1 Lorentz transformations

Let us consider two observers, $\mathcal{O}$ with coordinates x^μ and $\tilde{\mathcal{O}}$ with coordinates $\tilde{x}^{\tilde{\alpha}}$. If $\tilde{\mathcal{O}}$ moves with velocity v^i relative to $\mathcal{O}$, then their coordinates are related by the relation

$$\tilde{x}^{\tilde{\alpha}} = \Lambda^{\tilde{\alpha}}{}_\mu x^\mu + \tilde{x}^{\tilde{\alpha}}_0 \qquad \Leftrightarrow \qquad x^\mu = \Lambda^\mu{}_{\tilde{\alpha}} \tilde{x}^\alpha - x^\mu_0\,, \tag{1.1}$$

where the Lorentz transformation is given by

$$\Lambda^{\tilde{\alpha}}{}_\mu = \left(\begin{array}{c|c} \gamma & -\gamma v_j \\ \hline -\gamma v^i & \delta^i{}_j + (\gamma-1)\frac{v^i v_j}{|\vec{v}|^2} \end{array}\right) \qquad \Leftrightarrow \qquad \Lambda^\mu{}_{\tilde{\alpha}} = \left(\begin{array}{c|c} \gamma & \gamma v_j \\ \hline \gamma v^i & \delta^i{}_j + (\gamma-1)\frac{v^i v_j}{|\vec{v}|^2} \end{array}\right). \tag{1.2}$$

Note that here we raise and lower indices of purely spatial quantities with the flat three-dimensional Euclidean metric δ_{ij}, so that $v_j = \delta_{jk} v^k$, $\delta^i{}_j = \delta^{ik}\delta_{kj}$ etc. A straightforward calculation confirms that $\Lambda^{\tilde{\alpha}}{}_\mu \Lambda^\mu{}_{\tilde{\beta}} = \delta^\alpha{}_{\tilde{\beta}}$ and $\Lambda^\mu{}_{\tilde{\alpha}} \Lambda^{\tilde{\alpha}}{}_\nu = \delta^\mu{}_\nu$. Physically, the inverse Lorentz transformation corresponds to merely inverting the sign of the velocity $v^i \to -v^i$, as one would expect. Equation (1.1) furthermore implies that $x^\mu_0 = \Lambda^\mu{}_{\tilde{\alpha}} \tilde{x}^{\tilde{\alpha}}_0$ and $\tilde{x}^{\tilde{\alpha}}_0 = \Lambda^{\tilde{\alpha}} x^\mu_0$, and, without loss of generality, we can set $\tilde{x}^{\tilde{\alpha}}_0 = 0 = x^\mu_0$.
When the velocity points along one of the coordinate directions as for example in $v^i = (0,\,0,\,v)$, the Lorentz transformation matrices simplify to

$$\Lambda^{\tilde{\alpha}}{}_\mu = \left(\begin{array}{c|ccc} \gamma & 0 & 0 & -\gamma v \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma v & 0 & 0 & \gamma \end{array}\right) \qquad \Leftrightarrow \qquad \Lambda^\mu{}_{\tilde{\alpha}} = \left(\begin{array}{c|ccc} \gamma & 0 & 0 & \gamma v \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma v & 0 & 0 & \gamma \end{array}\right), \tag{1.3}$$

which gives us the simple coordinate relations

$$\begin{array}{l} \tilde{t} = \gamma(t - vz) \\ \tilde{z} = \gamma(z - vt) \end{array} \qquad \Leftrightarrow \qquad \begin{array}{l} t = \gamma(\tilde{t} + v\tilde{z}) \\ z = \gamma(\tilde{z} + v\tilde{t}) \end{array}. \tag{1.4}$$

These relations also exemplify that $\tilde{x}^{\tilde{\alpha}}$ are moving with velocity v relative to x^μ, as in the Newtonian limit $\tilde{z} = z - vt$.
In the next section we take the viewpoint of passive transformations. That is, we regard a physical system in the frame x^μ and wish to see how it looks like for observer $\tilde{\mathcal{O}}$ with coordinates $\tilde{x}^{\tilde{\alpha}}$. If the system is at rest in x^μ, then its description in the frame $\tilde{x}^\alpha$ is that of an object boosted with a velocity $-v^i$. The minus sign is a consequence of our convention and is trivially taken into account by inverting the user specified parameters for the velocity components v^x, v^y and v^z. This way a user who specifies, say, a positive v^z will be provided with an object boosted in the direction of increasing z.

2 Boosting general metrics

We shall pursue two approaches in these notes. (i) We will consider general spacetime metrics and compute the 3+1 variables $\tilde{\gamma}_{\tilde{i}\tilde{j}}$, $\tilde{K}_{\tilde{i}\tilde{j}}$, $\tilde{\alpha}$ and $\tilde{\beta}^{\tilde{i}}$ in terms of the spacetime metric components $g_{\mu\nu}$ in the rest frame. (ii) We will explicitly start with the Schwarzschild metric in isotropic coordinates and analytically compute spatial metric, extrinsic curvature, lapse and shift of its boosted counterpart in the coordinate system $\tilde{x}^{\tilde{\alpha}}$. In this section we discuss the first case.
In practice, we will have available as input the coordinates $\tilde{x}^\alpha$ of the space time point where we wish to evaluate the boosted metric. The analytically known metric *in the rest frame* is, of course, given

in terms of the rest frame coordinates x^μ, so our first task is to transform the coordinates from the boosted frame to the rest frame,

$$x^\mu = \Lambda^\mu{}_{\tilde{\alpha}} \tilde{x}^{\tilde{\alpha}} \,. \tag{2.1}$$

Next, we calculate the metric components $g_{\mu\nu}(x^\rho)$, its inverse $g^{\mu\nu}$ and the first derivatives $\partial_\lambda g_{\mu\nu}$. In the remainder of this section, we assume that we have all these quantities available in closed analytic form. For isotropic Schwarzschild, for instance, we have

$$g_{\mu\nu} = \begin{pmatrix} -\Omega^2\Psi^{-2} & 0 & 0 & 0 \\ 0 & \Psi^4 & 0 & 0 \\ 0 & 0 & \Psi^4 & 0 \\ 0 & 0 & 0 & \Psi^4 \end{pmatrix} \quad \Leftrightarrow \quad g^{\mu\nu} = \begin{pmatrix} -\Psi^2\Omega^{-2} & 0 & 0 & 0 \\ 0 & \Psi^{-4} & 0 & 0 \\ 0 & 0 & \Psi^{-4} & 0 \\ 0 & 0 & 0 & \Psi^{-4} \end{pmatrix} , \tag{2.2}$$

where

$$\Psi = 1 + \frac{m}{2r} \,, \qquad \Omega = 1 - \frac{m}{2r} \,, \qquad r = \sqrt{x^2+y^2+z^2} \,. \tag{2.3}$$

The non-zero derivatives of the metric are given by

$$\partial_i g_{jk} = 4\Psi^3 \, \delta_{jk} \, \partial_i \Psi \,, \tag{2.4}$$

$$\partial_i g_{00} = 4\Omega\Psi^{-3} \partial_i \Psi \,, \tag{2.5}$$

$$\partial_i \Psi = -\frac{m x^i}{2r^3} = -\partial_i \Omega \,, \tag{2.6}$$

where we have used that $\Psi + \Omega = 2$. We emphasize once more that the derivative $\partial_i \Psi = -m x^i/(2r^3)$ is evaluated in the rest frame coordinates x^μ which are *not* the coordinates used in the numerical code!

Next, we transform the metric components and their derivatives into the frame of the moving observer $\tilde{x}^{\tilde{\alpha}}$ according to

$$\tilde{g}_{\tilde{\alpha}\tilde{\beta}} = \Lambda^\mu{}_{\tilde{\alpha}} \Lambda^\nu{}_{\tilde{\beta}} g_{\mu\nu} \,, \tag{2.7}$$

$$\tilde{g}^{\tilde{\alpha}\tilde{\beta}} = \Lambda^{\tilde{\alpha}}{}_\mu \Lambda^{\tilde{\beta}}{}_\nu g^{\mu\nu} \,, \tag{2.8}$$

$$\tilde{\partial}_{\tilde{\gamma}} \tilde{g}_{\tilde{\alpha}\tilde{\beta}} = \Lambda^\lambda{}_{\tilde{\gamma}} \Lambda^\mu{}_{\tilde{\alpha}} \Lambda^\nu{}_{\tilde{\beta}} \partial_\lambda g_{\mu\nu} \,. \tag{2.9}$$

Now we can start extracting the ADM variables in the boosted frame $\tilde{\mathcal{O}}$. Through the 3+1 decomposition of the metric (valid in any frame),

$$g_{\alpha\beta} = \left(\begin{array}{c|c} -\alpha^2 + \beta_m \beta^m & \beta_j \\ \hline \beta_i & \gamma_{ij} \end{array} \right) \quad \Leftrightarrow \quad g^{\alpha\beta} = \left(\begin{array}{c|c} -\alpha^{-2} & \alpha^{-2}\beta^j \\ \hline \alpha^{-2}\beta^i & \gamma^{ij} - \alpha^{-2}\beta^i\beta^j \end{array} \right) , \tag{2.10}$$

we directly obtain

$$\tilde{\alpha} = 1/\sqrt{-\tilde{g}^{\tilde{0}\tilde{0}}} \,, \tag{2.11}$$

$$\tilde{\beta}_{\tilde{i}} = \tilde{g}_{\tilde{0}\tilde{i}} \,, \tag{2.12}$$

$$\tilde{\gamma}_{\tilde{i}\tilde{j}} = \tilde{g}_{\tilde{i}\tilde{j}} \,. \tag{2.13}$$

We then invert the three metric to obtain $\tilde{\gamma}^{\tilde{i}\tilde{j}}$ and raise the index of the shift vector,

$$\tilde{\beta}^{\tilde{i}} = \tilde{\gamma}^{\tilde{i}\tilde{m}}\beta_{\tilde{m}}\,. \tag{2.14}$$

There remains the extrinsic curvature which can be written as

$$\tilde{K}_{\tilde{i}\tilde{j}} = -\frac{1}{2\tilde{\alpha}}\left(\tilde{\partial}_{\tilde{0}}\tilde{\gamma}_{\tilde{i}\tilde{j}} - \tilde{\beta}^{\tilde{m}}\tilde{\partial}_{\tilde{m}}\tilde{\gamma}_{\tilde{i}\tilde{j}} - \tilde{\gamma}_{\tilde{m}\tilde{j}}\tilde{\partial}_{\tilde{i}}\tilde{\beta}^{\tilde{m}} - \tilde{\gamma}_{\tilde{i}\tilde{m}}\tilde{\partial}_{\tilde{j}}\tilde{\beta}^{\tilde{m}}\right)\,. \tag{2.15}$$

For the calculation of the derivatives of the shift vector, we use the following trick,

$$\begin{aligned}
&\tilde{\partial}_{\tilde{i}}\tilde{\beta}_{\tilde{j}} = \tilde{\partial}_{\tilde{i}}\tilde{g}_{\tilde{0}\tilde{j}} = \tilde{\partial}_{\tilde{i}}(\tilde{\gamma}_{\tilde{j}\tilde{k}}\tilde{\beta}^{\tilde{k}}) = \tilde{\gamma}_{\tilde{j}\tilde{k}}\,\tilde{\partial}_{\tilde{i}}\tilde{\beta}^{\tilde{k}} + \tilde{\beta}^{\tilde{k}}\,\tilde{\partial}_{\tilde{i}}\tilde{\gamma}_{\tilde{j}\tilde{k}} \\
\Rightarrow\quad &\tilde{\gamma}_{\tilde{m}\tilde{j}}\tilde{\partial}_{\tilde{i}}\tilde{\beta}^{\tilde{m}} = \tilde{\partial}_{\tilde{i}}\tilde{g}_{\tilde{0}\tilde{j}} - \tilde{\beta}^{\tilde{k}}\,\tilde{\partial}_{\tilde{i}}\tilde{g}_{\tilde{j}\tilde{k}}\,,
\end{aligned} \tag{2.16}$$

where in the very last term we replaced $\tilde{\gamma}_{\tilde{i}\tilde{k}} = \tilde{g}_{\tilde{i}\tilde{k}}$ according to Eq. (2.13) which remains valid inside the derivative operator. Note that the two derivatives of $\tilde{\beta}^{\tilde{i}}$ on the right-hand side of Eq. (2.15) are exactly of the form appearing on the left-hand side of (2.16), so that are now able to calculate the extrinsic curvature in terms of the transformed metric expressions we evaluated in Eqs. (2.7)-(2.9). This method works in principle for any rest-frame metric we start with. It has one caveat, however. It may fail to identify divisions 0/0 in the evaluation of the extrinsic curvature. Applying this formalism, for instance, to the isotropic Schwarzschild metric, we find that $\tilde{\alpha} \propto \Omega$ and, likewise, the second factor on the right-hand side of Eq. (2.15), i.e. the whole sum of derivatives inside the parentheses, also ends up being $\propto \Omega$. On the BH horizon, however, $\Omega = 0$ which leads to a devision 0/0 and, thus, a non-assigned number for the value of the extrinsic curvature components. In a complete analytic calculation, this problem is trivially solved by canceling the Ω factors appearing in the numerator and denominator of the extrinsic curvature and we therefore consider this calculation next.

3 Isotropic Schwarzschild boosted in the z direction

The expressions derived in the previous section are generally valid and we can evaluate the boosted version of essentially any metric we wish to specify. By performing this calculation analytically rather than simply implementing the transformation results of the previous section inside a subroutine, we are also able to cancel terms which might otherwise result in divisions 0/0. In this section, we consider isotropic Schwarzschild as the rest frame metric and apply a boost in the z direction. The calculation can be done inside the linear algebra packages of MAPLE or MATHEMATICA or by pen on paper. Starting with (2.2), the result is

$$\tilde{g}_{\tilde{\alpha}\tilde{\beta}} = \begin{pmatrix} -\gamma^2(\Omega^2\Psi^{-2} - v^2\Psi^4) & 0 & 0 & \gamma^2 v(\Psi^4 - \Omega^2\Psi^{-2}) \\ 0 & \Psi^4 & 0 & 0 \\ 0 & 0 & \Psi^4 & 0 \\ \gamma^2 v(\Psi^4 - \Omega^2\Psi^{-2}) & 0 & 0 & \gamma^2(\Psi^4 - v^2\Omega^2\Psi^{-2}) \end{pmatrix}, \tag{3.1}$$

$$\tilde{g}^{\tilde{\alpha}\tilde{\beta}} = \begin{pmatrix} -\gamma^2(\Psi^2\Omega^{-2} - v^2\Psi^{-4}) & 0 & 0 & \gamma^2 v(\Psi^2\Omega^{-2} - \Psi^{-4}) \\ 0 & \Psi^{-4} & 0 & 0 \\ 0 & 0 & \Psi^{-4} & 0 \\ \gamma^2 v(\Psi^2\Omega^{-2} - \Psi^{-4}) & 0 & 0 & \gamma^2(\Psi^{-4} - v^2\Psi^2\Omega^{-2}) \end{pmatrix}. \tag{3.2}$$

After a straightforward calculation, we read off the ADM variables as

$$\tilde{\alpha} = \frac{1}{\gamma} \frac{\Omega\Psi^2}{\sqrt{\Psi^6 - v^2\Omega^2}}\,, \tag{3.3}$$

$$\tilde{\beta}^{\tilde{i}} = \left(0,\ 0,\ v\frac{\Psi^6 - \Omega^2}{\Psi^6 - v^2\Omega^2}\right)\,, \tag{3.4}$$

$$\tilde{\gamma}_{\tilde{i}\tilde{j}} = \begin{pmatrix} \Psi^4 & 0 & 0 \\ 0 & \Psi^4 & 0 \\ 0 & 0 & \gamma^2\Psi^{-2}(\Psi^6 - v^2\Omega^2) \end{pmatrix}\,. \tag{3.5}$$

For the extrinsic curvature, we also need to evaluate and transform the derivatives. This yields the result

$$\tilde{K}_{\tilde{i}\tilde{j}} = \begin{pmatrix} -2v\frac{\Psi\Omega\,\partial_z\Psi}{\sqrt{\Psi^6-v^2\Omega^2}} & 0 & \gamma v\Psi\frac{3\Omega+\Psi}{\sqrt{\Psi^6-v^2\Omega^2}}\partial_x\Psi \\ \text{sym} & -2v\frac{\Omega\Psi}{\sqrt{\Psi^6-v^2\Omega^2}}\partial_z\Psi & \gamma v\Psi\frac{3\Omega+\Psi}{\sqrt{\Psi^6-v^2\Omega^2}}\partial_y\Psi \\ \text{sym} & \text{sym} & \gamma^2 v\frac{2\Psi^6(\Psi+2\Omega)-2v^2\Omega^2}{\Psi^5\sqrt{\Psi^6-v^2\Omega^2}}\partial_z\Psi \end{pmatrix}\,. \tag{3.6}$$

4 Isotropic Schwarzschild boosted in general directions

We next consider the ADM variables obtained for a boost with arbitrary velocity $v = v^i$. The calculation proceeds along the same way as that of the previous section but is rather lengthy and is performed more conveniently with Maple's LinearAlgebra package or a Mathematica equivalent or, at least, using such a package to verify pencil calculations. One obtains the result

$$\tilde{\gamma}_{\tilde{i}\tilde{j}} = \Psi^4\delta_{\tilde{i}\tilde{j}} + v_{\tilde{i}}v_{\tilde{j}}\frac{\Psi^6 - \Omega^2}{(1-v^2)\Psi^2}\,, \tag{4.1}$$

$$\tilde{\alpha} = \frac{1}{\gamma}\frac{\Omega\Psi^2}{\sqrt{\Psi^6 - v^2\Omega^2}}\,, \tag{4.2}$$

$$\tilde{\beta}^{\tilde{i}} = v^{\tilde{i}}\frac{\Psi^6 - \Omega^2}{\Psi^6 - v^2\Omega^2}\,, \tag{4.3}$$

$$\tilde{K}_{\tilde{i}\tilde{j}} = \frac{\gamma}{\Psi^5\sqrt{\Psi^6 - v^2\Omega^2}}\left\{(\Psi + 3\Omega)\Psi^6\left[(v_{\tilde{i}}\,\partial_{\tilde{j}}\Psi + v_{\tilde{j}}\,\partial_{\tilde{i}}\Psi) - 2\hat{v}_{\tilde{i}}\hat{v}_{\tilde{j}}\,\vec{v}\cdot\vec{\nabla}\Psi\right]\right. \tag{4.4}$$

$$\left. -2\gamma\,\vec{v}\cdot\vec{\nabla}\Psi\left[\hat{v}_{\tilde{i}}\hat{v}_{\tilde{j}}(v^2\Psi^6\Omega - \Psi^7 - 3\Psi^6\Omega + 2v^2\Omega^2) + \delta_{\tilde{i}\tilde{j}}(1-v^2)\Psi^6\Omega\right]\right\}\,. \tag{4.5}$$

The notation in the last expression for the extrinsic curvature requires some subtle explanation. The indices i and $\tilde{i}$ both run from 1 to 3 and the tilde has merely been introduced to better distinguish the Lorentz transformation matrix and its inverse. The velocity components v_i and $v_{\tilde{i}}$ thus have the same values. Likewise, the partial derivative $\partial_{\tilde{i}}$ is the same as ∂_i, i.e. the derivative $\partial/\partial x^i$. This is why in places such as Eq. (2.9) we also introduced a tilde on the ∂ symbol to denote differentiation with respect to $\tilde{x}^{\tilde{i}}$. We summarize this delicate point with the following definitions,

$$
\begin{aligned}
\partial_i &:= \frac{\partial}{\partial x^i}\,, \\
\tilde{\partial}_{\tilde{i}} &:= \frac{\partial}{\partial \tilde{x}^{\tilde{i}}}\,, \\
\partial_{\tilde{i}} &:= \frac{\partial}{\partial x^{\tilde{i}}} := \left(\frac{\partial}{\partial x},\, \frac{\partial}{\partial y},\, \frac{\partial}{\partial z}\right) \stackrel{!}{=} \frac{\partial}{\partial x^i}\,.
\end{aligned}
\tag{4.6}
$$

This is not entirely fortunate, but the only alternative would be to have $\tilde{i}, \tilde{j}$ on the left-hand side of Eq. (4.5) and i, j on the right-hand side.
Note that these expressions for general boost directions have not yet been tested and will need to be checked.

5 Superposing two boosted black-hole solutions

The superposition of two boosted black-hole solutions only yields constraint satisfying initial data in the limit of infinite separation or of vanishing velocity $v \to 0$. As a consequence, there is freedom in the precise combination of two boosted metrics and it will be through practical testing that we hopefully will identify particularly benevolent ways to superpose the data. Here we employ the procedure used for the superposition of two Kerr-Schild black holes in Refs. [3, 2].
Let A and B denote two single-black-hole solutions centered at $\tilde{x}^{\tilde{i}}_A$ and $\tilde{x}^{\tilde{i}}_B$ with velocities v^i_A and v^i_B, respectively. Let $\tilde{\alpha}_A$, $\tilde{\alpha}_B$, $\tilde{\beta}^{\tilde{i}}_A$, $\tilde{\beta}^i_B$, $\tilde{\gamma}^A_{\tilde{i}\tilde{j}}$, $\tilde{\gamma}^B_{\tilde{i}\tilde{j}}$, $\tilde{K}^A_{\tilde{i}\tilde{j}}$ and $\tilde{K}^B_{\tilde{i}\tilde{j}}$ denote the ADM variables of the two black holes' on their two respective initial slices. The binary-black-hole metric is then constructed according to

$$\tilde{\gamma}_{\tilde{i}\tilde{j}} = \tilde{\gamma}^A_{\tilde{i}\tilde{j}} + \tilde{\gamma}^B_{\tilde{i}\tilde{j}} - \delta_{\tilde{i}\tilde{j}}\,, \tag{5.1}$$

$$\tilde{\alpha} = \frac{1}{\sqrt{\tilde{\alpha}_A^{-2} + \tilde{\alpha}_B^{-2} - 1}}\,, \tag{5.2}$$

$$\tilde{\beta}^{\tilde{i}} = \tilde{\gamma}^{\tilde{i}\tilde{j}} \left(\tilde{\gamma}^A_{\tilde{j}\tilde{m}} \tilde{\beta}^{\tilde{m}}_A + \tilde{\gamma}^B_{\tilde{j}\tilde{m}} \tilde{\beta}^{\tilde{m}}_B\right)\,, \tag{5.3}$$

$$
\begin{aligned}
\tilde{K}^{\tilde{i}}{}_{\tilde{j}}^{M} &= \tilde{K}^{\tilde{i}}{}_{\tilde{j}}^{A} + \tilde{K}^{\tilde{i}}{}_{\tilde{j}}^{B}\,, \\
\tilde{K}_{\tilde{i}\tilde{j}} &= \frac{1}{2}\left(\tilde{\gamma}_{\tilde{i}\tilde{m}} \tilde{K}^{\tilde{m}}{}_{\tilde{j}}^{M} + \tilde{\gamma}_{\tilde{j}\tilde{m}} \tilde{K}^{\tilde{m}}{}_{\tilde{i}}^{M}\right)\,.
\end{aligned}
\tag{5.4}
$$

6 The Tangherlini solution in isotropic coordinates

In $D > 4$ spacetime dimensions, the Schwarzschild solution generalizes to the Tangherlini black hole given in Schwarzschild coordinates by

$$ds^2 = -\left(1 - \frac{\mu}{R^{D-3}}\right) dt^2 + \left(1 - \frac{\mu}{R^{D-3}}\right)^{-1} dR^2 + R^2 d\omega^2_{D-2}\,, \tag{6.1}$$

where $d\omega^2_{D-2}$ is the line element on the unit $D-2$ sphere; for $D=4$, for instance, $d\omega_2^2 = d\theta^2 + \sin\theta^2 d\phi^2$. We can transform the metric (6.1) into isotropic coordinates by changing the radial coordinate according to

$$R = r\left(1+\frac{\mu}{4r^{D-3}}\right)^{\frac{2}{D-3}} . \tag{6.2}$$

A straightforward, albeit slightly tedious, calculation shows that with this transformation, the line element becomes

$$ds^2 = -\Omega^2\Psi^{-2}dt^2 + \Psi^{\frac{4}{D-3}}(dr^2 + r^2 d\omega^2_{D-2}) , \tag{6.3}$$

where now

$$\begin{aligned}
\Omega &= 1-\frac{\mu}{4r^{D-3}} , && (6.4)\\
\Psi &= 1+\frac{\mu}{4r^{D-3}} , && (6.5)\\
\partial_i\Psi &= \frac{3-D}{4}\mu\frac{x^i}{r^{D-1}} = -\partial_i\Omega . && (6.6)
\end{aligned}$$

Switching to Cartesian coordinates $x^A = (t,\, x^i,\, x^a)$ and using our modified cartoon notation, the ADM variables for this metric are

$$\begin{aligned}
&\alpha = \Omega\Psi^{-1} , && (6.7)\\
&\beta^i = \beta^a = 0 , && (6.8)\\
&\gamma_{ij} = \Psi^{\frac{4}{D-3}}\delta_{ij} , && (6.9)\\
&\gamma_{ia} = \gamma_{ai} = 0 , && (6.10)\\
&\gamma_{ab} = \gamma_{ww}\delta_{ab} , \qquad \gamma_{ww} = \Psi^4 , && (6.11)
\end{aligned}$$

7 Lorentz boosting a $D > 4$ metric with $SO(D-3)$ isometry

In this case, we consider boost velocities in the x, y or z directions but not in the extra dimensions collectively denoted by w^a [1]. Strictly speaking, the z component of the velocity will also vanish for all applications to binary black holes, but we keep it in the formalism discussed in this section to make the similarity with the $D=4$ case as clear as possible. The higher-dimensional Lorentz transformation matrix for $SO(D-3)$ isometry becomes

$$\Lambda^{\tilde{A}}{}_M = \left(\begin{array}{c|c} \Lambda^{\tilde{\alpha}}{}_\mu & 0 \\ \hline 0 & \delta^{\tilde{a}}{}_m \end{array}\right) \qquad \Leftrightarrow \qquad \Lambda^M{}_{\tilde{A}} = \left(\begin{array}{c|c} \Lambda^{\mu}{}_{\tilde{\alpha}} & 0 \\ \hline 0 & \delta^{m}{}_{\tilde{a}} \end{array}\right) , \tag{7.1}$$

where $\delta^{\tilde{a}}{}_m$, $\delta^m{}_{\tilde{a}}$ denote the flat Euclidean metric in the extra dimensions. The coordinate transformation is given by

$$\tilde{x}^{\tilde{A}} = \Lambda^{\tilde{A}}{}_M x^M + \tilde{x}_0^{\tilde{A}} \qquad \Leftrightarrow \qquad x^M = \Lambda^M{}_{\tilde{A}}\tilde{x}^{\tilde{A}} - x_0^M . \tag{7.2}$$

With rotational symmetry in the extra dimensions, we have $x^m = 0$, $\tilde{x}^{\tilde{a}} = 0$, so that with (7.1) we obtain

$$\left.\begin{array}{l} \tilde{x}^{\tilde{\alpha}} = \Lambda^{\tilde{\alpha}}{}_\mu x^\mu + \tilde{x}_0^{\tilde{\alpha}} \\ \tilde{x}^{\tilde{a}} = x^{\tilde{a}} + \tilde{x}_0^{\tilde{a}} \end{array}\right\} \qquad \Leftrightarrow \qquad \left\{\begin{array}{l} x^\mu = \Lambda^\mu{}_{\tilde{\alpha}}\tilde{x}^\alpha - x_0^\mu \\ x^m = \tilde{x}^m - x_0^m \end{array}\right. \tag{7.3}$$

As before, we can set $\tilde{x}_0^{\tilde{A}} = 0 = x_0^M$ without loss of generality.

In the $D > 4$ case, the z axis becomes the quasi-radial direction in the dimensional reduction and the BHs can therefore only move in the xy plane. For Lorentz boosts along one coordinate axis, we therefore now take boosts in the x direction rather than z as we did in $D = 4$ dimensions. For this case, the Lorentz transformations are

$$\Lambda^{\tilde{\alpha}}{}_{\mu} = \left(\begin{array}{c|ccc} \gamma & -\gamma v & 0 & 0 \\ \hline -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \quad \Leftrightarrow \quad \Lambda^{\mu}{}_{\tilde{\alpha}} = \left(\begin{array}{c|ccc} \gamma & \gamma v & 0 & 0 \\ \hline \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right), \tag{7.4}$$

which gives us the coordinate relations

$$\begin{array}{l} \tilde{t} = \gamma(t - vx) \\ \tilde{x} = \gamma(x - vt) \end{array} \quad \Leftrightarrow \quad \begin{array}{l} t = \gamma(\tilde{t} + v\tilde{x}) \\ x = \gamma(\tilde{x} + v\tilde{t}) \end{array}. \tag{7.5}$$

We next transform the metric and its derivatives into the new frame according to

$$\tilde{g}_{\tilde{A}\tilde{B}} = \Lambda^{M}{}_{\tilde{A}}\Lambda^{\tilde{N}}{}_{\tilde{B}}g_{MN}\,, \tag{7.6}$$

$$\tilde{\partial}_{\tilde{C}}\tilde{g}_{\tilde{A}\tilde{B}} = \Lambda^{L}{}_{\tilde{C}}\Lambda^{M}{}_{\tilde{A}}\Lambda^{\tilde{N}}{}_{\tilde{B}}\partial_{L}g_{MN}\,. \tag{7.7}$$

We consider separately the components inside the computational domain (including the time coordinate) and the off-domain components. Bearing in mind the relations for off-domain components listed in Appendix A of [1], we obtain after some calculation,

$$\tilde{g}_{\tilde{\alpha}\tilde{\beta}} = \Lambda^{\mu}{}_{\tilde{\alpha}}\Lambda^{\nu}{}_{\tilde{\beta}}g_{\mu\nu}\,, \tag{7.8}$$

$$\tilde{g}_{\tilde{\alpha}\tilde{b}} = 0\,, \tag{7.9}$$

$$\tilde{g}_{\tilde{a}\tilde{b}} = \delta_{\tilde{a}\tilde{b}}g_{ww}\,, \tag{7.10}$$

$$\tilde{g}^{\tilde{\alpha}\tilde{\beta}} = \Lambda^{\tilde{\alpha}}{}_{\mu}\Lambda^{\tilde{\beta}}{}_{\nu}g^{\mu\nu}\,, \tag{7.11}$$

$$\tilde{g}^{\tilde{\alpha}\tilde{b}} = 0\,, \tag{7.12}$$

$$\tilde{g}^{\tilde{a}\tilde{b}} = \delta^{\tilde{a}\tilde{b}}g^{ww}\,, \tag{7.13}$$

$$\tilde{\partial}_{\tilde{\gamma}}\tilde{g}_{\tilde{\alpha}\tilde{\beta}} = \Lambda^{\lambda}{}_{\tilde{\gamma}}\Lambda^{\mu}{}_{\tilde{\alpha}}\Lambda^{\nu}{}_{\tilde{\beta}}\partial_{\lambda}g_{\mu\nu}\,, \tag{7.14}$$

$$\tilde{\partial}_{\tilde{c}}\tilde{g}_{\tilde{\alpha}\tilde{\beta}} = 0\,, \tag{7.15}$$

$$\tilde{\partial}_{\tilde{\gamma}}\tilde{g}_{\tilde{a}\tilde{b}} = \delta_{\tilde{a}\tilde{b}}\Lambda^{\lambda}{}_{\tilde{\gamma}}\,\partial_{\lambda}g_{ww}\,. \tag{7.16}$$

All other components of the metric derivative vanish.

A general metric in D dimensions with $SO(D-3)$ isometry can be written as

$$g_{AB}=\left(\begin{array}{c|c} -\alpha^2+\beta_M\beta^M & \beta_J \\ \hline \beta_I & \gamma_{IJ}\end{array}\right)=\left(\begin{array}{c|c|c} -\alpha^2+\beta_m\beta^m & \beta_j|0 & \\ \hline \beta_i & \gamma_{ij} & 0 \\ \hline 0 & 0 & \gamma_{ww}\delta_{ab}\end{array}\right),$$

$$g^{AB}=\left(\begin{array}{c|c} -\alpha^{-2} & \alpha^{-2}\beta^J \\ \hline \alpha^{-2}\beta^I & \gamma^{IJ}-\alpha^{-2}\beta^I\beta^J\end{array}\right)=\left(\begin{array}{c|c|c} \alpha^{-2} & \alpha^{-2}\beta^j & 0 \\ \hline \alpha^{-2}\beta^i & \gamma^{ij}-\alpha^{-2}\beta^i\beta^j & 0 \\ \hline 0 & 0 & \gamma^{ww}\delta^{ab}\end{array}\right), \tag{7.17}$$

where γ^{ij} is the inverse of γ_{ij}, $\gamma^{ww}=1/\gamma_{ww}$ and $\beta_i=\gamma_{ij}\beta^j$.
From Eq. (7.17) we obtain the expressions for the ADM variables as (restoring the tilde)

$$\tilde{\alpha} = \frac{1}{\sqrt{-\tilde{g}^{\tilde{0}\tilde{0}}}}, \tag{7.18}$$

$$\tilde{\beta}_{\tilde{i}} = \tilde{g}_{\tilde{0}\tilde{i}}, \tag{7.19}$$

$$\tilde{\gamma}_{\tilde{i}\tilde{j}} = \tilde{g}_{\tilde{i}\tilde{j}}, \tag{7.20}$$

$$\tilde{\gamma}_{\tilde{a}\tilde{b}} = \tilde{\gamma}_{\tilde{w}\tilde{w}}\,\delta_{\tilde{a}\tilde{b}}, \qquad \tilde{\gamma}_{\tilde{w}\tilde{w}}=\tilde{g}_{\tilde{w}\tilde{w}}, \tag{7.21}$$

$$\tilde{K}_{\tilde{i}\tilde{j}} = -\frac{1}{2\tilde{\alpha}}\left(\tilde{\partial}_{\tilde{0}}\tilde{\gamma}_{\tilde{i}\tilde{j}}-\tilde{\beta}^{\tilde{m}}\tilde{\partial}_{\tilde{m}}\tilde{\gamma}_{\tilde{i}\tilde{j}}-\tilde{\gamma}_{\tilde{m}\tilde{j}}\tilde{\partial}_{\tilde{i}}\tilde{\beta}^{\tilde{m}}-\tilde{\gamma}_{\tilde{i}\tilde{m}}\tilde{\partial}_{\tilde{j}}\tilde{\beta}^{\tilde{m}}\right), \tag{7.22}$$

$$\tilde{K}_{\tilde{a}\tilde{b}} = \tilde{K}_{\tilde{w}\tilde{w}}\delta_{\tilde{a}\tilde{b}}, \qquad \tilde{K}_{\tilde{w}\tilde{w}}=-\frac{1}{2\tilde{\alpha}}\left(\tilde{\partial}_{\tilde{0}}\tilde{g}_{\tilde{w}\tilde{w}}-\tilde{\beta}^{\tilde{m}}\tilde{\partial}_{\tilde{m}}\tilde{\gamma}_{\tilde{w}\tilde{w}}-2\tilde{\gamma}_{\tilde{w}\tilde{w}}\frac{\tilde{\beta}^{\tilde{z}}}{\tilde{z}}\right), \tag{7.23}$$

where in the last line we have used that [1]

$$\partial_a\beta^c=\frac{\beta^z}{z}\delta_a{}^c\,. \tag{7.24}$$

Note, however, that $\tilde{\beta}^{\tilde{z}}=0$ since we have no velocity component in the z direction. This is a manifestion of the $SO(D-3)$ isometry which does not accomodate BHs moving off the xy plane.
We compute the derivatives of the shift vector using the same trick as in Eq. (2.16) above, i.e. (dropping the tilde for simplicity)

$$\begin{aligned}&\partial_i\beta_j=\partial_i g_{0j}=\partial_i(\gamma_{jk}\beta^k)=\gamma_{jk}\partial_i\beta^k+\beta^k\partial_i\gamma_{jk}\\ \Rightarrow\ \ &\gamma_{jk}\partial_i\beta^k=\partial_i g_{0j}-\beta^k\partial_i g_{jk}\,.\end{aligned} \tag{7.25}$$

We finally obtain for the ADM variables of the boosted BH solution, setting $n := 4/(D-3)$,

$$\tilde{\alpha} = \frac{1}{\gamma}\frac{\Omega}{\sqrt{\Psi^2 - v^2\Omega^2\Psi^{-n}}}\,, \tag{7.26}$$

$$\tilde{\beta}^{\tilde{i}} = \left(v\frac{\Psi^2-\Omega^2\Psi^{-n}}{\Psi^2 - v^2\Omega^2\Psi^{-n}},\ 0,\ 0\right)\,, \qquad \tilde{\beta}^{\tilde{a}} = 0\,, \tag{7.27}$$

$$\tilde{\gamma}_{\tilde{i}\tilde{j}} = \begin{pmatrix} \gamma^2(\Psi^n - v^2\Omega^2\Psi^{-2}) & 0 & 0 \\ 0 & \Psi^n & 0 \\ 0 & 0 & \Psi^n \end{pmatrix}\,,$$

$$\tilde{\gamma}_{\tilde{w}\tilde{w}} = \gamma_{ww} = \Psi^n\,. \tag{7.28}$$

For the extrinsic curvature, we also need to evaluate and transform the derivatives. This yields the result

$$\tilde{K}_{\tilde{i}\tilde{j}} = \begin{pmatrix} \gamma^2 v\frac{\Psi^{2+n}(8+2n-n\Psi)-4v^2\Omega^2}{2\Psi^{3+n}\sqrt{\Psi^2-v^2\Omega^2\Psi^{-n}}}\partial_x\Psi & \gamma v\frac{4+n\Omega}{2\Psi\sqrt{\Psi^2-v^2\Omega^2\Psi^{-n}}}\partial_y\Psi & \gamma v\Psi\frac{4+n\Omega}{2\Psi\sqrt{\Psi^2-v^2\Omega^2\Psi^{-n}}}\partial_z\Psi \\ \text{sym} & -v\frac{n\Omega}{2\Psi\sqrt{\Psi^2-v^2\Omega^2\Psi^{-n}}}\partial_x\Psi & 0 \\ \text{sym} & \text{sym} & -v\frac{n\Omega}{2\Psi\sqrt{\Psi^2-v^2\Omega^2\Psi^{-n}}}\partial_x\Psi \end{pmatrix}\,.$$

$$\tilde{K}_{\tilde{w}\tilde{w}} = -v\frac{n\Omega}{2\Psi\sqrt{\Psi^2-v^2\Omega^2\Psi^{-n}}}\partial_x\Psi\,. \tag{7.29}$$

Note that this result reduces to the $D=4$ case if we set $n=4$.

References

[1] William G. Cook, Pau Figueras, Markus Kunesch, Ulrich Sperhake, and Saran Tunyasuvunakool. Dimensional reduction in numerical relativity: Modified cartoon formalism and regularization. *Int. J. Mod. Phys. D*, 25:1641013, 2016. arXiv:1603.00362 [gr-qc].

[2] U. Sperhake. Binary black-hole evolutions of excision and puncture data. *Phys. Rev. D*, 76:104015, 2007. gr-qc/0606079.

[3] U. Sperhake, B. Kelly, P. Laguna, K. L. Smith, and E. Schnetter. Black-hole head-on collisions and gravitational waves with fixed mesh-refinement and dynamic singularity excision. *Phys. Rev. D*, 71:124042, 2005. gr-qc/0503071.