bar{\nabla}^\mu\bar{\nabla}_\mu\psi = \frac{1}{F}\nabla^\mu\nabla_\mu\psi + \bar{g}^{\mu\nu}\frac{\partial_\mu F}{F}\partial_\nu\psi = \frac{1}{F}V_{,|\psi|^2}\psi + \bar{g}^{\mu\nu}\frac{\partial_\mu F}{F}\partial_\nu\psi\,, \tag{3.11}$$

or

$$\boxed{\bar{\nabla}^\mu\bar{\nabla}_\mu\psi = \bar{g}^{\mu\nu}\frac{\partial_\mu F}{F}\partial_\nu\psi + \frac{1}{F}V_{,|\psi|^2}\psi\,.} \tag{3.12}$$

4 Stationary, pherically symmetric boson stars

4.1 The field equations in areal gauge and polar slicing

We now have a wide range of choices to compute the complete set of equations for the metric and matter variables. Consider, for example the line element in radial gauge and polar slicing in the Jordan frame,

$$\mathrm{d}s^2 \coloneqq g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -N^2\mathrm{d}t^2 + B^2\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,, \tag{4.1}$$

and plug these metric functions into the field equations (2.8) for the physical frame. We would end up with a set of partial differential equations for N and B plus the matter equations. Alternatively, we could rescale the metric components into the Einstein frame and write the line element as

$$\mathrm{d}\bar{s}^2 \coloneqq \bar{g}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = Fg_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -FN^2\mathrm{d}t^2 + FB^2\mathrm{d}r^2 + Fr^2\mathrm{d}\Omega^2\,, \tag{4.2}$$

and plug this line element into the field equations (2.6) for the Einstein frame. This would give us exactly the same differential equations for N and B, just derived from a different (yet equivalent) set of covariant equations. We could also introduce Einstein-frame metric functions $\bar{N} := FN$ and $\bar{B} := FB$ and would get other differential equations for the functions $\bar{N}$ and $\bar{B}$. Likewise, we have the freedom to represent the gravitational scalar field by the variable ϕ or the variable φ which are related by Eq. (2.3).

Note that we have two different types of freedom here. First, we can choose the Einstein or Jordan line element and then use the covariant equations (2.6) or (2.8). Second, and independently, we can rescale the actual variables we use for the metric components and the gravitational scalar field. Only the boson-star scalar field remains ψ throughout. I guess, we might even play with that one, but I see no point in doing this; life's hard enough with the freedom we already have.

So what choices are we going to make? Possibly, some choices may end up simplifying the calculations a bit, but I doubt it; this looks too much like conservation of misery. Only in the vacuum case, there is a preference for the Einstein frame, since in that case we have no matter and the gravitational scalar field appears merely in minimally coupled form. This has the enormous benefit that a single numerical simulation represents the result for any choice of the conformal factor function F which just doesn't appear anywhere in the equations but can be reconstructed from the scalar field using one and the same simulation for any choice of F. But here we have matter and no such blessing of any frame. We therefore follow our notes for the core collapse calcuations and the formalism used in Ref. [10]. This gives us an independent check of the metric sector of the field equations we wish to calculate. Of course, the matter equations will be totally different, but it's better than no check at all.

So we start with the line element

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\alpha^2 \mathrm{d}t^2 + X^2 \mathrm{d}r^2 + \frac{r^2}{F}\mathrm{d}\Omega^2 \,. \tag{4.3}$$

where α and X are functions of the radius r only. This means that for now we work in the Jordan frame although we shall use the gravitational scalar-field variable φ of the Einstein frame. At some point, we might also introduce the alternative metric functions

$$\Phi := \ln(\sqrt{F}\alpha)\,, \qquad\qquad m := \frac{r}{2}\left(1 - \frac{1}{FX^2}\right)\,. \tag{4.4}$$

The gravitational scalar field will be $\varphi = \varphi(r)$ but the boson-star field is time dependent and we simplify the calculation by assuming harmonic time dependence with a constant frequency,

$$\psi(t, r) = A(r)e^{\mathrm{i}\omega t}\,. \tag{4.5}$$

In the Jordan frame, the energy momentum tensor for a boson star is the same as in general relativity,

$$T_{\alpha\beta} = \frac{1}{2}\partial_\alpha\psi^*\partial_\beta\psi + \frac{1}{2}\partial_\alpha\psi\partial_\beta\psi^* - \frac{1}{2}g_{\alpha\beta}\left[g^{\mu\nu}\partial_\mu\psi^*\partial_\nu\psi + V(\psi)\right]\,. \tag{4.6}$$

Most likely, the equations will be a bit easier to derive in the Einstein frame, so we will be using the covariant equations (2.6) with the line element

$$\mathrm{d}\bar{s}^2 = -F\alpha^2\mathrm{d}t^2 + FX^2\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,. \tag{4.7}$$

We will also need the energy momentum tensor in the Einstein frame which is given by

$$\bar{T}_{\alpha\beta} = \frac{1}{F}T_{\alpha\beta} = \frac{1}{F}\left\{\frac{1}{2}\partial_\alpha\psi^*\partial_\beta\psi + \frac{1}{2}\partial_\alpha\psi\partial_\beta\psi^* - \frac{1}{2F}\bar{g}_{\alpha\beta}\left[F\bar{g}^{\mu\nu}\partial_\mu\psi^*\partial_\nu\psi + V(\psi)\right]\right\}\,. \tag{4.8}$$

With this $\bar{T}_{\alpha\beta}$, we can evaluate the covariant field equations (2.6) for the line element (4.7). More specifically, we use the first two equations of (2.6), but replace the third one by the wave equation (3.12) that we have already derived directly from varying the action with respect to the scalar field. In total, we obtain one wave equation for the complex ψ, one wave equation for the gravitational scalar field φ and 4 equations for the metric components, one each for the non-trivial components G_{tt}, G_{rr}, $G_{\theta\theta}$ and $G_{\phi\phi}$ (sorry for double using the letter ϕ here as azimuthal angle; the distinction from the scalar field in the Jordan frame should be clear enough to avoid confusion). The equation from $G_{\phi\phi}$ is the same as that for $G_{\theta\theta}$ except for an overall factor $\sin^2\theta$. G_{rr} and G_{tt} give us the main field equations for α and X. Together with the two wave equations, we thus obtain 4 equations for 4 variables which in explicit form are

$$\frac{\partial_r\alpha}{\alpha} = \frac{FX^2-1}{2r} - rFX^2W - \frac{\partial_r F}{2F} + \frac{r}{2}(\partial_r\varphi)^2 + 2\pi r\frac{X^2}{F}\left[\frac{(\partial_r A)^2}{X^2} + \frac{A^2\omega^2}{\alpha^2} - V\right], \tag{4.9}$$

$$\frac{\partial_r X}{X} = -\frac{FX^2-1}{2r} + rFX^2W - \frac{\partial_r F}{2F} + \frac{r}{2}(\partial_r\varphi)^2 + 2\pi r\frac{X^2}{F}\left[\frac{(\partial_r A)^2}{X^2} + \frac{A^2\omega^2}{\alpha^2} + V\right], \tag{4.10}$$

$$\partial_r^2\varphi = \partial_r\varphi\left(\frac{\partial_r X}{X} - \frac{\partial_r\alpha}{\alpha}\right) - \frac{2}{r}\partial_r\varphi + FX^2W_{,\varphi} + \frac{2\pi X^2F_{,\varphi}}{F^2}\left[\frac{A^2\omega^2}{\alpha^2} - \frac{(\partial_r A)^2}{X^2} - 2V\right] \tag{4.11}$$

$$\partial_r^2 A = -2\frac{\partial_r A}{r} + \partial_r A\left(\frac{\partial_r X}{X} - \frac{\partial_r\alpha}{\alpha} + \frac{\partial_r F}{F}\right) - \frac{X^2\omega^2 A}{\alpha^2} + X^2V_{,|\psi|^2}A\,. \tag{4.12}$$

The final field equation for $G_{\theta\theta}$ can then be shown to be a consequence of these four differential equations. This is a non-trivial calculation, though, and represents a valuable test to ensure consistency of the equations. In fact, we have identified several bugs in the equations and their derivations when the $G_{\theta\theta}$ equation could *not* be reproduced from Eqs. (4.9)-(4.12). Once these got fixed, an appropriate linear combination, involving also derivatives of the main equations, ended up identical to the $G_{\theta\theta}$ one. Still, readers should be on the watch for possible errors in (4.9)-(4.12) that might also have crept in through latexing.

In Eq. (4.4) we introduced the alternative variables Φ and m for the metric components g_{tt} and g_{rr}. A straightforward calculation using

$$\partial_r m = \frac{1}{2}\left(1 - \frac{1}{FX^2}\right) + \frac{r}{2FX^2}\left(2\frac{\partial_r X}{X} + \frac{\partial_r F}{F}\right) = \frac{m}{r} + \frac{r}{2FX^2}\left(2\frac{\partial_r X}{X} + \frac{\partial_r F}{F}\right), \tag{4.13}$$

$$\partial_r\Phi = \frac{\partial_r F}{2F} + \frac{\partial_r\alpha}{\alpha}, \tag{4.14}$$

shows that this substitution replaces Eqs. (4.9), (4.10) by

$$\partial_r\Phi = \frac{FX^2-1}{2r} - rFX^2W + \frac{r}{2}(\partial_r\varphi)^2 + 2\pi r\frac{X^2}{F}\left[\frac{(\partial_r A)^2}{X^2} + \frac{A^2\omega^2}{\alpha^2} - V\right], \tag{4.15}$$

$$\partial_r m = \frac{r^2}{2FX^2}(\partial_r\varphi)^2 + r^2W + 2\pi r^2\frac{1}{F^2}\left[\frac{(\partial_r A)^2}{X^2} + \frac{A^2\omega^2}{\alpha^2} + V\right]. \tag{4.16}$$

Finally, we may find the following relations helpful to replace derivatives on the right-hand side of

Eqs. (4.9)-(4.16),

$$\frac{\partial_r X}{X} - \frac{\partial_r \alpha}{\alpha} = -\frac{FX^2-1}{r} + 2rFX^2W + 4\pi r\frac{X^2}{F}V\,, \tag{4.17}$$

$$\frac{\partial_r F}{F} = \frac{F_{,\varphi}}{F}\partial_r\varphi\,. \tag{4.18}$$

We'll have to see which specific set of equations turns out most convenient for our numerical studies. At present, I am inclined to work with Φ but stick to X instead of using the mass m which has this pesky $\propto r^3$ behaviour at the origin. But experience and pain will lead us to the best pastures...

4.2 The asymptotic behaviour

We next explore the asymptotic behaviour of the variables and, in particular, the scalar field variables. For this purpose, let us replace any derivatives on the right-hand sides of Eqs. (4.11), (4.12). For this purpose we use

$$\frac{\partial_r X}{X} - \frac{\partial_r \alpha}{\alpha} + \frac{\partial_r F}{F} = -\frac{FX^2-1}{r} + 2rFX^2W + 4\pi r\frac{X^2}{F}V + \frac{\partial_r F}{F}\,, \tag{4.19}$$

which gives us the scalar-field equations in the form

$$\partial_r^2 A = -2\frac{\partial_r A}{r} + \partial_r A\left(-\frac{FX^2-1}{r} + 2rFX^2W + 4\pi r\frac{X^2}{F}V + \frac{\partial_r F}{F}\right) - \frac{X^2\omega^2A^2}{\alpha^2} + X^2V_{,|\psi|^2}A\,. \tag{4.20}$$

To simplify the asymptotic analysis, we now make the following assumptions without proof.

1. We assume the scalar fields fall off at least as fast as $\propto \frac{1}{r}$.
2. The conformal factor F approaches unity at infinity, i.e. $F(\infty)=1$.
3. The metric approaches the Schwarzschild limit as $r\to\infty$, i.e. $X^2 \to \left(1-\frac{2M}{r}\right)^{-1} \approx \left(1+\frac{2M}{r}\right)$ and $\alpha^2 \to \left(1-\frac{2M}{r}\right)$.

Inserting $A\sim r^n$ into Eq. (4.20), we obtain an equation which at leading order (in each term) can be written as

$$a_0 r^{n-2} = a_1 r^{n-2} + a_2 r^{n-3} + (\mu^2-\omega^2)a_3 r^n\,, \tag{4.21}$$

for non-zero coefficients a_i. Unless $\omega^2=\mu^2$, the last term cannot be cancelled by any other term for any value of n, so that A cannot fall off like any power of r but has to fall off faster. In the $r\to\infty$ limit, Eq. (4.20) simplifies to

$$\partial_r^2 A + \frac{2}{r}\partial_r A = (\mu^2-\omega^2)A$$

$$\Rightarrow\quad \partial_r(r^2\partial_r A) = r^2(\mu^2-\omega^2)A \qquad \Big|\quad B := rA,\quad \Rightarrow\quad \partial_r B = A + r\partial_r A$$

$$\Rightarrow\quad \partial_r[r(\partial_r B - A)] = (\mu^2-\omega^2)rB$$

$$\Rightarrow\quad \partial_r B - A + r\partial_r^2 B - r\partial_r A = (1-\omega^2)rB$$

$$\Rightarrow\quad r\partial_r^2 B = (\mu^2-\omega^2)rB$$

$$\Rightarrow\quad \partial_r^2 B = (\mu^2-\omega^2)B\,, \tag{4.22}$$

which gives us the asymptotic behaviour

$$B \sim e^{\pm\sqrt{\mu^2-\omega^2}r} \quad \Rightarrow \quad A \sim \frac{e^{\pm\sqrt{\mu^2-\omega^2}r}}{r}\,. \tag{4.23}$$

Note that we would replace ω with ω/α if working in a gauge where α does not asymptote to one.

For the gravitational scalar φ, we consider Eq. (4.11) and make again an ansatz $\varphi \sim r^n$ to leading order at $r \to \infty$. We furthermore assume that $X^2 \to 1 + \frac{2M}{r}$ and $\alpha^2 \to 1 - \frac{2M}{r}$, so that

$$\frac{\partial_r X}{X} - \frac{\partial_r \alpha}{\alpha} \to -\frac{2M}{r^2}\,.$$

Finally, we consider a non-interacting potential $W = \frac{1}{2}\mu_\varphi \varphi^2$ and bear in mind the exponential falloff of A[4] The leading order behaviour of the terms in Eq. (4.11) can then be written as

$$n(n-1)r^{n-2} = -2Mnr^{n-3} - 2nr^{n-2} + \mu_\varphi^2 r^n\,, \tag{4.24}$$

where the last (and crucial) term arises from $FX^2W_{,\varphi}$ in Eq. (4.11). Unless $\mu_\varphi^2 = 0$, this term cannot be cancelled by any of the other terms, so that we conclude a non-polymonial falloff of φ in r^{-1} for a massive gravitational scalar field.

The analysis of Eq. (4.11) now presents us with a subtlety that might require more exploration, but let us first consider the simpler case where we assume that in Eq. (4.11), the term

$$2\pi X^2 \frac{F_{,\varphi}}{F^2}\left[\frac{A^2\omega^2}{\alpha^2} - \frac{(\partial_r A)^2}{X^2} - 2V\right]\,, \tag{4.25}$$

can be ignored. For the neutron star case, this was manifestly true since the matter distribution had compact support. For a boson star, this is not the case, since A extends to infinity, albeit with exponential decay. But let us ignore this term for now. Then to leading order around infinity, Eq. (4.11) becomes

$$\partial_r^2\varphi + 2\frac{\partial_r\varphi}{r} = \mu_\varphi^2\varphi\,, \tag{4.26}$$

which is exactly the same for as Eq. (4.22), merely with μ_φ^2 replacing $\mu^2 - \omega^2$ We then obtain the asymptotic behaviour

$$\varphi \sim \frac{e^{\pm\mu_\varphi r}}{r}\,. \tag{4.27}$$

So far so good; this is the asymptotic behaviour we have for neutron stars in ST theory and is what we might also expect here. The subtlety arises from the term (4.25) above. In contrast to Eq. (4.12), equation (4.11) for φ is not linear in its main variable, i.e. in φ, and it is precisely the term (4.25) which ruins this linearity. As we discuss further below, we will use a coupling function of the form $F(\varphi) = e^{-2\alpha_0\varphi - \beta_0\varphi^2}$ which implies that

$$\frac{F_{,\varphi}}{F} = \frac{\partial \ln F}{\partial\varphi} = -2\alpha_0 - 2\beta_0\varphi\,. \tag{4.28}$$

Here, α_0 and β_0 are parameters that have nothing to do with the lapse function α (again, sorry, out of Greek letters). For $\alpha_0 = 0$, all is fine and we have the asymptotic behaviour (4.27). But if $\alpha_0 \neq 0$,

[4]A more rigorous calculation should probably treat expansions of both scalar fields and all metric functions simultaneously, but that's very lengthy and we satisfy ourselves here with a more heuristic derivation.

we have in Eq. (4.11) two competing exponential falloffs, one from φ and the other from A or, more accurately, the terms $\propto A^2$ in the source term (4.25). Specifically,we have two competing terms of the form

$$\varphi \sim e^{-\mu_\varphi r} \qquad \text{and} \qquad e^{-2\sqrt{\mu^2-\omega^2}r}\,, \tag{4.29}$$

where we only keep the dominant exponential dependencies, and dropped factors of $1/r$. If $\mu_\varphi < 2\sqrt{\mu^2-\omega^2}$, then φ falls off more slowly than the A^2 terms sourcing it and the asymptotic behvaviour should be as for $\alpha_0 = 0$. But if $\mu_\varphi > 2\sqrt{\mu^2-\omega^2}$, this does not work. Possibly, φ then inherits the behaviour of A^2, maybe we have no solutions or maybe something really interesting happens. I do not know...

4.3 Boundary conditions

The boundary conditions are the main challenge for our numerical implementation. We know from the boson-star calculations in GR and the neutron-star modeling in scalar-tensor gravity that both scalar fields, φ and ψ have a tendency to blow up at large radii due to the presence of exponentially growing modes as formal solutions to the PDEs. It wasn't easy to control these fields individually, now we'll have both acting together.

So let us gather the boundary conditions that we know. At the origin $r = 0$, we have from regularity that

$$\partial_r\varphi = 0\,, \qquad \partial_r A = 0\,. \tag{4.30}$$

Finite density at the origin immediately implies that at $r = 0$

$$m = 0 \qquad \Rightarrow \qquad X^2 = \frac{1}{F(\varphi)}\,. \tag{4.31}$$

The central scalar field $A(0) =: A_\mathrm{c}$ is the free parameter that selects our boson-star model. The gravitational scalar at $r = 0$ is a bit trickier; we will likely not be able to choose φ_c freely, but need to choose a specific value to ensure asymptotic flatness at infinity. The situation would be similar for the boson-star scalar field if we had decided to fix the boson-star frequency. We remain open minded, but will probably stick to varying the frequency instead of A_c. This frequency ω thus enters our calculations as the Eigenvalue. Our expectation is that only for discrete values of ω *and* appropriate choices of φ_c regular solutions will be found. Depending on the choice of the function $F(\varphi)$, these may include the general relativistic boson stars we already know, but we expect that there will also be boson-star solutions with a nontrivial profile $\varphi(r)$. These latter solutions should end up being the scalarized boson-star models we will ultimately be interested in. Finally, we notice that the lapse function Φ or α only enters in the form $\partial_r\Phi$ on the left-hand side and ω^2/α^2 on the right-hand side (the $\partial_r\alpha/\alpha$ are substituted in terms of other functions without further ado). We are therefore free to add a constant to Φ together with a simultaneous rescaling of ω without affecting anything else. For starters we therefore will set $\Phi(r = 0) = 1$ but may eventually rescale this to obtain a lapse function $\alpha = 1$ at infinity. In GR, this simply implies matching to the Schwarzschild metric. In scalar-tensor gravity, this will probably be more complicated but likely work in the same way we have handled it for the core collapse in Ref. [10].

To summarize the boundary conditions, we have four main variables, Φ, X, φ, A. The equations for the latter two are second-order in r, so we expect a total of 6 boundary conditions plus the value of the frequency ω. The 6 boundary conditions are

$$\Phi(0) = 1\,, \qquad A(0) = A_{\rm c} = \text{free}\,, \qquad \varphi(0) = \varphi_{\rm c} = \text{not free}\,, \qquad X(0) = \frac{1}{\sqrt{F(\varphi(0))}}\,,$$
$$A(\infty) = 0\,, \qquad \varphi(\infty) = 0\,. \tag{4.32}$$

We expect that the conditions at ∞ will only be reachable for selected values of $\varphi_{\rm c}$ and ω, so these two parameters are not free. Another way to look at this is to consider the regularity requirements (4.30) on top of the 6 boundary conditions listed. We then have an overdetermined system that we expect to yield solutions only if the two parameters $\varphi_{\rm c}$ and ω have appropriate values. Finding these values is gonna be our main challenge...

5 Units, potentials and the coupling function

The above field equations are expressed in natural units where $G = c = 1$ and we now discuss how everything can ultimately be translated into SI units. This naturally involves the potential functions and we therefore bundle the discussion of units and the free functions into one here.

From the boson-star calculations in general relativity [11], we recall that dimensionless code units can be translated into SI units by using the (bosonic) scalar field mass μ and the gravitational constant according to

$$\tilde{t} = \frac{t}{\mu}\,, \qquad \tilde{r} = \frac{r}{\mu}\,, \qquad \tilde{m} = \frac{m}{\mu}\,, \qquad \tilde{A} = \frac{A}{\sqrt{G}} = AM_{\rm Pl}\,, \qquad \tilde{\omega} = \omega\mu\,. \tag{5.1}$$

where $M_{\rm Pl}$ denotes the Planck mass and μ is the mass of the boson-star's scalar field. For reference, we recall here that a mass of $10^{-14}\,$eV corresponds to a frequency of 2.418 Hz or an angular frequency of $15.19\,{\rm s}^{-1}$. Using the speed of light and the gravitational constant, we can directly convert these numbers into SI units for length or mass. The relation scalar mass versus frequency is linear, so that we can easily rescale to any other number of electron volts, such as $10^{-13}\,$eV being 24.18 Hz and so on. In general, we allow for the boson star's scalar field to be massive *and* self-interacting according to a potential function

$$\tilde{V}(\tilde{\psi}) = \tilde{V}(|\tilde{\psi}|^2) = \tilde{V}(\tilde{A}^2) = \mu^2\tilde{A}^2 + \Lambda_4\tilde{A}^4 + \Lambda_6\tilde{A}^6 + \Lambda_8\tilde{A}^8\,. \tag{5.2}$$

Inside the code, however, we work with rescaled units which results in the rescaled potential function

$$V(A^2) = A^2 + \lambda_4 A^4 + \lambda_6 A^6 + \lambda_8 A^8 \quad \text{with} \quad \lambda_n = \frac{M_{\rm Pl}^{n-2}}{\mu^2}\Lambda_n\,. \tag{5.3}$$

The specific solitonic potential

$$V(A^2) = A^2\left(1 - 2\frac{A^2}{\sigma_0^2}\right)^2\,, \tag{5.4}$$

corresponds to the values

$$\lambda_4 = -\frac{4}{\sigma_0^2}\,, \qquad \lambda_6 = \frac{4}{\sigma_0^4}\,, \qquad \lambda_8 = 0\,. \tag{5.5}$$

The gravitational scalar field may also have a mass. Since we have already used the letter m, we will denote this mass temporarily by $\mathscr{M}$, but quickly replace it, following our notation in Refs. [17, 13], by $\mu_\varphi := \mathscr{M}/\hbar$. The potential function for a massive but non-interacting gravitational scalar field is then given by

$$W(\varphi) = \frac{1}{2}\mu_\varphi^2 \varphi^2 \,. \tag{5.6}$$

Note the inconsistent conventions used in Eqs. (5.3), (5.6) where the latter has a factor 1/2 absent in the former. This is inherited and for now let's keep it for compatibility with our previous work on boson stars and scalar-tensor gravity.

The final function we need to specify is the conformal factor $F(\varphi)$. Here, we follow Damour & Esposito-Farèse and use

$$F(\varphi) = e^{-2\alpha_0\varphi - \beta_0\varphi^2} \,. \tag{5.7}$$

We have a more in-depth discussion of this coupling function in our notes on the core-collapse work. For massless gravitational scalars, the parameters α_0 and β_0 are rather tightly constrained to about $\alpha_0 \lesssim 10^{-3}$ and $\beta_0 \gtrsim -4.5$. Here lies one of the main motivations to consider *massive* scalar-tensor gravity.

So, how can we interprete the numbers generated by our codes? The first step consists in using Eq. (5.1); we choose a mass value μ and can use this to convert dimensionless time, radius etc into dimensionfull quantities. Note that the mass function m is defined in terms of the Einstein frame metric component $\bar{g}_{rr}$, so that it represents the Einstein frame mass. At infinity, however, physical and Einstein mass are the same since $F(\varphi = 0) = 1$. Next, let us consider the mass parameter μ_φ. Its meaning is best understood by considering the wave equation for φ in Eq. (2.6),

$$\bar{\Box}\varphi = 2\pi \frac{F_{,\varphi}}{F}\bar{T} + W_{,\varphi} \,. \tag{5.8}$$

At infinite radius, we have $\bar{T} = 0$ and $W_{,\varphi} = \mu_\varphi^2\varphi$, so that we obtain the plain and simple massive wave equation $\bar{\Box}\varphi = \mu_\varphi^2\varphi$. In spherical symmetry, this becomes a 1+1 wave equation for $\sigma := r\varphi$ with the dispersion relation $\omega^2 = k^2 + \mu_\varphi^2$ for individual Fourier modes. The parameter μ_φ therefore corresponds to the mass or frequency of the gravitational scalar field. Like all other quantities, it is measured in units of the boson star's scalar field mass μ, so that we can interprete the dimensionles μ_φ as the gravitational scalar mass measured in units of μ or

$$\tilde{\mu}_\varphi = \mu_\varphi\,\mu \,. \tag{5.9}$$

For instance, setting $\mu_\varphi = 1$ means that both scalar fields have the same mass whereas $\mu_\varphi = 0$ recovers the case of massless scalar-tensor gravity.

Let us summarize the potential and coupling functions as they will likely be used in the codes,

$$
\begin{aligned}
V(A^2) &= A^2 + \lambda_4 A^4 + \lambda_6 A^6 + \lambda_8 A^8\,, \qquad \lambda_4 = -\frac{4}{\sigma_0^2}\,, \quad \lambda_6 = \frac{4}{\sigma_0^4}\,, \\
V_{,|\psi^2|} &= 1 + 2\lambda_4 A^2 + 3\lambda_6 A^4 + 4\lambda_8 A^6\,, \\
W(\varphi) &= \frac{1}{2}\mu_\varphi^2 \varphi^2\,, \\
W_{,\varphi} &= \mu_\varphi^2 \varphi\,, \\
F(\varphi) &= e^{-2\alpha_0\varphi - \beta_0\varphi^2}\,, \\
\frac{F_{,\varphi}}{F} &= -2\alpha_0 - 2\beta_0\varphi\,.
\end{aligned} \tag{5.10}
$$

6 Brief description of previous works

[***TE: Some references:***]

1. *"Generation of scalar-tensor gravity effects in equilibrium state boson stars"* [6] - BS solutions in Brans-Dicke and calculation is done in the Einstein frame. Section 3.6 goes over their procedure for finding appropriate $\varphi(0)$ so that $\varphi(\infty) = 0$ is satisfied. They find that the gravitational scalar field falls off more slowly than the matter scalar field. Note that their gravitational scalar field is massless, so as such it is expected to fall off slower. In the case both fields are massive, they are expected to fall off exponentially. However, because their case is massless, they suggest that it is better to set at the numerical boundary r_∞ that $\varphi(r_\infty) = \varphi(\infty) + C/r_\infty$, where C is a constant such that $C = -r_\infty^2 \frac{d\varphi}{dr}(r = r_\infty)$.

2. *"Dynamical evolution of boson stars in Brans-Dicke theory"* [2] - dynamical evolution for BSs in Brans-Dicke in the Einstein frame again. Section C contains scalar field evolution equations, which might be useful proxies.

7 The asymptotic behaviour of boson-start solutions in GR

7.1 The problem

The behaviour of a boson star's scalar field near infinity is rather tricky and we discuss it here in more detail for the simpler case of general relativity. In the case of stationary boson stars, we are dealing with a complex scalar field decomposed into amplitude and phase according to

$$
\psi(t, r) = A(r) e^{i\omega t}\,, \tag{7.1}
$$

and the line element, chosen to be in radial gauge and polar slicing here,

$$\mathrm{d}s^2 = -\alpha^2 \mathrm{d}t^2 + X^2 \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2 \,. \tag{7.2}$$

The metric can also be represented by the alternative functions

$$\Phi := \ln\alpha \quad \Leftrightarrow \quad \alpha = e^{\Phi}\,, \qquad \text{and} \qquad X^2 = \left(1 - \frac{2m}{r}\right)^{-1} \quad \Leftrightarrow \quad m = \frac{r}{2}\frac{X^2-1}{X^2}\,. \tag{7.3}$$

The field equations for the boson star then become

$$\frac{\partial_r \alpha}{\alpha} = \frac{X^2-1}{2r} + 2\pi r X^2 \left(\eta^2 + \frac{\omega^2 A^2}{\alpha^2} - V\right)\,, \tag{7.4}$$

$$\frac{\partial_r X}{X} = -\frac{X^2-1}{2r} + 2\pi r X^2 \left(\eta^2 + \frac{\omega^2 A^2}{\alpha^2} + V\right)\,, \tag{7.5}$$

$$\partial_r A = X\eta\,, \tag{7.6}$$

$$\partial_r \eta = -2\frac{\eta}{r} - \eta\frac{\partial_r \alpha}{\alpha} + X\left(V' - \frac{\omega^2}{\alpha^2}\right) A\,, \tag{7.7}$$

where

$$V' := \frac{\mathrm{d}V}{\mathrm{d}(A^2)}\,. \tag{7.8}$$

The equations for the alternative variables Φ and m are

$$\partial_r \Phi = \frac{X^2-1}{2r} + 2\pi r X^2 (\eta^2 + \omega^2 e^{-2\Phi} A^2 - V)\,, \tag{7.9}$$

$$\partial_r m = 2\pi r^2 (\eta^2 + \omega^2 e^{-2\Phi} A^2 + V)\,. \tag{7.10}$$

For completeness, we also note the second-order equation for the scalar field amplitude,

$$\partial_r^2 A = -2\frac{\partial_r A}{r} + \partial_r A\left(-\frac{X^2-1}{r} + 4\pi r X^2 V\right) - \frac{X^2}{\alpha^2}\omega^2 A + X^2 V' A\,. \tag{7.11}$$

We will only consider potentials with a leading order term $\sim A^2$ and higher-order terms of even power in A, i.e.

$$V(A) = A^2 + \lambda_4 A^4 + \lambda_6 A^6 + \ldots\,. \tag{7.12}$$

Note that we are using here units normalized by the scalar field's mass μ, which enables to set the coefficient of the quadratic term to unity. In other notes, we have labeled these rescaled coordinates with an overbar as in $\bar{r}$, but we will exclusively use rescaled coordinates here, so that no confusion arises and we can therefore drop this notational inconvenience. The key point about the potential (7.12) is that for expoentially decaying scalar fields, all higher-order terms become irrelevant in any series expansions.

We have seen above that the flat-field equation for the scalar field implies an asymptotic behaviour

$$A \sim \frac{e^{-kr}}{r} \tag{7.13}$$

in the limit $r \to \infty$. This is indeed confirmed by the asymptotic expansion we will carry out below. But, and that is the strange result, we arrive at a contradiction in the series expansion if we postulate the same asymptotic behaviour with gravity even though we recover the Minkowski metric as $r \to \infty$. That is, unless there is some mistake in the below calculation... Finding that out is the goal of this section.

Our first step is to translate Eqs. (7.4)-(7.7) and (7.9), (7.10) to the radial variable $y = 1/r$ where spatial infinity is represented by $y = 0$ and which enables us to apply a series expansion in y around $y = 0$. Hypothecizing for the moment the asymptotic behaviour given by the flat-field limit 7.13), or,

$$A \sim y e^{-k/y}\,, \tag{7.14}$$

we define the rescaled variables

$$\sigma := A e^{k/y} \quad \Leftrightarrow \quad A = \sigma e^{-k/y} \qquad \text{and} \qquad \lambda := -e^{k/y}\eta \quad \Leftrightarrow \quad \eta = -\lambda e^{-k/y}\,. \tag{7.15}$$

Here $k = \sqrt{1-\omega^2}$ according to our flat-field analysis, but we will regard k as an otherwise free positive constant for now. It turns out to be determined by the asymptotic expansion. Using

$$\partial_r = -y^2 \partial_y\,, \tag{7.16}$$

it is then straightforward to show that the field equations for the boson star become

$$\partial_y \Phi = -\frac{X^2-1}{2y} - \frac{2\pi}{y^3 e^{2k/y}} X^2 \left[\lambda^2 + \sigma^2(\omega^2 e^{-2\Phi} - \hat{V})\right]\,, \tag{7.17}$$

$$\partial_y m = -\frac{2\pi}{y^4 e^{2k/y}} \left[\lambda^2 + \sigma^2(\omega^2 e^{-2\Phi} + \hat{V})\right]\,, \tag{7.18}$$

$$\partial_y \sigma = \frac{X\lambda - \sigma k}{y^2}\,, \tag{7.19}$$

$$\partial_y \lambda = \frac{2y-k}{y^2}\lambda - \lambda \partial_y \Phi + \frac{X\sigma}{y^2}(V' - \omega^2 e^{-2\Phi})\,, \tag{7.20}$$

where $\hat{V} := V/\sigma^2$. In Eq. (7.20), we have not substituted for the derivative $\partial_y \Phi$, but either way will be fine for the asymptotic analysis. The key effect of using the rescaled variables σ and λ is that the potentially troublesome term $1/y^4$ in Eq. (7.18) is accompanied by an extra factor $e^{2k/y}$ which goes to infinity faster than any polynomial in y goes to zero. In the spirit of a series expansion, the right-hand side of Eq. (7.18) will therefore vanish at any order y^n. The series expansion for m is therefore given by a constant $m = M$. This is, of course, what we expect for an exponentially decaying matter source which gets arbitrarily close to vacuum at large radii. The series expansion for X follows directly from its definition

$$X = \left(1 - \frac{2m}{r}\right)^{-1/2}\,, \tag{7.21}$$

and for the lapse function we use the fact that $\alpha X = 1$ at infinity for a Schwarzschild metric at infinity, i.e.

$$\alpha = \left(1 - \frac{2m}{r}\right)^{1/2} \quad \Rightarrow \quad \Phi = \frac{1}{2}\ln\left(1 - \frac{2m}{r}\right) . \tag{7.22}$$

This gives us the series expansion for the metric variables as

$$m(y) = M , \tag{7.23}$$

$$\alpha(y) = 1 - My - \frac{1}{2}M^2y^2 - \frac{1}{2}M^3y^3 - \frac{5}{8}M^4y^4 + \mathcal{O}(y^5) , \tag{7.24}$$

$$\Phi(y) = -My - M^2y^2 - \frac{4}{3}M^3y^3 - 2M^4y^4 - \frac{16}{5}M^5y^5 + \mathcal{O}(y^6) , \tag{7.25}$$

$$X(y) = 1 + My + \frac{3}{2}M^2y^2 + \frac{5}{2}M^3y^3 + \frac{35}{8}M^4y^4 + \mathcal{O}(y^5) . \tag{7.26}$$

We have been quite exuberant here in including higher-order terms; for the actual calculations, the leading and next-to-leading order terms should suffice, but to make sure we don't loose anything by accident, we threw a good deal more into the calculation (`Maple` is quite forgiving in this context).

The series for the scalar field variables σ and λ are not determined without further use of the equations, but we see from Eq. (7.13) that $\sigma \sim y$ at leading order and then Eq. (7.19) gives us the leading-order behaviour of λ as $\lambda \sim y$ as long as $k \neq 0$. We allow for higher-order terms in both these variables and therefore consider series expansions of σ and λ given by

$$\begin{aligned} \sigma(y) &= ay + by^2 + cy^3 + dy^4 + \mathcal{O}(y^5) , \\ \lambda(y) &= Ay + By^2 + Cy^3 + Dy^4 + \mathcal{O}(y^5) . \end{aligned} \tag{7.27}$$

Again, we are using here more terms than necessary, just out of paranoia to avoid any risk of loosing out on potentially higher-order terms becoming relevant through derivatives or unexpected cancelations.

We then plug all these series expansions into the field equations (7.19) and (7.20) and compare the coefficients for the resulting set of terms at equal order in y. We list the relations thus obtained as follows labeled by the equation used – $\partial_y\sigma$ for Eq. (7.19) and $\partial_y\lambda$ for Eq. (7.20) – and the respective order in y, starting with the lowest. This gives us,

$$\begin{aligned} \partial_y\sigma,\quad y^{-1}: &\qquad -ak + A = 0 , \\ \partial_y\sigma,\quad y^{0}: &\qquad MA - kb + B - a = 0 , \\ \partial_y\sigma,\quad y: &\qquad -2b + C + BM + \frac{3}{2}M^2A - kc = 0 , \\ \partial_y\lambda,\quad y^{-1}: &\qquad -a\omega^2 - Ak + a = 0 , \\ \partial_y\lambda,\quad y^{0}: &\qquad -3Ma\omega^2 - b\omega^2 - Bk + Ma + A + b = 0 , \\ \partial_y\lambda,\quad y: &\qquad -c\omega^2 - \frac{15}{2}\omega^2M^2 - kC - 3Mb\omega^2 + MA + c + \frac{3}{2}M^2a + Mb = 0 . \end{aligned} \tag{7.28}$$

From the first relation, we immediately conclude that

$$A = ak\,, \tag{7.29}$$

and plugging this into the y^{-1} result for $\partial_y \lambda$, we get

$$a(1 - k^2 - \omega^2) = 0 \quad \Rightarrow \quad k^2 = 1 - \omega^2\,, \tag{7.30}$$

which is exactly what we expect to obtain for the exponential coefficient k. So far so good. The problem arises when we look at the y^0 relations for the two equations. From the first, we obtain

$$B = a - akM + bk\,, \tag{7.31}$$

which, together with $k^2 = 1 - \omega^2$, we readily plug into the second to find

$$2Ma(1 - 2\omega^2) = 0\,. \tag{7.32}$$

This is problematic, because it implies that at order y^0, Eqs. (7.19) and (7.20) can only be solved simultaneously if $\omega^2 = 1/2$. We have checked this result by rewriting the second-order equation (7.11) in terms of the rescaled variable σ and the compactified coordinate y where it becomes

$$\partial_y^2 \sigma = -\frac{1 - X^2}{y}\partial_y \sigma + \sigma k \frac{y - k}{y^4} - 4\pi \partial_y \sigma \frac{X^2 V}{y^3} - \frac{2k}{y^2}\partial_y \sigma + \frac{X^2}{y^4}\sigma\left(ky + V' - \frac{\omega^2}{\alpha^2} - 4\pi\frac{kV}{y}\right)\,. \tag{7.33}$$

We can then plug in the same series expansions as above and obtain the same leading-order relations (7.30) and (7.32).

In the absence of gravity, we have $M = 0$ and all is well since Eq. (7.32) vanishes identically, but as soon as $M \neq 0$, the problem reappears. What does this mean? Here are some possible answers to this question

- Does this fix the boson-star frequency as $\omega = 1/\sqrt{2}$? This might not be quite as dumb a conclusion as it appears at first glance. The actual frequency is not necessarily ω but may also depend on $\alpha(r \to \infty)$. Could it be that $\omega = 1/\sqrt{2}$ and all actual frequencies are obtained by just varying the asymptotic value of α? We can add a constant to Φ without changing anything in the equations except for a rescaling of ω since this is the only place where Φ appears on the right-hand sides. However, adding such an offset to Φ would also appear in Eq. (7.32), so what we actually fix is $\omega e^{-\Phi}$, so if we change Φ at infinity, the physical frequency $\omega e^{-\Phi}$ would still be $1/\sqrt{2}$ and this does not look acceptable.

- Is the series expansion for the lapse or Φ correct? We obtained it from the condition $\alpha X = 1$. Could this equality be corrupted by the scalar matter? In that case, we might need to modify the series expansion for Φ, allowing for another linear coefficient. But again, I have a hard time accepting this, since the mass M is constant at any level of the series expansion and I do not see how $\alpha X = 1$ could be violated at any level in the series expansion.

- Is the asymptotic behaviour of the scalar field different from Eq. (7.13) once we switch on gravity? If so, the correction cannot be an analytic function since any such modification would be accommodated by our series expansion for σ. But then, what could it be? Extra terms in the exponent of $e^{-k/y}$? I cannot rule this out and it may be worth playing with that, but it would imply a very bizarre limit as we approach light boson stars with virtually no self-gravity.

- Of course, there is always the chance of an error in the calculation, but all has been checked with `Maple`. Any errors in these notes are far likelier to be LaTeXerrors than errors in the actual equations. Nonetheless, everything here needs to be checked. Could it be that the expansion of the scalar field needs to be given an extra factor y^ϵ with non-integer ϵ? At the moment, this looks the least bad candidate solution to me; possibly the exponent ϵ just becomes integer in the limit $M \to 0$. This is the case I am most eager to explore next.

- Boson stars don't exist for frequencies other than $\omega^2 = 1/2$. That would be most bizarre and I won't accept this possibility unless everything else has been 100% excluded.

7.2 The solution

The least-bad candidate solution, alluded to in the fourth bullet item above, seems to be the solution to our conundrum. We have checked this by modifying the series expansion (7.27) as follows,

$$\begin{aligned} \sigma &= ay^{1+\epsilon} + by^{2+\epsilon} + cy^{3+\epsilon} + dy^{4+\epsilon} + \ldots \,, \\ \lambda &= Ay^{1+\chi} + By^{2+\chi} + Cy^{3+\chi} + Dy^{4+\chi} + \ldots \,. \end{aligned}$$

Upon plugging this expansion into Eq. (7.19) – multiplied by y^2 – the leading-order terms of our series expansions for σ, λ as well as Eq. (7.26) for X give us

$$(1 + My)(Ay^{1+\chi} + By^{2+\chi}) - kay^{1+\epsilon} - kby^{2+\epsilon} = (1 + \epsilon)ay^{2+\epsilon} + (2 + \epsilon)by^{3+\epsilon} \,. \tag{7.34}$$

We need $\chi = \epsilon$ because otherwise either the $y^{1+\chi}$ or the $y^{1+\epsilon}$ term cannot be cancelled. This simplifies our ensuing calculations considerably, since we now can use the series expansion

$$\sigma = ay^{1+\epsilon} + by^{2+\epsilon} + cy^{3+\epsilon} + dy^{4+\epsilon} + \ldots \,, \tag{7.35}$$

$$\lambda = Ay^{1+\epsilon} + By^{2+\epsilon} + Cy^{3+\epsilon} + Dy^{4+\epsilon} + \ldots \,. \tag{7.36}$$

We can now plug these expansions together with Eqs. (7.23)-(7.26) for the metric variables into Eqs. (7.19) and (7.20). Analyzing the terms of Eq. (7.19) order by order, we obtain relations between the lower and upper case coefficients,

$$\begin{aligned} A &= ka \,, \\ B &= (1 + \epsilon)a + kb - Mka \,, \\ C &= ck + b\epsilon - \frac{1}{2}M^2ak - Ma\epsilon - Mbk - Ma + 2b \,, \\ D &= -Mck - \frac{1}{2}M^3ak - \frac{1}{2}M^2a\epsilon - \frac{1}{2}M^2bk - \frac{1}{2}M^2a - 2Mb + \epsilon c + kd + 3c \,. \end{aligned} \tag{7.37}$$

Using these relations in the series expansion of Eq. (7.20) and comparing the resulting terms order by order, we can determine the free coefficients. At first and second order, respectively, we find

$$k^2 = 1 - \omega^2 \,, \tag{7.38}$$

$$\epsilon = M\frac{2k^2 - 1}{k} = M\frac{1 - 2\omega^2}{\sqrt{1 - \omega^2}} \,, \tag{7.39}$$

while the corresponding relations for third, fourth and higher order each determine one further coefficient in the expansion of σ in terms of already known coefficients. For example, the third-order comparison gives us b in terms of a, k and ϵ, the fourth order gives us c in terms of a, b, k and ϵ and so forth. Readers will have noticed that we do not have any relation that determines a; this is expected since we will need one free parameter in order to solve the eigenvalue problem for a *regular* boson-star solution.

This result is quite remarkable. First, we notice that in a flat spacetime $M = 0$, so that $\epsilon = 0$ and we recover the asymptotic behaviour (4.27) of the flat-spacetime limit. In the case of gravity, however, we do not recover this limit even at infinite radius. We have here an example where the full solution to the approximate field equations does not recover the limiting behaviour of the solution to the full field equations. This case bears similarity to the discussion of gravitational waves prior to the early 1960ss; people quite rightly argued that the solution to the linearized Einstein equations is not necessarily equal to the weak-field limit of the solutions to the full Einstein equations. This issue was eventually resolved by Bondi, Sachs and collaborators [3, 15] who clarified the character of gravitational waves in the framework of the fully non-linear Einstein equations.

But let us return to our more humble study. Something readers will likely also have noticed is that the modified behaviour of our functions σ and λ creates an additional challenge in our numerical attempts to solve the system of equations (7.17)-(7.20). Since we cannot evaluate the right-hand sides of Eqs. (7.19) and (7.20) at $y = 0$, we need to series expand them to be able to integrate the equations out of the singularity $y = 0$. For $M \neq 0$, however, the series expansion will give us $\partial_y \lambda = 0$ and $\partial_y \sigma = 0$ at $y = 0$ if $\epsilon > 0$ or ∞ if $\epsilon < 0$. Either case will not allow us to integrate out of the singularity. The solution should consider in yet another change of variables, namely we need to factor out the y^ϵ behaviour from both σ and λ. I fear we are out of Greek letters by now, but let us have a go at such a reformulation of the equations in the next subsection.

7.3 Rescaling the equations

The series expansions (7.35) and (7.36) suggest that we apply a further rescaling of our scalar-field variables σ and λ. Let's do that and derive the corresponding equations. We therefore define

$$\zeta := \frac{\sigma}{y^{\epsilon}}\,, \quad \Pi := \frac{\lambda}{y^{\epsilon}} \qquad \Leftrightarrow \qquad \sigma = y^{\epsilon}\zeta\,, \quad \lambda = y^{\epsilon}\Pi\,, \tag{7.40}$$

which modified Eqs. (7.17)-(7.20) to

$$\partial_y \Phi = -\frac{X^2-1}{2y} - \frac{2\pi X^2}{y^{3-2\epsilon}e^{2k/y}}\left[\Pi^2 + \zeta^2(\omega^2 e^{-2\Phi} - \hat{V})\right]\,, \tag{7.41}$$

$$\partial_y m = -\frac{2\pi}{y^{4-2\epsilon}e^{2k/y}}\left[\Pi^2 + \zeta^2(\omega^2 e^{-2\Phi} + \hat{V})\right]\,, \tag{7.42}$$

$$\frac{\partial_y X}{X} = \frac{X^2-1}{2y} - \frac{2\pi X^2}{y^{3-2\epsilon}e^{2k/y}}\left[\Pi^2 + \zeta^2(\omega^2 e^{-2\Phi} + \hat{V})\right] \tag{7.43}$$

$$\partial_y \zeta = \frac{X\Pi - \zeta(k+\epsilon y)}{y^2}\,, \tag{7.44}$$

$$\partial_y \Pi = \frac{(2-\epsilon)y - k}{y^2}\Pi - \Pi\partial_y\Phi + \frac{X\zeta}{y^2}(V' - \omega^2 e^{-2\Phi})\,. \tag{7.45}$$

Using the series expansions

$$\begin{aligned}
m(y) &= M\,, \\
\Phi(y) &= -My - M^2y^2 + \dots\,, \\
X(y) &= 1 + My + \frac{3}{2}M^2y^2 + \dots\,, \\
\zeta(y) &= ay + by^2 + \dots\,, \\
\Pi(y) &= Ay + By^2 + \dots\,,
\end{aligned}$$

then gives us for the two leading orders the following four relations,

$$\begin{aligned}
A &= ak\,, \\
k^2 &= 1-\omega^2\,, \\
B &= a(1+\epsilon-kM) + bk\,, \\
\epsilon &= M\frac{2k^2-1}{k} = M\frac{1-2\omega^2}{\sqrt{1-\omega^2}}\,,
\end{aligned} \tag{7.46}$$

in agreement with the asymptotic behaviour that we have found in the previous subsection.

8 The asymptotic behaviour of boson-start solutions in ST theory

[***US: Just a word of caution: The calculations in this section have not been as well checked as those in the previous one, so errors are more likely. I'll do a check with*** `Maple` ***eventually.***] We now generalize the discussion of the previous section to the case of scalar-tensor gravity. For this purpose we consider Eqs. (4.9)-(4.12), i.e. use the second-order version of the scalar-field equations. The asymptotic behaviour is not affected if we translate into first-order systems for the scalar fields and the second-order version is a bit easier to handle algebraically. We start by transforming to a compactified variable $y = 1/r$, so that Eqs. (4.9)-(4.12) become

$$\frac{\partial_y \alpha}{\alpha} = -\frac{FX^2-1}{2y} + \frac{FX^2}{y^3}W - \frac{F_{,\varphi}}{2F}\partial_y\varphi - \frac{y}{2}(\partial_y\varphi)^2 - 2\pi\frac{X^2}{F}\left[y\frac{(\partial_y A)^2}{X^2} + \frac{\omega^2 A^2}{\alpha^2 y^3} - \frac{V}{y^3}\right] \tag{8.1}$$

$$\frac{\partial_y X}{X} = \frac{FX^2-1}{2y} - \frac{FX^2}{y^3}W - \frac{F_{,\varphi}}{2F}\partial_y\varphi - \frac{y}{2}(\partial_y\varphi)^2 - 2\pi\frac{X^2}{F}\left[y\frac{(\partial_y A)^2}{X^2} + \frac{\omega^2 A^2}{\alpha^2 y^3} + \frac{V}{y^3}\right], \tag{8.2}$$

$$\partial_y^2\varphi = \partial_y\varphi\left(\frac{\partial_y X}{X} - \frac{\partial_y\alpha}{\alpha}\right) + \frac{FX^2}{y^4}W_{,\varphi} + 2\pi X^2\frac{F_{,\varphi}}{F^2}\left[\frac{\omega^2A^2}{\alpha^2y^4} - \frac{(\partial_y A)^2}{X^2} - 2\frac{V}{y^4}\right], \tag{8.3}$$

$$\partial_y^2 A = \partial_y A\left(\frac{\partial_y X}{X} - \frac{\partial_y\alpha}{\alpha} + \frac{F_{,\varphi}}{F}\partial_y\varphi\right) - \frac{X^2\omega^2}{\alpha^2y^4}A + X^2V_{,A^2}A\,. \tag{8.4}$$

We now make the assumption that the scalar fields fall off exponentially as in the flat-space case, but we leave the exponential coefficients k and h free. We thus define

$$\sigma := Ae^{k/y} \quad\wedge\quad \varrho := \varphi e^{h/y} \qquad\Leftrightarrow\qquad A = \sigma e^{-k/y} \quad\wedge\quad \varphi = \varrho e^{-h/y}\,, \tag{8.5}$$

and assume that these have at leading order a power-law dependence on y, though not necessarily the same. Equations (8.3) and (8.4) are given in terms of the new rescaled variables by

$$\begin{aligned}\partial_y^2\varrho = {}& \left(\partial_y\varrho + \frac{h}{y^2}\varrho\right)\left(\frac{\partial_y X}{X} - \frac{\partial_y\alpha}{\alpha}\right) - \frac{2h}{y^2}\partial_y\varrho + \frac{X^2F}{y^4}m^2\varrho + \frac{2y-h}{y^4}h\varrho \\ & -\frac{F_{,\varphi}}{F}e^{(h-2k)/y}\left[(\partial_y\sigma)^2 + 2\frac{k\sigma}{y^2}\partial_y\sigma + \frac{k^2+2X^2}{y^4}\sigma^2\right]\end{aligned} \tag{8.6}$$

$$\begin{aligned}\partial_y^2\sigma = {}& \frac{e^{-h/y}}{y^4}\frac{F_{,\varphi}}{F}\left[y^4\partial_y\sigma\,\partial_y\varrho + y^2h\varrho\partial_y\sigma + y^2k\sigma\partial_y\varrho + kh\varrho\sigma + \left(\partial_y\sigma + \frac{k}{y^2}\sigma\right)\left(\frac{\partial_y X}{X} - \frac{\partial_y\alpha}{\alpha}\right)\right] \\ & -\frac{2k}{y^2}\partial_y\sigma + \frac{X^2\sigma}{y^4}V_{,A^2} + \frac{2y-k^2}{y^4}\sigma - \frac{X^2\omega^2}{\alpha^2y^4}\sigma\,.\end{aligned} \tag{8.7}$$

We furthermore assume that our metric functions still approach their Schwarzschild values at infinity. In summary, this gives us the series expansions[5]

[5] We apologize here for reusing the letter A as a coefficient in the series expansion of ϱ; it has nothing to do with the amplitude of the complex boson-star scalar which we denoted above by the same letter.

$$X = 1 + My + \frac{3}{2}M^2y^2 + \frac{5}{2}M^3y^3 + \frac{35}{8}M^4y^4 + \dots\,, \tag{8.8}$$

$$\alpha = 1 - My - \frac{1}{2}M^2y^2 - \frac{1}{2}M^3y^3 - \frac{5}{8}M^4y^4 + \dots\,, \tag{8.9}$$

$$\varrho = Ay^{1+\delta} + By^{2+\delta} + Cy^{3+\delta} + Dy^{4+\delta} + \dots\,, \tag{8.10}$$

$$\sigma = ay^{1+\epsilon} + by^{2+\epsilon} + cy^{3+\epsilon} + d^{4+\epsilon} + \dots\,. \tag{8.11}$$

For the moment, we impose no conditions on the free parameters δ, ϵ, h, k, A, B, a, $b, \dots$. We note, however, that the two scalar equations (8.6) and (8.7) differ in one important regard. Whereas the equation for σ contains exclusively terms that are at least linear in σ, the equation for ϱ is explicitly sourced by terms $\sim F_{,\varphi}\sigma^2$ with no ϱ factor; hence the overall exponential $e^{(h-2k)/y}$. If $\alpha_0 = 0$. then $F_{,\varphi} \propto \varphi$ and we have no problem since then these extra terms also acquire a factor ϱ. For non-zero α_0, however, the series expansion of Eq. (8.6) will depend on the relative size of h and k. We distinguish three cases.

(1) $h < 2k$: In that case, the terms $e^{h-2k}/y\big[\dots\big]$ are exponentially suppressed relative to the other terms in Eq. (8.6) and do not contribute in the series expansion.

(2) $h = 2k$: Then the exponential factor becomes 1 and we need to include the terms in the series expansion.

(3) $h > 2k$: The terms in question then diverge and the equation becomes ferociously singular at $y = 0$. We discard this case as unphysical. Note that we do not relate h and k, respectively, to the scalar field mass and frequency parameters at this stage. So we do not exclude any combinations of m_φ, ω and M at this point. We merely require that h be sufficiently small relative to k.

Let us start with the first case which is very similar to the GR scenario. In that case, the factors $e^{-h/y}$ and $e^{(h-2k)/y}$ both drop to zero very quickly as $y \to 0$, faster than any power of y. We can then ignore all terms with such an exponential factor in the series expansion of Eqs. (8.6) and (8.7). If we furthermore insert the series expansions (8.8)-(8.11) into Eqs. (8.6), (8.7), we can schematically write the resulting relevant parts of these equations as

$$\underbrace{\partial_y^2\varrho}_{\sim y^{\delta-1}} = \underbrace{\partial_y\varrho\left(\frac{\partial_y X}{X} - \frac{\partial_y\alpha}{\alpha}\right)}_{\sim y^{\delta}} - \underbrace{\frac{2h}{y^2}\partial_y\varrho}_{\sim y^{\delta-2}} + \underbrace{\frac{h}{y^2}\varrho\left(\frac{\partial_y X}{X} - \frac{\partial_y\alpha}{\alpha}\right)}_{\sim y^{\delta-1}} + \underbrace{\frac{X^2F}{y^4}m^2\varrho}_{\sim y^{\delta-3}} + \underbrace{\frac{2y-h}{y^4}h\varrho}_{\sim y^{\delta-3}}\,,$$

$$\underbrace{\partial_y^2\sigma}_{\sim y^{\epsilon-1}} = \underbrace{\partial_y\sigma\left(\frac{\partial_y X}{X} - \frac{\partial_y\alpha}{\alpha}\right)}_{\sim y^{\epsilon}} - \underbrace{\frac{2k}{y^2}\partial_y\sigma}_{\sim y^{\epsilon-2}} + \underbrace{\frac{k}{y^2}\sigma\left(\frac{\partial_y X}{X} - \frac{\partial_y\alpha}{\alpha}\right)}_{\sim y^{\epsilon-1}} + \underbrace{\frac{X^2}{y^4}\sigma}_{\sim y^{\epsilon-3}} - \underbrace{\frac{k^2}{y^4}\sigma}_{\sim y^{\epsilon-3}} + \underbrace{\frac{2k}{y^3}\sigma}_{\sim y^{\epsilon-2}} - \underbrace{\frac{X^2\omega^2}{y^4\alpha^2}\sigma}_{\sim y^{\epsilon-3}}\,.$$

We are interested here in the two leading order contributions only in each equation, i.e. terms $\sim y^{\delta-3}$, $\sim y^{\delta-2}$, $\sim y^{\epsilon-3}$ and $\sim y^{\epsilon-2}$. This simplifies our task, since some terms drop out altogether and in the others we need at most two terms of the series expansions (8.8)-(8.11). This gives us the following four relations.

$$\begin{aligned}
&\partial_y^2\varrho, \quad y^{\epsilon-3}: \qquad && h = m\,,\\
&\partial_y^2\sigma, \quad y^{\epsilon-3}: \qquad && k^2 = 1-\omega^2\,,\\
&\partial_y^2\varrho, \quad y^{\epsilon-2}: \qquad && \delta = Mm = Mh\,,\\
&\partial_y^2\sigma, \quad y^{\epsilon-2}: \qquad && \epsilon = M\frac{1-2\omega^2}{k} = M\frac{2k^2-1}{k}\,.
\end{aligned} \tag{8.12}$$

For the boson-star scalar this is the same behaviour as in GR and we now merely augment it with the fall off of the gravitational scalar which has the (by now) expected asymptotic behaviour

$$\varrho \sim y^{1+Mm} \qquad \Rightarrow \qquad \varphi \sim \frac{e^{-m/y}}{y^{1+Mm}}\,. \tag{8.13}$$

The second case, $h = 2k$, $\alpha_0 \neq 0$ results in $e^{(h-2k)/y} = 1$ and we now need to include in our series expansion the additional terms in Eq. (8.6). Bearing in mind that asymptotically $F_{,\varphi}/F = -2\alpha_0$ and $F \to 1$ exponentially fast, we can write these extra terms schematically as

$$4\pi\alpha_0\Big[\underbrace{(\partial_y\sigma)^2}_{\sim y^{2\epsilon}}\underbrace{+\frac{2k}{y^2}\sigma\partial_y\sigma}_{\sim y^{2\epsilon-1}}\underbrace{+\frac{k^2}{y^4}\sigma^2}_{\sim y^{2\epsilon-2}}\underbrace{+2\frac{X^2}{y^4}\sigma^2}_{\sim y^{2\epsilon-2}}\underbrace{-\frac{X^2\omega^2}{\alpha^2 y^4}\sigma^2}_{\sim y^{2\epsilon-2}}\Big] \tag{8.14}$$

These need to be added on the right-hand side of the above schematic equation for $\partial_y^2\varrho$. At leading order, these terms contribute a term $\propto (2k^2+1)/y^{2\epsilon-2}$ which is manifestly non-zero. The only chance to cancel this term is through the the rest of the equation for $\partial_y^2\varrho$. This rest has leading-order terms $\propto y^{\delta-3}$, which suggest that

$$\delta - 3 = 2\epsilon - 2 \qquad \Rightarrow \qquad \epsilon = \frac{\delta-1}{2}\,. \tag{8.15}$$

Strictly speaking, we may also have the scenario where the $y^{2\epsilon-2}$ terms are canceled by lower order terms of the remainder of Eq. (8.6) in which case, we would arrive at the relation $\delta-2 = 2\epsilon-2$ or $\delta-1 = 2\epsilon-2$ etc. In these cases we would inevitably recover the condition $h = m$ from the leading order behaviour of Eq. (8.6). We now recall, however, that m is a free parameter and we could choose it as large as we want; in particular, we can choose it so large that $m = h > 2k$, in violation of our assumption that $h = 2k$. I cannot 100% rule out that there is some emergence exit for this scenario, but for now let us drop this case and proceed with $\delta-3 = 2\epsilon-2$.

Inserting the series expressions for α, X, ϱ and σ works as before and we get the same result for the σ equation as above; only the result for the ϱ equation gives us different conditions now.

$$\begin{aligned}
&\partial_y^2\varrho, \quad y^{\epsilon-3}: \qquad && h^2 = m^2 + 4\pi\alpha_0\frac{a^2}{A}(3-2\omega^2) \quad \Rightarrow \quad A = 4\pi\alpha_0 a^2\frac{2k^2+1}{4k^2-m^2}\,,\\
&\partial_y^2\sigma, \quad y^{\epsilon-3}: \qquad && k^2 = 1-\omega^2\,,\\
&\partial_y^2\varrho, \quad y^{\epsilon-2}: \qquad && B = F(a,A,b,k,h)\,,\\
&\partial_y^2\sigma, \quad y^{\epsilon-2}: \qquad && \epsilon = M\frac{1-2\omega^2}{k} = M\frac{2k^2-1}{k}\,,
\end{aligned} \tag{8.16}$$

where $F(a, A, b, k, h)$ is a lengthy but straightforward expression that we do not need in detail. Clearly, we have now a qualititatively different scenario for the series of ϱ. Its leading-order coefficient is no longer free, but determined by the leading-order behaviour of σ. In physical terms it looks like a too slowly decaying boson-star scalar forces the gravitational scalar to also decay more slowly.

There remains one open question: If $4k^2 - m^2$, we seem to obtain a diverging A. This may require more investigation, but before spending time on this, I want to check the algebra of these equations more carefully. I would not at all be surprised if there is a sign error and this seemingly problematic case disappears. Alternatively, it might be that this special case needs to be discussed separately.

Finally, I had a brief chat with Chris (Moore) about these asymptotics, either in GR or ST gravity. He suggested the intriguing possibility that the modified powerlaw in the scalar field's falloff might alternatively be interpreted in terms of radially dependent coefficients $k(r)$, $h(r)$ in the exponentials. I have not looked at this at all, but it could be an interesting way to reinvestigate these equations if someone has enough time...

9 Non-perturbative perturbations

9.1 Motivation

In this section, we explore a different numerical approach inspired by Ref. [9] where non-linear oscillations of neutron stars were modelled in terms of deviations from an equilibrium state without, however, truncating at any order. The key benefit of this scheme is that one can in this way eliminate the background terms from the equations at analytic level which results in a lengthier set of equations where all terms are at least linear in the deviations. This drastically increases the numerical accuracy in the regime where the deviations are small but still significant enough to trigger non-linear effects.

Here we are not dealing with a time dependent problem, so one may ask what plays the role of the equilibrium state on our case? The answer is that this role is taken over by the extremely challenging thin-shell states or, rather, the approximately thin-shell states that we obtain for a solitonic potential with small but not yet catastrophically small σ_0. To see this, we first recall the equations for a boson star in plain and simple radial gauge and polar slicing,

$$\frac{\partial_r \alpha}{\alpha} = \frac{X^2-1}{2r} + 2\pi r X^2 \left(\eta^2 + \frac{\omega^2 A^2}{\alpha^2} - V\right), \tag{9.1}$$

$$\frac{\partial_r X}{X} = -\frac{X^2-1}{2r} + 2\pi r X^2 \left(\eta^2 + \frac{\omega^2 A^2}{\alpha^2} + V\right), \tag{9.2}$$

$$\partial_r A = X\eta, \tag{9.3}$$

$$\partial_r \eta = -2\frac{\eta}{r} - \eta\frac{\partial_r \alpha}{\alpha} + X\left(V' - \frac{\omega^2}{\alpha^2}\right) A, \tag{9.4}$$

$$\partial_r \Phi = \frac{X^2-1}{2r} + 2\pi r X^2(\eta^2 + \omega^2 e^{-2\Phi} A^2 - V), \tag{9.5}$$

$$\partial_r m = 2\pi r^2(\eta^2 + \omega^2 e^{-2\Phi} A^2 + V). \tag{9.6}$$

First, let us consider the trivial case where $A = \text{const} \neq 0$. Is this a solution to this set of ODEs? Clearly, we then have $\eta = 0$ and this automatically implies by Eq. (9.4) that

$$V' = \omega^2 e^{-2\Phi} \,. \tag{9.7}$$

Since A is constant, V' also remains constant in r. ω is a constant anyway, so we must have $\Phi = \text{const}$ or, equivalently, $\alpha = \text{const}$. This can indeed be achieved if $X = 1$ everywhere and the remaining sources in Eqs. (9.1) and (9.2) vanish, i.e.

$$\frac{\omega^2 A^2}{\alpha^2} = 0 \qquad \wedge \qquad V = 0 \,. \tag{9.8}$$

This is definitely achieved if we have $\omega = 0$ and $V(A) = 0$. In summary, we can find our trivial solution if we manage to satisfy

$$V'(A) = V(A) = \omega = 0 \,. \tag{9.9}$$

For the solitonic potential (5.4), this can indeed be achieved since

$$V(A) = A^2 \left(1 - 2\frac{A^2}{\sigma_0^2}\right)^2 \qquad \Rightarrow \qquad V'(A) = \left(1 - 2\frac{A^2}{\sigma_0^2}\right)\left(1 - 6\frac{A^2}{\sigma_0^2}\right) \,, \tag{9.10}$$

and both vanish if $A = \sigma_0/\sqrt{2}$. Of course, this solution is just vacuum Minkowski where we have a constant scalar field with zero energy and momentum density and no stress, $T_{\alpha\beta} = 0$.

This solution is neither numerically challenging nor physically interesting, so why did we bother deriving it in so much detail? The answer is that the thin-shell solutions can be regarded as an approximation to this extreme yet trivial case. Inside the shell, we recover flat space (though typically with non-unit lapse) and as we reduce σ_0, we push the shell radius further and further out. I am not sure whether it occurs at $\sigma = 0$ that the shell radius is pushed to infinity, but I think it is likely. It is also likely that the limit is not continuous; the thin-shell BSs likely will become more and more massive as $\sigma_0 \to 0$ and therefore not recover the massless Minkowski limit in a smooth process. I am not certain here, but our numerical results and those of [5] indicate a divergence of the mass.

Of course, we are interested in the case where the shell resides at finite radius and we therefore are interested in boson stars that only approximate a constant scalar field amplitude over some radial range around the origin. Let us therefore consider the case where σ_0 is small but finite. Considering again Eq. (9.4), we see that a flat scalar profile is obtained if $\partial_r \eta \approx 0$. Since $\eta = 0$ at $r = 0$ for regularity, we see that for sufficiently small radii, the only non-vanishing terms on the right-hand side of Eq. (9.4) are

$$X\left(V' - \omega^2 e^{-2\Phi}\right) A \,.$$

For an approximately thin-shell model, we want this term to vanish and that gives us the condition

$$\boxed{V_0' = \left(1 - 2\frac{A_0^2}{\sigma_0^2}\right)\left(1 - 6\frac{A_0^2}{\sigma_0^2}\right) = \omega_0^2 e^{-2\Phi_0} \,.} \tag{9.11}$$

We have used the subscript 0 here to emphasize that these are background values that the fields A, V' and Φ acquire for a thin-shell configuration at small r for a frequency ω_0. Of course, this is a condition we can impose for any boson star, thin-shell or not. The key point is that for a thin-shell model, the fields will stay very close to these background values over a significant radial range and

ω_0 will be close to the true frequency of the boson star, period. And these are, of course, precisely the models that are so difficult to calculate numerically. Our hope is that we achieve much better accuracy if we compute for these models the frequency difference $\delta\omega := \omega - \omega_0$ rather than ω itself; $\delta\omega$ will be small for the "tough" models and we may be able to determine it numerically with much higher precision that ω itself. Let us get started...

9.2 The equations

This approach is tailored to boson stars with a solitonic potential, so in this section, we assume the potential is given by

$$\begin{aligned} V(A) &= A^2\left(1-2\frac{A^2}{\sigma_0^2}\right)^2, \\ \hat{V} := \frac{V}{A^2} &= \left(1-2\frac{A^2}{\sigma_0^2}\right)^2, \\ V'(A) := \frac{\mathrm{d}V}{\mathrm{d}(A^2)} &= \left(1-2\frac{A^2}{\sigma_0^2}\right)\left(1-6\frac{A^2}{\sigma_0^2}\right), \\ \hat{V}-V' &= 4\frac{A^2}{\sigma_0^2}\left(1-2\frac{A^2}{\sigma_0}\right). \\ \hat{V}+V' &= 2\left(1-4\frac{A^2}{\sigma_0^2}\right)\left(1-2\frac{A^2}{\sigma_0^2}\right). \end{aligned} \tag{9.12}$$

We furthermore describe three variables in terms of background plus deviation,

$$\begin{aligned} \Phi &= \Phi_0 + \delta\Phi, \\ A &= A_0 + \delta A, \\ \omega &= \omega_0 + \delta\omega. \end{aligned} \tag{9.13}$$

Here, Φ_0 and A_0 are the field values at $r = 0$ whereas ω_0 is the frequency of a background model. One of these parameters merely represents a choice for the scaling of the time coordinate, one is a free parameter that determines our model and the third is the Eigenvalue that needs to be finetuned to obtain a physical solution. Typically, we choose Φ_0 to scale the time such that $\Phi_0 = 0$ at infinity, we use A_0 to control the model we want and then finetune ω. But we remain open minded and leave the purposes of the three parameters open for now. Note that Φ_0 and A_0 are parameters and *simultaneously* also the background values of Φ and A whereas ω_0 is merely the background value of the parameter ω. The notation isn't perfect, but any other concerves misery.

For the equations, it turns out useful to also introduce background and deviations for the potential functions. These are given by (I don't think we'll need V and V_0, but let's see...)

$$\begin{aligned}
\hat{V}_0 := \hat{V}(A_0) &= \left(1 - 2\frac{A_0^2}{\sigma_0^2}\right)^2 , \qquad (9.14)\\
V_0' := V'(A_0) &= \left(1 - 2\frac{A_0^2}{\sigma_0^2}\right)\left(1 - 6\frac{A_0^2}{\sigma_0^2}\right) ,\\
\delta\hat{V} := \hat{V}(A) - \hat{V}(A_0) &= -4\frac{\delta A}{\sigma_0^2}(2A_0 + \delta A) + 4\frac{\delta A}{\sigma_0^4}\left(4A_0^3 + 6A_0^2\delta A + 4A_0\delta A^2 + \delta A^3\right) ,\\
\delta V' := V'(A) - V'(A_0) &= -8\frac{\delta A}{\sigma_0^2}(2A_0 + \delta A) + 12\frac{\delta A}{\sigma_0^4}\left(4A_0^3 - 6A_0^2\delta A + 8A_0\delta A^2 + \delta A^3\right) ,\\
\delta & \qquad (9.15)
\end{aligned}$$

Our system of differential equations in this new notation becomes

$$\begin{aligned}
\partial_r\delta\Phi &= \frac{X^2-1}{2r} + 2\pi r X^2\Big\{\eta^2 + A^2\Big[\omega_0^2 e^{-2\Phi_0}(e^{-2\delta\Phi} - 1) + \delta\omega(2\omega_0 + \delta\omega)e^{-2\Phi} + V_0' - \hat{V}\Big]\Big\}\\
\frac{\partial_r X}{X} &= -\frac{X^2-1}{2r} + 2\pi r X^2\Big\{\eta^2 + A^2\Big[\omega_0^2 e^{-2\Phi_0}(e^{-2\delta\Phi} - 1) + \delta\omega(2\omega_0 + \delta\omega)e^{-2\Phi} + V_0' + \hat{V}\Big]\Big\}\\
\partial_r\delta A &= X\eta ,\\
\partial_r\eta &= -2\frac{\eta}{r} - \eta\partial_r\delta\Phi + XA\Big[\delta V' - \omega_0^2(e^{-2\delta\Phi} - 1) + \delta\omega(2\omega_0 + \delta\omega)e^{-2\Phi}\Big] \qquad (9.16)
\end{aligned}$$

If we impose $\Phi_0 = 0$, these equations slightly simplify to

$$\begin{aligned}
\partial_r\delta\Phi &= \frac{X^2-1}{2r} + 2\pi r X^2\Big\{\eta^2 + A^2\Big[\omega_0^2(e^{-2\delta\Phi} - 1) + \delta\omega(2\omega_0 + \delta\omega)e^{-2\delta\Phi} + V_0' - \hat{V}\Big]\Big\}\\
\frac{\partial_r X}{X} &= -\frac{X^2-1}{2r} + 2\pi r X^2\Big\{\eta^2 + A^2\Big[\omega_0^2(e^{-2\delta\Phi} - 1) + \delta\omega(2\omega_0 + \delta\omega)e^{-2\delta\Phi} + V_0' + \hat{V}\Big]\Big\}\\
\partial_r\delta A &= X\eta ,\\
\partial_r\eta &= -2\frac{\eta}{r} - \eta\partial_r\delta\Phi + XA\Big[\delta V' - \omega_0^2(e^{-2\delta\Phi} - 1) + \delta\omega(2\omega_0 + \delta\omega)e^{-2\delta\Phi}\Big] \qquad (9.17)
\end{aligned}$$

10 Exponential Rosenbrock Methods

We summarize in this section a numerical method specifically adapted to handling exponential behaviour in solutions to differential equations. I do not think it will solve our problem which simply is of a different type, but it might still be interesting to implement it to try and see how it is working, if only to get some hands-on experience with more sophisticated integration techniques. I got this information from the Wikipedia page [1] which looks remarkably comprehensive to get a main understanding of these methods; likely there are more advanced articles and books, but this reference looks enough to let us have a try.

The main idea of this method seems to be to handle stiff ODEs which often involve exponential behaviour in solutions that grow or decay on drastically different "time" scales. Solutions of the type $Ae^{1000t} + Be^{0.001t}$. In this case, the "time" (in quotes because t need not be time) step needs to be adapted to the shortest physical scale which then becomes very inefficient for the long scale.

In simple terms, the numerical schemes adapted to this identify that part of the ODE responsible for the exponential behaviour and solve it analytically with the remaining contribution of the ODE added as a correction. This turns out to be simpler than it may sound at first glance and looks quite elegant. The key point is that exponential behaviour of the solutions typically arises from the linear part of the ODE, essentially a consequence of the fact that $e^{\lambda t}$ solves the linear ODE $f' = \lambda f$, i.e. a linear differential equation.

So let us consider the differential equation

$$u'(t) = f\big(u(t)\big)\,, \qquad u(t_0) = u_0\,, \tag{10.1}$$

where u may be vector values, $u : \mathbb{R} \to \mathbb{R}^n$ and f would then be a $n \times n$ matrix. For most of our calculation, it does not make a difference and we therefore do not switch to bold or sans serif fonts we usually use for vectors and matrices. But keep your mind open that u need not be scalar. In fact the above problem of stiffness seems to come about only when we have a system of at least 2 ODEs; I guess you just cannot get two different exponentials with just one scalar ODE, though I am not sure stiffness might not arise in a scalar ODE through other means.

Given Eq. (10.1), we next consider some value u^* of the solution (it doesn't matter which) and define the linear and nonlinear parts of the source,

$$L := \frac{\partial f}{\partial u}(u^*)\,, \qquad\qquad N := f(u) - Lu\,. \tag{10.2}$$

Note that L is an $n \times n$ matrix for a system of n ODEs, the Jaobian of $f(u)$ evaluated at u^*. Before going on, we recap an elementary property of integrals,

$$\int_0^{x-x_0} f(\tilde{x})\mathrm{d}\tilde{x} \;=\; \int_{x_0}^{x-x_0} f(s-x_0)\mathrm{d}s\,. \qquad \Big| \;\; s = \tilde{x} + x_0 \;\Rightarrow\; \mathrm{d}s = \mathrm{d}\tilde{x} \tag{10.3}$$

This is pretty trivial, but the shifting of integrations plays an important role in the following calculations and it can be confusing to not keep accurate track of this.

Proposition: The solution $u(t)$ to the ODE (10.1) satisfies

$$u(t) = e^{L(t-t_0)}u_0 + \int_0^{t-t_0} e^{L(t-t_0-\tau)} N\big(u(t_0+\tau)\big)\mathrm{d}\tau\,, \tag{10.4}$$

where $u_0 := u(t_0)$. Note that we are not doing numerics yet, this is totally analytic, exact and holds for any t, t_0 inside the ODE's domain.

Proof. We show that the right-hand side of (10.4) satisfies the ODE (10.1). For this purpose, we first write using the integral shift (10.3)

$$\begin{aligned} u(t) &= e^{L(t-t_0)}u_0 + \int_{t_0}^{t} e^{L[t-t_0-(s-t_0)]} N\big(u(t_0+s-t_0)\big)\mathrm{d}s \\ &= e^{L(t-t_0)}u_0 + \int_{t_0}^{t} e^{L(t-s)} N\big(u(s)\big)\mathrm{d}s \\ &= e^{L(t-t_0)}u_0 + e^{Lt}\int_{t_0}^{r} e^{-Ls} N\big(u(s)\big)\mathrm{d}s\,. \end{aligned} \tag{10.5}$$

Now we differentiate and obtain

$$\begin{aligned} u'(t) &= Le^{L(t-t_0)}u_0 + Le^{Lt}\underbrace{\int_{t_0}^{t} e^{-Ls} N\big(u(s)\big)\mathrm{d}s}_{=\left[u(t)-e^{L(t-t_0)}u_0\right]e^{-Lt}} + e^{Lt}e^{-Lt}N\big(u(t)\big) \\ &= Le^{L(t-t_0)}u_0 + L\left[u(t)-e^{L(t-t_0)}u_0\right] + N\big(u(t)\big)\,. \\ &= Lu(t) + N\big(u(t)\big) \;=\; f\big(u(t)\big)\,. \end{aligned} \tag{10.6}$$

Furthermore, $u(t_0) = u_0$ follows directly by setting $t = t_0$. □

Now we start considering numerics and, in particular, how we can advance a solution from t_n to $t_{n+1} := t_n + \Delta t$. Setting $t_0 = t_n$, $t = t_{n+1}$, $u^* = u_0 = u_n$ in Eq. (10.4), and defining $h_n := t_{n+1} - t_n$, we obtain

$$u(t_{n+1}) = e^{Lh_n}u(t_n) + \int_0^{h_n} e^{L(h_n-\tau)} N\big(u(t_n+\tau)\big)\mathrm{d}\tau\,. \tag{10.7}$$

Note how we have split the change of u into an analytic piece due to the linear part of the source + an integral correction due to the non-linear part. The linear part of the change is analytic and therefore free of discretization error. Only the non-linear part, i.e. the integral on the right-hand side, needs a numerical approximation.

The Exponential Rosenbrock method discretizes this integral using a set of functions φ_k defined by

$$\varphi_0(z) = e^z\,, \qquad \varphi_k(z) = \int_0^1 e^{(1-\theta)z}\frac{\theta^{k-1}}{(k-1)!}\mathrm{d}\theta \quad \text{for} \quad k \geq 1\,. \tag{10.8}$$

This looks more complicated than it is; let's just compute φ_1 to see how this works,

$$\begin{aligned} \varphi_1(z) &= \int_0^1 e^{(1-\theta)z}\mathrm{d}\theta \;=\; \int_0^1 e^z e^{-\theta z}\mathrm{d}\theta \;=\; e^z\left[-\frac{1}{z}e^{-\theta z}\right]_{\theta=0}^1 \;=\; e^z\left(-\frac{1}{z}e^{-z}+\frac{1}{z}\right) \\ &= \frac{e^z-1}{z}\,. \end{aligned} \tag{10.9}$$

For generic discretization schemes, one first chooses how many stages one wants (this is analogous to the number of Runge-Kutta steps). Say we want s stages, then one chooses s "nodes" c_i and a bunch of coefficients $b_i(z)$ and $a_{ij}(z)$ that depend on $z \coloneqq L_n h_n$, where

$$L_n \coloneqq \frac{\partial f}{\partial u}(u_n)\,.$$

These coefficients that are typically given in terms of the $\varphi_k(z)$ functions (one of their tasks) and they are tabulated for different schemes; see for a few examples [1]. A numerical solution is then obtained from

$$\boxed{\begin{aligned} U_{n,i} &= u_n + c_i h_n \varphi_1(c_i L_n h_n)\, f(u_n) + h_n \sum_{j=2}^{i-1} a_{ij}(L_n h_n) D_{n,j}\,, \\ u_{n+1} &= u_n + h_n \varphi_1(L_n h_n)\, f(u_n) + h_n \sum_{i=2}^{s} b_i(L_n h_n) D_{n,i},\,, \end{aligned}} \tag{10.10}$$

where $D_{n,i} \coloneqq N_n\big(U_{n,i} - N_n(u_n)\big)$ is the difference in the non-linear source.

One point where the notes are not entirely clear is that the functions φ_k must be matrix valued for sets of ≥ 2 ODEs. These matrices eat the vector f and then return a vector valued contribution to u_{n+1}. I assume that the φ_k functions are applied component wise to the argument $L_n h_n$, but the notes do not say so explicitly.

While this scheme (10.10) looks quite complicated, its simplest version, the Exponential-Rosenbrock-Euler scheme

$$\boxed{u_{n+1} = u_n + h_n \varphi_1(h_n L_n) f(u_n)\,,} \tag{10.11}$$

does not require any additional nodes or coefficients and is reported as second-order accurate. This may sound surprising at first glance, since an Euler scheme without intermediate points is first-order only in normal discretization. I believe the explanation is that we have already integrated the ODE here and thereby gain one order of accuracy: a standard integration $f_{n+1} = f_n + f_n' \Delta x$ has a second-order truncation error.

To get a little more intuition for this method, let us see how it would handle the flat-spacetime limit of our boson star equations. For this purpose, we recall the field equations for the scalar amplitude A and its gradient function η,

$$\begin{aligned} \partial_r A &= X\eta\,, \\ \partial_r \eta &= -2\frac{\eta}{r} - \eta\frac{\partial_r \alpha}{\alpha} + X\left(V' - \frac{\omega^2}{\alpha^2}\right) A\,. \end{aligned} \tag{10.12}$$

The flat space limit is

$$\partial_r A = \eta\,, \qquad \partial_r \eta = (V' - \omega^2) A\,, \tag{10.13}$$

or, in vectorial form with $u = \binom{A}{\eta}$,

$$\partial_r u = \partial_r \begin{pmatrix} A \\ \eta \end{pmatrix} = f(u) = \begin{pmatrix} \eta \\ (V' - \omega^2) A \end{pmatrix}\,. \tag{10.14}$$

The linear part is given by the matrix

$$L = \begin{pmatrix} \frac{\partial f_A}{\partial A} & \frac{\partial f_A}{\partial \eta} \\ \frac{\partial f_\eta}{\partial A} & \frac{\partial f_\eta}{\partial \eta} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ V' - \omega^2 & 0 \end{pmatrix} . \tag{10.15}$$

Let us analyse how we would obtain an exponential solution to the ODE that consists only of the linear part with $V' = 1$, i.e.

$$\partial_r \begin{pmatrix} A \\ \eta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 - \omega^2 & 0 \end{pmatrix} \begin{pmatrix} A \\ \eta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ k^2 & 0 \end{pmatrix} \begin{pmatrix} A \\ \eta \end{pmatrix} =: \mathbf{Q} \begin{pmatrix} A \\ \eta \end{pmatrix} , \tag{10.16}$$

where we have introduced $\mathbf{Q}$ as the matrix for ease of notation. The exponential solution is given by

$$\begin{pmatrix} A \\ \eta \end{pmatrix} = e^{\mathbf{Q}r} \begin{pmatrix} A_0 \\ \eta_0 \end{pmatrix} = \left(\sum_{n=0}^{\infty} \mathbf{Q}^n r^n \right) \begin{pmatrix} A_0 \\ \eta_0 \end{pmatrix} = \left[1 + \mathbf{Q}r + \frac{1}{2}\mathbf{Q}^2 r^2 + \frac{1}{6}\mathbf{Q}^3 r^3 + \ldots \right] \begin{pmatrix} A_0 \\ \eta_0 \end{pmatrix} \tag{10.17}$$

One straightforwardly finds

$$\mathbf{Q} = \begin{pmatrix} 0 & 1 \\ k^2 & 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{Q}^2 = \begin{pmatrix} k^2 & 0 \\ 0 & k^2 \end{pmatrix} , \quad \mathbf{Q}^3 = \begin{pmatrix} 0 & k^2 \\ k^4 & 0 \end{pmatrix} = k^2 \mathbf{Q}$$

$$\Rightarrow \quad \mathbf{Q}^n = k^2 \mathbf{Q}^{n-2} . \tag{10.18}$$

Equation (10.17) then gives us

$$\begin{aligned} A &= A_0 + \eta_0 r + \frac{1}{2} k^2 A_0 r^2 + \frac{1}{6} k^2 \eta_0 r^3 + \frac{1}{4!} k^4 r^4 + A_0 + \ldots \\ \eta &= \eta_0 + k^2 A_0 r + \frac{1}{2} k^2 \eta_0 r^2 + \frac{1}{6} k^4 A_0 r^3 + \frac{1}{4!} k^4 \eta_0 r^4 + \ldots . \end{aligned} \tag{10.19}$$

With $\eta_0 = kA_0$, this gives us the expected $A = A_0 e^{kr}$ and $\eta = \eta_0 e^{kr}$. Note that here $k = \pm\sqrt{1 - \omega^2}$ can be either positive of negative. Unfortunately, this apprpach does not enable us to suppress the $k > 0$ contributions, since all the ODE sees is k^2. Possibly, we can change variables like $\eta \to k\eta$ or so to acquire a matrix of the form

$$\tilde{\mathbf{Q}} = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} , \tag{10.20}$$

rather than $\mathbf{Q}$itself. But I am not sure about this...

11 A first-order system for boson stars in scalar-tensor gravity

We now return to the boson-star case in scalar-tensor gravity. We recall the line element in the Einstein frame,

$$\mathrm{d}\bar{s}^2 = -F\alpha^2\mathrm{d}t^2 + FX^2\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,. \tag{11.1}$$

and the resulting field equations (4.9)-(4.12) in spherical symmetry. These equations have been checked carefully and should be reliable. In this section, we formulate them as a first-order system and then rescale the variables for a formulation in the compactified exterior. Introducing the variables

$$\begin{aligned}
\partial_r A &= \frac{XF}{\alpha}\theta \qquad &\Leftrightarrow\qquad \theta &= \frac{\alpha}{XF}\partial_r A\,,\\
\partial_r\varphi &= \frac{X}{\alpha}\kappa \qquad &\Leftrightarrow\qquad \kappa &= \frac{\alpha}{X}\partial_r\varphi\,,\\
\alpha = \frac{e^\Phi}{\sqrt{F}} \;\Leftrightarrow\; \Phi &= \ln(\sqrt{F}\alpha)\,, \qquad & m = \frac{r}{2}\left(1-\frac{1}{FX^2}\right) \;\Leftrightarrow\; X^2 &= \frac{1}{F}\left(1-\frac{2m}{r}\right)^{-1}\,,
\end{aligned} \tag{11.2}$$

we can write the complete set of equations in first-order form as

$$\begin{aligned}
\partial_r\Phi &= \frac{FX^2-1}{2r} - rFX^2W + \frac{r}{2}\frac{X^2}{\alpha^2}\kappa^2 + \frac{2\pi rX^2}{F\alpha^2}\left(\omega^2A^2-\alpha^2V+F^2\theta^2\right)\\
\frac{\partial_r X}{X} &= -\frac{FX^2-1}{2r} + rFX^2W - \frac{X}{2\alpha}\frac{F'}{F}\kappa + \frac{r}{2}\frac{X^2}{\alpha^2}\kappa^2 + \frac{2\pi rX^2}{F\alpha^2}\left(\omega^2A^2+\alpha^2V+F^2\theta^2\right)\\
\partial_r m &= \frac{r^2\kappa^2}{2F\alpha^2} + r^2W + \frac{2\pi r^2}{F^2\alpha^2}\left(\omega^2A^2+\alpha^2V+F^2\theta^2\right)\,,\\
\partial_r\varphi &= \frac{X}{\alpha}\kappa\,,\\
\partial_r\kappa &= -2\frac{\kappa}{r} + 2\alpha XFW_{,\varphi^2}\varphi + 2\pi\frac{XF'}{\alpha F^2}\left(\omega^2A^2-2\alpha^2V-F^2\theta^2\right)\,,\\
\partial_r A &= \frac{XF}{\alpha}\theta\,,\\
\partial_r\theta &= -2\frac{\theta}{r} + \frac{X}{\alpha F}A\left(\alpha^2V_{,A^2}-\omega^2\right)\,.
\end{aligned} \tag{11.3}$$

In the exterior, we use $y=1/r$ and anticipate the asymptotic behaviour by switching to rescaled variables

$$\zeta := \frac{e^{k/y}}{y^{\epsilon}} A\,, \qquad \Pi := -\frac{e^{k/y}}{y^{\epsilon}} \theta\,,$$
$$\varrho := \frac{e^{h/y}}{y^{\delta}} \varphi\,, \qquad \Lambda := -\frac{e^{h/y}}{y^{\delta}} \kappa\,. \tag{11.4}$$

Here, h, k, ϵ and δ are as yet free parameters which we will explore in more detail in the asymptotic analysis. The field equations (11.3) expressed in the inverse radius y and these new variables become

$$\begin{aligned}
\partial_y \Phi &= -\frac{FX^2-1}{2y} + \frac{y^{2\delta-3}}{e^{2h/y}} X^2 \left(F\hat{W}\varrho^2 - \frac{\Lambda^2}{2\alpha^2} \right) + 2\pi y^{2\epsilon-3} \frac{X^2}{F\alpha^2} e^{-2k/y} \left(\alpha^2 \hat{V}\zeta^2 - \omega^2\zeta^2 - F^2\Pi^2 \right) \\
\partial_y m &= -\frac{y^{2\delta-4}}{e^{2h/y}} \left(\hat{W}\varrho^2 + \frac{\Lambda^2}{2F\alpha^2} \right) - \frac{2\pi y^{2\epsilon-4}}{F^2\alpha^2} e^{-2k/y} \left(\omega^2\zeta^2 + \alpha^2\hat{V}\zeta^2 + F^2\Pi^2 \right)\,, \\
\partial_y \varrho &= \frac{X\Lambda/\alpha - (y\delta + h)\varrho}{y^2}\,, \\
\partial_y \Lambda &= \frac{(2-\delta)y - h}{y^2}\Lambda + 2\frac{\alpha X F}{y^2} W_{,\varphi^2}\varrho + \frac{2\pi}{y^{2+\delta-2\epsilon}} e^{(h-2k)/y} \frac{X}{\alpha}\frac{F'}{F^2} \left(\omega^2\zeta^2 - 2\alpha^2\hat{V}\zeta^2 - F^2\Pi^2 \right)\,, \\
\partial_y \zeta &= \frac{FX\Pi/\alpha - (\epsilon y + k)\zeta}{y^2}\,, \\
\partial_y \Pi &= \frac{(2-\epsilon)y - k}{y^2}\Pi + \frac{X\zeta}{\alpha F y^2} \left(\alpha^2 V_{,A^2} - \omega^2 \right)\,.
\end{aligned} \tag{11.5}$$

where

$$W(\varphi) = \frac{1}{2}\mu^2\varphi^2\,, \qquad V(A) = A^2 \left(1 - 2\frac{A^2}{\sigma_0^2} \right)^2\,, \qquad F(\varphi) = e^{-2\alpha_0\varphi - \beta_0\varphi^2}\,, \tag{11.6}$$
$$\hat{W} := \frac{W}{\varphi^2} = \frac{\mu^2}{2}\,, \qquad \hat{V} := \frac{V}{A^2}\,, \qquad W_{,\varphi^2} := \frac{\mathrm{d}W}{\mathrm{d}(\varphi^2)} = \frac{\mu^2}{2}\,, \qquad F' := \frac{\mathrm{d}F}{\mathrm{d}\varphi}\,, \qquad V_{,A^2} := \frac{\mathrm{d}V}{\mathrm{d}(A^2)}\,.$$

For the asymptotic expansion, we have to distinguish a few cases.

11.1 A massive φ that decays more slowly than A

In this case, the boson-star scalar becomes subdominant relative to the gravitational scalar at infinity which implies

$$h < 2k \qquad \text{or} \qquad \alpha_0 = 0\,, \tag{11.7}$$

and slightly simplifies the analysis. Both, φ and A decay exponentially in this case, so that their series expansion in y is constant. This also implies that the functions V, W and F only have a constant in their expansions. From Eqs. (4.9) and (4.10), we also see that their right-hand sides differ only by exponentially decaying terms, so in a series expansion, we obtain $X = 1/\alpha = \exp(-\Phi) = (1-2m/r)^{-1}$. This leads to the following Ansatz around $y = 0$,

$$
\begin{aligned}
m(y) &= M + m_1 y + m_2 y^2 + m_3 y^3 + m_4 + y^4 + \dots\,, \\
\Phi(y) &= \Phi_1 y + \Phi_2 y^2 + \Phi_3 y^3 + \Phi_4 y^4 + \dots\,, \\
\hat{W}(y) &= \mu^2\,, \\
W_{,\varphi^2} &= \frac{\mu^2}{2}\,, \\
\hat{V}(y) &= 1\,, \\
V'(y) \coloneqq V_{,A^2} &= 1\,, \\
\varrho(y) &= \varrho_1 y + \varrho_2 y^2 + \varrho_3 y^3 + \varrho_4 y^4 + \dots\,, \\
\zeta(y) &= \zeta_1 y + \zeta_2 y^2 + \zeta_3 y^3 + \zeta_4 y^4 + \dots\,, \\
\Lambda(y) &= \Lambda_1 y + \Lambda_2 y^2 + \Lambda_3 y^3 + \Lambda_4 y^4 + \dots\,, \\
\Pi(y) &= \Pi_1 y + \Pi_2 y^2 + \Pi_3 y^3 + \Pi_4 y^4 + \dots\,.
\end{aligned} \tag{11.8}
$$

Here the coefficients ϱ_1, ϱ_2, ζ_1, Λ_1, Π_1, ... are free and determined by plugging the expansion into Eqs. (11.5). This is most conveniently done in Maple or Mathematica and first demonstrates that in the equations for $\partial_y \Phi$ and $\partial_y m$ only terms of order y^3 or higher remain. This demonstrates that our expansion for α, Φ, m and X are consistent with the equations. The remaining scalar-field equations determine the free coefficients. The expansion then yields the following conditions,

$$
\begin{aligned}
\partial_y m: \quad & y^0 && : && m_1 = 0\,,\\
& y^1 && : && m_2 = 0\,,\\
& y^2 && : && m_3 = 0\,,\\
\partial_y \Phi: \quad & y^0 && : && \Phi_1 = -M\,,\\
& y^1 && : && \Phi_2 = -M^2\,,\\
& y^2 && : && \Phi_3 = -\frac{4}{3}M^3\,,\\
\partial_y \varrho: \quad & y^{-1} && : && \Lambda_1 = h\varrho_1\,,\\
& y^0 && : && \Lambda_2 = (1+\delta)\varrho_1 + h\varrho_2 - 2Mh\varrho_1\,,\\
& y^1 && : && \Lambda_3 = h\varrho_3 - 4M^2 h\varrho_1 + (2+\delta)\varrho_2 - 2M\Lambda_2\,,\\
\partial_y \Lambda: \quad & y^{-1} && : && h^2 = \mu^2\,,\\
& y^0 && : && \Lambda_2 = \varrho_1(1-\delta) + h\varrho_2\,,\\
& y^1 && : && \Lambda_3 = h\varrho_3 + M(Mh-1)\varrho_1 - Mh\varrho_2\,,\\
\partial_y \zeta: \quad & y^{-1} && : && \Pi_1 = k\zeta_1\,,\\
& y^0 && : && \Pi_2 = \zeta_1 + \epsilon\zeta_1 + k\zeta_2 - 2Mk\zeta_1\,,\\
& y^1 && : && \Pi_3 = k\zeta_3 + (2+\epsilon-2Mk)\zeta_2 - 2M(1+\epsilon)\zeta_1\,,\\
\partial_y \Pi: \quad & y^{-1} && : && k^2 = 1-\omega^2\,,\\
& y^0 && : && \epsilon = M\frac{2k^2-1}{k} = M\frac{1-2\omega^2}{\sqrt{1-\omega^2}}\\
& y^1 && : && \zeta_2 = \left(4M^2k + \frac{M}{2k^2} - \frac{M^2}{2k^3} - 2\frac{M^2}{k}\right)\zeta_1\,.
\end{aligned}
\tag{11.9}
$$

Specifically, at order y^{-1} and y^0, we obtain

$$
\Lambda_1 = h\varrho_1\,,\quad \Pi_1 = k\zeta_1\,,\quad h^2 = \mu^2\,,\quad k^2 = 1-\omega^2\,,\quad \delta = Mh\,,\quad \epsilon = M\frac{2k^2-1}{k} = M\frac{1-2\omega^2}{\sqrt{1-\omega^2}}\,.
$$

This is, of course, identical to our derivation in Sec. 8 that used the second-order versions of the scalar-field equations.

11.2 A massive φ that decays as fast as A

The case $h > 2k$ clearly leads to divergent terms in the equation for $\partial_y\Lambda$, since $e^{(h-2k)/y}$ diverges. Physically, my interpretation of this scenario is that the bosonic scalar drives the gravitational scalar all the way to infinity and simply prevents φ from decaying too fast. The limiting case $h = 2k$ where both fields decay in synchrony, however, is mathematically regular and we consider it here. The key difference in this calculation is that we need to include the term involving $e^{(h-2k)/y}$ on the right-hand side of $\partial_y \Lambda$ in Eq. (11.5) with the exponential factor set to 1. These terms can be written as

$$\frac{2\pi}{y^{2+\delta-2\epsilon}} \frac{X}{\alpha} \frac{F'}{F^2} \left(\omega^2 \zeta^2 - 2\alpha^2 \hat{V} \zeta^2 - F^2 \Pi^2 \right) \tag{11.10}$$

The terms in parentheses have the asymptotic behaviour

$$\omega^2 \zeta_1^2 y^2 - 2y^2 \zeta_1 - \Pi_1 y^2 = -\zeta_1^2 y^2 (2k^2 + 1) \,, \tag{11.11}$$

which has a manifestly positive coefficient. Combining this with the other terms in the $\partial_y \Lambda$ equation give the leading-order behaviour

$$0 = -\frac{h\Lambda_1}{y} + \frac{\varrho_1 \mu^2}{y} + \frac{4\pi}{y^{\delta-2\epsilon}} \alpha_0 \zeta_1 (2k^2 + 1) \,. \tag{11.12}$$

Now the term $\propto y^{-\delta+2\epsilon}$ can either be cancelled by the leading order term $\propto 1/y$ of the other terms or by some higher-order term $\propto 1/y^n$, $n \geq 2$. In general, this gives us the relation

$$\delta - 2\epsilon = n \qquad \Rightarrow \qquad \epsilon = \frac{\delta - n}{2} \,, \quad n \in \mathbb{N} \,. \tag{11.13}$$

It is not clear at this point whether $n = 1$ or solutions with $n > 1$ are possible. We might do more case distinctions, but for now, we assume that $n = 1$ and consider the result.

11.3 A massless φ

For a massless φ, we do not have an exponential falloff involving $e^{h/y}$ and therefore have $h = 0$ in Eqs. (11.4). We also anticipate that we need no y^δ correction in this case and therefore set $\delta = 0$ (but keep k and ϵ for the bosonic field). The main difference to the case of a massive φ is that we not need to account for the polynomial falloff of ϱ which actually makes the asymptotic behaviour a bit more complicated in the massless case. We can see this by considering Eqs. (8.1) and (8.2), where for an exponentially decaying φ, the right-hand sides $\partial_r \alpha / \alpha$ and $\partial_r X / X$ cancel each other under addition to any order in a series expansion in y, so that

$$\partial_r \ln \alpha + \partial_r \ln X = 0 \qquad \Rightarrow \qquad X\alpha = \text{const} \,. \tag{11.14}$$

For a massless φ, however, the four terms involving $\partial_y \varphi$ in Eqs. (8.1), (8.2) change this behaviour because these non-canceling terms will contribute in a series expansion in y. A detailed analysis gives us the asymptotic behaviours

$$X^2(y) = 1 + 2(M + \alpha_0 \Lambda_2) y + \dots \,, \qquad \alpha^2(y) = 1 - 2(M - \alpha_0 \Lambda_2) y + \dots \,, \tag{11.15}$$

where S_0 is a free parameter that we will investigate more closely below. For $\alpha_0 \neq 0$, we therefore deviate at order y. In Eq. (11.3) for $\partial_r X$, we can see this by noticing that the first term on the right-hand side is $\propto r^{-2}$, the second vanishes (either because $W = 0$ or φ decays exponentially) and the third term is $\propto \alpha_0 r^{-2}$. So $\alpha_0 \neq 0$ adjusts the $1/r$ part of X. For $\alpha_0 = 0$, on the other hand, the third and fourth term in the right-hand side are both $\propto r^{-3}$ and therefore affect X at order $r^{\geq 2}$ only. The same arguments hold for α or (equivalently) Φ.

After this segue, let us continue with the systematic series expansion of all our variables. We use the same variables as in the massive case and expand these variables as before, but we are now agnostic about the expansions of Φ and m, respectively α and X. We add free coefficients for these and write

$$\begin{aligned}
\Phi(y) &= \Phi_1 y + \Phi_2 y^2 + \Phi_3 y^3 + \Phi_4 y^4 + \dots\,, \\
m(y) &= M + m_1 y + m_2 y^2 + m_3 y^3 + m_4 y^4 + \dots\,, \\
\varrho(y) &= \varrho_1 y + \varrho_2 y^2 + \varrho_3 y^3 + \varrho_4 y^4 + \dots\,, \\
\Lambda(y) &= \Lambda_1 y + \Lambda_2 y^2 + \Lambda_3 y^3 + \Lambda_4 y^4 + \dots\,, \\
\zeta(y) &= \zeta_1 y + \zeta_2 y^2 + \zeta_3 y^3 + \zeta_4 y^4 + \dots\,, \\
\Pi(y) &= \Pi_1 y + \Pi_2 y^2 + \Pi_3 y^3 + \Pi_4 y^4 + \dots\,,
\end{aligned} \tag{11.16}$$

The expansion then yields the following conditions,

$$\begin{aligned}
\partial_y\Lambda: \quad & y^0 &:\quad & \Lambda_1 = 0\,, \\
& y^1 &:\quad & \Lambda_2 = \text{free}\,, \\
& y^2 &:\quad & \Lambda_3 = 0\,, \\
\partial_y m: \quad & y^{-1} &:\quad & \Lambda_1^2\Phi_1 - \Lambda_1\Lambda_2 = 0\,, \\
& y^0 &:\quad & m_1 = -\frac{1}{2}\Lambda_2^2\,, \\
& y^1 &:\quad & m_2 = -\frac{M}{2}\Lambda_2^2\,, \\
\partial_y\varrho: \quad & y^0 &:\quad & \varrho_1 = \Lambda_2\,, \\
& y^1 &:\quad & \varrho_2 = M\Lambda_2\,, \\
& y^2 &:\quad & \varrho_3 = \frac{4}{3}M^2\Lambda_2 - \frac{1}{6}\Lambda_2^3\,, \\
\partial_y\Phi: \quad & y^0 &:\quad & \Phi_1 = -M\,, \\
& y^1 &:\quad & \Phi_2 = -M^2\,, \\
& y^2 &:\quad & \Phi_3 = -\frac{4}{3}M^3 + \frac{1}{6}M\Lambda_2^2 \\
\partial_y\zeta: \quad & y^{-1} &:\quad & \Pi_1 = k\zeta_1\,, \\
& y^0 &:\quad & \Pi_2 = \zeta_1 + \epsilon\zeta_1 + k\zeta_2 + 2\alpha_0\Lambda_2\zeta_1 k - 2M\zeta_1 k\,, \\
\partial_y\Pi: \quad & y^{-1} &:\quad & k^2 = 1 - \omega^2\,, \\
& y^0 &:\quad & \epsilon = M\frac{2k^2-1}{k} + \frac{\alpha_0\Lambda_2}{k} = M\frac{1-2\omega^2}{\sqrt{1-\omega^2}} + \frac{\alpha_0\Lambda_2}{\sqrt{1-\omega^2}}\,.
\end{aligned} \tag{11.17}$$

The leading order terms in the series expansion are thus given by

$$
\begin{aligned}
\Phi &= -My - M^2y^2 + \dots\,, \\
m &= M - \frac{1}{2}\Lambda_2^2 y - \frac{M}{2}\Lambda_2^2 y^2 \dots\,, \\
\varrho &= \Lambda_2 y + M\Lambda_2 y^2 \dots\,, \\
\Lambda &= \Lambda_2 y^2 + \mathcal{O}(y^5)\,, \\
\zeta &= \zeta_1 y + \dots\,, \\
\Pi &= k\zeta_1 y + \dots\,,
\end{aligned}
\tag{11.18}
$$

with two free parameters, ζ_1 and Λ_2. Note that a non-zero α_0 changes the exponent ϵ in our polynomial adjustment. This is not too surprising, given that we already know the dependence of ϵ on the mass, i.e. the $1/r$ falloff coefficient in GR. A non-zero α_0 simply changes this $1/r$ coefficient in ST gravity and we should expect it to then also adjust ϵ.

A final word on the series expansions for ζ and Π. One may at first glance be surprised that we do not obtain at order y^0 sufficient conditions that determine ζ_2 *and* Π_2. Instead, we obtain one relation between ζ_2 and Π_2 and the value for ϵ. Casual evaluation of the order y^1 for these two variables results in an involvement of ζ_3 and Π_3, so that one may be tempted to believe that either ζ_2 or Π_3 is a free parameter. That looks like a contradiction since ζ and Π obey first-order in y ODEs, so their second derivatives are not free parameters. Indeed, a closer investigation of the series expansion of ζ and Π at order y^1 in their equations shows that ζ_3 and Π_3 cancel in an appropriate linear combination of the two conditions one obtains at this order. This linear combination determines ζ_2 in terms of ζ_1, k and ϵ as we would expect.

In summary, we have the following parameters that are not determined by the ODE system itself,

$$
M,\ \zeta_1,\ \Lambda_2,\ k,\ \alpha_0,\ \beta_0\,. \tag{11.19}
$$

In place of k we can choose ω and we could also relabel $\Lambda_2 = \varrho_1$ since the two are equal and equivalent.

In a two-way shooting algorithm, we also have two specify parameters at the origin. By regularity, we have $m = 0 \;\Leftrightarrow\; X = 1$ and we also have the scalar-field values $\varphi(r = 0)$ and $A(r = 0)$. Finally, we need to choose $\Phi(r = 0)$; we typically use the freedom to rescale Φ by requiring $\Phi(y = 0) = 0$ or we use the value obtained from a previous solution. With these inner boundary conditions in place, we can start the integration outwards up to some radius. Likewise, we integrate inwards from $y = 0$ with specified ζ_1, Λ_2 and M. The remaining free parameter is ω (or k) which is constant. This makes 7 parameters in total, 3 at $r = 0$, 3 at $y = 0$ and 1 constant. One of these is the *control parameter* that selects our model (or models since we may sometimes have more than one solution). Any of the 7 parameters can be the control parameter; in GR we typically chose $A_0 := A(r = 0)$ or ω, although $\Phi(r = 0)$ turned out the only one which uniquely determined a single boson star. In scalar-tensor gravity, we have less experience and may have to play with this. Let's factor this freedom in the coding, so that we can change the control parameter with as little hassle as possible.

After having acquired some experience, we have noticed that no parameter enables us to uniquely determine a model; there are different branches and we have to separately identify them. For this purpose, we use A_0 along the gr-like branch as well as A_0 and Φ_0 along the scalarized branches. For some cases, there is an ineversion in A_0 and here Φ_0 works better.

11.4 $\Phi_0 \neq 0$: A massive φ that decays more slowly than A

We now repeat the asymptotic analysis without assuming that $\Phi_0 = 0$, i.e. for the case where the lapse function has not yet been normalized to 1 at infinity. These expressions may become important for supplementing the one-way shooting code with an inward integration to meet the end of the r integration. The series Ansatz only differs by adding Φ_0 to the Φ expansion, but this factor propagates into the resulting relations; fortunately not in too complicated a form. We also slightly adjust our notation by using ϱ_1 as the free parameter instead of Λ_2; this eliminates some e^{Φ_0} terms and is also the terminology used in the code. We thus start with

$$
\begin{aligned}
m(y) &= M + m_1 y + m_2 y^2 + m_3 y^3 + m_4 + y^4 + \dots\,, \\
\Phi(y) &= \Phi_0 + \Phi_1 y + \Phi_2 y^2 + \Phi_3 y^3 + \Phi_4 y^4 + \dots\,, \\
\hat{W}(y) &= \mu^2\,, \\
W_{,\varphi^2} &= \frac{\mu^2}{2}\,, \\
\hat{V}(y) &= 1\,, \\
V'(y) := V_{,A^2} &= 1\,, \\
\varrho(y) &= \varrho_1 y + \varrho_2 y^2 + \varrho_3 y^3 + \varrho_4 y^4 + \dots\,, \\
\zeta(y) &= \zeta_1 y + \zeta_2 y^2 + \zeta_3 y^3 + \zeta_4 y^4 + \dots\,, \\
\Lambda(y) &= \Lambda_1 y + \Lambda_2 y^2 + \Lambda_3 y^3 + \Lambda_4 y^4 + \dots\,, \\
\Pi(y) &= \Pi_1 y + \Pi_2 y^2 + \Pi_3 y^3 + \Pi_4 y^4 + \dots\,.
\end{aligned}
\tag{11.20}
$$

Inserting these into the equations yields the following conditions,

$$
\begin{aligned}
\partial_y m: \quad & y^0 \quad : \quad m_1 = 0\,, \\
& y^1 \quad : \quad m_2 = 0\,, \\
& y^2 \quad : \quad m_3 = 0\,, \\
\partial_y \Phi: \quad & y^0 \quad : \quad \Phi_1 = -M\,, \\
& y^1 \quad : \quad \Phi_2 = -M^2\,, \\
& y^2 \quad : \quad \Phi_3 = -\frac{4}{3}M^3\,, \\
\partial_y \varrho: \quad & y^{-1} \quad : \quad \Lambda_1 = e^{\Phi_0} h\varrho_1\,, \\
& y^0 \quad : \quad \Lambda_2 = e^{\Phi_0}\left[(1+\delta)\varrho_1 + h\varrho_2 - 2Mh\varrho_1\right]\,, \\
& y^1 \quad : \quad \Lambda_3 = e^{\Phi_0}\left[h\varrho_3 - 4M^2 h\varrho_1 + (2+\delta)\varrho_2\right] - 2M\Lambda_2\,, \\
\partial_y \Lambda: \quad & y^{-1} \quad : \quad h^2 = \mu^2\,, \\
& y^0 \quad : \quad \Lambda_2 = e^{\Phi_0}\left[\varrho_1(1-\delta) + h\varrho_2\right] \quad \Rightarrow \quad \delta = Mh\,, \\
& y^1 \quad : \quad \Lambda_3 = e^{\Phi_0}\left[h\varrho_3 + M(Mh-1)\varrho_1 - Mh\varrho_2\right]\,, \\
\partial_y \zeta: \quad & y^{-1} \quad : \quad \Pi_1 = e^{\Phi_0} k\zeta_1\,, \\
& y^0 \quad : \quad \Pi_2 = e^{\Phi_0}\left[(1+\epsilon)\zeta_1 + k\zeta_2 - 2Mk\zeta_1\right]\,, \\
& y^1 \quad : \quad \Pi_3 = e^{\Phi_0}\left[k\zeta_3 + (2+\epsilon-2Mk)\zeta_2 - 2M(1+\epsilon)\zeta_1\right]\,, \\
\partial_y \Pi: \quad & y^{-1} \quad : \quad k^2 = 1 - \frac{\omega^2}{e^{2\Phi_0}} \quad \Leftrightarrow \quad \omega^2 = e^{2\Phi_0}(1-k^2)\,, \\
& y^0 \quad : \quad \epsilon = M\frac{2k^2-1}{k}\,, \\
& y^1 \quad : \quad \zeta_2 = \left(4M^2 k + \frac{M}{2k^2} - \frac{M^2}{2k^3} - 2\frac{M^2}{k}\right)\zeta_1\,.
\end{aligned}
\tag{11.21}
$$

Specifically, at order y^{-1} and y^0, we obtain

$$
\Lambda_1 = e^{\Phi_0} h\varrho_1\,, \qquad \Pi_1 = e^{\Phi_0} k\zeta_1\,, \qquad h^2 = \mu^2\,, \qquad k^2 = 1 - \frac{\omega^2}{e^{2\Phi_0}}\,, \qquad \delta = Mh\,, \qquad \epsilon = M\frac{2k^2-1}{k}\,.
$$

This is, of course, identical to our derivation in Sec. 8 that used the second-order versions of the scalar-field equations.

11.5 $\Phi_0 \neq 0$: A massless φ

Keeping a non-zero Φ_0 in the case of a massless φ gives us

$$
\begin{aligned}
\Phi(y) &= \Phi_0 + \Phi_1 y + \Phi_2 y^2 + \Phi_3 y^3 + \Phi_4 y^4 + \dots\,, \\
m(y) &= M + m_1 y + m_2 y^2 + m_3 y^3 + m_4 y^4 + \dots\,, \\
\varrho(y) &= \varrho_1 y + \varrho_2 y^2 + \varrho_3 y^3 + \varrho_4 y^4 + \dots\,, \\
\Lambda(y) &= \Lambda_1 y + \Lambda_2 y^2 + \Lambda_3 y^3 + \Lambda_4 y^4 + \dots\,, \\
\zeta(y) &= \zeta_1 y + \zeta_2 y^2 + \zeta_3 y^3 + \zeta_4 y^4 + \dots\,, \\
\Pi(y) &= \Pi_1 y + \Pi_2 y^2 + \Pi_3 y^3 + \Pi_4 y^4 + \dots\,,
\end{aligned}
\tag{11.22}
$$

The expansion then yields the following conditions,

$$
\begin{array}{rccl}
\partial_y\Lambda: & y^0 & : & \Lambda_1 \;=\; 0\,, \\
 & y^1 & : & \Lambda_2 \;=\; \text{free} \quad \Leftrightarrow \quad \varrho_1 \;=\; \text{free}\,, \\
 & y^2 & : & \Lambda_3 \;=\; 0\,, \\
\partial_y m: & y^{-1} & : & \Lambda_1^2\Phi_1 - \Lambda_1\Lambda_2 = 0\,, \\
 & y^0 & : & m_1 \;=\; -\dfrac{1}{2}\dfrac{\Lambda_2^2}{e^{2\Phi_0}}\,, \\
 & y^1 & : & m_2 = \dfrac{\Phi_1}{2}\dfrac{\Lambda_2^2}{e^{2\Phi_0}}\,, \\
\partial_y\Phi: & y^0 & : & \Phi_1 \;=\; -M\,, \\
 & y^1 & : & \Phi_2 \;=\; -M^2\,, \\
 & y^2 & : & \Phi_3 \;=\; -\dfrac{4}{3}M^3 + \dfrac{M}{6}\dfrac{\Lambda_2^2}{e^{2\Phi_0}} \\
\partial_y\varrho: & y^0 & : & \Lambda_2 \;=\; e^{\Phi_0}\varrho_1\,, \\
 & y^1 & : & \varrho_2 \;=\; M\varrho_1\,, \\
 & y^2 & : & \varrho_3 \;=\; \dfrac{4}{3}M^2\varrho_1 - \dfrac{1}{6}\varrho_1^3\,, \\
\partial_y\zeta: & y^{-1} & : & \Pi_1 \;=\; e^{\Phi_0}k\zeta_1\,, \\
 & y^0 & : & \Pi_2 \;=\; e^{\Phi_0}\left[(1+\epsilon)\zeta_1 + k\zeta_2 + 2\alpha_0 k\varrho_1\zeta_1 - 2Mk\zeta_1\right]\,, \\
\partial_y\Pi: & y^{-1} & : & k^2 \;=\; 1 - \dfrac{\omega^2}{e^{2\Phi_0}} \quad \Leftrightarrow \quad \omega^2 = e^{2\Phi_0}(1-k^2)\,, \\
 & y^0 & : & \epsilon \;=\; M\dfrac{2k^2-1}{k} + \dfrac{\alpha_0\varrho_1}{k}
\end{array}
\tag{11.23}
$$

12 Jordan frame diagnostics

Recalling the line elements

$$
\mathrm{d}\bar{s}^3 = -F\alpha^2\mathrm{d}t^2 + FX^2\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,, \qquad \mathrm{d}s^2 = -\alpha^2\mathrm{d}t^2 + X^2\mathrm{d}r^2 + \frac{r^2}{F}\mathrm{d}\Omega^2\,, \tag{12.1}
$$

we see that the Jordan metric is obtained from its Einstein-frame counterpart through division by F. We note, however, that our radial coordinate r represents the areal radius in the Einstein frame. There is no specific reason why we have chosen it that way; it just seems to have migrated historically. Whether it is a good choice or not remains to be seen, but we're definitely not going to recode everything. We have to take it into account in our diagnostics, however.

It appears natural to define a mass function in terms of the g_{rr} and $\bar{g}_{rr}$ components of the metric in accordance with the Schwarzschild metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)^2 \mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2 \,, \tag{12.2}$$

where M appears as the gravitational mass in the Newtonian limit for test particles. For the Einstein metric, this directly leads to the definition

$$FX^2 = \left(1 - \frac{2m}{r}\right)^{-1} \qquad \Rightarrow \qquad m = \frac{r}{2}\left(1 - \frac{1}{FX^2}\right) \,, \tag{12.3}$$

which we have used throughout these notes. In the Jordan frame, one would in analogy proceed from the metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2\tilde{m}}{\tilde{r}}\right) \mathrm{d}t^2 + \left(1 - \frac{2\tilde{m}}{\tilde{r}}\right)^{-1} \mathrm{d}\tilde{r}^2 + \tilde{r}^2 \mathrm{d}\Omega^2 \,, \tag{12.4}$$

and define

$$\tilde{m} := \frac{\tilde{r}}{2}\left(1 - \frac{1}{X^2}\right) \,. \tag{12.5}$$

This is where some complication comes in: the areal radius in the Jordan frame $\tilde{r}$ is not our radial coordinate r, but

$$\boxed{\tilde{r} = \frac{r}{\sqrt{F}} \,.} \tag{12.6}$$

If we want to follow this analogy, we therefore need to transform our Jordan metric

$$\mathrm{d}s^2 = -\alpha^2 \mathrm{d}t^2 + X^2 \mathrm{d}r^2 + \frac{r^2}{F} \mathrm{d}\Omega^2 \,,$$

to the Jordan areal radius $\tilde{r}$. This is not difficult, but leads to some extra terms, since

$$\mathrm{d}\tilde{r} = \frac{\partial \tilde{r}}{\partial r} \mathrm{d}r = \frac{\partial}{\partial r}\left(\frac{r}{\sqrt{F}}\right) \mathrm{d}r = \frac{1}{\sqrt{F}}\left[1 - \frac{r}{2}\frac{F'}{F}\partial_r \varphi\right] \mathrm{d}r = \underbrace{\frac{1}{\sqrt{F}}\left[1 - \frac{r}{2}\frac{F'}{F}\frac{X}{\alpha}\kappa\right]}_{=:Q} \mathrm{d}r \,. \tag{12.7}$$

Our Jordan-fram metric in areal gauge and polar slicing therefore becomes

$$\mathrm{d}s^2 = -\alpha^2 \mathrm{d}t^2 + \frac{X^2}{Q^2} \mathrm{d}\tilde{r}^2 + \tilde{r}^2 \mathrm{d}\Omega^2 \,, \tag{12.8}$$

and our Jordan-fram mass function would become

$$\frac{X^2}{Q^2} = \left(1 - \frac{2\tilde{m}}{\tilde{r}}\right)^{-1} \qquad \Rightarrow \qquad \tilde{m} = \frac{\tilde{r}}{2}\left(1 - \frac{Q^2}{X^2}\right) \,. \tag{12.9}$$

Using

$$\frac{F'}{F} = (\ln F)' = -2\alpha_0 - 2\beta_0 \varphi \,, \tag{12.10}$$

the function

$$Q = \frac{1}{\sqrt{F}}\left[1 - \frac{r}{2}\frac{F'}{F}\frac{X}{\alpha}\kappa\right], \tag{12.11}$$

is straightforward to evaluate in the interior. In the exterior, we have to rephrase in terms of our rescaled function $\Lambda = -y^{-\delta}e^{h/y}\kappa$ and obtain

$$Q = \frac{1}{\sqrt{F}}\left[1 + \frac{y^{\delta-1}}{2}\frac{F'}{F}\frac{X}{\alpha}e^{-h/y}\Lambda\right]. \tag{12.12}$$

The good news is that at infinity, the Einstein- and Jordan-frame mass functions agree, since $F \to 1$ and inside the function Q, the second term in parentheses disappears either because of the exponential factor $e^{-h/y}$ for massive φ or the $\Lambda \propto y^2$ fall-off for massless φ.

The slightly less satisfactory news is that computing the Jordan-mass in practice seems to be non-monotonic at medium radii. This appears to be a consequence of the local peak in the profile $\varphi(r)$ which leads to a non-monotonic behaviour of $1 - Q^2/X^2$. This is not necessarily a problem, since the Jordan-frame mass function is not an observable at finite radius, but we should check whether a bug may have crept into our calculations or numerics...

13 Boson-star models in scalar-tensor gravity

In this section, we collect figures that display the properties of boson-star models in scalar-tensor gravity. For starters, we summarize our main parameters and diagnostics. Our computational models are characterized by the following parameters,

$$\begin{aligned}
\alpha_0 &= \text{linear parameter of the scalar-tensor theory}\,,\\
\beta_0 &= \text{quadratic parameter of the scalar-tensor theory}\,,\\
\mu_\varphi &= \text{mass of the gravitational scalar}\,,\\
\sigma_0 &= \text{solitonic potential parameter}\,,\\
\Phi_0 &= \text{logarithm of the Einstein-frame lapse at } r=0\,,\\
A_0 &= \text{amplitude of the bosonic scalar at } r=0\,,\\
\varphi_0 &= \text{gravitational scalar at } r=0\,,\\
M &= \text{ADM mass}\,,\\
\zeta_1 &= \text{linear coefficient in } y \text{ of the rescaled bosonic scalar at infinity}\,,\\
\varrho_1 &= \text{linear coefficient in } y \text{ of the rescaled gravitational scalar at infinity}\,,\\
\omega &= \text{boson-star frequency}\,.
\end{aligned}$$

Some of these quantities also act as diagnostics, but we add the following further diagnostic variables,

$$\begin{aligned}
\varphi_{\text{max}} &= \text{maximal absolute value of the gravitational scalar}\,,\\
R_{\text{E}} &= \text{Einstein radius containing 99\,\% of the Einstein mass function}\,,\\
\mathcal{C}_{\text{max}} &= \text{maximum of the ratio Einstein mass to Einstein radius, } m(r)/r\,,\\
\mathcal{Q} &= \text{Noether charge}\,,
\end{aligned} \tag{13.1}$$

13.1 Boson-star models for $\mu_\varphi = 0$, $\alpha_0 = 0$, $\sigma_0 = 0.2$

We start with boson-star models for a massless gravitational scalar and vanishing linear term in the coupling function. For each value of the quadratic coupling parameter β_0, we thus obtain a set of boson-star models composed of two one-parameter families, the GR branch and a scalarized branch. For simplicity, we have fixed the solitonic parameter at $\sigma_0 = 0.2$ here. We will explore different potentials at a later stage.

For the fixed parameters $\mu_\varphi = 0$, $\alpha_0 = 0$ and $\sigma_0 = 0.2$, we have empirically found scalarized branches only for $\beta_0 \leq -6.2$ which is a more negative threshold that the value -4.35 commonly found for neutron stars. For this β_0 value, the scalarized branch is very small in the mass-radius diagram, so that we expect the exact threshold value to be possibly marginally less negative than $\beta_0 = -6.2$ but not much.

For each β_0, we present the resulting boson-star models in the form of mass-radius or M_{ADM} vs. R_{E} diagrams where M_{ADM} denotes the ADM mass (which is the same for the Jordan and Einstein frame) and R_{E} is the Einstein radius value where the monotonically increasing Einstein mass function $m(r)$ has reached 99 % of its value M_{ADM} at infinity. Each figure has 4 panels, each using a different diagnostic encoded in the color of the curves: φ_0, φ_{max}, Φ_0 and A_0. We will summarize some first results at the end of this subsection.

13.2 Boson-star models for $\mu_\varphi = 0.05$, $\alpha_0 = 0$, $\sigma_0 = 0.2$

We now repeat the very same analysis for a massive gravitational scalar with $\mu_\varphi = 0.05$. need to be a bit careful with our interpretation of the mass values here, since our potential for the gravitational and bosonic scalar follow different conventions,

$$W(\varphi) = \frac{1}{2}\mu_\varphi^2\varphi^2\,, \qquad V(A) = A^2 + (O)(A^4)\,.$$

Of course, we have the additional freedom of embedding these potential functions with any constant coefficients in the Lagrangian (2.1). Rather than torturing out brains over all these purely conventional factors, it is much easier to let physics have the final word and compare the mass parameters by looking at the flat-space limit of the wave equations for the two respecitve scalar fields. We have static equations, of course, but we in our minds add the time derivatives $-\partial_t^2\varphi$ and $-\partial_t^2 A$ to the left-hand sides of the equations for $\partial_r\kappa$ and $\partial_r\theta$ in Eq. (11.3). In the case of A, the $\partial_t^2 A$ term is actually present in the form of $-\omega^2 A$. Setting $X = \alpha = F = 1$ and bearing in mind that $\kappa \sim \partial_r\varphi$ and $\theta \sim \partial_r A$, this

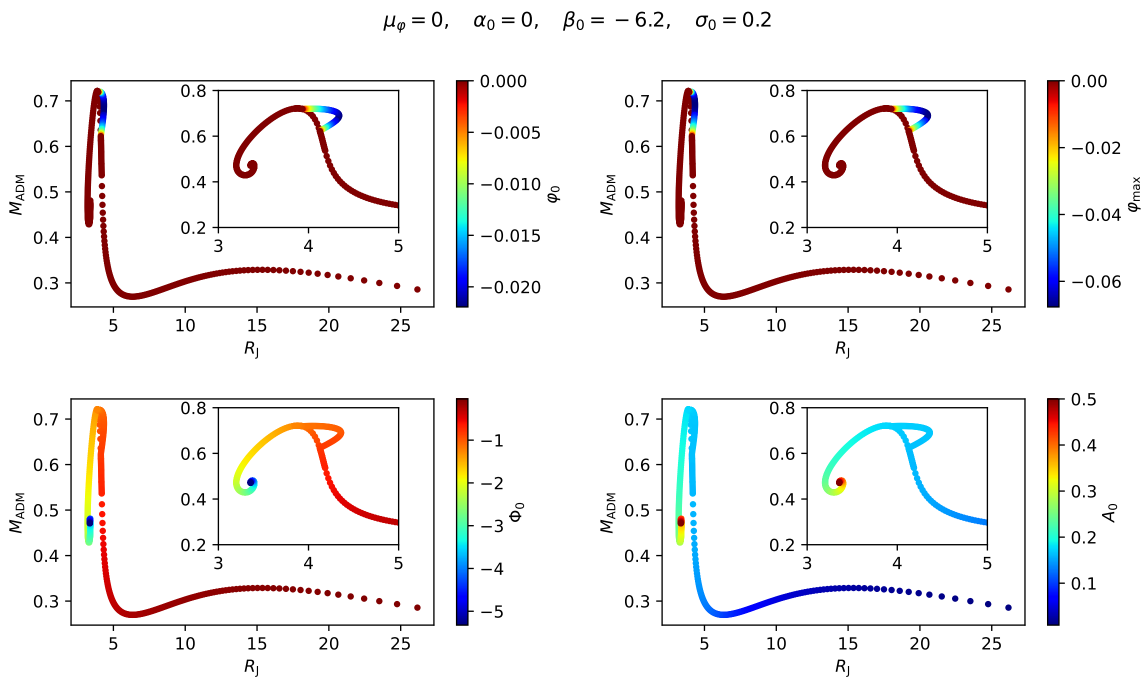

Figure 1: Boson-star models in scalar-tensor gravity with $\alpha_0 = 0$, $\mu_\varphi = 0$ and $\beta_0 = -6.2$ for a solitonic potential with $\sigma_0 = 0.2$. Each point in these mass-radius diagrams represents a boson-star model. The location of the point represents the star's mass $M_{\rm ADM}$ and Einstein radius $R_{\rm E}$ and the color coding in the respective panels displays the central gravitational scalar φ_0, the maximum (over radius) gravitational scalar $\varphi_{\rm max}$, the central value of the logarithmic Einstein lapse Φ_0, and the central boson scalar value A_0.

gives us to linear order in φ and A,

$$\begin{aligned}
-\partial_t^2\varphi + \partial_r^2\varphi &= -2\frac{\partial_r\varphi}{r} + 2W_{,\varphi^2}\varphi \;=\; -2\frac{\partial_r\varphi}{r} + \mu_\varphi^2\varphi\,,\,, \\
-\partial_t^2 A + \partial_r^2 A &= -2\frac{\partial_r A}{r} + V_{,A^2}A = -2\frac{\partial_r A}{r} + A\,. \qquad (13.2)
\end{aligned}$$

We see that the bosonic scalar mass is normalized to 1 and μ_φ is the mass of the gravitational scalar (the above mentioned factor of 1/2 has indeed been compensated for by adding W with a factor 2 to the action.

With that potential source of confusion out of our way, we now show results for boson-star models in massive scalar-tensor gravity. The diagnostics and parameters are the same as in the massive case. The least negative value of β_0 for which we have found scalarized models is $\beta_0 = -7$ which we display in Fig. 5. This case resembles the $\beta_0 = -6.2$ set of models for massless scalar-tensor gravity in Fig. 1, indicating that the gravitational scalar mass tends to weaken the scalarization as has also been observed for neutron-star models [12, 14].

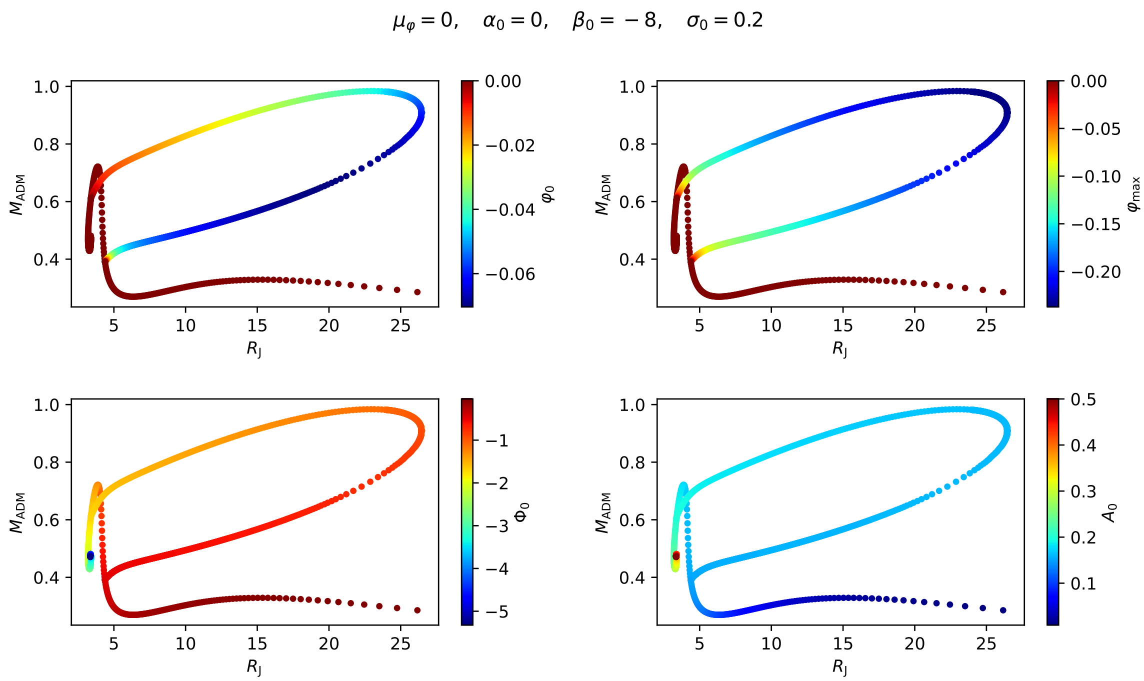

Figure 2: Same as Fig. 1 but for $\beta_0 = -8$.

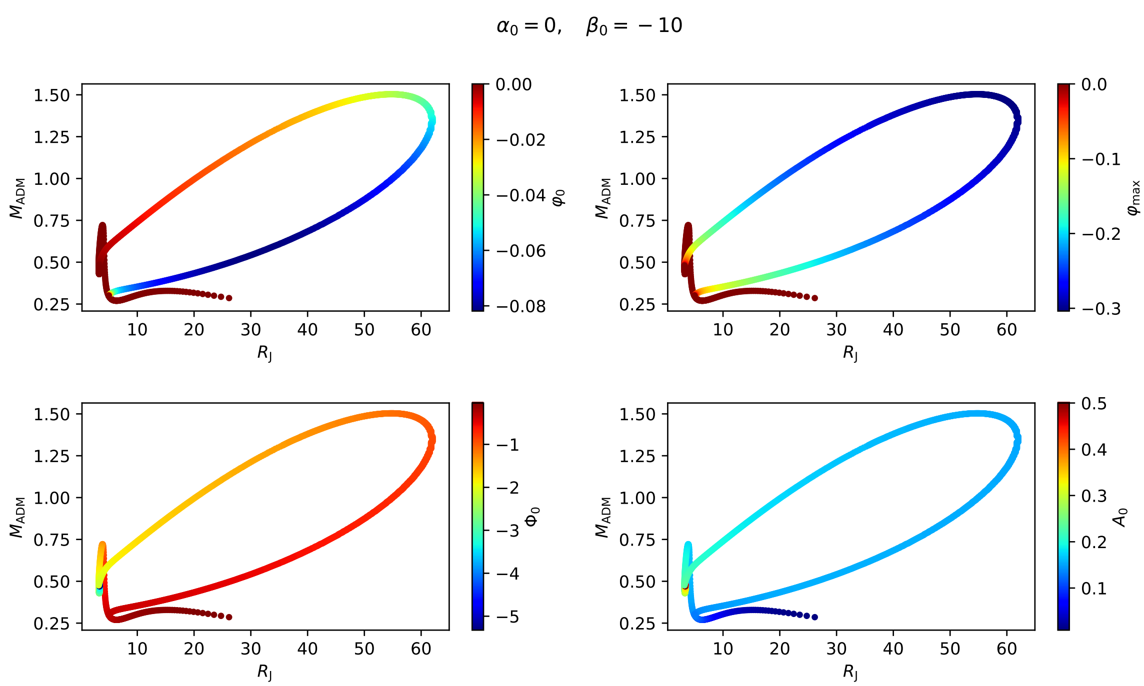

Figure 3: Same as Fig. 1 but for $\beta_0 = -10$.

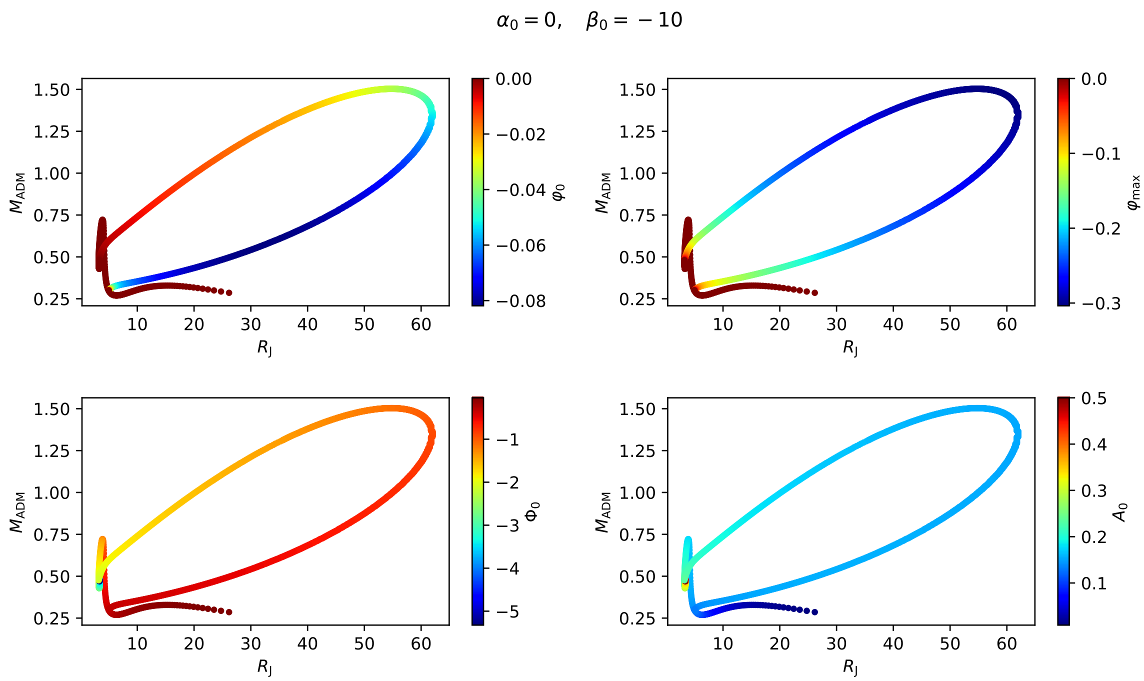

Figure 4: Same as Fig. 1 but for $\beta_0 = -12$.

13.3 Preliminary results

In this subsection, we gather systematic features of the scalarized models obtained across the parameter space. Naturally, our coverage of the parameter space is limited at present, so this is a *white section* which will keep evolving as we computer more models. For the time being, we regard the following parameters as our sandbox: α_0, β_0 and μ_φ for the ST theory and σ_0 for the bosonic scalar potential.

The first insights we can infer from Figs. 1-7 are as follows.

- the general structure of the scalarized and GR like branches is similar to that observed for neutron star models; cf. e.g. Ref. [14].
- The onset of spontaneous scalarization in terms of the β_0 parameter, however, differs from the neutron-star case: Whereas scalarized neutron stars are found for $\beta_0 \lesssim -4.35$ over a wide range of equations of state, the cases studied above require $\beta_0 \lesssim -6.2$ and $\beta_0 \lesssim -7$, respectively. They differ both from the neutron-star value and from each other. This could be a consequence of neutron stars having quite similar compactness for different EOSs whereas boson stars can vary quite significantly in terms of their compactness and, probably more importantly, of the maximum compactness they can reach for a given potential.
- A non-zero mass of the gravitational scalar appears to weaken the scalarization as also observed for GR [12, 14].
- Especially for $\mu_\varphi = 0$, we observe rather large radii of the scalarized boson-star models. This should be taken with a grain of salt, however, since a boson-star radius is not a rigorously defined quantity. The 99 % mass criterion used here may indicate that the long-range of a

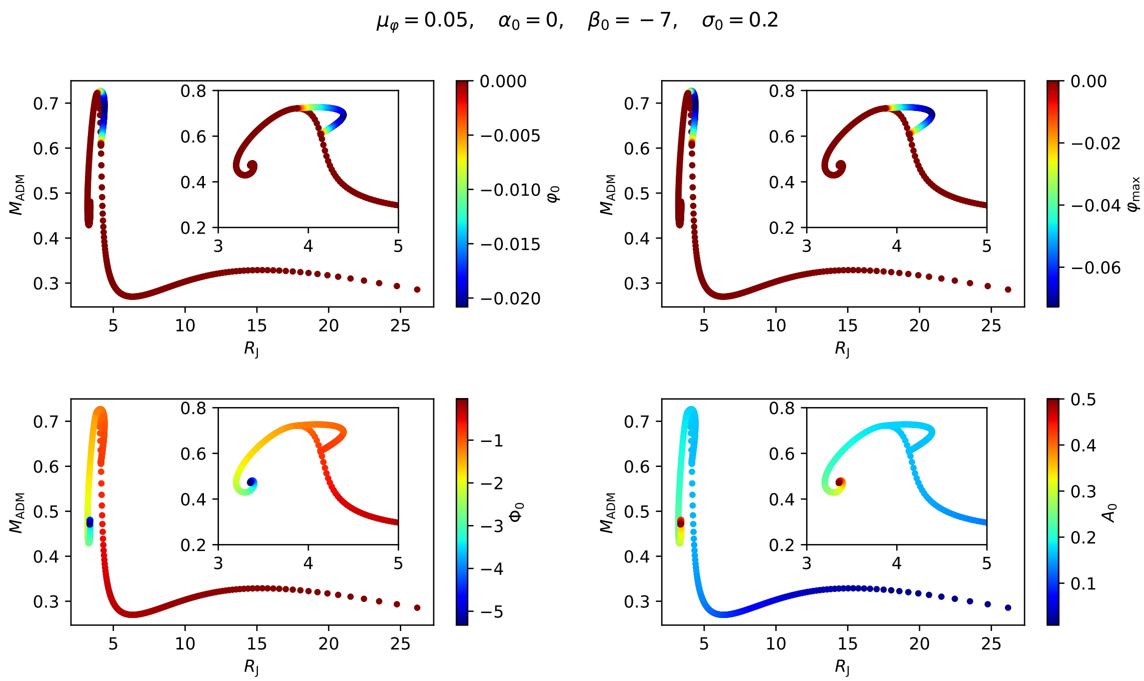

Figure 5: Boson-star models in scalar-tensor gravity with $\alpha_0 = 0$, $\mu_\varphi = 0.05$ and $\beta_0 = -7$ for a solitonic potential with $\sigma_0 = 0.2$. Each point in these mass-radius diagrams represents a boson-star model. The location of the point represents the star's mass $M_{\rm ADM}$ and Einstein radius $R_{\rm E}$ and the color coding in the respective panels displays the central gravitational scalar φ_0, the maximum (over radius) gravitational scalar $\varphi_{\rm max}$, the central value of the logarithmic Einstein lapse Φ_0, and the central boson scalar value A_0.

massless gravitational scalar field could drive the bosonic scalar to non-negligible radii out to large radii. We may need to investigate the equations in more detail, however, to identify how such a coupling may be realized mathematically.

- A first, tentative analysis of the boson star's stability comparing the gravitational mass of stars with equal Neother charge (i.e. boson mass) indicates that along the scalarized branches, stars to the right of the maximum mass (i.e. with larger radius) are stable and those to the left are not. Scalarized stars, in general appear to be energetically favored over their GR-like counter parts, confirming what is observed for neutron stars [7].

- In agreement with Ref. [14], we find plenty of models along the scalarized branch where the maximum gravitational scalar is realized away from the origin, resulting in a shell like profile of φ.

- Strongly scalarized become more challenging to compute numerically, revealing a beahviour similar to the difficulties encountered for thin-shell boson stars in GR [5]. This is confirmed by first explorations of the profiles $A(r)$ of the bosonic scalar which displays roughly constant magnitude out to larger radii, as well as a decrease in the frequency ω of these models. This behaviour might be related to the fact that for $\alpha_0 = 0$, $F(\varphi) > 0$ for non-zero φ which decreases the magnitude of the last term on the right-hand side of the $\partial_y \Pi$ equation in Eqs. (11.5): This

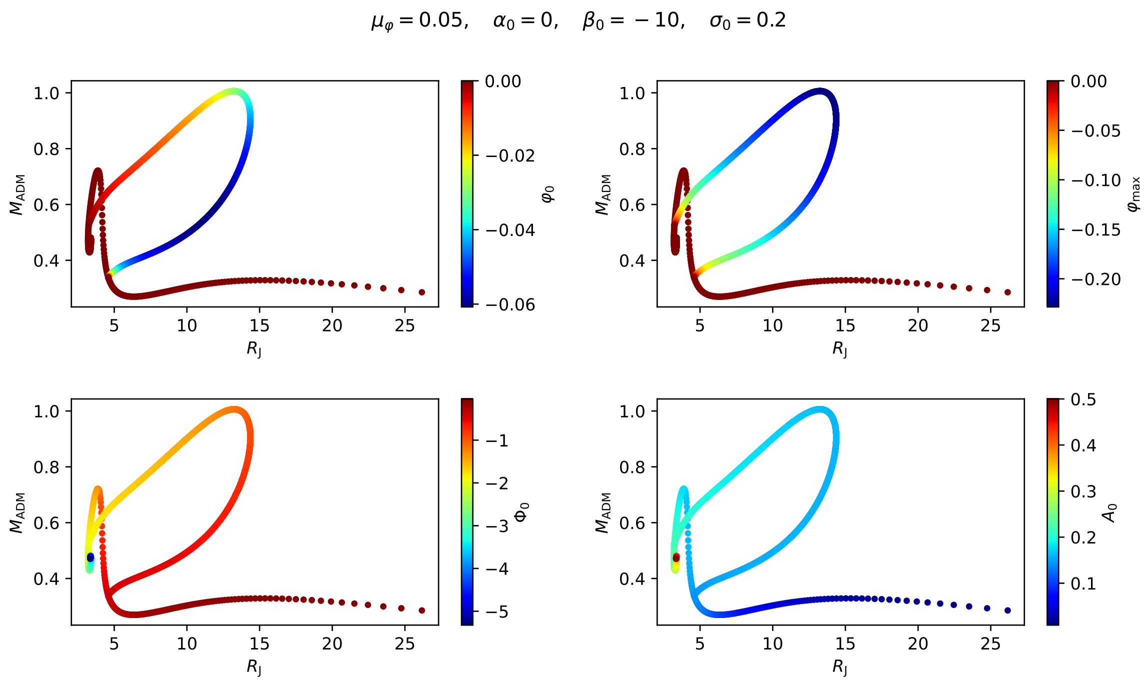

Figure 6: Same as Fig. 5 but for $\beta_0 = -10$.

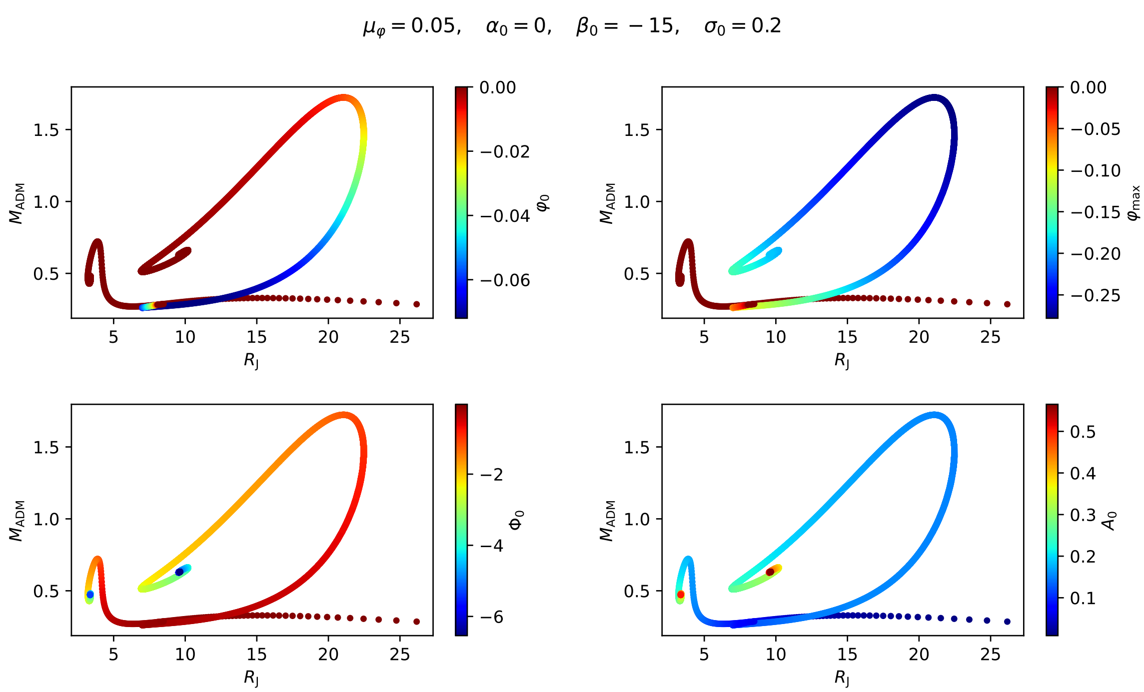

Figure 7: Same as Fig. 5 but for $\beta_0 = -15$.

term determines the exponential behaviour of the scalar field $A(r)$ and very small values of this term approach the limit where the exponential behaviour $e^{\pm kr}$ approaches a constant.

Possible TODOs:

- Check for which μ_φ scalarization disappears or how strongly it is suppressed.
- Compute $\mathrm{tr}T_{\alpha\beta}$. Can we use something to predict at which β_0 scalarization starts? Explore whether the Tachyonic instability gives us predictions.
- Go to smaller σ_0. Maybe 0.15 and 0.1 or 0.12... Check if the β_0 threshold changes. Also explore other potentials (repulsive and mini-BS).
- Units! What are realistic values for μ_φ.
- Is the $\max(\varphi)$ always away from $r = 0$? And why?
- Find best ways of graphically illustrating stability based on Noether charge.

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