Contents

1 The action and covariant equations

Our starting point is the action [4]

$$S=\int\left\{\frac{1}{16\pi G}R-\frac{1}{2}\left[g^{\alpha\beta}\nabla_\alpha\bar{\varphi}\nabla_\beta\varphi+V(\varphi)\right]\right\}\sqrt{-g}\,d^4x\,, \tag{1.1}$$

where φ denotes a complext scalar field $\varphi=\varphi_1+\mathrm{i}\varphi_2$, g is the determinant of the spacetime metric $g_{\alpha\beta}$ and V a potential function which we leave free for now.

The field equations are obtained by varying the action with respect to the metric and the scalar field. For this purpose we write

$$\begin{aligned} I_{\mathrm{H}} &= \int R\sqrt{-g}\,d^4x\,,\\ I_{\mathrm{M}} &= \int -\frac{1}{2}\left[g^{\alpha\beta}\nabla_\alpha\bar{\varphi}\nabla_\beta\varphi+V(\varphi)\right]\sqrt{-g}\,d^4x\,, \end{aligned} \tag{1.2}$$

for the Hilbert and matter contributions to the action.

Let us first vary with respect to the metric and, more specifically, the inverse metric which is a bit simpler. We first recall that in general

$$\delta\sqrt{-g}=-\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}\,, \tag{1.3}$$

and then write in normal coordinates (where $\Gamma^\alpha_{\beta\gamma}=0$),

$$\begin{aligned} & R_{\alpha\beta}\overset{*}{=}\Gamma^\mu_{\alpha\beta;\mu}-\Gamma^\mu_{\alpha\mu;\beta}\\ \Rightarrow\quad & \delta R_{\alpha\beta}\overset{*}{=}\delta\Gamma^\mu_{\alpha\beta;\mu}-\delta\Gamma^\mu_{\alpha\mu;\beta}\,. \end{aligned} \tag{1.4}$$

The difference between two connections, however, is a tensor, so that the last equation is tensorial and, hence, valid in any coordinate system. This helps us considerably in simplifying the variation. We can now write

$$\begin{aligned} \delta I_{\mathrm{H}} &= \int\delta(g^{\alpha\beta}R_{\alpha\beta}\sqrt{-g})\,d^4x=\int\left\{R_{\alpha\beta}\sqrt{-g}\delta g^{\alpha\beta}+g^{\alpha\beta}\sqrt{-g}\delta R_{\alpha\beta}+R\delta\sqrt{-g}\right\}d^4x\\ &= \int\left(R_{\alpha\beta}-\frac{1}{2}Rg_{\alpha\beta}\right)\sqrt{-g}\delta g^{\alpha\beta}\,d^4x+\int g^{\alpha\beta}\left(\delta\Gamma^\mu_{\alpha\beta;\mu}-\delta\Gamma^\mu_{\alpha\mu;\beta}\right)\sqrt{-g}\,d^4x\\ &= \int G_{\alpha\beta}\sqrt{-g}\delta g^{\alpha\beta}\,d^4x+\int X^\mu{}_{;\mu}\sqrt{-g}\,d^4x\,, \qquad \Big|\; X^\mu:=g^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta}-g^{\alpha\mu}\delta\Gamma^\beta_{\alpha\beta}\\ &= \int G_{\alpha\beta}\sqrt{-g}\delta g^{\alpha\beta}\,d^4x+\oint X^\mu\tilde{n}_\mu\sqrt{|\gamma|}\,d^3y\,, \end{aligned} \tag{1.5}$$

where we have used the divergence theorem to convert the volume integral into an integral over a surface with outgoing normal $\tilde{n}_\mu$ and induced metric $\gamma_{\alpha\beta}$. We will drop the surface integral, since it does not contribute to the equations of motion. A more rigorous treatment requires the addition of a surface term to the action (1.1) which cancels this surface integral and, after regularization through a constant term, leads to the ADM momenta of the spacetime; for details see e.g. [8].

The variation of the matter action proceeds in analogy. Noting that $\nabla_\alpha\varphi = \partial_\alpha\varphi$ does not depend on the metric, we obtain

$$\begin{aligned} \delta I_{\rm M} &= \int \left\{ -\frac{1}{2}\left[\delta g^{\alpha\beta}\nabla_\alpha\bar{\varphi}\nabla_\beta\varphi\sqrt{-g} - \frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}\left(g^{\mu\nu}\nabla_\mu\bar{\varphi}\nabla_\nu\varphi + V(\varphi)\right)\right]\right\} d^4x \\ &= -\frac{1}{2}\int\left[\nabla_\alpha\bar{\varphi}\nabla_\beta\varphi - \frac{1}{2}g_{\alpha\beta}\left(g^{\mu\nu}\nabla_\mu\bar{\varphi}\nabla_\nu\varphi + V(\varphi)\right)\right]\delta g^{\alpha\beta}\sqrt{-g}\,d^4x\,. \end{aligned} \tag{1.6}$$

Restoring the factor $16\pi G$, we obtain the Einstein equations

$$\boxed{\begin{aligned} G_{\alpha\beta} &= 4\pi G\left[\nabla_\alpha\bar{\varphi}\nabla_\beta\varphi + \nabla_\alpha\varphi\nabla_\beta\bar{\varphi} - g_{\alpha\beta}g^{\mu\nu}\nabla_\mu\bar{\varphi}\nabla_\nu\varphi - g_{\alpha\beta}V(\varphi)\right] \stackrel{!}{=} 8\pi G T_{\alpha\beta} \\ &= 8\pi G\left[\nabla_{(\alpha}\bar{\varphi}\nabla_{\beta)}\varphi - \frac{1}{2}g_{\alpha\beta}\big(g^{\mu\nu}\nabla_\mu\bar{\varphi}\nabla_\nu\varphi + V(\varphi)\big)\right], \end{aligned}} \tag{1.7}$$

where we have symmetrized $\nabla_\alpha\bar{\varphi}\nabla_\beta\varphi$ in order to maintain the symmetry of this term in the action (1.1).

Next, we vary the action with respect to the scalar field which gives us

$$\begin{aligned} \delta S &= -\frac{1}{2}\int\Big[g^{\alpha\beta}\nabla_\alpha(\bar{\varphi}+\delta\bar{\varphi})\nabla_\beta(\varphi+\delta\varphi) + V(\varphi+\delta\varphi) - g^{\alpha\beta}\nabla_\alpha\bar{\varphi}\nabla_\beta\varphi - V(\varphi)\Big]\sqrt{-g}\,d^4x \\ &= -\frac{1}{2}\int\Big[g^{\alpha\beta}\nabla_\alpha\delta\bar{\varphi}\,\nabla_\beta\varphi + g^{\alpha\beta}\nabla_\alpha\bar{\varphi}\,\nabla_\beta\delta\varphi + V'(\varphi)\delta\varphi\Big]\sqrt{-g}\,d^4x \\ &= -\frac{1}{2}\int\left[\underbrace{g^{\alpha\beta}\nabla_\alpha(\delta\bar{\varphi}\nabla_\beta\varphi)}_{\to 0} - g^{\alpha\beta}\delta\varphi\nabla_\alpha\nabla_\beta\varphi + \underbrace{g^{\alpha\beta}\nabla_\beta(\delta\varphi\nabla_\alpha\bar{\varphi})}_{\to 0} - g^{\alpha\beta}\delta\varphi\nabla_\beta\nabla_\alpha\bar{\varphi} + V'(\varphi)\,\delta\varphi\right]\sqrt{-g}\,d^4x\,, \end{aligned} \tag{1.8}$$

where we have used integration by parts to convert terms into total derivatives which lead to surface integral which reduce to zero for matter fields with compact support or sufficiently rapid falloff (exponential in our case).

Next, we notice that

$$\begin{aligned} &\delta\bar{\varphi}\,\nabla_\alpha\nabla_\beta\varphi = \delta\varphi_1\,\nabla_\alpha\nabla_\beta\varphi_1 + \delta\varphi_2\,\nabla_\alpha\nabla_\beta\varphi_2 + \mathrm{i}\left\{-\delta\varphi_2\,\nabla_\alpha\nabla_\beta\varphi_1 + \delta\varphi_1\nabla_\alpha\nabla_\beta\varphi_2\right\} \\ &\delta\varphi\,\nabla_\alpha\nabla_\beta\bar{\varphi} = \delta\varphi_1\,\nabla_\alpha\nabla_\beta\varphi_1 + \delta\varphi_2\,\nabla_\alpha\nabla_\beta\varphi_2 + \mathrm{i}\left\{\delta\varphi_2\,\nabla_\alpha\nabla_\beta\varphi_1 - \delta\varphi_1\,\nabla_\alpha\nabla_\beta\varphi_2\right\} \\ \Rightarrow\ &g^{\alpha\beta}\delta\bar{\varphi}\,\nabla_\alpha\nabla_\beta\varphi + g^{\alpha\beta}\delta\varphi\,\nabla_\alpha\nabla_\beta\bar{\varphi} = 2\delta\varphi_1\,\nabla_\alpha\nabla_\beta\varphi_1 + 2\delta\varphi_2\,\nabla_\alpha\nabla_\beta\varphi_2\,. \end{aligned} \tag{1.9}$$

Combining this with Eq. (1.8), we obtain

$$\delta S = -\frac{1}{2}\int\left[-2g^{\alpha\beta}(\delta\varphi_1\,\nabla_\alpha\nabla_\beta\varphi_1 + \delta\varphi_2\,\nabla_\alpha\nabla_\beta\varphi_2) + \frac{\partial V}{\partial\varphi_1}\delta\varphi_1 + \frac{\partial V}{\partial\varphi_2}\delta\varphi_2\right]\sqrt{-g}\,d^4x \tag{1.10}$$

The variations of φ_1 and φ_2 are independent, so that both,

$$\boxed{-2g^{\alpha\beta}\nabla_\alpha\nabla_\beta\varphi_1 + \frac{\partial V}{\partial\varphi_1} = 0\,, \qquad -2g^{\alpha\beta}\nabla_\alpha\nabla_\beta\varphi_2 + \frac{\partial V}{\partial\varphi_2} = 0\,,} \tag{1.11}$$

must be satisfied. We can combine these into one equation,

$$\nabla^\mu \nabla_\mu \varphi = \frac{1}{2}\left(\frac{\partial V}{\partial \varphi_1} + \mathrm{i}\frac{\partial V}{\partial \varphi_2}\right) . \tag{1.12}$$

In practice, the potential will typically depend on the magnitude only of the scalar field, $V = V(|\varphi|^2)$, so that

$$\frac{\partial V}{\partial \varphi_1} = \frac{\partial V}{\partial |\varphi|^2} 2\varphi_1 \,, \qquad \frac{\partial V}{\partial \varphi_2} = \frac{\partial V}{\partial |\varphi|^2} 2\varphi_2 \,, \tag{1.13}$$

and the equation for the scalar field becomes

$$\boxed{\nabla^\mu \nabla_\mu \varphi = \frac{dV}{d|\varphi|^2}\varphi =: V'\varphi =: V_{,|\varphi|^2}\varphi \,. \tag{1.14}}$$

Note that we use the metric signature $-+++$, so that the second time derivative in the last equation has the minus sign. Note also that in this convention, the potential for a massive scalar field without self interaction is

$$V(\varphi) = \mu^2 |\varphi|^2 \,, \tag{1.15}$$

and does not have the factor 1/2 which we have used in our studies on massive scalar-tensor gravity [7]

2 Units

As is often the case with scalar fields, the units must be considered with some care. In our formulation, we have followed the literature and set $\hbar = c = 1$, but keep the gravitational constant G. With

$$\hbar = 1.0545718 \times 10^{-34}\ \frac{\mathrm{kg\,m^2}}{\mathrm{s}} \,, \qquad c = 2.99792458 \times 10^{8}\ \frac{\mathrm{m}}{\mathrm{s}} \,, \tag{2.1}$$

we obtain

$$1\ \mathrm{m} = \frac{1}{3.51767288 \times 10^{-43}\ \mathrm{kg}} = \frac{1}{1.973269788 \times 10^{-7}\ \mathrm{eV}} \,, \tag{2.2}$$

and the gravitational constant becomes

$$G = 6.67408 \times 10^{-11}\ \frac{\mathrm{m^3}}{\mathrm{kg\ s^2}} = 2.612194153 \times 10^{-70}\ \mathrm{m^2} \,. \tag{2.3}$$

Comparing this with the action (1.1), we notice that the scalar field now has dimension length^{-1} or, equivalently, mass. Note that the Planck mass is defined as $M_{\mathrm{Pl}} = \sqrt{\hbar c/G}$, so that we can replace the gravitational constant by the Planck mass according to $G = 1/M_{\mathrm{Pl}}^2$. We can interprete this by measuring the scalar field, or its amplitude, in units of the Planck mass.

3 Spherically symmetric stationary models

We now focus on single, stationary boson star models in spherical symmetry. Using polar slicing and radial gauge, we can write the line element as

$$ds^2 = -\alpha^2 dt^2 + X^2 dr^2 + r^2 d\Omega^2 \,, \tag{3.1}$$

where α and X are functions of radius r only, whereas the scalar field may also depend on time t. Plugging this into Eqs. (1.7) and (1.11), we find

$$\begin{aligned}
\frac{\partial_r \alpha}{\alpha} &= \frac{X^2-1}{2r} + 2\pi r G \left\{ (\partial_r\varphi_1)^2 + (\partial_r\varphi_2)^2 + \frac{X^2}{\alpha^2}[(\partial_t\varphi_1)^2 + (\partial_t\varphi_2)^2] - X^2 V \right\} , \\
\frac{\partial_r X}{X} &= -\frac{X^2-1}{2r} + 2\pi r G \left\{ (\partial_r\varphi_1)^2 + (\partial_r\varphi_2)^2 + \frac{X^2}{\alpha^2}[(\partial_t\varphi_1)^2 + (\partial_t\varphi_2)^2] + X^2 V \right\} , \\
-\frac{X^2}{\alpha^2}\partial_t\partial_t\varphi_i + \partial_r\partial_r\varphi_i &= \partial_r\varphi_i \frac{\partial_r X}{X} - \partial_r\varphi_i \frac{\partial_r\alpha}{\alpha} - 2\frac{\partial_r\varphi_i}{r} + \frac{1}{2}X^2\frac{\partial V}{\partial\varphi_i} \,,
\end{aligned} \tag{3.2}$$

where φ_i in the last equation stands for either of φ_1 or φ_2.

Furthermore, we can introduce the potential and mass functions

$$\Phi := \ln(\alpha)\,, \qquad m := \frac{r}{2}\frac{X^2-1}{X^2} = \frac{r}{2}\left(1 - \frac{1}{X^2}\right) \quad \Leftrightarrow \quad X^2 = \left(1 - \frac{2m}{r}\right)^{-1} , \tag{3.3}$$

which obey the equations

$$\begin{aligned}
\partial_r \Phi &= \frac{\partial_r\alpha}{\alpha} \,, \\
\partial_r m &= 2\pi G r^2 \left[\frac{(\partial_r\varphi_1)^2 + (\partial_r\varphi_2)^2}{X^2} + \frac{(\partial_t\varphi_1)^2 + (\partial_t\varphi_2)^2}{\alpha^2} + V \right] .
\end{aligned} \tag{3.4}$$

The $\propto r^3$ falloff of the mass m near the origin can cause some difficulties, however, so that we work with X in the interior and with m in the exterior. Note that with our definition, the mass has physical dimension length and it might be more correct to call it *length function*. We keep the term *mass* as it is the way we usually interprete it, but as it stands, it is half of the Schwarzschild radius associated with a mass and needs to be converted into SI units or solar masses accordingly.

In these expressions, we have worked with the individual scalar field components φ_1 and φ_2. Alternatively, we can keep the complex scalar field. Bearing in mind that

$$\begin{aligned}
&\varphi = \varphi_1 + \mathrm{i}\varphi_2 \\
\Rightarrow\quad &\partial_r\varphi\,\partial_r\bar\varphi = (\partial_r\varphi_1 + \mathrm{i}\partial_r\varphi_2)(\partial_r\varphi_1 - \mathrm{i}\partial_r\varphi_2) = (\partial_r\varphi_1)^2 + (\partial_r\varphi_2)^2 \,,
\end{aligned} \tag{3.5}$$

and likewise for the time derivative, Eqs. (3.2) and (3.4) become

$$\begin{aligned}
\frac{\partial_r \alpha}{\alpha} &= \frac{X^2-1}{2r} + 2\pi G r \left[(\partial_r\varphi)(\partial_r\bar\varphi) + \frac{X^2}{\alpha^2}(\partial_t\varphi)(\partial_t\bar\varphi) - X^2 V \right] , \\
\frac{\partial_r X}{X} &= -\frac{X^2-1}{2r} + 2\pi G r \left[(\partial_r\varphi)(\partial_r\bar\varphi) + \frac{X^2}{\alpha^2}(\partial_t\varphi)(\partial_t\bar\varphi) + X^2 V \right] , \\
\partial_r m &= 2\pi G r^2 \left[\frac{(\partial_r\varphi)(\partial_r\bar\varphi)}{X^2} + \frac{(\partial_t\varphi)(\partial_t\bar\varphi)}{\alpha^2} + V \right] , \\
\partial_r\partial_r\varphi &= \frac{X^2}{\alpha^2}\partial_t\partial_t\varphi + \partial_r\varphi\frac{\partial_r X}{X} - \partial_r\varphi\frac{\partial_r\alpha}{\alpha} - 2\frac{\partial_r\varphi}{r} + X^2 V'\varphi \,,
\end{aligned} \tag{3.6}$$

where we have defined

$$V' := \frac{dV}{d|\varphi|^2}\,. \tag{3.7}$$

For stationary boson star models, we assume that the scalar field has the form

$$\begin{aligned} &\varphi(t,r) = A(r)e^{i\omega t}\,, \qquad \omega = \text{const} \in \mathbb{R}\,. \\ \Rightarrow\quad &\partial_r\varphi = \partial_r A e^{i\omega t}\,, \quad \partial_r\partial_r\varphi = \partial_r\partial_r A e^{i\omega t}\,, \quad \partial_t\varphi = i\omega A e^{i\omega t}\,, \quad \partial_t\partial_t\varphi = -\omega^2 e^{i\omega t}\,. \end{aligned} \tag{3.8}$$

Replacing in (3.6), we obtain

$$\begin{aligned} \frac{\partial_r\alpha}{\alpha} &= \frac{X^2-1}{2r} + 2\pi G r\left[(\partial_r A)^2 + \frac{X^2}{\alpha^2}\omega^2 A^2 - X^2 V\right]\,, \\ \frac{\partial_r X}{X} &= -\frac{X^2-1}{2r} + 2\pi G r\left[(\partial_r A)^2 + \frac{X^2}{\alpha^2}\omega^2 A^2 + X^2 V\right]\,, \\ \partial_r m &= 2\pi G r^2\left[\frac{(\partial_r A)^2}{X^2} + \frac{\omega^2 A^2}{\alpha^2} + V\right]\,, \\ \partial_r\partial_r A &= -\frac{X^2}{\alpha^2}\omega^2 A + \partial_r A\frac{\partial_r X}{X} - \partial_r A\frac{\partial_r\alpha}{\alpha} - 2\frac{\partial_r A}{r} + X^2 V' A\,. \end{aligned} \tag{3.9}$$

Next, we reduce these equations to a first order system by introducing

$$\partial_r A = X\eta \qquad \Leftrightarrow \qquad \eta = \frac{\partial_r A}{X}\,. \tag{3.10}$$

This results in

$$\boxed{\begin{aligned} \frac{\partial_r\alpha}{\alpha} &= \frac{X^2-1}{2r} + 2\pi G r X^2\left(\eta^2 + \frac{\omega^2 A^2}{\alpha^2} - V\right)\,, \\ \frac{\partial_r X}{X} &= -\frac{X^2-1}{2r} + 2\pi G r X^2\left(\eta^2 + \frac{\omega^2 A^2}{\alpha^2} + V\right)\,, \\ \partial_r m &= 2\pi G r^2\left(\eta^2 + \frac{\omega^2 A^2}{\alpha^2} + V\right)\,, \\ \partial_r A &= X\eta\,, \\ \partial_r\eta &= -2\frac{\eta}{r} - \eta\frac{\partial_r\alpha}{\alpha} + X\left(V' - \frac{\omega^2}{\alpha^2}\right)A\,. \end{aligned} \quad (3.11)}$$

4 Mini boson stars

The simplest class of boson stars is obtained for a massive but non self-interacting scalar field with potential

$$V = \mu^2|\varphi|^2 = \mu^2 A^2 \qquad \Rightarrow \qquad V' = \mu^2\,. \tag{4.1}$$

The important consquence of this potential is that the scalar field terms in each of the field equations (3.11) are of the same power in η or A. This allows us to eliminate all free parameters from the equations by introducing dimensionless variables

$$\bar{t} := \mu t\,, \qquad \bar{r} := \mu r\,, \qquad \bar{m} := \mu m\,, \qquad \bar{A} := \sqrt{G}A\,, \qquad \bar{\eta} := \frac{\sqrt{G}}{\mu}\eta\,, \qquad \bar{\omega} := \frac{\omega}{\mu}\,. \tag{4.2}$$

In terms of these variables, Eqs. (3.11) become

$$\frac{\partial_{\bar{r}}\alpha}{\alpha} = \frac{X^2-1}{2\bar{r}} + 2\pi\bar{r}X^2\left[\bar{\eta}^2 + \left(\frac{\bar{\omega}^2}{\alpha^2}-1\right)\bar{A}^2\right]\,, \tag{4.3}$$

$$\frac{\partial_{\bar{r}}X}{X} = -\frac{X^2-1}{2\bar{r}} + 2\pi\bar{r}X^2\left[\bar{\eta}^2 + \left(\frac{\bar{\omega}^2}{\alpha^2}+1\right)\bar{A}^2\right]\,, \tag{4.4}$$

$$\partial_{\bar{r}}\bar{m} = 2\pi\bar{r}^2\left[\bar{\eta}^2 + \left(\frac{\bar{\omega}^2}{\alpha^2}+1\right)\bar{A}^2\right]\,, \tag{4.5}$$

$$\partial_{\bar{r}}\bar{A} = X\bar{\eta}\,, \tag{4.6}$$

$$\partial_{\bar{r}}\bar{\eta} = -2\frac{\bar{\eta}}{\bar{r}} - \bar{\eta}\frac{\partial_{\bar{r}}\alpha}{\alpha} + X\left(1-\frac{\bar{\omega}^2}{\alpha^2}\right)\bar{A}\,. \tag{4.7}$$

Put another way, if we solve Eqs. (4.3)-(4.7), we can reconstruct the solution for arbitrary scalar mass μ by deriving from the solution

$$r = \frac{\bar{r}}{\mu}\,, \qquad m = \frac{\bar{m}}{\mu}\,, \qquad \omega = \mu\bar{\omega}\,, \qquad A = \frac{\bar{A}}{\sqrt{G}}\,, \qquad \eta = \frac{\mu}{\sqrt{G}}\bar{\eta}\,. \tag{4.8}$$

Recall that $\sqrt{G}$ has dimension length and μ has dimension length^{-1}. Note also that we do *not* rescale the metric functions α and X. Both are dimensionless and we keep them that way.

Before continuing, let us explore the asymptotic behaviour of the scalar-field amplitude at large distances $r\to\infty$. Note that the exponential fall-off we will thus obtain holds for stationary fields but not radiative fields. For stationary fields, it is easiest to consider the second-order equation (3.9) for the scalar amplitude and note that by Eqs. (4.3), (4.4),

$$\frac{\partial_{\bar{r}}X}{X} - \frac{\partial_{\bar{r}}\alpha}{\alpha} = -\frac{X^2-1}{\bar{r}} + 4\pi\bar{r}X^2\bar{A}^2\,. \tag{4.9}$$

With this substitution, we can write Eq. (3.9) as

$$\partial^2_{\bar{r}}\bar{A} = \partial_{\bar{r}}\bar{A}\left[-\frac{2}{\bar{r}} - \frac{X^2-1}{\bar{r}} + 4\pi\bar{r}X^2\bar{A}^2\right] + X^2\left(1-\frac{\bar{\omega}^2}{\alpha^2}\right)\bar{A}\,. \tag{4.10}$$

Inserting $\bar{A}\sim\bar{r}^n$ into this equation, the leading-order terms give us

$$n(n-1)\bar{r}^{n-2} = a_0\bar{r}^{n-2} + a_1\bar{r}^{3n} + a_2\bar{r}^n\,, \tag{4.11}$$

which yields no solution for any n, not even $n=0$ (then the left-hand side vanishes and thus does not cancel the first term on the right-hand side). We conclude that the scalar field does not obey any polynomian fall-off. It might, of course, grow as $r \to \infty$, but that case would lead to an irregular spacetime with infinite energy density. We conclude that $4\pi\bar{r}X^2\bar{A}^2$ falls off faster with increasing $\bar{r}$ than the preceding $1/\bar{r}$ terms inside the square brackets of Eq. (4.10). Furthermore, at spatial infinity we have $X \to 1$ and $\alpha \to 1$, so that for $\bar{r} \to \infty$, Eq. (4.10) becomes

$$\partial_{\bar{r}}^2 \bar{A} + \frac{2}{\bar{r}}\partial_{\bar{r}}\bar{A} = (1-\bar{\omega}^2)\bar{A}$$

$$\Rightarrow \quad \partial_{\bar{r}}(\bar{r}^2\partial_{\bar{r}}\bar{A}) = \bar{r}^2(1-\bar{\omega}^2)\bar{A}\,. \tag{4.12}$$

To continue, it turns out convenient to intordUce the new variable

$$B := \bar{r}\bar{A} \qquad \Rightarrow \qquad \partial_{\bar{r}}B = \bar{A} + \bar{r}\partial_{\bar{r}}\bar{A}\,, \tag{4.13}$$

which enables to write the asymptotic equation for the scalar field amplitude as

$$\begin{aligned}
&\partial_{\bar{r}}\left[\bar{r}(\partial_{\bar{r}}B - \bar{A})\right] = (1-\bar{\omega}^2)\bar{r}B \\
\Rightarrow \quad &\partial_{\bar{r}}B - \bar{A} + \bar{r}\partial_{\bar{r}}^2 B - \bar{r}\partial_{\bar{r}}\bar{A} = (1-\bar{\omega}^2)\bar{r}B \\
\Rightarrow \quad &\bar{r}\partial_{\bar{r}}^2 B = (1-\bar{\omega}^2)\bar{r}B \\
\Rightarrow \quad &\partial_{\bar{r}}^2 B = (1-\bar{\omega}^2)B\,,
\end{aligned} \tag{4.14}$$

which gives us the asymptotic behaviour

$$B = e^{\pm\sqrt{1-\bar{\omega}^2}\bar{r}} \qquad \Rightarrow \qquad \bar{A} = \frac{e^{\pm\sqrt{1-\bar{\omega}^2}\bar{r}}}{\bar{r}}\,. \tag{4.15}$$

Note that we would have to replace $\bar{\omega}$ with $\bar{\omega}/\alpha$ in these expressions if we work in a gauge where α does not approach 1 at infinity. This may indeed be the case if we start shooting with the initial condition $\alpha(0) = 1$ (i.e. we have unit lapse at the origin but *not* at infinity).

In the exterior region, we introduce a compactified coordinate $\bar{y} = 1/\bar{r}$ and also take into account the exponential falloff of the scalar,

$$\varphi \sim \frac{e^{-\sqrt{\mu^2-\omega^2/\alpha^2}r}}{r} \sim \frac{e^{-\sqrt{1-\bar{\omega}^2/\alpha^2}\bar{r}}}{\bar{r}} = \bar{y}e^{-\sqrt{1-\bar{\omega}^2/\alpha^2}/\bar{y}}\,, \tag{4.16}$$

which follows from a perturbative expansion of all field variables around $\bar{y} = 0$.

Specifically, we use

$$\begin{aligned}
&\bar{y} = \frac{1}{\bar{r}} \quad \Leftrightarrow \quad \bar{r} = \frac{1}{\bar{y}} \qquad \Rightarrow \qquad \partial_{\bar{r}} = -y^2\partial_{\bar{y}}\,, \qquad \partial_{\bar{y}} = -\bar{r}^2\partial_{\bar{r}} \\
&\bar{A} := \bar{\sigma}e^{-M/\bar{y}} \quad \Leftrightarrow \quad \bar{\sigma} = \bar{A}e^{M/\bar{y}}\,, \qquad \bar{\eta} = -\bar{\kappa}e^{-M/\bar{y}} \quad \Leftrightarrow \quad \bar{\kappa} = -\bar{\eta}e^{M/\bar{y}}\,,
\end{aligned} \tag{4.17}$$

where

$$M = \sqrt{1-\bar{\omega}^2}\,, \qquad \text{requiring} \quad \lim_{r\to\infty}\alpha = 1\,. \tag{4.18}$$

A short calculation shows that Eqs. (4.3)-(4.7) are given by

$$
\begin{aligned}
\frac{\partial_{\bar y}\alpha}{\alpha} &= -\frac{\bar m}{1-2\bar m\bar y} - \frac{2\pi}{(1-2\bar m\bar y)\,\bar y^3\,e^{2M/\bar y}}\left[\bar\kappa^2 + \left(\frac{\bar\omega^2}{\alpha^2}-1\right)\bar\sigma^2\right],\\
\partial_{\bar y}\bar m &= -\frac{2\pi}{\bar y^4\,e^{2M/\bar y}}\left[\bar\kappa^2 + \left(\frac{\bar\omega^2}{\alpha^2}+1\right)\bar\sigma^2\right],\\
\partial_{\bar y}\bar\sigma &= \frac{X\bar\kappa-\bar\sigma}{\bar y^2},\\
\partial_{\bar y}\bar\kappa &= \frac{2\bar y-1}{\bar y^2}\bar\kappa - \bar\kappa\frac{\partial_{\bar y}\alpha}{\alpha} + \frac{X\bar\sigma}{\bar y^2}\left(1-\frac{\bar\omega^2}{\alpha^2}\right).
\end{aligned}
\tag{4.19}
$$

Let us finally rewrite the equations using $\Phi = \ln\alpha$ which gives us for the interior

$$
\boxed{
\begin{aligned}
\partial_{\bar r}\Phi &= \frac{X^2-1}{2\bar r} + 2\pi\bar r X^2\left[\bar\eta^2 + \left(\bar\omega^2 e^{-2\Phi}-1\right)\bar A^2\right],\\
\frac{\partial_{\bar r}X}{X} &= -\frac{X^2-1}{2\bar r} + 2\pi\bar r X^2\left[\bar\eta^2 + \left(\bar\omega^2 e^{-2\Phi}+1\right)\bar A^2\right],\\
\partial_{\bar r}\bar A &= X\bar\eta,\\
\partial_{\bar r}\bar\eta &= -2\frac{\bar\eta}{\bar r} - \bar\eta\partial_{\bar r}\Phi + X\left(1-\bar\omega^2 e^{-2\Phi}\right)\bar A.
\end{aligned}
\qquad (4.20)}
$$

and for the exterior,

$$
\begin{aligned}
\partial_{\bar y}\Phi &= -\frac{\bar m}{1-2\bar m\bar y} - \frac{2\pi}{(1-2\bar m\bar y)\,\bar y^3\,e^{2M/\bar y}}\left[\bar\kappa^2 + \left(\bar\omega^2 e^{-2\Phi}-1\right)\bar\sigma^2\right],\\
\partial_{\bar y}\bar m &= -\frac{2\pi}{\bar y^4\,e^{2M/\bar y}}\left[\bar\kappa^2 + \left(\bar\omega^2 e^{-2\Phi}+1\right)\bar\sigma^2\right],\\
\partial_{\bar y}\bar\sigma &= \frac{X\bar\kappa-M\bar\sigma}{\bar y^2},\\
\partial_{\bar y}\bar\kappa &= \frac{2\bar y-M}{\bar y^2}\bar\kappa - \bar\kappa\partial_{\bar y}\Phi + \frac{X\bar\sigma}{\bar y^2}\left(1-\bar\omega^2 e^{-2\Phi}\right).
\end{aligned}
\tag{4.21}
$$

The final version of the equations we are using is (4.20); we keep the expressions in the exterior region here for completeness, but will not use them in the eventual numerical computation.

5 Attempts for a relaxation algorithm

Our attempts to compute boson star models with a relaxation algorithm that covers the entire (compactified) computational domain from $\bar r = 0$ to $\bar y = 0$ has not succeeded; the code just did not manage to find the special solutions regular at infinity. Here we keep some relations derived for our relaxation attempts, but this section can be ignored for all other purposes of this document.

A few helpful relations are

$$
M = \sqrt{1-\bar\omega^2}\,,\qquad \frac{dM}{d\bar\omega} = -\frac{\bar\omega}{M}\,,\qquad \frac{d}{d\bar\omega}\frac{2\pi}{\bar y^3 e^{2M\bar y}} = \frac{2\bar\omega}{M\bar y}\,\frac{2\pi}{\bar y^3 e^{2M\bar y}}\,.
\tag{5.1}
$$

Following convergence difficulties, we try an alternative set of variables in the exterior. In the interior, we leave

$$\partial_{\bar r}\Phi = \frac{X^2-1}{2\bar r} + 2\pi\bar r X^2\left[\bar\eta^2 + \left(\bar\omega^2 e^{-2\Phi} - 1\right)\bar A^2\right] , \tag{5.2}$$

$$\frac{\partial_{\bar r}X}{X} = -\frac{X^2-1}{2\bar r} + 2\pi\bar r X^2\left[\bar\eta^2 + \left(\bar\omega^2 e^{-2\Phi} + 1\right)\bar A^2\right] , \tag{5.3}$$

$$\partial_{\bar r}\bar A = X\bar\eta\,, \tag{5.4}$$

$$\partial_{\bar r}\bar\eta = -2\frac{\bar\eta}{\bar r} - \bar\eta\partial_{\bar r}\Phi + X\left(1-\bar\omega^2 e^{-2\Phi}\right)\bar A\,. \tag{5.5}$$

unchanged. In the exterior, however, define

$$\begin{aligned} \sigma &:= \bar y^2 \ln \bar A &\quad\Leftrightarrow\quad& A = e^{\bar\sigma/\bar y^2}\,, \\ \bar\kappa &:= \frac{\bar\eta}{\bar A} &\quad\Leftrightarrow\quad& \bar\eta = e^{\bar\sigma/\bar y^2}\bar\kappa\,. \end{aligned} \tag{5.6}$$

These definitions lead to the exterior field equations

$$\begin{aligned} \partial_{\bar y}\Phi &= -\frac{\bar m}{1-2\bar m\bar y} - 2\pi\frac{e^{2\bar\sigma/\bar y^2}}{\bar y^3(1-2\bar m\bar y)}\left(\bar\kappa^2 + \bar\omega^2 e^{-2\Phi} - 1\right) , \\ \partial_{\bar y}\bar m &= -2\pi\frac{e^{2\bar\sigma/\bar y^2}}{\bar y^4}\left(\bar\kappa^2 + \bar\omega^2 e^{-2\Phi} + 1\right) , \\ \partial_{\bar y}\bar\sigma &= -X\bar\kappa + 2\frac{\bar\sigma}{\bar y}\,, \\ \partial_{\bar y}\bar\kappa &= 2\frac{\bar\kappa}{\bar y} - \bar\kappa\partial_{\bar y}\Phi + \frac{X}{\bar y^2}\left(\bar\kappa^2 + \bar\omega^2 e^{-2\Phi} - 1\right) . \end{aligned} \tag{5.7}$$

6 Boson star models computed with a shooting algorithm

We repeat here the set of ODEs (4.20) we need to integrate in order to obtain a spherically symmetric, stationary boson star model,

$$\begin{aligned} \partial_{\bar r}\Phi &= \frac{X^2-1}{2\bar r} + 2\pi\bar r X^2\left[\bar\eta^2 + \left(\bar\omega^2 e^{-2\Phi}-1\right)\bar A^2\right], \\ \frac{\partial_{\bar r}X}{X} &= -\frac{X^2-1}{2\bar r} + 2\pi\bar r X^2\left[\bar\eta^2 + \left(\bar\omega^2 e^{-2\Phi}+1\right)\bar A^2\right], \\ \partial_{\bar r}\bar A &= X\bar\eta, \\ \partial_{\bar r}\bar\eta &= -2\frac{\bar\eta}{\bar r} - \bar\eta\partial_{\bar r}\Phi + X\left(1-\bar\omega^2 e^{-2\Phi}\right)\bar A. \end{aligned} \tag{6.1}$$

Auxiliary variables suitable for diagnostics and other purposes are

$$m = \frac{r}{2}\left(1-\frac{1}{X^2}\right), \qquad \alpha = e^{\Phi}, \tag{6.2}$$

and we recall the dimensionless variables defined here according to

$$r = \frac{\bar r}{\mu}, \qquad m = \frac{\bar m}{\mu}, \qquad \omega = \mu\bar\omega, \qquad A = \frac{\bar A}{\sqrt{G}}, \qquad \eta = \frac{\mu}{\sqrt{G}}\bar\eta. \tag{6.3}$$

The boundary conditions for the system (6.1) are

$$\bar A(0) = \bar A_0, \quad X(0) = 1, \quad \bar\eta(0) = 0, \quad \Phi(\infty) = 0, \quad \bar A(\infty) = 0. \tag{6.4}$$

More specifically, $X(0) = 1$ avoids a conical singularity, $\bar A(\infty) = 0$ guarantees a regular solution with finite ADM mass, $\Phi(\infty) = 0$ implies that coordinate time coincides with propertime of an observer at infinity and $\bar\eta(0) = 0$ follows from the requirement of finite density at the origin. $\bar A_0$, on the other hand represents a free parameter and determines the overall mass and size of the star. We thus have a one-parameter family of solutions.

Our construction of boson star models through a shooting algorithm relies on the following properties of solutions to (6.1).

- A solution obtained from integrating (6.1) outwards will lead to a diverging scalar field $\bar A \to \infty$ for all values of ω except for a countable number of eigenvalues ω_n, $n = 0, \ldots, \infty$. For these ω_n, the scalar field decays asymptotically in accordance with Eq. (4.16). We seek these regular solutions, i.e. values of ω_n.
- For a diverging solution, the scalar field amplitude $\bar A$ has a number $n \in \mathbb{N}_0$ of zero crossings.
- This number of zero crossings monotonically increases with ω.
- The value $\omega = \omega_n$ that marks an increase of this number of zero crossings from n to $n + 1$ is precisely the value for a regular boson star model with n zero crossings of $\bar A$.

Our search for the frequencies ω_n of regular solutions therefore consists in identifying the boundary between irregular solutions with n and $n+1$ zero crossings. This feature is at the heart of our shooting algorithm which proceeds as follows.

(1) The user specifies $\bar{A}_0$, a target number $n_{\rm tar}$ of zero crossings or, equivalently, the excitation state n_0 of the boson star model and an initial guess $\tilde{\omega}_0$. The only requirement for a successful search is that ω_0 is larger than the correct frequency.

(2) Starting with $\tilde{\omega}_0$, the code computes a sequence of models for a decreasing series of frequencies $\tilde{\omega}_i < \tilde{\omega}_{i-1}$. According to the above list of features, the number of zero crossings n_i in these models is a non-increasing function of i.

(3) Once $n_i = n_{\rm tar}$ is reached, we know that the correct $\omega \in [\tilde{\omega}_{i-1}, \tilde{\omega}_i]$.

(4) We continue iterating by setting $\tilde{\omega}_{i+1} = (\tilde{\omega}_i + \tilde{\omega}_{i-1})/2$. If the model for this new frequency contains n $(n+1)$ zero crossings, we replace the lower (upper) bound of the interval with the new frequency. This bisection process is repeated until the upper and lower bound agree to within roundoff precision, typically $\mathcal{O}(10^{-15})$.

(5) This value is our estimate for the frequency of a regular boson star model with n zero crossings.

Even for these high precision estimates of the frequencies, the scalar field will still diverge at a radius of the order $\mathcal{O}(100)$ with typical values of the amplitude $\bar{A}$ being of order $\mathcal{O}(10^{-10})$ before the exponentially growing mode takes over. Details depend on the specific configuration.

In order to obtain a regular model on a larger domain, the code proceeds as follows.

(1) Find the last point where the scalar field was diverging but still had a regular value (i.e. not a "nan").

(2) Searching radially inwards from that point, find the first location of a local minima in abs($\bar{A}$). Note that $\bar{A}$ might have diverged towards negative values, hence the "abs".

(3) Starting at this point, either set the scalar field $\bar{A}$ and its derivative $\bar{\eta}$ to zero ("vacuum" exterior) or match the interior solution to the asymptotic falloff (4.16) in the exterior. For both, $\bar{A}$ and $\bar{\eta}$, we use the free factor in front of the asymptotic term to ensure continuity across the matching radius. For further flexibility, we also introduce a parameter `rmatchfac` that multiplies the matching radius thus found. In practice, setting this parameter to 0.9 smoothens the transition between the interior and exterior profiles.

(4) Integrate the metric components Φ and X according to the system (6.1), but do not use the differential equations for the scalar field.

Finally, we employ the freedom to add a constant offset to the variable Φ such that we satisfy its boundary condition (6.4) at infinity. We cannot directly impose $\Phi(\infty) = 0$ because our integration does not stretch to spatial infinity, but we can impose the Schwarzschild condition

$$\alpha^2 X^2 = 1 \qquad \Rightarrow \qquad \Phi = -\ln X \tag{6.5}$$

at the outer edge of our grid. This condition is exact in vacuum and an excellent approximation for the small scalar field values we encounter at typical radii of $\bar{r} = 200$. For diagnostic purposes, we also compute the mass function m from Eq. (6.2) and estimate the radius of the boson star as the value r where m reaches a user specified percentage of its asymptotic value. In this context, we note that at

$\bar{A}_c$	$\bar{m}$	$\bar{r}$	$\bar{\omega}$
0.0764	0.63300	7.875	0.85314

Table 1: Physical parameters of the maximum mass mini boson star model.

radii $\bar{r} = 200$ the mass function varies at the level of $\mathcal{O}(10^{-12})$ which is likely due to roundoff error. Note that the mass m converges way more rapidly towards its asymptotic value than X converges to 1.

7 Results

In order to interprete our results, it is necessary to consider once more the units. From Sec. 2 we recall that the gravitational constant and the scalar mass correspond to length and inverse length, respectively, according to

$$\begin{aligned} \frac{1}{\text{eV}} &= 1.973269788 \times 10^{-7}\text{m} = 6.582119511 \times 10^{-16}\ \text{s}\,, \\ \sqrt{G} &= 1.616228373 \times 10^{-35}\ \text{m} = \frac{1}{1.220910251 \times 10^{28}\ \text{eV}}\,. \end{aligned} \tag{7.1}$$

An intuitively more accessible form of translating from dimensionless to dimensional variables is then given by

$$\begin{aligned} r &= \bar{r}\ \text{km} \times \left(\frac{\mu}{1.937 \times 10^{-10}\ \text{eV}}\right)^{-1}, \\ \omega &= \bar{\omega}\ \text{Hz} \times \frac{\mu}{6.582 \times 10^{-16}\ \text{eV}} \\ A &= \bar{A}\ 1.221 \times 10^{28}\ \text{eV}\,. \end{aligned} \tag{7.2}$$

For illustration, we show in Fig. 1 the scalar field amplitude $\bar{A}$, the metric functions Φ and X and the mass m as functions of the radius $\bar{r}$ for three boson star models characterized by the central scalar amplitude values $\bar{A}_0 = 0.05$, 0.074 and 0.15. From the figure, we clearly see that for larger central amplitude $\bar{A}_c$, the star becomes more compact: the scalar field is more concentrated towards the origin and the lapse function decreases to smaller values near $\bar{r} = 0$. The mass, however, does not exhibit a monotonic dependency on the scalar amplitude. This behaviour is also known from neutron star models [6] and reflects the stability properties of the models. In order to illustrate this point further, we show in Fig. 2 the boson star mass as a function of the central scalar field value and of the boson star radius – evaluated here as the radius containing 99 % of the ADM mass. For reference we summarize in Table 7 the physical properties of the maximum mass mini boson star model. These values are the result of a sequence of 51 models uniformly distributed in $\bar{A}_c \in [0.7,\ 0.8]$ and computed over a grid of 3201 points stretching from $\bar{r} = 0$ to $\bar{r} = 200$ using a 4th order Runge-Kutta integrator.

8 The 3+1 equations for a complex scalar field

The 3+1 decomposition for the evolution of a real scalar field minimally coupled to Einstein's gravity has been developed in Sec. IV of Ref. [1]. The conventions differ from ours, however, and we will

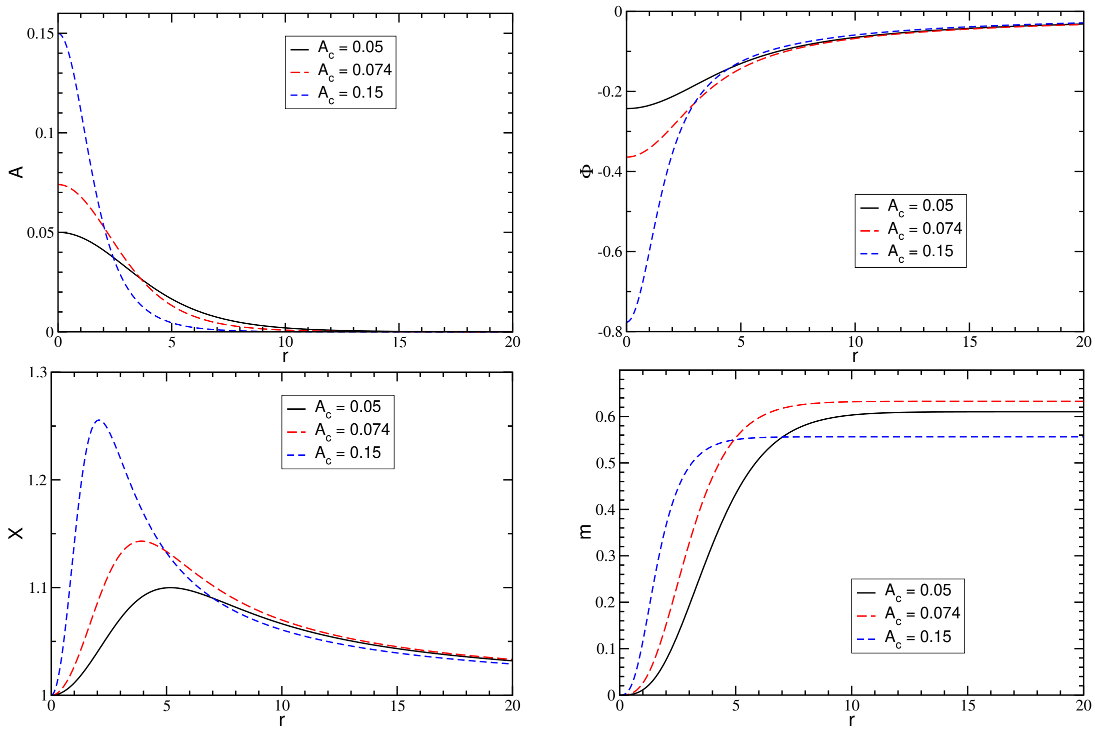

Figure 1: Scalar amplitude $\bar{A}$, mass function $\bar{m}$ and the metric variables Φ and X as functions of the radius $\bar{r}$ for three selected ground state boson star models with central amplitude values $\bar{A}_c = 0.05$, 0.074 and 0.15. Note that we ommit the bars from the variable name in the figure.

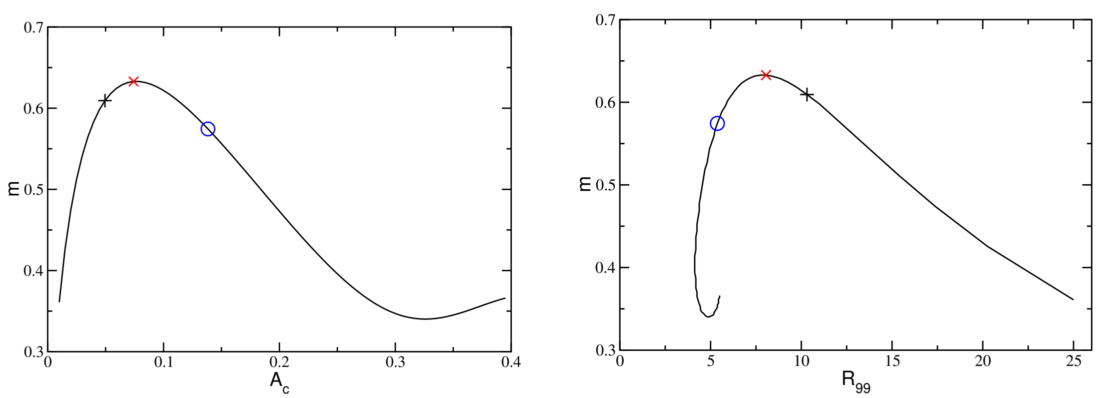

Figure 2: The boson star mass as a function of the central scalar amplitude $\bar{A}_c$ (left panel) and as a function of the radius $\bar{r}$ (right panel). The radius is defined here to contain 99 % of the stellar mass. The maximum mass $\bar{m}_{\max} \approx 0.633$ is the Kaup [3] limit for mini boson stars. The black +, the red $\times$ symbol and the blue circle mark the three models shown in Fig. 1.

highlight the relations between their variables and ours. In this comparison, we will always denote their variables with a caret, as in $\hat{\varphi}$ or $\hat{K}_\varphi$.

The starting point of our discussion is the action of Eq. (1.1) which gives us the energy momentum tensor [cf. the right-hand side of Eq. (1.7)] as

$$T_{\alpha\beta} = \frac{1}{2}\partial_\alpha\bar{\varphi}\,\partial_\beta\varphi + \frac{1}{2}\partial_\alpha\varphi\,\partial_\beta\bar{\varphi} - \frac{1}{2}g_{\alpha\beta}\left[g^{\mu\nu}\partial_\mu\bar{\varphi}\,\partial_\nu\varphi + V(\varphi)\right] . \tag{8.1}$$

Note that in Ref. [1] we call this quantity $8\pi\hat{T}_{\alpha\beta}$, i.e. the right-hand side equals $8\pi\hat{T}_{\alpha\beta}$ in [1]. This factor of 8π carries through the entire remainder of the calculation in [1]. The 3+1 decomposition of the energy-momentum tensor is given by

$$\rho := T_{\mu\nu}n^\mu n^\nu\,, \qquad j_\alpha := -\perp^\nu{}_\alpha T_{\mu\nu}n^\mu\,, \qquad S_{\alpha\beta} := \perp^\mu{}_\alpha\perp^\nu{}_\beta T_{\mu\nu}\,, \tag{8.2}$$

Before evaluating these projections, we define the momentum of the scalar field following [1],

$$K_\varphi := -\frac{1}{2}\mathcal{L}_{\boldsymbol{n}}\varphi = -\frac{1}{2}n^\mu\partial_\mu\varphi \qquad \Leftrightarrow \qquad \partial_t\varphi = \mathcal{L}_{\boldsymbol{\beta}}\varphi - 2\alpha K_\varphi = \beta^m\partial_m\varphi - 2\alpha K_\varphi\,, \tag{8.3}$$

where we have used $n^\mu = (\boldsymbol{\partial}_t^\mu - \beta^\mu)/\alpha$. The projections of the energy-momentum tensor then become

$$
\begin{aligned}
\rho &= 2K_\varphi\bar{K}_\varphi + \frac{1}{2}\partial^m\bar{\varphi}\,\partial_m\varphi + \frac{1}{2}V\,, && (8.4)\\
j_i &= \bar{K}_\varphi\partial_i\varphi + K_\varphi\partial_i\bar{\varphi}\,, && (8.5)\\
S_{ij} &= \partial_{(i}\bar{\varphi}\,\partial_{j)}\varphi - \frac{1}{2}\gamma_{ij}\left(\gamma^{mn}\partial_m\bar{\varphi}\,\partial_n\varphi - 4\bar{K}_\varphi K_\varphi + V\right)\,, && (8.6)\\
S_{ij}^{\rm TF} &= \frac{1}{2}\partial_i\bar{\varphi}\,\partial_j\varphi + \frac{1}{2}\partial_i\varphi\,\partial_j\bar{\varphi} - \frac{1}{3}\tilde{\gamma}_{ij}\tilde{\gamma}^{mn}\partial_m\bar{\varphi}\partial_n\varphi\,, && (8.7)\\
S+\rho &= 8\bar{K}_\varphi K_\varphi - V(\varphi)\,, && (8.8)\\
\text{where}\quad & \frac{1}{2}(\partial_i\bar{\varphi}\,\partial_j\varphi + \partial_i\varphi\,\partial_j\bar{\varphi}) = \partial_{(i}\bar{\varphi}\,\partial_{j)}\varphi = \partial_i\varphi_R\,\partial_j\varphi_R + \partial_i\varphi_I\,\partial_j\varphi_I\,, && (8.9)\\
& \bar{K}_\varphi K_\varphi = K_{\varphi R}^2 + K_{\varphi I}^2\,, \qquad \bar{K}_\varphi\partial_i\varphi + K_\varphi\partial_i\bar{\varphi} = 2K_{\varphi R}\partial_i\varphi_R + 2K_{\varphi I}\partial_i\varphi_I\,. && (8.10)
\end{aligned}
$$

Here, we have written the components in the form required in the BSSN equations and in Ref. [1], we would have again a factor 8π in front of each term on the right-hand side. By denoting their quantities with a caret, we can establish the following relations between their variables and ours (ignoring here the fact that our scalar field is complex),

$$
\begin{aligned}
\hat{\varphi} &= \sqrt{8\pi}\varphi\,, && (8.11)\\
\hat{K}_\varphi &= \sqrt{8\pi}K_\varphi\,, && (8.12)\\
\hat{\rho} &= \frac{1}{16\pi}\left[4\hat{K}_\varphi^2 + \partial^m\hat{\varphi}\,\partial_m\hat{\varphi} + 2W\right] \stackrel{!}{=} \rho\,, && (8.13)\\
\hat{j}_i &= \frac{1}{16\pi}4\hat{K}_\varphi\partial_i\hat{\varphi} \stackrel{!}{=} j_i\,, && (8.14)\\
\hat{S}_{ij}^{\rm TF} &= \hat{S}_{ij} - \frac{1}{3}\gamma_{ij}\hat{S} = \frac{1}{8\pi}\left(\partial_i\hat{\varphi}\,\partial_j\hat{\varphi} - \frac{1}{3}\tilde{\gamma}_{ij}\partial^m\hat{\varphi}\,\partial_m\hat{\varphi}\right) \stackrel{!}{=} S_{ij}^{\rm TF}\,, && (8.15)\\
\hat{S}+\hat{\rho} &= \frac{1}{4\pi}(4\hat{K}_\varphi^2 - W) \stackrel{!}{=} S+\rho = 8\bar{K}_\varphi K_\varphi - V\,. && (8.16)
\end{aligned}
$$

Here, we have added the potential W that is not contained in Ref. [1], but has been added to the code in this form in a later modification. There we have used an expression that differs from our potential definition by a factor 1/2 that we need to take into account when comparing with our $V(\varphi)$. The relation between the two according to Eqs. (8.13) and (8.16) is

$$W = \frac{1}{2}\hat{\mu}^2\hat{\varphi}^2 \stackrel{!}{=} 4\pi V = 4\pi\mu^2\,|\varphi|^2 \qquad \Rightarrow \qquad \hat{\mu} = \mu \qquad \text{since} \qquad \hat{\varphi} = \sqrt{8\pi}\varphi\,. \tag{8.17}$$

Returning now to the BSSN equations, we can informally write the addition of the matter terms as

$$\begin{aligned}
\partial_t\chi &\quad\rightarrow\quad \partial_t\chi\,, && (8.18)\\
\partial_t\tilde{\gamma}_{ij} &\quad\rightarrow\quad \partial_t\tilde{\gamma}_{ij}\,, && (8.19)\\
\partial_t K &\quad\rightarrow\quad \partial_t K + 4\pi\alpha(S+\rho)\,, && (8.20)\\
\partial_t\tilde{A}_{ij} &\quad\rightarrow\quad \partial_t\tilde{A}_{ij} - 8\pi\chi\alpha S^{\mathrm{TF}}_{ij}\,, && (8.21)\\
\partial_t\tilde{\Gamma}^i &\quad\rightarrow\quad -16\pi\frac{\alpha}{\chi}j^i\,. && (8.22)
\end{aligned}$$

Finally, the evolution of the scalar field is determined by the conservation of energy-momentum, $\nabla_\mu T^{\mu\alpha} = 0$. For the energy-momentum tensor (8.1), this gives us the familiar wave equation (1.14) for the scalar field, which we repeat here for completeness,

$$\nabla^\mu\nabla_\mu\varphi = V_{,|\varphi|^2} = V'\,. \tag{8.23}$$

This wave equation can be conveniently written in 3+1 form by using our definition (8.3) for the momentum K_φ. We take this opportunity, however, to highlight a subtlety that can easily introduce errors in a 3+1 decomposition. Contrary to what one might believe at first glance, the following two expressions are *not* equal,

$$D_\mu D_\nu\varphi \neq \perp\nabla_\mu\nabla_\nu\varphi := \perp^\rho{}_\mu\perp^\sigma{}_\nu\nabla_\rho\nabla_\sigma\varphi\,. \tag{8.24}$$

Instead, it is imperative to rigorously follow the rules for computing projections of tensors and only then take the derivatives. Explicitly, this leads to

$$\begin{aligned}
D_\mu D_\nu\varphi &= D_\mu(\perp^\gamma{}_\nu\nabla_\gamma\varphi) = \perp^\rho{}_\mu\perp^\sigma{}_\nu\nabla_\rho\left(\perp^\gamma{}_\sigma\nabla_\gamma\varphi\right) = \perp^\rho{}_\mu\perp^\sigma{}_\nu\nabla_\rho\left[(\delta^\gamma{}_\sigma + n^\gamma n_\sigma)\nabla_\gamma\varphi\right]\\
&= \perp^\rho{}_\mu\perp^\sigma{}_\nu\big[\nabla_\rho\nabla_\sigma\varphi + \nabla_\rho\big(n_\sigma\underbrace{n^\gamma\nabla_\gamma\varphi}_{=-2K_\varphi}\big)\big]\\
&= \perp^\rho{}_\mu\perp^\sigma{}_\nu\big[\nabla_\rho\nabla_\sigma\varphi - 2\big(K_\varphi\nabla_\rho n_\sigma + n_\sigma\nabla_\rho K_\varphi\big)\big]\\
&= (\delta^\rho{}_\mu + n^\rho n_\mu)(\delta^\sigma{}_\nu + n^\sigma n_\nu)\nabla_\rho\nabla_\sigma\varphi - 2K_\varphi\perp^\rho{}_\mu\perp^\sigma{}_\nu\nabla_\rho n_\sigma + 0\\
&= \nabla_\mu\nabla_\nu\varphi + n^\rho n_\mu\nabla_\rho\nabla_\nu\varphi + n^\sigma n_\nu\nabla_\mu\nabla_\sigma\varphi + n^\rho n_\mu n^\sigma n_\nu\nabla_\rho\nabla_\sigma\varphi + 2K_\varphi K_{\mu\nu}\,,
\end{aligned} \tag{8.25}$$

where we have used in the last line that $K_{\alpha\beta} = -\perp\nabla_\alpha n_\beta = -\perp\nabla_\beta n_\alpha$. The last term would have been missed with the naive approach of equating the two sides in (8.24).

Using Eq. (8.25), we trade $\nabla_\mu \nabla_\nu \varphi$ for $D_\mu D_\nu \varphi$ plus extra terms which will also give us the second time dervative or φ or, equivalently, the first time derivative of K_φ. First, we contract Eq. (8.25) which gives us

$$\begin{aligned} g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi &= (\perp^{\mu\nu} - n^\mu n^\nu)\nabla_\mu\nabla_\nu\varphi = \perp^{\mu\nu}\nabla_\mu\nabla_\nu\varphi - n^\mu n^\nu\nabla_\mu\nabla_\nu\varphi \\ &\overset{(8.25)}{=} \perp^{\mu\nu}D_\mu D_\nu\varphi - 0 - 0 - 0 - 2\perp^{\mu\nu}K_\varphi K_{\mu\nu} - n^\mu n^\nu\nabla_\mu\nabla_\nu\varphi\,. \end{aligned}$$

For the time-time projection, we notice that

$$\begin{aligned} n^\rho n^\sigma \nabla_\rho\nabla_\sigma\varphi &= n^\rho\nabla_\rho\big[n^\sigma\nabla_\sigma\varphi\big] - n^\rho(\nabla_\sigma\varphi)\,\nabla_\rho n^\sigma = n^\rho\nabla_\rho\big[-2K_\varphi\big] - (\nabla_\sigma\varphi)\,n^\rho\nabla_\rho n^\sigma \\ &= -2n^\rho\nabla_\rho K_\varphi - a^\sigma\nabla_\sigma\varphi\,, \qquad \text{where} \quad a^\mu := n^\rho\nabla_\rho n^\mu\,. \end{aligned} \tag{8.26}$$

Note that $a^\mu n_\mu = 0$ since $m_\mu n^\mu = -1 = \text{const}$, so that the acceleration is spatial. Finally, we need the Lie derivative of K_φ,

$$\begin{aligned} &\mathcal{L}_{\boldsymbol{n}}K_\varphi = n^\mu\nabla_\mu K_\varphi = n^\mu\partial_\mu K_\varphi\,, \qquad n^\mu = \frac{1}{\alpha}\,(\boldsymbol{\partial}_t^\mu - \beta^\mu) \\ \Rightarrow\quad &\mathcal{L}_{\boldsymbol{n}}K_\varphi = \frac{1}{\alpha}\partial_t K_\varphi - \frac{1}{\alpha}\mathcal{L}_{\boldsymbol{\beta}}K_\varphi = \frac{1}{\alpha}\partial_t K_\varphi - \frac{1}{\alpha}\beta^m\partial_m K_\varphi \end{aligned} \tag{8.27}$$

Combining this with Eq. (8.26), we obtain

$$n^\rho n^\sigma\nabla_\rho\nabla_\sigma\varphi = -2\left[\frac{1}{\alpha}\partial_t K_\varphi - \frac{1}{\alpha}\beta^m\partial_m K_\varphi\right] - a^m\partial_m\varphi\,. \tag{8.28}$$

Recalling $a^m = \partial^m \ln\alpha$, Eq. (1.14) can now be written as

$$\begin{aligned} &g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi = -2K\,K_\varphi + D^\mu D_\mu\varphi - n^\rho n^\sigma\nabla_\rho\nabla_\sigma\varphi \\ \Rightarrow\quad &g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi = -2K\,K_\varphi + D^\mu D_\mu\varphi + \frac{2}{\alpha}\left(\partial_t K_\varphi - \beta^m\partial_m K_\varphi\right) + \frac{\partial^m\alpha}{\alpha}\partial_m\varphi \overset{!}{=} V_{,|\varphi|^2}\varphi \\ \Rightarrow\quad &\frac{2}{\alpha}\left(\partial_t K_\varphi - \beta^m\partial_m K_\varphi\right) = -D^m D_m\varphi - \frac{1}{\alpha}\partial^m\alpha\,\partial_m\varphi + V_{,|\varphi|^2}\varphi + 2K\,K_\varphi\,, \end{aligned} \tag{8.29}$$

which gives us the eventual evolution equation for K_φ that also appears in the code of [1],

$$\boxed{\partial_t K_\varphi = \beta^m\partial_m K_\varphi + \alpha K_\varphi K + \frac{1}{2}\alpha V_{,|\varphi|^2}\varphi - \frac{1}{2}\chi\tilde{\gamma}^{mn}(\partial_m\varphi\partial_n\alpha + \alpha\tilde{D}_m\tilde{D}_n\varphi) + \frac{1}{4}\alpha\tilde{\gamma}^{mn}\partial_m\varphi\,\partial_n\chi\,. \quad (8.30)}$$

Together with the evolution equation (8.3) for φ, this completes our 3+1 evolution of the scalar field.

9 3+1 Initial data

9.1 Isotropic spherical coordinates

We next consider the conversion of our boson star profiles into intial data analogous to the isotropic Schwarzschild solution. It is not entirely clear whether this step is necessary; after all we do not have the Schwarzschild coordinate singularity at $r = 2M$ in our boson star models and the initial line element (3.1), after conversion to Cartesian coordinates, may give us perfectly suitable initial data. But let us convert to isotropic data, nonetheless, since we are used to working with them.

In isotropic coordinates, the line element has the form

$$ds^2 = -\alpha^2 dt^2 + \psi^4 (dR^2 + R^2 d\Omega^2)\,. \tag{9.1}$$

Comparing this with the polar-areal line element (3.1), we obtain two equations,

$$\psi^4 R^2 = r^2 \qquad \wedge \qquad \psi^4 dR^2 = X^2 dr^2\,. \tag{9.2}$$

This results in a differential equation that relates the two radii,

$$\frac{dR}{dr} = X\frac{R}{r}\,. \tag{9.3}$$

This equation could in principle be integrated, but its singular nature at the origin requires integration from the outside inwards. Integrating into a singularity is typically not a good thing to do and we can avoid this by introducing a new variable,

$$f(r) := \frac{R}{r} \qquad \Rightarrow \qquad \frac{df}{dr} = \frac{f}{r}(X-1)\,. \tag{9.4}$$

Now we can integrate out of the singularity by assuming that near $r = 0$, we have a linear relation, say, $R \propto r$. This indeed solves the differential equation for small r, since we already know that near the origin $X = 1 + \mathcal{O}(r^2)$. The proportionality factor is not determined by the differential equation, since any solution $f(r)$ remains a solution after multiplication by a constant, i.e. $c\,f$ also solves (9.4). The constant c will eventually be fixed by requiring that both radii agree at infinity; for now, however, we simply set $f = 1$ at the origin and solve (9.4) for $f(r)$ using a fourth-order Runge-Kutta integration.

In order to rescale the resulting solution, we consider the outer region of the boson-star spacetime where it is to very high precision a vacuum spacetime. In that case, we know the conformal factor from the Birkhoff theorem,

$$\psi = 1 + \frac{m}{2r}\,, \tag{9.5}$$

where m is the mass of the boson star; at sufficiently large radii, this mass is indeed a constant within roundoff precision. We then consider spheres of constant radius r and evaluate the area of this sphere using either the line element (3.1) or (9.1). This directly leads to the equation,

$$\begin{aligned}
& 4\pi\psi^4 R^2 = 4\pi r^2 \\
\Rightarrow \quad & \left(1 + \frac{m}{2R}\right)^4 R^2 = r^2 \\
\Rightarrow \ldots \Rightarrow \quad & R = \frac{r-m}{2}\left\{1 \pm \sqrt{1 - \frac{m^2}{(r-m)^2}}\right\} \\
\Rightarrow \quad & R \approx r - m - \frac{m^2}{4(r-m)} \qquad \wedge \qquad r \approx R + m + \frac{m^2}{4R}\,.
\end{aligned} \tag{9.6}$$

We have verified the last relation using a 3D calculation of the areal radius for schwarzschild initial data in isotropic coordinates: The results agree to within $\mathcal{O}(10^{-5})$ even at relatively low resolution.

We can now take r and m on the outer edge of our grid and compute the expected value of the isotropic radius and accordingly rescale the function $R(r)$ using the same constant factor on the entire grid.

The final ingredient is the conformal factor which we obtain from Eq. (9.2),

$$\psi^4 = \frac{r^2}{R^2} = \frac{1}{f^2} \qquad \Rightarrow \qquad \psi = \frac{1}{\sqrt{f}}\,. \tag{9.7}$$

We now have the line element

$$ds^2 = -\alpha^2 dt^2 + \psi^4 (dR^2 + R^2 d\Omega^2)\,, \tag{9.8}$$

together with the scalar field profile

$$\varphi(t, R) = A(R) e^{\mathrm{i}\omega t + \phi_0}\,, \tag{9.9}$$

where we have introduced ϕ_0 as an arbitrary (but constant, i.e. R independent) phase offset. In prractice, we have the profile available on a non-uniform grid, since our numerical grid is uniform in the areal radius r. We could generate a uniform grid in R and fill its values with fourth-order interpolation, but for now we will employ second-order interpolation onto the Cartesian grid, assuming that we have used so many points for the radia grid that its error is negligible.

9.1.1 Translation to a Cartesian 3D grid

For completeness, we give here a summary of the transformation between spherical and Cartesian coordinates, even though the result is trivial on our case. We define the two coordinate systems by

$$\left.\begin{array}{l} x = R\sin\theta\,\cos\phi \\[2ex] y = R\sin\theta\,\sin\phi \\[2ex] z = R\cos\theta \end{array}\right\} \qquad \Leftrightarrow \qquad \left\{\begin{array}{l} R = \sqrt{x^2+y^2+z^2} \\[2ex] \theta = \arctan\frac{\sqrt{x^2+y^2}}{z} \\[2ex] \phi = \arctan\frac{y}{x} \end{array}\right. \tag{9.10}$$

We thus obtain the transformation matrices

$$\begin{pmatrix} \dfrac{\partial x}{\partial R} = \sin\theta\,\cos\theta & \dfrac{\partial x}{\partial \theta} = R\cos\theta\,\cos\phi & \dfrac{\partial x}{\partial \phi} = -R\sin\theta\,\sin\phi \\ \dfrac{\partial y}{\partial R} = \sin\theta\,\sin\phi & \dfrac{\partial y}{\partial \theta} = R\cos\theta\,\sin\phi & \dfrac{\partial y}{\partial \phi} = R\sin\theta\,\cos\phi \\ \dfrac{\partial z}{\partial R} = \cos\theta & \dfrac{\partial z}{\partial \theta} = -R\sin\theta & \dfrac{\partial z}{\partial \phi} = 0 \end{pmatrix}\,, \tag{9.11}$$

and $[(\arctan x)' = 1/(1+x^2)]$

$$\begin{pmatrix} \dfrac{\partial R}{\partial x} = \dfrac{x}{R} & \dfrac{\partial R}{\partial y} = \dfrac{y}{R} & \dfrac{\partial R}{\partial z} = \dfrac{z}{R} \\ \dfrac{\partial \theta}{\partial x} = \dfrac{zx}{\rho R^2} & \dfrac{\partial \theta}{\partial y} = \dfrac{zy}{\rho R^2} & \dfrac{\partial \theta}{\partial z} = -\dfrac{\rho}{R^2} \\ \dfrac{\partial \phi}{\partial x} = -\dfrac{y}{\rho^2} & \dfrac{\partial \phi}{\partial y} = \dfrac{x}{\rho^2} & \dfrac{\partial \phi}{\partial z} = 0 \end{pmatrix}\,, \tag{9.12}$$

where we have defined $\rho = \sqrt{x^2+y^2}$. The metric in Cartesian coordinates is then given by

$$g_{xx} = \frac{\partial R}{\partial x}\frac{\partial R}{\partial x} g_{RR} + \frac{\partial \theta}{\partial x}\frac{\partial \theta}{\partial x} g_{\theta\theta} + \frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial x}\,, \tag{9.13}$$

and likewise for the other coordinates. Rather than working this out explicitly, however, our case is simplified by the isotropic character of the metric. We have

$$ds^2 = -\alpha^2 dt^2 + \psi^4 h_{ij} dx^i dx^j\,, \tag{9.14}$$

where h_{ij} is just the flat Euclidean metric in either spherical or Cartesian coordinates (or any other spatial coordinates adapted to the same 3+1 split with the specific time coordinate t). In Cartesian coordinates, $h_{ij} = \delta_{ij}$ and we have our isotropic boson star metric in Cartesian coordinates,

$$ds^2 = -\alpha^2 dt^2 + \psi^4 \delta_{ij} dx^i\, dx^j = -\alpha^2 dt^2 + \psi^4 (dx^2 + dy^2 + dz^2)\,. \tag{9.15}$$

Furthermore, this metric is straightforward to invert, so that

$$g_{\alpha\beta} = \left(\begin{array}{c|c} -\alpha^2 & 0 \\ \hline 0 & \psi^4 \delta_{ij} \end{array}\right) \qquad \Leftrightarrow \qquad g^{\alpha\beta} = \left(\begin{array}{c|c} -\alpha^{-2} & 0 \\ \hline 0 & \psi^{-4}\delta^{ij} \end{array}\right) \tag{9.16}$$

Comparing this with the standard 3+1 metric

$$g_{\alpha\beta} = \left(\begin{array}{c|c} -\alpha^2 + \beta_m \beta^m & \beta_j \\ \hline \beta_i & \gamma_{ij} \end{array}\right) \quad \Leftrightarrow \quad g^{\alpha\beta} = \left(\begin{array}{c|c} -\alpha^{-2} & \alpha^{-2}\beta^j \\ \hline \alpha^{-2}\beta^i & \gamma^{ij} - \alpha^{-2}\beta^i\beta^j \end{array}\right)\,, \tag{9.17}$$

we find

$$\alpha = \alpha\,, \qquad \beta^i = 0\,, \qquad \gamma_{ij} = \psi^4 \delta_{ij}\,, \qquad K_{ij} = 0\,, \tag{9.18}$$

where the last identity follows from the ADM equation

$$\partial_t \gamma_{ij} = \beta^m \partial_m \gamma_{ij} + \gamma_{mi}\partial_j \beta^m + \gamma_{mj}\partial_i \beta^m - 2\alpha K_{ij}\,, \tag{9.19}$$

the vanishing shift vector and the time independence of the metric. Of course, this will change when we consider boosted boson stars next, and will obtain a non-zero extrinsic curvature.

10 Boosted boson stars

Our implementation of boosted boson star initial data in large parts follows that for the boosted Schwarzschild black holes in isotropic coordinates. We consider two frames: the boson star is at rest in frame $\mathcal{O}$, whereas frame $\tilde{\mathcal{O}}$ moves with velocity v^i relative to this rest frame. The Lorentz transformation matrices are

$$\Lambda^{\tilde{\alpha}}{}_{\mu} = \left(\begin{array}{c|c} \gamma & -\gamma v_j \\ \hline -\gamma v^i & \delta^i{}_j + (\gamma-1)\frac{v^i v_j}{|\vec{v}|^2} \end{array}\right) \quad \Leftrightarrow \quad \Lambda^{\mu}{}_{\tilde{\alpha}} = \left(\begin{array}{c|c} \gamma & \gamma v_j \\ \hline \gamma v^i & \delta^i{}_j + (\gamma-1)\frac{v^i v_j}{|\vec{v}|^2} \end{array}\right)\,, \tag{10.1}$$

with the simplified limit for velocities in the z direction,

$$\Lambda^{\tilde{\alpha}}{}_{\mu} = \left(\begin{array}{c|ccc} \gamma & 0 & 0 & -\gamma v \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma v & 0 & 0 & \gamma \end{array}\right) \quad \Leftrightarrow \quad \Lambda^{\mu}{}_{\tilde{\alpha}} = \left(\begin{array}{c|ccc} \gamma & 0 & 0 & \gamma v \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma v & 0 & 0 & \gamma \end{array}\right)\,. \tag{10.2}$$

The coordinate systems are then related by

$$\tilde{x}^{\tilde{\alpha}} = \Lambda^{\tilde{\alpha}}{}_{\mu} x^{\mu} + \tilde{x}_0^{\tilde{\alpha}} \qquad \Leftrightarrow \qquad x^{\mu} = \Lambda^{\mu}{}_{\tilde{\alpha}} \tilde{x}^{\alpha} - x_0^{\mu} \,. \tag{10.3}$$

Note that the coordinates of a 3D evolution correspond to $\tilde{x}^i$, since we wish to obtain a star that is moving relative to our 3D coordinate frame; so this is $\tilde{\mathcal{O}}$. The restframe, on the other hand, is the frame where we calculate our spherically symmetric boson star model.

Our starting point is the single boson star in isotropic coordinates with metric

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 & 0 & 0 & 0 \\ 0 & \psi^4 & 0 & 0 \\ 0 & 0 & \psi^4 & 0 \\ 0 & 0 & 0 & \psi^4 \end{pmatrix} , \qquad g^{\mu\nu} = \begin{pmatrix} -\alpha^{-2} & 0 & 0 & 0 \\ 0 & \psi^{-4} & 0 & 0 \\ 0 & 0 & \psi^{-4} & 0 \\ 0 & 0 & 0 & \psi^{-4} \end{pmatrix} . \tag{10.4}$$

Here, α and ψ are functions of the isotropic radius R only. The scalar field is given by

$$\varphi(t, r) = A(r) e^{\mathrm{i}\omega t + \theta_0} \,, \tag{10.5}$$

where θ_0 is the initial phase and ω the frequency of the scalar field.

Let us next recall the transformation between areal radius r and isotropic radius R. We defined

$$f := \frac{R}{r} \qquad \Rightarrow \qquad \psi = \frac{1}{\sqrt{f}} \qquad \Rightarrow \qquad \frac{\mathrm{d}\psi}{\mathrm{d}R} = -\frac{1}{2} f^{-3/2} \frac{\mathrm{d}f}{\mathrm{d}R} \,, \tag{10.6}$$

and also concluded that

$$\frac{\mathrm{d}R}{\mathrm{d}r} = X \frac{R}{r} = X f \,, \qquad \frac{\mathrm{d}f}{\mathrm{d}r} = \frac{f}{r}(X - 1) \,. \tag{10.7}$$

For the transformation we will need the derivatives of the scalar field and the metric components. Given that x^i denote isotropic Cartesian coordinates, we have

$$\partial_i \psi = \frac{\mathrm{d}\psi}{\mathrm{d}R} \frac{\partial R}{\partial x^i} = \frac{\mathrm{d}\psi}{\mathrm{d}R} \frac{x^i}{R} \tag{10.8}$$

$$\text{with} \quad \frac{\mathrm{d}\psi}{\mathrm{d}R} = \frac{\mathrm{d}r}{\mathrm{d}R} \frac{\mathrm{d}\psi}{\mathrm{d}r} = \frac{1}{Xf} \frac{\mathrm{d}}{\mathrm{d}r} \frac{1}{\sqrt{f}} = -\frac{1}{2} \frac{1}{X f^{5/2}} \frac{\mathrm{d}f}{\mathrm{d}r} \tag{10.9}$$

$$\Rightarrow \quad \boxed{\frac{\mathrm{d}\psi}{\mathrm{d}R} = -\frac{1}{2} \frac{X - 1}{X} \frac{\psi}{R}} \,. \tag{10.10}$$

Likewise, we obtain for $\alpha =: e^{\Phi}$ using Eq. (4.20) for the function Φ,

$$\frac{\mathrm{d}\Phi}{\mathrm{d}R} = \frac{\mathrm{d}r}{\mathrm{d}R} \frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{1}{Xf} \left[\frac{X^2 - 1}{2r} + 2\pi r X^2 (\eta^2 + \omega^2 e^{-2\Phi} A^2 - V) \right] \tag{10.11}$$

$$\Rightarrow \quad \boxed{\frac{\mathrm{d}\Phi}{\mathrm{d}R} = \frac{X^2 - 1}{2XR} + \frac{2\pi R}{f^2} X (\eta^2 + \omega^2 e^{-2\Phi} A^2 - V)} \,. \tag{10.12}$$

We have dropped here the overbars for dimensionless variables, since we are only working with these at this stage in the code. Combined with Eq. (10.4), this gives us the first derivative of the metric in the rest frame,

$$\partial_i g_{00} = -2 e^{2\Phi} \frac{\mathrm{d}\Phi}{\mathrm{d}R} \frac{x^i}{R} \,, \qquad \partial_i g^{00} = 2 e^{-2\Phi} \frac{\mathrm{d}\Phi}{\mathrm{d}R} \frac{x^i}{R} \,,$$

$$\partial_i g_{kk} = 4\psi^3 \frac{\mathrm{d}\psi}{\mathrm{d}R} \frac{x^i}{R} \,, \qquad \partial_i g^{kk} = -4\psi^{-5} \frac{\mathrm{d}\psi}{\mathrm{d}R} \frac{x^i}{R} \,. \tag{10.13}$$

Here we do *not* sum over the repeated index k in the last line. For the scalar field, we have

$$\boxed{\begin{aligned} \varphi_R &= A\cos(\omega t + \phi_0)\,, & \partial_t \varphi_R &= -\omega\varphi_I\,, \\ \varphi_I &= A\sin(\omega t + \phi_0)\,, & \partial_t \varphi_I &= \omega\varphi_R\,. \end{aligned} \qquad (10.14)}$$

Recalling that $X\eta = \partial_r A$ and using again $\mathrm{d}r/\mathrm{d}R = 1/(Xf)$, we also obtain the spatial derivatives

$$\boxed{\partial_i\varphi_R = \frac{\eta}{f}\cos(\omega t + \phi_0)\,\frac{x^i}{R}\,, \qquad \partial_i\varphi_I = \frac{\eta}{f}\sin(\omega t + \phi_0)\,\frac{x^i}{R}\,. \qquad (10.15)}$$

We are now in the position to apply the Lorentz transformation. With the transformation matrices (10.1), we have the following relations

$$\boxed{\begin{aligned} &\tilde{g}_{\tilde{\alpha}\tilde{\beta}} = g_{\mu\nu}\Lambda^{\mu}{}_{\tilde{\alpha}}\Lambda^{\nu}{}_{\tilde{\beta}}\,, \qquad \tilde{g}^{\tilde{\alpha}\tilde{\beta}} = g^{\mu\nu}\Lambda^{\tilde{\alpha}}{}_{\mu}\Lambda^{\tilde{\beta}}{}_{\nu}\,, \\ &\tilde{\partial}_{\tilde{\gamma}}\tilde{g}_{\tilde{\alpha}\tilde{\beta}} = \Lambda^{\lambda}{}_{\tilde{\gamma}}\Lambda^{\mu}{}_{\tilde{\alpha}}\Lambda^{\nu}{}_{\tilde{\beta}}\partial_{\lambda}g_{\mu\nu}\,. \end{aligned} \qquad (10.16)}$$

Since $\gamma_{ij} = g_{ij}$ in any frame, we directly conclude from these expressions that

$$\tilde{\gamma}_{\tilde{k}\tilde{l}} = \tilde{g}_{\tilde{k}\tilde{l}}\,, \qquad \tilde{\partial}_{\tilde{0}}\tilde{\gamma}_{\tilde{k}\tilde{l}} = \tilde{\partial}_{\tilde{0}}\tilde{g}_{\tilde{k}\tilde{l}}\,, \qquad \tilde{\partial}_{\tilde{m}}\tilde{\gamma}_{\tilde{k}\tilde{l}} = \tilde{\partial}_{\tilde{m}}\tilde{g}_{\tilde{k}\tilde{l}}\,. \qquad (10.17)$$

Next, we can invert the spatial metric $\tilde{\gamma}_{\tilde{k}\tilde{l}}$ to obtain $\tilde{\gamma}^{\tilde{m}\tilde{n}}$, and we obtain our first set of 3+1 variables in the frame $\tilde{\mathcal{O}}$,

$$\boxed{\begin{aligned} &\tilde{\beta}_{\tilde{k}} = \tilde{g}_{\tilde{0}\tilde{k}}\,, \qquad \tilde{\alpha} = \sqrt{-\tilde{g}^{\tilde{0}\tilde{0}}}^{\,-1}\,, \qquad \tilde{\gamma}_{\tilde{k}\tilde{l}} = \tilde{g}_{\tilde{k}\tilde{l}}\,, \\ &\tilde{\beta}^{\tilde{k}} = \tilde{\gamma}^{\tilde{k}\tilde{m}}\tilde{\beta}_{\tilde{m}}\,. \end{aligned} \qquad (10.18)}$$

For the extrinsic curvature, we need to transform spatial derivatives of the shift vector. This is more complicated, but can be achieved quite conveniently by considering

$$\begin{aligned} &\tilde{\partial}_{\tilde{l}}\tilde{\beta}_{\tilde{k}} = \tilde{\partial}_{\tilde{l}}\tilde{g}_{\tilde{0}\tilde{k}} = \tilde{\partial}_{\tilde{l}}(\tilde{\gamma}_{\tilde{k}\tilde{m}}\tilde{\beta}^{\tilde{m}}) = \tilde{\gamma}_{\tilde{k}\tilde{m}}\tilde{\partial}_{\tilde{l}}\tilde{\beta}^{\tilde{m}} + \tilde{\beta}^{\tilde{m}}\tilde{\partial}_{\tilde{l}}\tilde{\gamma}_{\tilde{k}\tilde{m}} \\ \Rightarrow\quad &\tilde{\gamma}_{\tilde{k}\tilde{m}}\tilde{\partial}_{\tilde{l}}\tilde{\beta}^{\tilde{m}} = \tilde{\partial}_{\tilde{l}}\tilde{g}_{\tilde{0}\tilde{k}} - \tilde{\beta}^{\tilde{m}}\tilde{\partial}_{\tilde{l}}\tilde{g}_{\tilde{k}\tilde{m}}\,. \end{aligned} \qquad (10.19)$$

We know the right-hand-side of this expression from our transformation (10.16), which gives us the derivative of the shift on the left which, in turn, allows us to compute the extrinsic curvature from the general expression (true in any frame)

$$\boxed{\tilde{K}_{\tilde{k}\tilde{l}} = -\frac{1}{2\tilde{\alpha}}\left(\tilde{\partial}_{\tilde{0}}\tilde{\gamma}_{\tilde{k}\tilde{l}} - \tilde{\beta}^{\tilde{m}}\tilde{\partial}_{\tilde{m}}\tilde{\gamma}_{\tilde{k}\tilde{l}} - \tilde{\gamma}_{\tilde{m}\tilde{l}}\tilde{\partial}_{\tilde{k}}\tilde{\beta}^{\tilde{m}} - \tilde{\gamma}_{\tilde{k}\tilde{m}}\tilde{\partial}_{\tilde{l}}\tilde{\beta}^{\tilde{m}}\right)\,. \qquad (10.20)}$$

The scalar field transforms straightforwardly, but a little care is needed for the derivative variable $\tilde{K}_{\varphi}$. By definition,

$$K_{\varphi} = -\frac{1}{2\alpha}\left(\partial_0\varphi - \beta^m\partial_m\varphi\right)\,, \qquad (10.21)$$

so that we need to Lorentz transform the derivatives of the scalar field. The resulting transformations are summarized as

$$\varphi = \tilde{\varphi}\,, \qquad \tilde{\partial}_{\tilde{\alpha}}\varphi = \Lambda^{\mu}{}_{\tilde{\alpha}}\partial_{\mu}\varphi\,, \qquad \tilde{K}_{\varphi} = -\frac{1}{2\tilde{\alpha}}\left(\tilde{\partial}_{\tilde{0}}\varphi - \tilde{\beta}^{\tilde{m}}\tilde{\partial}_{\tilde{m}}\varphi\right)\,. \tag{10.22}$$

Thanks to their linearity, these three relations directly apply to the real and imaginary parts of φ and K_{φ} individually.

In addition to these expressions, we need to bear in mind the coordinate transformation (10.3): The computational domain of our 3D code consists of points $\tilde{x}^{\tilde{\alpha}}$. which need to be converted into coordinates x^{μ} in the rest frame. These x^{μ} are then inserted into computing the scalar field according to Eqs. (10.14), (10.15) and the metric (10.4) and its derivatives. In particular, the initial time $\tilde{t}=0$ on the initial slice of our 3D evolution in general corresponds to points with $t\neq 0$ in the rest frame. This makes no difference for the time independent metric, but needs to be taken into account when evaluating the scalar field.

11 Superposition of boson stars

We now assume that we have two individual boson star solutions, boosted or not, located at $x^i_{\rm A}$ and $x^i_{\rm B}$ and described by the ADM variables $\gamma^{\rm A}_{ij}$, $\alpha_{\rm A}$, $\beta^i_{\rm A}$ and $K^{\rm A}_{ij}$ and likewise for boson star B. There are various ways of superposing these two metrics. For now, we superpose according to the following relations,

$$\begin{aligned}
\gamma_{ij} &= \gamma^{\rm A}_{ij} + \gamma^{\rm B}_{ij} - \delta_{ij}\,, && (11.1)\\
\mathcal{K}^i{}_j &:= \gamma^{im}_{\rm A}K^{\rm A}_{mj} + \gamma^{im}_{\rm B}K^{\rm B}_{mj}\,, && (11.2)\\
\alpha &= \frac{1}{\sqrt{\alpha_{\rm A}^{-2} + \alpha_{\rm B}^{-2}}}\,, && (11.3)\\
\beta^i &= \gamma^{im}(\beta^{\rm A}_m + \beta^{\rm B}_m)\,, && (11.4)\\
K_{ij} &= \frac{1}{2}\left[\gamma_{im}\mathcal{K}^m{}_j + \gamma_{jm}\mathcal{K}^m{}_i\right]\,, && (11.5)\\
\varphi &= \varphi_{\rm A} + \varphi_{\rm B}\,, && (11.6)\\
K_{\varphi} &= K^{\rm A}_{\varphi} + K^{\rm B}_{\varphi}\,. && (11.7)
\end{aligned}$$

12 Boson stars with self interaction

In this section we discuss how the equations or the rescaling may change when we consider scalar potentials with self-interaction. Even though we will allow for general potential functions $V(A^2)$, we will start by writing this in terms of a series expansion of the form

$$V = V(A^2) = \mu^2 A^2 + \Lambda_4 A^4 + \Lambda_6 A^6 + \Lambda_8 A^8 + \ldots = \sum_{n=1}^{N}\Lambda_{2n}A^{2n}\,, \qquad \text{with} \qquad \Lambda_2 = \mu^2\,. \tag{12.1}$$

We definte the derivative of the potential as

$$V' := \frac{dV}{d(A^2)} = \mu^2 + 2\Lambda_4 A^2 + 3\Lambda_6 A^4 + 4\Lambda_8 A^6 + \ldots = \sum_{n=1}^{N} n\Lambda_{2n} A^{2(n-1)}\,. \tag{12.2}$$

Now let us return to our set of field equations (3.11) expressed in physical units. We shall use the same rescaling as in Eq. (4.2), namely

$$\bar{t} := \mu t\,, \qquad \bar{r} := \mu r\,, \qquad \bar{m} := \mu m\,, \qquad \bar{A} := \sqrt{G} A\,, \qquad \bar{\eta} := \frac{\sqrt{G}}{\mu}\eta\,, \qquad \bar{\omega} := \frac{\omega}{\mu}\,, \tag{12.3}$$

but now complement these expressions by introducing a rescaled version of the potential (12.1),

$$\begin{aligned}
\bar{V} := \frac{G}{\mu^2} V &= \bar{A}^2 + \lambda_4 \bar{A}^4 + \lambda_6 \bar{A}^6 + \lambda_8 \bar{A}^8 + \ldots = \bar{A}^2 + \sum_{n=2}^{N} \lambda_{2n} \bar{A}^{2n} \\
\Rightarrow \quad \bar{V}' := \frac{d\bar{V}}{d(\bar{A}^2)} \stackrel{!}{=} \frac{V'}{\mu^2} &= 1 + 2\lambda_4 \bar{A}^2 + 3\lambda_4 \bar{A}^4 + 4\lambda_8 \bar{A}^6 + \ldots = 1 + \sum_{n=2}^{N} n\lambda_{2n} \bar{A}^{2(n-1)}\,.
\end{aligned} \tag{12.4}$$

In practice, we will work with the rescaled potential (12.4) in the simulations and can reconstruct the corresponding physical potential through the rescaling (12.3) and the conversions between the coefficients

$$\Lambda_4 = \mu^2 G \lambda_4 = \frac{\mu^2}{M_{\rm Pl}^2}\lambda^4\,, \quad \Lambda_6 = \mu^2 G^2 \lambda_6 = \frac{\mu^2}{M_{\rm Pl}^4}\lambda_6\,, \quad \lambda_8 = \mu^2 G^3 \lambda_8 = \frac{\mu^2}{M_{\rm Pl}^6}\lambda_8\,, \quad \ldots\,, \qquad \Lambda_n = \frac{\mu^2}{M_{\rm Pl}^{n-2}}\lambda_n\,. \tag{12.5}$$

Note that by construction all λ_n are dimensionless but Λ_n has dimension $1/\text{mass}^{n-4}$. Since the scalar field has dimension mass, we conclude that the physical potential V has dimension mass^4 or, equivalently, $1/\text{length}^4$ as it must when we consider the action (1.1); recall that $R \sim \text{length}^{-2}$, $G \sim \text{length}^2$, $\varphi \sim \text{mass} \sim \text{length}^{-1}$.

We can now apply the rescaling (12.3) and (12.4) to the field equations (3.11) and obtain

$$\boxed{\begin{aligned}
\partial_{\bar{r}}\Phi &= \frac{X^2-1}{2\bar{r}} + 2\pi\bar{r}X^2\left(\bar{\eta}^2 + \bar{\omega}^2 e^{-2\Phi}\bar{A}^2 - \bar{V}\right)\,, \\
\frac{\partial_{\bar{r}}X}{X} &= -\frac{X^2-1}{2\bar{r}} + 2\pi\bar{r}X^2\left(\bar{\eta}^2 + \bar{\omega}^2 e^{-2\Phi}\bar{A}^2 + \bar{V}\right)\,, \\
\partial_{\bar{r}}\bar{A} &= X\bar{\eta}\,, \\
\partial_{\bar{r}}\bar{\eta} &= -2\frac{\bar{\eta}}{\bar{r}} - \bar{\eta}\partial_{\bar{r}}\Phi + X\left(\bar{V}' - \bar{\omega}^2 e^{-2\Phi}\right)\bar{A}\,,
\end{aligned}} \tag{12.6}$$

where the potential is now given by Eq. (12.4). For the non-self-interacting case $\bar{V} = \bar{A}^2$, $\bar{V} = 1$, we recover the mini-boson star equations (6.1), as expected.

A special case of the potential (12.1) is the solitonic potential (see for example Eq. (7) in Ref. [5]),

$$\begin{aligned}
V &= \mu^2 A^2 \left(1 - 2\frac{A^2}{\sigma_0^2}\right)^2\,, \\
V' &= \mu^2 \left(1 - 2\frac{A^2}{\sigma_0^2}\right)\left(1 - 6\frac{A^2}{\sigma_0^2}\right)\,.
\end{aligned} \tag{12.7}$$

Using our rescaling together with $\bar{\sigma}_0 = \sqrt{G}\sigma_0$, we obtain,

$$\bar{V} := \frac{G}{\mu^2}V = \bar{A}^2\left(1-2\frac{\bar{A}^2}{\bar{\sigma}_0}\right)^2, \qquad \text{where} \qquad \bar{A}=\sqrt{G}A\,, \quad \bar{\sigma}_0=\sqrt{G}\sigma_0\,.$$

$$\bar{V}' := \frac{d\bar{V}}{d(\bar{A}^2)} \stackrel{!}{=} \frac{V'}{\mu^2} = \left(1-2\frac{\bar{A}^2}{\bar{\sigma}_0^2}\right)\left(1-6\frac{\bar{A}^2}{\bar{\sigma}_0^2}\right)\,. \tag{12.8}$$

13 Fluxes and integration

13.1 3+1 and 2+1 split

As a preliminary for some of our diagnostics, it is instructive to recall some properties of the spacetime split, integration over volumes and surfaces, and the behaviour of flux conservation laws. Let us start with the basic 3+1 split. All we will say here, generalizes straightforwardly to any number of spacetime dimensions, but for simplicity of notation we work with four dimensions. We start with the standard slicing of spacetime using a function $t(x^\alpha)$ with non-zero gradient $\mathbf{d}t$ everywhere. We ultimately wish t to be a timelike coordinate, and therefore require $\boldsymbol{g}(\mathbf{d}t,\mathbf{d}t)<0$, i.e. $\mathbf{d}t$ be timelike. We next consider coordinates adapted to this slicing. We let Σ_t denote the hypersurfaces $t=\text{const}$ with spatial coordinates x^i labeling points inside each Σ_t. Adapted coordinates are then given by $x^\alpha=\{t,\,x^i\}$, and the coordinate basis vector $\boldsymbol{\partial}_t$ is defined as the directional derivative operator along curves $x^i=\text{const}$ parametrized with t. We then define

$$\begin{aligned}
\text{Lapse} \qquad & \alpha := \frac{1}{||\mathbf{d}t||}\,, \\
\text{Timelike unit normal} \qquad & \boldsymbol{n} := -\alpha\mathbf{d}t\,, \\
\text{Shift vector} \qquad & \boldsymbol{\beta} := \boldsymbol{\partial}_t - \alpha\boldsymbol{n}\,, \\
\text{Spatial metric} \qquad & \boldsymbol{\gamma} := \boldsymbol{g}+\boldsymbol{n}\otimes\boldsymbol{n} \qquad \Leftrightarrow \qquad \gamma_{\alpha\beta} := g_{\alpha\beta}+n_\alpha n_\beta\,.
\end{aligned} \tag{13.1}$$

From these definitions, we directly conclude the following.

$$\langle\mathbf{d}t,\boldsymbol{\partial}_t\rangle = 1 \qquad \text{(by construction)}\,, \tag{13.2}$$

$$\langle\mathbf{d}t,\boldsymbol{\partial}_i\rangle = \langle\boldsymbol{n},\boldsymbol{\partial_i}\rangle = 0 \qquad \text{by construction)}\,, \tag{13.3}$$

$$\beta^0 \stackrel{!}{=} \langle\mathbf{d}t,\boldsymbol{\beta}\rangle = \langle\mathbf{d}t,\boldsymbol{\partial}_t\rangle - \alpha\langle\mathbf{d}t,\boldsymbol{n}\rangle = 1+\langle\boldsymbol{n},\boldsymbol{n}\rangle = 0\,, \tag{13.4}$$

$$g_{00} := \boldsymbol{g}(\boldsymbol{\partial}_t,\boldsymbol{\partial}_t\rangle = \boldsymbol{g}(\boldsymbol{\beta}+\alpha\boldsymbol{n},\boldsymbol{\beta}+\alpha\boldsymbol{n}\rangle = \beta_m\beta^m-\alpha^2\,, \tag{13.5}$$

$$g_{0i} = \boldsymbol{g}(\boldsymbol{\partial}_t,\boldsymbol{\partial}_i) = \boldsymbol{g}(\boldsymbol{\beta}+\alpha\boldsymbol{n},\boldsymbol{\partial}_i) = \beta_i\,, \tag{13.6}$$

$$g_{ij} = (\boldsymbol{\gamma}-\boldsymbol{n}\otimes\boldsymbol{n})(\boldsymbol{\partial}_i,\boldsymbol{\partial}_i) = \gamma_{ij}\,. \tag{13.7}$$

The last three equations give us the 3+1 version of the metric which can be inverted, so that

$$g_{\mu\nu} = \left(\begin{array}{c|c} -\alpha^2+\beta_m\beta^m & \beta_j \\ \hline \beta_i & \gamma_{ij}\end{array}\right) \qquad \Leftrightarrow \qquad g^{\mu\nu} = \left(\begin{array}{c|c} -\alpha^{-2} & \alpha^{-2}\beta^j \\ \alpha^{-2}\beta^i & \gamma^{ij}-\alpha^{-2}\beta^i\beta^j\end{array}\right)\,. \tag{13.8}$$

Here, γ^{ij} is defined as the inverse of the 3-metric γ_{ij}. We can also evaluate the determinant of the spacetime metric. For this purpose, we recall the following method to construct the inverse of an invertible matrix. (i) We define by $\mathcal{C}_{\alpha\beta}$ the cofactor of $g_{\alpha\beta}$, i.e. $(-1)^{\alpha+\beta}$ times the determinant of the 3×3 matrix obtained by crossing out row α and column β in $g_{\alpha\beta}$. The inverse is then given by

$$g^{\mu\nu} = \mathcal{C}_{\nu\mu}\frac{1}{\det g} \qquad \Leftrightarrow \qquad \mathcal{C}_{\alpha\beta} = \det g\; g^{\beta\alpha}\,. \tag{13.9}$$

The transposition does not do anything in our case, since the metric $g_{\mu\nu}$ is symmetric. Bearing in mind this construction of the cofactor matric, we obtain by expanding Eq. (13.8) along the first row,

$$\begin{aligned}
\det g_{\mu\nu} &= (-\alpha^2+\beta_m\beta^m)\det\gamma_{ij} - \beta_1\begin{vmatrix}\beta_1 & \gamma_{12} & \gamma_{13}\\ \beta_2 & \gamma_{22} & \gamma_{23}\\ \beta_3 & \gamma_{32} & \gamma_{33}\end{vmatrix} + \beta_2\begin{vmatrix}\beta_1 & \gamma_{11} & \gamma_{13}\\ \beta_2 & \gamma_{21} & \gamma_{23}\\ \beta_3 & \gamma_{31} & \gamma_{33}\end{vmatrix} - \beta_3\begin{vmatrix}\beta_1 & \gamma_{11} & \gamma_{12}\\ \beta_2 & \gamma_{21} & \gamma_{22}\\ \beta_3 & \gamma_{31} & \gamma_{32}\end{vmatrix}\\
&= (-\alpha^2+\beta_m\beta^m)\mathcal{C}_{00} + \beta_1\mathcal{C}_{01} + \beta_2\mathcal{C}_{02} + \beta_3\mathcal{C}_{03}\\
&= (-\alpha^2+\beta_m\beta^m)\det\gamma + \beta_1\det g\; g^{10} + \beta_2\det g\; g^{20} + \beta_3\det g\; g^{30}\\
&= (-\alpha^2+\beta_m\beta^m)\det\gamma_{ij} + \det g_{\mu\nu}\Big(\beta_1\alpha^{-2}\beta^1 + \beta_2\alpha^{-2}\beta^2 + \beta_3\alpha^{-2}\beta^3\Big)\\
\Rightarrow\quad & \det g_{\mu\nu}(1-\alpha^{-2}\beta_m\beta^m) = \alpha^2(-1+\alpha^{-2}\beta_m\beta^m)\det\gamma_{ij}\\
\Rightarrow\quad & \det g_{\mu\nu} = -\alpha^2\det\gamma_{ij}\,,
\end{aligned}\tag{13.10}$$

provided $\alpha^2-\beta_m\beta^m\neq 0$.

It will come in quite handy to replay the same game on the 3-dimensional hypersurface Σ. The only difference is that we now introduce a spatial foliation. We start with our spatial coordinates x^i and consider a foliation defined in terms of level sets of the function $r(x^i)=\text{const}$. Again, we assume that the gradient $\mathbf{d}R\neq 0$ everywhere, but since we are now dealing with a Riemannian manifold, all our tensors are purely spacelike. We denote the 2-dimensional hypersurfaces by A_r and introduce coordinates R, x^a adapted to this foliation; early Latin indices run over 2, 3. In these coordinates, $\boldsymbol{\partial}_R$ is the directional derivative operator along the curves $x^a=\text{const}$ parametrized by R. In analogy to Eq. (13.1), we define

$$\begin{aligned}
\text{“Lapse”}\qquad & \lambda := \frac{1}{||\mathbf{d}R||}\,,\\
\text{Unit normal}\qquad & \boldsymbol{N} := \lambda\mathbf{d}R\,,\\
\text{“Shift” vector}\qquad & \boldsymbol{B} := \boldsymbol{\partial}_R - \lambda\boldsymbol{N}\,,\\
\text{2-metric}\qquad & \boldsymbol{\sigma} := \boldsymbol{\gamma} - \boldsymbol{N}\otimes\boldsymbol{N} \qquad\Leftrightarrow\qquad \sigma_{ij} := \gamma_{ij} - N_aN_b\,.
\end{aligned}\tag{13.11}$$

Note that a couple signs have changed relative to Eq. (13.1); this will propagate throughout our calculations. Defining the basis coordinate vectors on the 2-surfaces by $\boldsymbol{\partial}_a$, our definitions directly imply the following relations,

$$\langle\mathbf{d}R,\boldsymbol{\partial}_R\rangle = 1 \qquad \text{(by construction)}\,, \tag{13.12}$$

$$\langle\mathbf{d}R,\boldsymbol{\partial}_a\rangle = \langle\boldsymbol{N},\boldsymbol{\partial}_{\boldsymbol{a}}\rangle = 0 \qquad \text{by construction)}\,, \tag{13.13}$$

$$B^R \stackrel{!}{=} \langle\mathbf{d}R,\boldsymbol{B}\rangle = \langle\mathbf{d}R,\boldsymbol{\partial}_R\rangle - \lambda\langle\mathbf{d}R,\boldsymbol{N}\rangle = 1-\langle\boldsymbol{N},\boldsymbol{N}\rangle = 0\,, \tag{13.14}$$

$$\gamma_{11} := \boldsymbol{\gamma}(\boldsymbol{\partial}_R,\boldsymbol{\partial}_R\rangle = \boldsymbol{\gamma}(\boldsymbol{B}+\lambda\boldsymbol{N},\boldsymbol{B}+\lambda\boldsymbol{N}\rangle = B_aB^a+\lambda^2\,, \tag{13.15}$$

$$\gamma_{1a} = \boldsymbol{\gamma}(\boldsymbol{\partial}_R,\boldsymbol{\partial}_a) = \boldsymbol{\gamma}(\boldsymbol{B}+\lambda\boldsymbol{N},\boldsymbol{\partial}_a) = B_a\,, \tag{13.16}$$

$$\gamma_{ab} = (\boldsymbol{\sigma}-\boldsymbol{N}\otimes\boldsymbol{N})(\boldsymbol{\partial}_a,\boldsymbol{\partial}_b) = \sigma_{ab}\,. \tag{13.17}$$

This leads to the 2+1 decomposition of the spatial metric which we can invert as in the 3+1 case,

$$\gamma_{ij} = \left(\begin{array}{c|c} \lambda^2 + B_c B^c & B_b \\ \hline B_a & \sigma_{ab} \end{array}\right) \qquad \Leftrightarrow \qquad \gamma^{\mu\nu} = \left(\begin{array}{c|c} \lambda^{-2} & -\lambda^{-2}B^b \\ -\lambda^{-2}B^a & \sigma^{ab} + \lambda^{-2}B^a B^b \end{array}\right). \tag{13.18}$$

The decomposition of the determinant of γ_{ij} carries over directly from the above time-space split and now becomes

$$\begin{aligned} & \det\gamma_{ij} = (\lambda^2 + B_c B^c)\det\sigma_{ab} + B_2\mathcal{C}_{12} + B_3\mathcal{C}_{13} = (\lambda^2 + B_c B^c)\det\sigma_{ab} + B_2\gamma^{21}\det\gamma_{ij} + B_3\gamma^{31}\det\gamma_{ij} \\ \Rightarrow\quad & \det\gamma_{ij}\big(1 - B_2\gamma^{21} - B_3\gamma^{31}\big) = \det\gamma_{ij}\big(1 + B_2\lambda^{-2}B^2 + B_3\lambda^{-2}B^3\big) = \big(\lambda^2 + B_c B^c\big)\det\sigma_{ab} \\ \Rightarrow\quad & \det\gamma_{ij} = \lambda^2\det\sigma_{ab}\,. \end{aligned} \tag{13.19}$$

13.2 Gauss law in the 3+1 split

Our eventual flux integration will make use of Gauss' law to replace the volume integral of a divergence in terms of a surface integral. We therefore consider how this can be implemented in terms of the 3+1 split. Let us consider for this purpose a flux vector J^α that obeys the conservation law

$$\nabla_\mu J^\mu = 0\,. \tag{13.20}$$

Clearly, this implies

$$\int \nabla_\mu J^\mu \sqrt{-g}\,\mathrm{d}^4x = 0\,. \tag{13.21}$$

First, we follow Katy's work in Ref. [2] and rewrite this conservation law in the form of coordinate integrals by introducing the tensor density

$$\tilde{J}^\mu := \sqrt{-g}J^\mu\,. \tag{13.22}$$

For the Levi-Civita connection, we have the general result

$$\partial_\alpha\sqrt{-g} = \sqrt{-g}\,\Gamma^\mu_{\alpha\mu}\,, \tag{13.23}$$

which we derive, for instance in Eq. (4.50) of the `ccz4.pdf` notes. This allows us to rewrite the conservation law (13.21) in a form involving only partial derivatives,

$$\nabla_\mu J^\mu = \partial_\mu J^\mu + \Gamma^\mu_{\rho\mu}J^\rho = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}J^\mu) = \frac{1}{\sqrt{-g}}\partial_\mu\tilde{J}^\mu = 0\,, \tag{13.24}$$

or $\partial_\mu\tilde{J}^\mu = 0$ for short. The benefit of doing this consists in applying Gauss' law directly without having to worry about the fact that we operate in a curved spacetime. Of course, the density weight introduces factors of the determinant of the metric in exactly the same form as they would appear in the integration of differential forms in the integration on curved manifolds. The simple conservation law $\partial_\mu\tilde{J}^\mu = 0$ allows us to write

$$\partial_0\tilde{J}^0 = -\partial_i\tilde{J}^i\,, \tag{13.25}$$

and we can now consider the integral of $\tilde{J}^0$ over some region $V \subset \Sigma$ of a spatial hypersurface,

$$\partial_0\int_V \tilde{J}^0\mathrm{d}^3x = \int_V \partial_0\tilde{J}^0\mathrm{d}^3x = \int_V -\partial_i\tilde{J}^i\mathrm{d}^3x = -\int_{\partial V}\tilde{J}^i\tilde{N}_i d^2x\,. \tag{13.26}$$

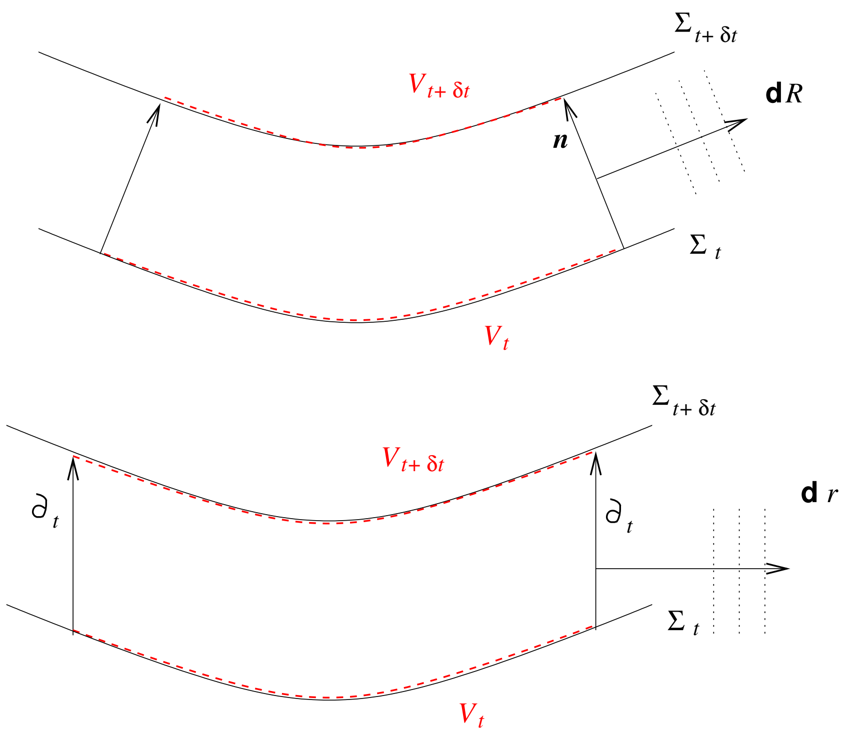

Figure 3: Two spatial hypersurfaces Σ_t and $\Sigma_{t+\delta t}$ are shown together with the unit timelike normal $\boldsymbol{n}$ (upper panel) and the coordinate time vector $\boldsymbol{\partial}_t$ (lower panel). The radial spacetime coordinate r and the spatial radius R denote identical points on the hypersurface Σ_t, but their associated 1-forms $\mathbf{d}r$ and $\mathbf{d}R$ do not even live in the same cotangent spaces; $\mathbf{d}r$ is a four-dimensional tensor and $\mathbf{d}R$ a three-dimensional one. When we evaluate the rate of change of volume integrals over V_t, we need to exercise care in choosing the correct radial 1-forms. We also need to make clear which observers we consider. Observers moving along $\boldsymbol{n}$ or $\boldsymbol{\partial}_t$ will consider different volumes at time $t+\delta t$ and therefore conclude different rates of change. We are interested in Eulerian observers moving along $\boldsymbol{\partial}_t$.

Here, $\tilde{N}_i$ is the outgoing normal of the surface ∂V normalized with a flat metric. In terms of the adapted coordinates (R, x^a) of our 2+1 split for surfaces of constant radius, we simply have

$$\tilde{\boldsymbol{N}} = \mathbf{d}R\,. \tag{13.27}$$

At this point, we need to voice a warning. We are totally free to apply both the 3+1 and a subsequent 2+1 split to obtain coordinates (t, r, x^a) adapted to a space-time-radial $2+1+1$ split. On the three-dimensional hypersurface Σ_t, the coordinates r and R are even identical in the sense that the point (R, x^a) is the same point as (t, r, x^a) in the spacetime $\mathcal{M}$. However, the 1-forms associated with r and R are not the same, they are not even unequal, since $\mathbf{d}r$ lives in the cotangent space $\mathcal{T}^*(\mathcal{M})$ and $\mathbf{d}R$ in the cotangent space $\mathcal{T}^*(\Sigma_t)$. This is the reason why we have used the capital R to denote the radial coordinate on Σ_t. We illustrate the difference of the one-forms in Fig. 3.

13.3 The flux law viewed from the spatial hypersurface

Let us now generalize the conservation law (13.20) by allowing for a sourceterm s,

$$\nabla_\mu J^\mu = \frac{1}{\sqrt{-g}}\partial_\mu \tilde{J}^\mu = s\,. \tag{13.28}$$

The derivation of the integral equation proceeds in complete analogy to that of Eq. (13.26),

$$\begin{aligned}
&\partial_0 \tilde{J}^0 = -\partial_i \tilde{J}^i + \sqrt{-g}s \\
\Rightarrow\quad &\partial_0 \int_V \tilde{J}^0 \mathrm{d}^3x = \int_V -\partial_i \tilde{J}^i \mathrm{d}^3x + \int_V \sqrt{-g}s\mathrm{d}^3x = -\int_{\partial V} \tilde{J}^i(\mathbf{d}R)_i \mathrm{d}^2x + \int_V \alpha\sqrt{\gamma}s\mathrm{d}^3x\,.
\end{aligned} \tag{13.29}$$

Here we have used that Eq. (13.27) for the outgoing coordinate unit normal $\tilde{\boldsymbol{N}}$ and Eq. (13.10) for the 3+1 split of $\det g_{\alpha\beta}$. Note that our application of Gauss' law here is a purely spatial, three-dimensional operation; we do not even "know" that there exists a four-dimensional spacetime. For this reason, we use the radius R and its associated 1-form $\mathbf{d}R$. Using Eqs. (13.10) and (13.19) for the decomposition of the metric determinants, our integration equation (13.29) then becomes (recall $\tilde{J}^\mu = \sqrt{-g}J^\mu$),

$$\begin{aligned}
&\partial_0 \int_V \alpha J^0 \sqrt{\gamma}\mathrm{d}^3x = -\int_{\partial V} \alpha\sqrt{\gamma}\frac{J^i N_i}{\lambda}\mathrm{d}^2x + \int_V \alpha s\sqrt{\gamma}\mathrm{d}^3x \qquad \Big|\ \gamma = \lambda^2\sigma\,,\quad n_\mu = -\alpha\langle \mathbf{d}t, \boldsymbol{J}\rangle = -\alpha J^0 \\
\Rightarrow\quad &-\partial_0 \int_V n_\mu J^\mu \sqrt{\gamma}\mathrm{d}^3x = -\int_{\partial V} \alpha\sqrt{\sigma}N_i J^i d^2x + \int_V \alpha s\sqrt{\gamma}\mathrm{d}^3x\,.
\end{aligned} \tag{13.30}$$

Following Robin's definitions, we can summarize this equation as

$$\boxed{\begin{aligned}
\mathcal{Q} &:= n_\mu J^\mu\,, \\
\mathcal{F} &:= \alpha N_i J^i\,, \\
\mathcal{S} &:= \alpha s\,, \\
\partial_0 \int_V \mathcal{Q}\sqrt{\gamma}\mathrm{d}^3x &= \int_{\partial V} \mathcal{F}\sqrt{\sigma}\mathrm{d}^2x + \int_V \mathcal{S}\sqrt{\gamma}\mathrm{d}^3x\,. \qquad (13.31)
\end{aligned}}$$

This appears to differ from Robin's Eq. (2), since the flux term $\mathcal{F}$ appears to be missing a factor $\sqrt{g^{rr}/\gamma^{rr}}$. We will see in the next section that this apparent discrepancy arises from the fact that Robin defines his flux in terms of a different radial 1-form, namely the one we call k_μ in the next section. We will leave the details on how to reconcile the two expressions until we have completed this alternative derivation, but note two subtle points related to this issue.

- The densitized components $\tilde{J}^i$ still carry the entire determinant factor $\sqrt{-g}$.
- The components J^i in Eq. (13.31) are the spatial components of the spacetime vector J^μ. They are in general **not** equal to the components of the spatial projection $\perp\boldsymbol{J}$. Using $n_\mu = -\alpha(\mathbf{d}t)_\mu$ and $n^\mu = \left(\frac{1}{\alpha}, -\frac{\beta^i}{\alpha}\right)$, we find

  $$(\perp\boldsymbol{J})^i = \perp^i{}_\mu J^\mu J^i + n^i n_\mu J^\mu = J^i + n^i(-\alpha)J^0 = J^i + \beta^i\,, \tag{13.32}$$

  and, in particular, $(\perp J)^r = J^r + \beta^r$ if we use radially adapted coordinates.

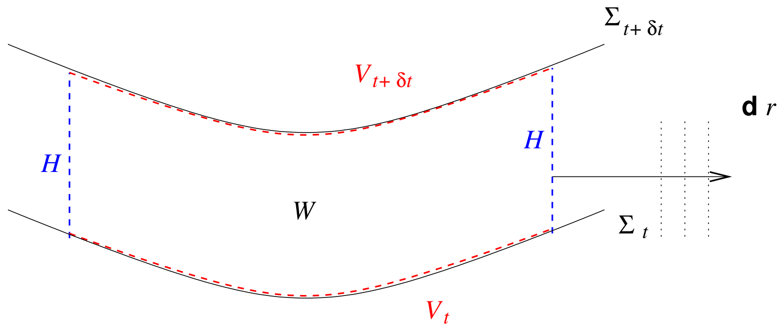

Figure 4: We graphically illustrate the four-dimensional volume $W \subset \mathcal{M}$ bounded by two spatial surfaces V_t and $V_{t+\delta t}$ in time and by the cylinder H of constant radius r. $\mathbf{d}r$ is the outgoing radial one form. Note that the time coordinate basis vector $\boldsymbol{\partial}_t$ of the bottom panel of Fig. 3 points along H where as the unit timelike normal $\boldsymbol{n}$ of the upper panel of Fig. 3 will in general not be tangential to H.

13.4 Eulerian observers in a spacetime approach

It is probably clearer to evaluate the rate of change of the volume integral on the left-hand side of Eq. (13.29) in a spacetime approach. For this purpose, let us first try and get some order into our notation. We use the following symbols.

- We denote the 4D spacetime by $\mathcal{M}$ and a 4D volume in this spacetime by W.
- Σ_t is the spatial hypersurface of constant time t, V_t is a bounded volume inside Σ_t, and ∂V_t is the 2D surface of constant time t and radius r.
- Finally, we let H denote the hyper cylinder made up of the various ∂V. In other words, H is the 3D surface $r = \text{const}$ in the time interval $[t, t+\delta t]$ under consideration.
- As before, $g_{\alpha\beta}$, γ_{ij} and σ_{ab} denote the spacetime metric, the spatial 3-metric on Σ_t and the induced 2-metric on the surface $t = \text{const}$, $r = \text{const}$.

We graphically illustrate this in Fig. 4. As before, we can write the flux conservation law in terms of tensor densities according to Eq. (13.28),

$$\partial_\mu \tilde{J}^\mu = \sqrt{-g}s \qquad \Rightarrow \qquad \int_W \partial_u \tilde{J}^\mu d^4x = \int_W s\sqrt{-g}\, \mathrm{d}^4x\,, \tag{13.33}$$

where once again our integration happens in coordinate space. This not only makes the analogy to the flat-space version of Gauss' law clearer, but also helps us in identifying minus signs that occur in Gauss' law when we consider spacelike hypersurfaces in a Lorentzian spacetime.

Considering the volume W and its surface in Fig. 4, we see that W is bounded by V_t, $V_{t+\delta t}$ and the hyper cylinder H. Gauss' law in coordinate space gives us

$$\int_W \partial_\mu \tilde{J}^\mu \mathrm{d}^4x = \int_W s\sqrt{-g}\, \mathrm{d}^4x \qquad \Rightarrow \qquad \oint_{\partial W} \tilde{J}^\mu \tilde{N}_\mu \mathrm{d}^3x = \int_W s\sqrt{-g}\, \mathrm{d}^4x\,,$$

where $\partial W = V_t \cup V_{t+\delta t} \cup H$, and $\tilde{N}_\mu$ denotes the outgoing coordinate unit one-form on the respective boundary sets. We decompose the left-hand side of this equation into three contributions

$$\oint_{\partial W} \tilde{J}^\mu \tilde{N}_\mu \mathrm{d}^3x = \underbrace{\int_{V_t} \tilde{J}^\mu \tilde{N}_\mu \mathrm{d}^3x}_{=:I_1} + \underbrace{\int_{V_{t+\delta t}} \tilde{J}^\mu \tilde{N}_\mu \mathrm{d}^3x}_{=:I_2} + \underbrace{\int_H \tilde{J}^\mu \tilde{N}_\mu \mathrm{d}^3x}_{=:I_3}, \tag{13.34}$$

and use coordinates adapted to the 3+1 split and with the radial coordinate that defines H via $r = \text{const}$.

(1) Let us start with I_1 on V_t. The outgoing coordinate normal in this case is $\tilde{\boldsymbol{N}} = -\mathbf{d}t$, since the outgoing direction points to the past on V_t. Clearly, the components are $\tilde{N}_\mu = (-\mathbf{d}t)_\mu = (-1, 0, 0, 0)$, so that $\tilde{J}^\mu \tilde{N}_\mu = -\tilde{J}^0$. We then obtain directly

$$I_1 = \int_{V_t} \tilde{J}^\mu \tilde{N}_\mu \mathrm{d}^3x = \int_{V_t} \tilde{J}^\mu (-\mathbf{d}t)_\mu \mathrm{d}^3x = \int_{V_t} J^\mu \alpha \sqrt{\gamma} (-\mathbf{d}t)_\mu \mathrm{d}^3x$$

$$\Rightarrow \quad I_1 = \int_{V_t} J^\mu n_\mu \sqrt{\gamma}\, \mathrm{d}^3x\,, \tag{13.35}$$

where we have used $\boldsymbol{n} = -\alpha \mathbf{d}t$ in the last step.

(2) The integral I_2 is evaluated in the same way and differs from I_1 only in two ways. First, the outgoing coordinate normal now points to the future, so that $\tilde{\boldsymbol{N}} = +\mathbf{d}t$ and, second, the integral is now evaluated over $V_{t+\delta t}$, so that

$$I_2 = -\int_{V_t+\delta t} J^\mu n_\mu \sqrt{\gamma}\, \mathrm{d}^3x\,. \tag{13.36}$$

(3) For the integral over H, the outgoing coordinate normal is $\tilde{\boldsymbol{N}} = \mathbf{d}r$. In analogy to the lapse function, we define the norm of this one form and the radial unit normal one-form

$$\mu := \frac{1}{||\mathbf{d}r||} = \frac{1}{\sqrt{\boldsymbol{g}(\mathbf{d}r, \mathbf{d}r)}} = \frac{1}{\sqrt{g^{rr}}}\,, \qquad \boldsymbol{k} := \mu \mathbf{d}r\,. \tag{13.37}$$

To linear order in δt, we can approximate this integral by

$$I_3 = \int_H \tilde{J}^\mu \tilde{N}_\mu \mathrm{d}^3x \approx \delta t \int_{\partial V_t} \tilde{J}^\mu \tilde{N}_\mu \mathrm{d}^2x\,. \tag{13.38}$$

Note that this is an integral in coordinate space and that all complicating features arising from the underlying curved manifold are safely contained in the density weights of the integrand. Now we return to tensors proper and obtain

$$I_3 \approx \delta t \int_{\partial V_t} \alpha \sqrt{\gamma} \frac{1}{\mu} k_\mu\, \mathrm{d}^2x\,. \tag{13.39}$$

Next, we recall from the first identity in Eq. (13.11) that $\lambda = 1/\sqrt{\gamma^{rr}}$ and from Eq. (13.19) that $\gamma = \lambda^2 \sigma$, so that

$$I_3 \approx \delta t \int_{\partial V_t} \alpha \frac{\lambda}{\mu} J^\mu k_\mu \sqrt{\sigma}\, \mathrm{d}^2x = \delta t \int_{\partial V_t} \alpha \frac{\sqrt{g^{rr}}}{\sqrt{\gamma^{rr}}} J^\mu k_\mu \sqrt{\sigma}\, \mathrm{d}^2x\,, \tag{13.40}$$

which is exactly the radial flux integral that Robin has obtained. There remain three subtleties, however, that we need to bear in mind when evaluating this integral in practice. First, we work with spatial vectors in our codes, not with four vectors like k_μ. In adapted coordinates, we obtain

$$
\begin{aligned}
& n_\mu = (-\alpha \mathbf{d}t)_\mu = (-\alpha,\, 0,\, 0,\, 0)\,, \qquad\qquad k_\mu = (\mu \mathbf{d}r)_\mu = (0,\, \mu,\, 0,\, 0) \\
\Rightarrow\quad & k_i = \perp^\mu{}_i k_\mu = \mu \perp^r{}_i = \mu(\delta^r{}_i + n^r \underbrace{n_i}_{=0}) = (\mu,\, 0,\, 0) \\
\Rightarrow\quad & J^\mu k_\mu = J^i k_i\,.
\end{aligned}
\tag{13.41}
$$

The second point is that J^i are not identical to the components of the spatial vector $(\perp J)^i$,

$$
\begin{aligned}
& \perp J^i = \perp^i{}_\mu J^\mu = (\delta^i{}_\mu + n^i n_\mu) J^\mu = J^i + n^i(-\alpha) J^0 \qquad \Big| \quad n^\mu = \left(\alpha^{-1},\, -\alpha^{-1}\beta^i\right) \\
\Rightarrow\quad & \perp J^i = J^i + \beta^i \\
\Rightarrow\quad & J^\mu k_\mu = J^i k_i = (\perp J)^i k_i - \beta^i k_i\,.
\end{aligned}
\tag{13.42}
$$

The third subtlety is more a trap than a subtle point: when we compute the components k_i of the radial unit normal, it is all too tempting to do so using the spatial metric γ_{ij}. That is not correct, however, since $\mu = 1/\sqrt{g^{rr}}$ and

$$
g^{rr} = \gamma^{rr} - \alpha^{-2}\beta^r\beta^r\,. \tag{13.43}
$$

We therefore also need to include the lapse and shift terms when we compute k_i in our simulations.

I am not sure the following lines will be helpful, but there seems no harm in including them. We can slightly simplify the final expression by by considering the complete integrand of I_3,

$$
\alpha \frac{\sqrt{g^{rr}}}{\sqrt{\gamma^{rr}}} \sqrt{\sigma} J^i k_i = \alpha \frac{\sqrt{g^{rr}}}{\sqrt{\gamma^{rr}}} \sqrt{\sigma} J^r \mu \overset{!}{=} \alpha \frac{\sqrt{g^{rr}}}{\sqrt{\gamma^{rr}}} \sqrt{\sigma} J^r \frac{1}{\sqrt{g^{rr}}} = \alpha \frac{J^r}{\sqrt{\gamma^{rr}}} \sqrt{\sigma}\,, \tag{13.44}
$$

which gives us the equivalent but simpler version

$$
I_3 \approx \delta t \int_{\partial V_t} \alpha \frac{J^r}{\sqrt{\gamma^{rr}}} \sqrt{\sigma} \mathrm{d}^2 x\,. \tag{13.45}
$$

(4) Of course, we also need to evaluate the right-hand side in our integral equation (13.34). We again approximate this to linear order in δt and define

$$
I_4 := \int_W s\sqrt{-g}\, \mathrm{d}^4 x \approx \delta t \int_{V_t} \alpha s \sqrt{\gamma}\, \mathrm{d}^3 x\,. \tag{13.46}
$$

Putting all together, Eq. (13.33) becomes

$$
\begin{aligned}
& I_1 + I_2 + I_3 = I_4 \\
\Rightarrow\quad & \int_{V_t} J^\mu n_\mu \sqrt{\gamma}\,\mathrm{d}^3x - \int_{V_{t+\delta t}} J^\mu n_\mu \sqrt{\gamma}\,\mathrm{d}^3x + \delta t \int_{\partial V_t} \alpha \frac{\sqrt{g^{rr}}}{\sqrt{\gamma^{rr}}} J^\mu k_\mu \sqrt{\sigma}\,\mathrm{d}^2x = \delta t \int_{V_t} \alpha s \sqrt{\gamma}\,\mathrm{d}^3x \\
\Rightarrow\quad & \partial_t \int_{V_t} \underbrace{J^\mu n_\mu}_{=:\mathcal{Q}} \sqrt{\gamma}\,\mathrm{d}^3x = \int_{\partial V_t} \underbrace{\alpha \frac{\sqrt{g^{rr}}}{\sqrt{\gamma^{rr}}} J^\mu k_\mu}_{=:\mathcal{F}} \sqrt{\sigma}\,\mathrm{d}^2x - \int_{V_t} \underbrace{\alpha s}_{=:\mathcal{S}} \sqrt{\gamma}\,\mathrm{d}^3x\,, \qquad (13.47)
\end{aligned}
$$

where we added Robin's definition of his variables $\mathcal{Q}$, $\mathcal{F}$ and $\mathcal{S}$ to emphasize that our equation for the conservation law is identical to his version.

In summary, we have now obtained the following conservation law, which is identical to Robin's Eq. (2) except that he denotes our V_t by S.

$$
\begin{aligned}
\mathcal{Q} &:= n_\mu J^\mu\,, \\
\mathcal{F} &:= \alpha \frac{\sqrt{g^{rr}}}{\sqrt{\gamma^{rr}}} k_\mu J^\mu\,, \\
\mathcal{S} &:= \alpha s\,, \\
\partial_t \int_{V_t} \mathcal{Q}\sqrt{\gamma}\,\mathrm{d}^3x &= \int_{\partial V_t} \mathcal{F}\sqrt{\sigma}\,\mathrm{d}^2x - \int_{V_t} \mathcal{S}\sqrt{\gamma}\,\mathrm{d}^3x\,. \qquad (13.48)
\end{aligned}
$$

In order to see that this is indeed equivalent to Eq. (13.31), we recall that the spatial unit normal was $\boldsymbol{N} = \lambda \mathbf{d}R = (1/\sqrt{\gamma^r r})\mathbf{d}R$, so that

$$
N_i J^i = \frac{1}{\sqrt{\gamma^{rr}}} J^R = \frac{1}{\sqrt{\gamma^{rr}}} J^r\,. \qquad (13.49)
$$

Note that we $J^R = J^r$, because R and r here merely denote the same radial coordinate on Σ_T, but that J^r is the radial component of the four-dimensional J^μ. In the spacetime approach, we have $\boldsymbol{k} = \mu \mathbf{d}r = (1/\sqrt{g^{rr}})\mathbf{d}r$, so that

$$
k_\mu J^\mu = \frac{1}{\sqrt{g^{rr}}} J^r \qquad \Rightarrow \qquad N_i J^i = \frac{\sqrt{g^{rr}}}{\sqrt{\gamma^{rr}}} k_\mu J^\mu\,, \qquad (13.50)
$$

and Eqs. (13.31) and (13.48) indeed are the same.

14 The Noether current

The Noether current associated with a complex scalar field gives us an important diagnostic in analyzing boson star simulations. The Noether current J^μ and the Noether charge N are defined in terms of the complex scalar field $\varphi = \varphi_R + \mathrm{i}\varphi_I$ by

$$
J^\mu := \frac{\mathrm{i}}{2} g^{\mu\nu} (\bar{\varphi}\partial_\nu\varphi - \varphi\partial_\nu\bar{\varphi}) \qquad (14.1)
$$

$$
N := -\int_\Sigma d^3x\,\sqrt{\gamma} n_\mu J^\mu\,, \qquad (14.2)
$$

where $\gamma = \det\gamma_{ij}$ and Σ is a spatial hypersurface with timelike unit normal n_μ. Recalling the definition (8.3) of the momentum of the scalar field, $K_\varphi = -\frac{1}{2}n^\mu\partial_\mu\varphi$, we can express the Noether charge in terms of the real and imaginary part of the scalar field as

$$\begin{aligned}
N &= -\int_\Sigma d^3x\,\sqrt{\gamma}\left(\frac{\mathrm{i}}{2}n_\mu g^{\mu\nu}\bar{\varphi}\,\partial_\nu\varphi - \frac{\mathrm{i}}{2}n_\mu g^{\mu\nu}\varphi\,\partial_\nu\bar{\varphi}\right) = -\int_\Sigma d^3x\sqrt{\gamma}\left[\frac{\mathrm{i}}{2}\bar{\varphi}(-2K_\varphi) - \frac{\mathrm{i}}{2}\varphi(-2\bar{K}_\varphi)\right] \\
&= \int_\Sigma d^3x\sqrt{\gamma}\,\mathrm{i}(\bar{\varphi}K_\varphi - \varphi\bar{K}_\varphi) \qquad \Big|\ \gamma = \det\gamma_{ij} = \chi^3\det\tilde{\gamma}_{ij} = \chi^3 \\
&= \int_\Sigma d^3x\chi^{3/2}\,\mathrm{i}\big[(\varphi_R - \mathrm{i}\varphi_I)(K_{\varphi R} + \mathrm{i}K_{\varphi I}) - (\varphi_R + \mathrm{i}\varphi_I)(K_{\varphi R} - \mathrm{i}K_{\varphi I})\big] \\
&= \int_\Sigma d^3x\chi^{3/2}\,\mathrm{i}\big[\underset{\sim}{\varphi_R K_{\varphi R}} - \mathrm{i}\varphi_I K_{\varphi R} + \mathrm{i}\varphi_R K_{\varphi I} + \underset{\sim}{\varphi_I K_{\varphi I}} - (\underset{\sim}{\varphi_R K_{\varphi R}} + \mathrm{i}\varphi_I K_{\varphi R} - \mathrm{i}\varphi_R K_{\varphi I} + \underset{\sim}{\varphi_I K_{\varphi I}}\big] \\
&= 2\int_\Sigma d^3x\chi^{3/2}(\varphi_I K_{\varphi R} - \varphi_R K_{\varphi I})\,. \qquad (14.3)
\end{aligned}$$

The spatial and time projections of the Noether current are defined by

$$\boxed{\begin{aligned}
&Q := -n_\mu J^\mu\,, \qquad Q^\alpha := \perp^\alpha{}_\mu J^\mu \\
\Rightarrow\ &Qn^\alpha + Q^\alpha = -n_\mu J^\mu n^\alpha + (\delta^\alpha{}_\mu + n^\alpha n_\mu)J^\mu = J^\alpha\,. \qquad (14.4)
\end{aligned}}$$

14.1 Continuity of the Noether current

The evolution equation (1.14) of the scalar field, $\nabla^\mu\nabla_\mu\varphi = V'\varphi$, implies that the Noether current obeys a continuity equation,

$$\begin{aligned}
\nabla_\mu J^\mu &= \frac{\mathrm{i}}{2}g^{\mu\nu}\nabla_\mu(\bar{\varphi}\partial_\nu\varphi - \varphi\partial_\nu\bar{\varphi}) = \frac{\mathrm{i}}{2}g^{\mu\nu}\big(\partial_\mu\bar{\varphi}\,\partial_\nu\varphi + \bar{\varphi}\nabla_\mu\partial_\nu\varphi - \partial_\mu\,\partial_\nu\bar{\varphi} - \varphi\nabla_\mu\partial_\nu\bar{\varphi}\big) \\
&= \frac{\mathrm{i}}{2}g^{\mu\nu}\big(\bar{\varphi}\nabla_\mu\partial_\nu\varphi - \varphi\nabla_\mu\partial_\nu\bar{\varphi}\big) = \frac{\mathrm{i}}{2}\bar{\varphi}V'\varphi - \frac{\mathrm{i}}{2}\varphi V'\bar{\varphi} = 0\,. \qquad (14.5)
\end{aligned}$$

The Noether current is therefore conserved according to

$$\boxed{\nabla_\mu J^\mu = 0\,. \qquad (14.6)}$$

Let us now apply a 3+1 decomposition of this continuity equation. Using the familiar 3+1 expressions

$$\nabla_\alpha n_\beta = -K_{\alpha\beta} - n_\alpha a_\beta\,, \qquad a_\mu := n^\rho\nabla_\rho n^\mu = \frac{\partial_\mu\alpha}{\alpha}\,, \qquad (14.7)$$

we find (recall that $n_\mu a^\mu = 0$) that

$$\begin{aligned}
D_\mu J_\nu &:= \perp^\alpha{}_\mu \perp^\beta{}_\nu \nabla_\alpha J_\beta = (\delta^\alpha{}_\mu + n^\alpha n_\mu)(\delta^\beta{}_\nu + n^\beta n_\nu)\nabla_\alpha J_\beta \\
&= \nabla_\mu J_\nu + n^\alpha n_\mu \nabla_\alpha J_\nu + n^\beta n_\nu \nabla_\mu J_\beta + n^\alpha n_\mu n^\beta n_\nu \nabla_\alpha J_\beta \\
\Rightarrow \quad g^{\mu\nu} D_\mu J_\nu = \gamma^{\mu\nu} D_\mu J_\nu &= (g^{\mu\nu} + n^\mu n^\nu)\nabla_\mu J_\nu + 0 = \underbrace{\nabla_\mu J^\mu}_{=0} + n^\mu n^\nu \nabla_\mu J_\nu \\
&= n^\mu \big[\nabla_\mu (n^\nu J_\nu) - J_\nu \nabla_\mu n^\nu\big] = n^\mu \big[-\nabla_\mu Q - J_\nu(-K_\mu{}^\nu - n_\mu a^\nu)\big] \\
&= -n^\mu \nabla_\mu Q + 0 - J_\nu a^\nu = -n^\mu \nabla_\mu Q - Q_\nu a^\nu \,,
\end{aligned} \tag{14.8}$$

where we have used $J_\nu a^\nu = J_\nu(\perp^\nu{}_\rho a^\rho) = J_\rho a^\rho$ in the last step. Now we are at the point of a trap we have encountered on other occasions already, as for example in the derivation of the CCZ4 equations. Contrary to what one might naively think, in general

$$D_\mu Q_\nu \;=\; D_\mu(\perp^\rho{}_\nu J_\rho) \;\overset{!}{\neq}\; D_\mu J_\nu \,. \tag{14.9}$$

Instead, we find

$$\begin{aligned}
D_\mu J_\nu &= D_\mu(Q_\nu + Q n_\nu) = D_\mu Q_\nu + D_\mu(Q n_\nu) = D_\mu Q_\nu + Q D_\mu n_\nu + n_\nu D_\mu Q \\
\Rightarrow \quad g^{\mu\nu} D_\mu J_\nu &= D_\mu Q^\mu + g^{\mu\nu}\rho D_\mu n_\nu + 0 = D_\mu Q^\mu + g^{\mu\nu}\rho \perp^\alpha{}_\mu \perp^\beta{}_\nu \nabla_\alpha n_\beta \\
&= D_\mu Q^\mu + g^{\mu\nu}\rho \perp^\alpha{}_\mu \perp^\beta{}_\nu (-K_{\alpha\beta} - n_\alpha a_\beta) = D_\mu Q^\mu - KQ \,.
\end{aligned} \tag{14.10}$$

Combining Eqs. (14.8) and (14.10), we can write the continuity equation as

$$D_\mu Q^\mu - KQ + n^\mu \nabla_\mu Q + Q^\mu a_\mu = 0 \,. \tag{14.11}$$

Next, we use the Lie derivative of Q along $\boldsymbol{n} = (\boldsymbol{\partial}_t - \boldsymbol{\beta})/\alpha$,

$$\mathcal{L}_{\boldsymbol{n}} Q = n^\mu \nabla_\mu Q = \frac{1}{\alpha}\partial_t Q - \frac{1}{\alpha}\mathcal{L}_{\boldsymbol{\beta}} Q = \frac{1}{\alpha}\partial_t Q - \frac{\beta^m}{\alpha}\partial_m Q \,, \tag{14.12}$$

where we have switched to coordinates adapted to the 3+1 split in the last equality. Combining this with Eq. (14.11), we obtain the continuity equation in 3+1 form,

$$\boxed{\partial_t Q = \beta^m \partial_m Q - Q^m \partial_m \alpha - \alpha(D_m Q^m - KQ) = \beta^m \partial_m Q - D_m(\alpha Q^m) + \alpha K Q \,. \qquad (14.13)}$$

14.2 The Noether charge

In terms of the 3+1 variables, we can write the Noether charge from Eq. (14.2) as

$$N = \int_V d^3x \sqrt{\gamma} Q \,, \tag{14.14}$$

where we now integrate over a volume V inside the hypersurface Σ. We wish to calculate how this charge changes in time. Recalling the evolution equation of the 3-metric (written here with the covariant spatial derivative) and the derivative of the metric determinant,

$$\partial_t \gamma_{ij} \;=\; \beta^m D_m \gamma_{ij} + \gamma_{mj} D_i \beta^m + \gamma_{mi} D_j \beta^m - 2\alpha K_{ij} \,, \tag{14.15}$$

$$\partial_t \sqrt{\gamma} \;=\; \frac{1}{2}\sqrt{\gamma}\,\gamma^{ij}\partial_t \gamma_{ij} \,, \tag{14.16}$$

we find

$$\begin{aligned}
\partial_t(\sqrt{\gamma}Q) &= Q\partial_t\sqrt{\gamma} + \sqrt{\gamma}\partial_t Q \overset{(14.13)}{=} \frac{1}{2}Q\sqrt{\gamma}\,\gamma^{ij}\partial_t\gamma_{ij} + \sqrt{\gamma}\,\beta^m\partial_m Q - \sqrt{\gamma}\,D_m(\alpha Q^m) + \sqrt{\gamma}\,\alpha K Q \\
&= \sqrt{\gamma}\left[\frac{Q}{2}\beta^m \underbrace{D_m(\gamma^{ij}\gamma_{ij})}_{=0} + \frac{Q}{2}D_m\beta^m + \frac{Q}{2}D_m\beta^m + \frac{Q}{2}\gamma^{ij}(-2\alpha K_{ij}) + \beta^m\partial_m Q - D_m(\alpha Q^m) + \alpha K Q\right] \\
&= \sqrt{\gamma}\left[Q D_m\beta^m + \beta^m\partial_m Q - D_m(\alpha Q^m)\right] = \sqrt{\gamma}D_m(Q\beta^m - \alpha Q^m)\,, \qquad (14.17)
\end{aligned}$$

which leads to the final expression

$$\partial_t(\sqrt{\gamma}Q) = \sqrt{\gamma}D_m(Q\beta^m - \alpha Q^m) \qquad (14.18)$$

$$\Rightarrow \quad \partial_t N = \int_V d^3x\,\partial_t\sqrt{\gamma}Q) = \int_V d^3x\,\sqrt{\gamma}D_m(\beta^m Q - \alpha Q^m) = \oint_{\partial V} d^2x\,\sqrt{q}\,s_m(\beta^m Q - \alpha Q^m)\,, \qquad (14.19)$$

where s_i is the spatial, outward unit normal of the surface ∂V and $q_{ij} = \gamma_{ij} - s_i s_j$ is the induced two-metric on ∂V.

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