An outgoing wave scalar $\phi(t, r)$ is recorded as a function of time at the extraction radius, $\phi(t, r_{\rm ex}) = u(t)$. Our task is to find time dependence of the field at larger radii, $\phi(t, r')$ with $r' > r_{\rm ex}$. For massless fields in flat spacetime this is trivial; evaluated at the retarded time, the field scales as $1/r$. The task is complicated by cosmological evolution and a field mass.

Klein-Gordon equation with mass term (I use metric signature $\{- + ++\}$)

$$\Box\phi - m^2\phi = 0 \tag{1}$$

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) - m^2\phi = 0 \tag{2}$$

FRW (spatially flat, $k = 0$) metric written in terms of conformal time coordinate $\mathrm{d}\eta = \mathrm{d}t/a(t)$

$$\mathrm{d}s^2 = a^2(\eta)\left(-\mathrm{d}\eta^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right). \tag{3}$$

Wave equation becomes

$$-\partial_\eta^2\phi - 2\frac{\dot{a}}{a}\partial_\eta\phi + \frac{\partial_r(r^2\partial_r\phi)}{r^2} - a^2m^2\phi = 0 \tag{4}$$

Looking for solutions that decay as $1/r$, substitute $\psi = r\phi$

$$-\partial_\eta^2\psi - 2\frac{\dot{a}}{a}\partial_\eta\psi + \partial_r^2\psi - a^2m^2\psi = 0 \tag{5}$$

Still looking for solutions that decay as $1/r$, substitute $\Phi = a(\eta)\psi$ gives

$$-\partial_\eta^2\Phi + \frac{\ddot{a}}{a}\Phi + \partial_r^2\Phi - a^2m^2\Phi = 0 \tag{6}$$

Case 1: $m = 0$ and $a =$constant

The familiar case of massless field in flat spacetime. Choose coords such that $a = 1$, then Eq. 6 becomes a simple wave equation

$$-\partial_\eta^2\Phi + \partial_r^2\Phi = 0 \tag{7}$$

Outgoing solutions of the form $\Phi(\eta - r)$ and $\eta = t$. Therefore we simply shift and scale.

$$\phi(t, r') = \frac{r_{\rm ex}}{r'}u(t + r_{\rm ex} - r') \tag{8}$$

Case 2: $m = 0$ and $a = (\eta/\eta_0)^2$

Massless field in a matter dominated universe with $a(\eta) = (\eta/\eta_0)^2$, so that $\ddot{a}/a = 2/\eta^2$.

Eq. 6 becomes

$$-\partial_\eta^2 \Phi + 2\eta^{-2}\Phi + \partial_r^2 \Phi = 0 \tag{9}$$

Taking the Fourier transform $\tilde{f}(\omega) = \int_{-\infty}^{\infty} \exp(-i\omega\eta) f(\eta)$ of this equation and approximating η^{-2} as constant over the duration of the signal gives

$$\partial_r^2 \tilde{\Phi} = -(\omega^2 + 2\eta^{-2})\tilde{\Phi} \tag{10}$$

Assuming $\omega\eta \gg 1$, we find approximate solutions of the form $\tilde{\Phi} \sim \exp(\pm i\omega r)$. Any function of the form $\Phi(\eta - r)$ will be an approximate solution provided its Fourier transform has negligible power at frequencies below $\omega \lesssim \eta^{-1}$.

$$\phi(t, r') = \frac{r_{\rm ex}}{a(t) r'} u\left(\frac{t}{a(t)} + r_{\rm ex} - r'\right) \tag{11}$$

Cosmological signals get redshifted as they propagate.

Case 3: $m \neq 0$ and $a =$constant

Massive field in flat spacetime; this is the case we tackled in our Roxana's long 2020 PRD paper; the punchline was that frequencies $\omega < m$ don't propagate and at large radii the remaining parts of the signal with $\omega > m$ get stretched out into an inverse chirp.

Case 4: $m \neq 0$ and $a = (\eta/\eta_0)^2$

1 Massive scalar fields on a FRW background

1.1 The wave equation

The case $m \neq 0$ and $a \neq$ const is the scenario we really are interested in. Let us therefore summarize our findings and notation obtained so far. We consider spatially flat Friedmann-Robertson-Walker spacetimes with metric[1]

$$\boxed{\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2)\,, \qquad (12)}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$ is the usual line element on the unit 2-sphere. We will need the letters θ and ϕ for other variables, however, and therefore quickly forget their presence in $\mathrm{d}\Omega^2$; the angles will play no further role in our discussion, so that they can safely be stored in the oubliette.

We consider the wave equation $\nabla^2\phi = V_{,\phi}$ for a scalar field ϕ with potential $V(\phi) = \frac{1}{2}m^2\phi^2$. For the metric (12), this equation becomes

$$-\partial_t^2\phi - 3\frac{\dot{a}}{a}\partial_t\phi + \frac{1}{a^2r^2}\partial_r(r^2\partial_r\phi) = m^2\phi\,. \tag{13}$$

In order to eliminate the first time derivative $\partial_t\phi$, it turns out helpful to switch to conformal time η defined by

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{1}{a}\,, \quad \frac{\mathrm{d}t}{\mathrm{d}\eta} = a \qquad \Rightarrow \qquad \partial_t = \frac{1}{a}\partial_\eta\,, \quad \partial_\eta = a\partial_t\,. \tag{14}$$

With this redefinition of time, Eq. (13) becomes

$$-\partial_\eta^2\phi - 2\frac{\partial_\eta a}{a}\partial_\eta\phi + \frac{1}{r^2}\partial_r(r^2\partial_r\phi) = m^2a^2\phi\,. \tag{15}$$

In analogy to our study of asymptotically flat spacetimes [?, ?], we work with the rescaled scalar field variable

$$\sigma := ar\phi\,, \tag{16}$$

which recasts the wave equation in a quasi 1+1 dimensional form,

$$-\partial_\eta^2\sigma + \frac{\partial_\eta^2 a}{a}\sigma + \partial_r^2\sigma = m^2a^2\sigma\,. \tag{17}$$

Using the standard definitions of the Hubble and deceleration parameters in cosmology,

$$H := \frac{\dot{a}}{a}\,, \qquad q := -\frac{a\ddot{a}}{\dot{a}^2}\,, \tag{18}$$

[1] Ignoring the k term in $\mathrm{d}r^2/(1-kr^2)$ appears a plausible approximation in our consideration of the local universe, but there remains the possibility that this term completely drops out of the calculation on its own, just like it does in the textbook red shift calculations for FRW universes. We leave an exploration of this point for future work and set $k = 0$ for now.

we can rewrite

$$\frac{\partial_\eta^2 a}{a} = (\partial_t a)^2 + a\partial_t^2 a = H^2(a^2 - a^2 q) = a^2 H^2(1-q) =: B\,. \tag{19}$$

The name B for this function may not be overly inspired, but it should serve conveniently as an abbreviation. We can then write Eq. (17) as

$$\boxed{-\partial_\eta^2\sigma(\eta,r) + B(\eta)\sigma(\eta,r) + \partial_r^2\sigma(\eta,r) = m^2 a(\eta)^2\sigma(\eta,r)\,; \qquad B = a^2H^2(1-q)\,. \quad (20)}$$

We have been quite explicit here with dependence of the individual functions on (conformal) time and/or radius. This issue will become a bit tricky for the functions a and B in a moment, but for now it is important to bear in mind that they will in general vary in time.

1.2 Fourier transforming the wave equation

We use the Fourier transformation in the form

$$\boxed{\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(\eta)e^{\mathrm{i}\omega\eta}\mathrm{d}\eta \qquad \Leftrightarrow \qquad 2\pi f(\eta) = \int_{-\infty}^{\infty}\tilde{f}(\omega)e^{-\mathrm{i}\omega\eta}\mathrm{d}\omega\,. \quad (21)}$$

The time derivative of a function transforms as

$$\mathcal{F}[\partial_\eta f](\omega) = \int_{-\infty}^{\infty}(\partial_\eta f)e^{\mathrm{i}\omega\eta}\mathrm{d}\eta = -\int_{-\infty}^{\infty} f\partial_\eta e^{\mathrm{i}\omega\eta}\mathrm{d}\eta = -\int_{-\infty}^{\infty}\mathrm{i}\omega f e^{\mathrm{i}\omega\eta}\mathrm{d}\eta = -\mathrm{i}\omega\tilde{f}(\omega)\,, \tag{22}$$

so that our wave equation (20) in Fourier space becomes

$$\omega^2\tilde{\sigma}(\omega,r) + \int_{-\infty}^{\infty} B\sigma e^{\mathrm{i}\omega\eta}\mathrm{d}\eta + \partial_r^2\tilde{\sigma}(\omega,r) = \int_{-\infty}^{\infty} a^2m^2\sigma e^{\mathrm{i}\omega\eta}\mathrm{d}\eta\,. \tag{23}$$

We now assume that at any given radius, the duration of the signal σ is short compared to the time scale of cosmological evolution. In other words, the scale factor $a(\eta)$ and its derivatives do not change significantly during the passage of the signal at a detector. This sounds a reasonable assumption even if we consider long-lived signals lasting centuries. While we cannot completely rule out signals with a lifetime of many millions of years, we can safely assume that there will not exist a funding program (let alone a species capable of executing it) of sufficient longevity to target the detection of such a hyperlong GW signal.

In Eq. (23), this allows us to treat the functions a and B as approximately constant throughout the interval of a non-negligible σ. Note, however, that these constants depend on the specific time at which the signal passes through the radial position r. In terms of our coordinate system (ω, r), this results in an implicit r dependence of a and B. We account for this implicit dependence of the respective "constant" factors with a subscript r. Within this approximation, Eq. (23) becomes

$$\partial_r^2\tilde{\sigma}(\omega,r) = -(\omega^2 - a_r^2m^2 + B_r)\tilde{\sigma}(\omega,r)\,. \tag{24}$$

In order to solve this equation, we make the usual Ansatz

$$\tilde{\sigma}(\omega,r) = \tilde{f}(\omega)e^{\mathrm{i}kr}\,, \qquad \text{with} \qquad k = \sqrt{\omega^2 - a_r^2m^2 + B_r}\,. \tag{25}$$

Evaluating the radial derivative, we obtain

$$\partial_r\tilde{\sigma} = \pm\tilde{f}(\omega)\left[\mathrm{i}ke^{\pm\mathrm{i}kr} + \mathrm{i}r(\partial_r k)e^{\pm\mathrm{i}kr}\right] = \pm\mathrm{i}(k + r\partial_r k)e^{\mathrm{i}kr}\tilde{f}(\omega)\,. \tag{26}$$

The $\partial_r k$ term makes our life difficult here and prevents (25) from being a genuine solution to Eq. (24). We overcome this difficulty with a second assumption, namely that $\omega \gg a_r^2m^2 - B_r$. This implies that our signal has negligible power near the critical frequencies $\omega \approx \pm\sqrt{a^2m^2 - B}$, i.e. near the non-propagating threshold. In other words, this assumption implies that our results are valid only for signals lasting much shorter than the distance to the source (measured in time). Even for sources "only" tens of kpc away, this sounds a plausible assumption.

With these assumptions in place, we have reduced the solution to individual wave modes quite similar to the Minkowski case. More specifically, the radial derivatives become

$$\begin{aligned} &\partial_r\tilde{\sigma}(\omega,r) \approx \pm\mathrm{i}ke^{\pm\mathrm{i}kr}\tilde{f}(\omega) \\ \Rightarrow\quad &\partial_r^2\tilde{\sigma}(\omega,r) \approx -k^2e^{\pm\mathrm{i}kr}\tilde{f}(\omega)\,, \end{aligned} \tag{27}$$

and we obtain the general solution for the wave signal (henceforth dropping the subscript r from a and B),

$$\tilde{\sigma}(\omega,r) = \tilde{f}(\omega)e^{\mathrm{i}k(r-r_\mathrm{e})} + \tilde{g}(\omega)e^{-\mathrm{i}k(r-r_\mathrm{e})}\,; \qquad k = \sqrt{\omega^2 - a^2m^2 + B}\,, \tag{28}$$

$$\Rightarrow\quad 2\pi\sigma(\eta,r) = \int_{-\infty}^{\infty}\tilde{f}(\omega)e^{\mathrm{i}[k(r-r_\mathrm{e})-\omega\eta]} + \tilde{g}(\omega)e^{-\mathrm{i}[k(r-r_\mathrm{e})+\omega\eta)]}\mathrm{d}\omega\,. \tag{29}$$

Note that this implies

$$\tilde{\sigma}(\omega,r_\mathrm{e}) = \tilde{f}(\omega) + \tilde{g}(\omega)\,. \tag{30}$$

Before we continue with the evaluation of this integral, it is helpful to consider some properties of the functions involved therein.

First, we define

$$\varrho = r - r_\mathrm{e}\,, \qquad \omega_* = \sqrt{a^2m^2 - B}\,, \tag{31}$$

so that the individual Fourier modes become

$$\tilde{f}(\omega)e^{\mathrm{i}(k\varrho-\omega\eta)}\,, \qquad \tilde{g}(\omega)e^{-\mathrm{i}(k\varrho+\omega\eta)}\,, \tag{32}$$

with $k=\sqrt{\omega^2-\omega_*^2}$. We now have three regimes of the frequency ω,

$$\begin{aligned}
\omega>\omega_*: &\quad k\in\mathbb{R}, \quad \tilde{f}\text{ outgoing}\,, \quad \tilde{g}\text{ ingoing}\,. \\
-\omega_*<\omega<\omega_*: &\quad \mathrm{i}k\in\mathbb{R}, \quad \tilde{f}(\omega)e^{\mathrm{i}(k\varrho-\omega\eta)}\propto e^{-|k|\varrho}\,, \quad \tilde{g}(\omega)e^{-\mathrm{i}(k\varrho+\omega\eta)}\propto e^{|k|\varrho}\,, \\
\omega<-\omega_*: &\quad k\in\mathbb{R}, \quad \tilde{f}\text{ ingoing}\,, \quad \tilde{g}\text{ outgoing}\,.
\end{aligned} \tag{33}$$

Imposing the requirement that our signal is real and finite everywhere (in particular at spatial infinity), we find in complete analogy to the Minkowskian case that

$$\begin{aligned}
\text{for } |\omega|>\omega_*: &\quad \tilde{g}^*(-\omega)=\tilde{f}(\omega) \quad \Leftrightarrow \quad \tilde{f}^*(-\omega)=\tilde{g}(\omega)\,, \\
\text{for } |\omega|<\omega_*: &\quad \tilde{g}(\omega)=0 \quad \wedge \quad \tilde{f}^*(-\omega)=\tilde{f}(\omega)\,.
\end{aligned} \tag{34}$$

We furthermore assume that we do not have a standing wave, so that $\tilde{f}(\pm\omega_*)=\tilde{g}(\pm\omega_*)=0$. Together with these conditions, Eq. (28) gives us the solution to the wave equation (24) subject to the approximation of short signals compared to the timescale $a/\dot{a}$ of the cosmological evolution. Our next task is the inverse Fourier transformation (29).

1.3 Inverse FT and the stationary phase approximation

With the conditions (34), we can write the inverse Fourier transform for the signal $\sigma(\eta,r)$ in the time domain as

$$2\pi\sigma(\eta,r)=\int_\Sigma \tilde{f}(\omega)e^{\mathrm{i}(k\varrho-\omega\eta)}+\tilde{g}(\omega)e^{-\mathrm{i}(k\varrho+\omega\eta)}\mathrm{d}\omega+\int_{\bar{\Sigma}}\tilde{f}(\omega)e^{-|k|\varrho}e^{-\mathrm{i}\omega\eta}\mathrm{d}\omega$$

$$\text{with}\quad \Sigma=(-\infty,-\omega_*]\cup[\omega_*,\infty)\,, \qquad \bar{\Sigma}=(-\omega_*,\omega_*)\,, \qquad k=\sqrt{\omega^2-\omega_*^2} \tag{35}$$

Here, the non-propagating contribution over $\bar{\Sigma}$ decays exponentially with ϱ and therefore becomes negligible at astrophysical distances.

For the propagating contributions, on the other hand, we can evoke the stationary phase approximation (SPA): For small ϵ, integrals of the form

$$I(\epsilon):=\int_{-\infty}^{\infty}A(\omega)e^{\mathrm{i}\vartheta(\omega)/\epsilon}\mathrm{d}\omega\,, \tag{36}$$

are dominated by frequencies ω_0 where $\vartheta'(\omega_0)=0$. All other contributions cancel out due to the rapid oscillation of the function $e^{\mathrm{i}\vartheta(\omega)/epsilon}$. This allows us to Taylor expand

$$\begin{aligned}
\vartheta(\omega) &= \vartheta(\omega_0)+\underbrace{\vartheta'(\omega_0)}_{=0}(\omega-\omega_0)+\frac{1}{2}\vartheta''(\omega_0)(\omega-\omega_0)^2+\dots \\
A(\omega) &= A(\omega_0)+\dots\,,
\end{aligned} \tag{37}$$

and approximate Eq. (36) by

$$I(\epsilon) = \int_{-\infty}^{\infty} A(\omega) e^{\mathrm{i}\vartheta(\omega)/\epsilon} \approx A(\omega_0) e^{\mathrm{i}\vartheta(\omega_0)/\epsilon} \int_{-\infty}^{\infty} \exp\left[\frac{\mathrm{i}\vartheta''(\omega_0)}{2\epsilon} s^2\right] \mathrm{d}s\,, \quad \text{with} \quad s = \omega - \omega_0\,. \tag{38}$$

The last integral can be written in terms of the Fresnel integral and evaluated (after appropriate change of variables) according to

$$\int_{-\infty}^{\infty} e^{\mathrm{i}t^2/\varepsilon} \mathrm{d}t = \sqrt{\frac{\pi|\varepsilon|}{2}}\,[1 + \mathrm{sig}(\varepsilon)]\,. \tag{39}$$

In case our function $\vartheta(\omega)$ has more than one extremum, we simply obtain one contribution of the form (38) for each ω_0 and sum the lot.

Returning to our wave signal (35), we notice two things. First, our integration range is Σ instead of $(-\infty, \infty)$. We can straightforwardly extend our integrals in Eq. (35) to all of $\mathbb{R}$, however, by formally setting $\tilde{f}(\omega) = \tilde{g}(\omega) = 0$ over $\bar{\Sigma}$. Second, we see that the smallness of ϵ in Eq. (36) corresponds to a large radius ϱ in (35)[2]. In order to make this role of the radius more explicit, we rewrite Eq. (35) as

$$2\pi\sigma(t, r) = \int_{-\infty}^{\infty} \left[\tilde{f}(\omega) e^{\mathrm{i}\vartheta(\omega)\varrho} + \tilde{g}(\omega) e^{\mathrm{i}\theta(\omega)\varrho}\right] \mathrm{d}\omega \tag{40}$$

$$\text{with} \quad \vartheta(\omega) = k - \omega\frac{\eta}{\varrho} = k - \frac{\omega}{v}\,, \quad \theta(\omega) = -k - \omega\frac{\eta}{\varrho} = -k - \frac{\omega}{v}\,, \quad k = \sqrt{\omega^2 - \omega_*^2}\,.$$

Here, the velocity $v = \varrho/\eta$ is determined by the spacetime location (η, r) of the observer. Below, we shall need the first two derivatives of θ and ϑ and therefore summarize them here using $\mathrm{d}k/\mathrm{d}\omega = \omega/k$,

$$\vartheta'(\omega) = \frac{\omega}{k} - \frac{1}{v}\,, \qquad \theta'(\omega) = -\frac{\omega}{k} - \frac{1}{v}\,,$$

$$\vartheta''(\omega) = \frac{k^2 - \omega^2}{k^3}\,, \qquad \theta''(\omega) = -\frac{k^2 - \omega^2}{k^3}\,. \tag{41}$$

Given that $k = k(\omega^2)$, we immediately see that

$$\theta(-\omega) = -\vartheta(\omega)\,, \quad \theta'(-\omega) = \vartheta'(\omega)\,, \quad \theta''(-\omega) = -\vartheta''(\omega)\,, \tag{42}$$

which will come in handy in our calculations below.

[2]Strictly speaking, we need a large *dimensionless* quantity for this purpose. This could be achieved, for example, by comparing the radius ϱ with the characteristic wavelength of the scalar radiation. The resulting expressions for the radiated signal are not affected by such a rescaling of the radial coordinate and we are quite satisfied here by noting its existence.

In the spirit of the SPA, we now look for the roots of $\vartheta'(\omega)$ and $\theta'(\omega)$. Clearly, these are given by

$$\vartheta'(\omega)=0 \quad \Rightarrow \quad v\omega=\sqrt{\omega^2-\omega_*^2} \quad \Rightarrow \quad \omega=\mathrm{sig}(v)\,\frac{\omega_*}{\sqrt{1-v^2}}=\mathrm{sig}(v)\,\omega_0\,,$$

$$\theta'(\omega)=0 \quad \Rightarrow \quad -v\omega=\sqrt{\omega^2-\omega_*^2} \quad \Rightarrow \quad \omega=-\mathrm{sig}(v)\,\frac{\omega_*}{\sqrt{1-v^2}}=-\mathrm{sig}(v)\,\omega_0\,, \tag{43}$$

where we have defined

$$\boxed{\omega_0 := \frac{\omega_*}{\sqrt{1-v^2}} \geq 0\,.} \tag{44}$$

Combining this with Eq. (40), we see that for a positive velocity $v=\varrho/t$, we have one contribution from $\tilde{f}(\omega)$ at $\omega=\omega_0$ and one from $\tilde{g}(\omega)$ at $\omega=-\omega_0$. This is in complete agreement with out earlier observation that outgoing radiation is encoded in $\tilde{f}$ for positive and in $\tilde{g}$ for negative frequencies. For $v<0$, it is the other way round and we have ingoing contributions from $\tilde{f}(\omega)$ for $\omega=-\omega_0$ and $\tilde{g}(\omega)$ for $\omega=\omega_0$.

1.4 Evaluating the wave signal at large distance

We have now assembled all ingredients for evaluating the integral (40) and thus compute the wave signal in the time domain at large distance as a function of the initial data $\tilde{f}(\omega)$ and $\tilde{g}(\omega)$. We need to distinguish 4 cases, two of them trivial and the other two nearly identical.

Case 1: $\boldsymbol{v>1}$

Clearly, Eq. (43) has no solutions in this case, since $\vartheta'(\omega)=0$ and $\theta'(\omega)=0$, respectively require

$$v=\pm\frac{\sqrt{\omega^2-\omega_*^2}}{\omega} \quad \Rightarrow \quad |v|<1 \;\text{ for }\; \omega_*>0\,. \tag{45}$$

The interpretation is simply that no signal propagates at superluminal group velocity. We also see that $v=1$ corresponds to the case of a massless scalar with $\omega_*=0$.

Case 2: $\boldsymbol{v<-1}$

Again, there exist no solutions to $\vartheta'(\omega)=0$ or $\theta'(\omega)=0$ and we have no superluminal ingoing signals either. This is a trivial result, but it is still nice that causality is completely borne out by our SPA calculation.

Case 3: $\boldsymbol{0 < v < 1}$

We have two contributions, $\vartheta'(\omega_0) = 0$ and $\theta'(-\omega_0) = 0$, and Eq. (40) with the SPA (38) becomes

$$2\pi\sigma(t,r) = \tilde{f}(\omega_0)e^{\mathrm{i}\vartheta(\omega_0)\varrho}I_f + \tilde{g}(-\omega_0)e^{\mathrm{i}\theta(-\omega_0)\varrho}I_g\,, \tag{46}$$

with

$$I_f = \int_{-\infty}^{\infty} e^{\mathrm{i}(\omega-\omega_0)^2/\varepsilon_f}\mathrm{d}\omega = \int_{-\infty}^{\infty} e^{\mathrm{i}\omega^2/\varepsilon_f}\mathrm{d}\omega \quad \text{with} \quad \varepsilon_f = \frac{2}{\vartheta''(\omega_0)\varrho}\,,$$

$$I_g = \int_{-\infty}^{\infty} e^{\mathrm{i}(\omega+\omega_0)^2/\varepsilon_g}\mathrm{d}\omega = \int_{-\infty}^{\infty} e^{\mathrm{i}\omega^2/\varepsilon_g}\mathrm{d}\omega \quad \text{with} \quad \varepsilon_g = \frac{2}{\theta''(-\omega_0)\varrho}\,. \tag{47}$$

Using

$$\vartheta''(\omega_0) = \frac{-\omega_*^2}{k_0^3}\,, \qquad \theta''(-\omega_0) = \frac{\omega_*^2}{k_0^3}\,, \qquad k_0 := k(\omega_0) = \sqrt{\omega_0^2 - \omega_*^2}\,, \tag{48}$$

$\varepsilon_f = -\varepsilon_g < 0$ and, with the Fresnel integral (39),

$$I_f = \sqrt{\frac{\pi}{2}}\sqrt{\frac{2}{|\vartheta''(\omega_0)|\varrho}}(1-\mathrm{i}) = \sqrt{\frac{\pi k_0^3}{\varrho\omega_*^2}}\,(1-\mathrm{i})\,,$$

$$I_g = \sqrt{\frac{\pi}{2}}\sqrt{\frac{2}{|\theta''(\omega_0)|\varrho}}(1+\mathrm{i}) = \sqrt{\frac{\pi k_0^3}{\varrho\omega_*^2}}\,(1+\mathrm{i})\,. \tag{49}$$

Combining Eqs. (46)-(49), our wave signal in the time domain becomes, using $z + z^* = 2\mathsf{Re}(z)$ for $z \in \mathbb{C}$,

$$\begin{aligned} 2\pi\sigma(\eta,r) &= \tilde{f}(\omega_0)e^{\mathrm{i}\vartheta(\omega_0)\varrho}\sqrt{\frac{\pi k_0^3}{\varrho\omega_*^2}}(1-\mathrm{i}) + \underbrace{\tilde{g}(-\omega_0)}_{=\tilde{f}^*(\omega_0)}\ \underbrace{e^{\mathrm{i}\theta(-\omega_0)\varrho}}_{=e^{-\mathrm{i}\vartheta(\omega_0)\varrho}}\sqrt{\frac{\pi k_0^3}{\varrho\omega_*^2}}(1+\mathrm{i}) \\ &= 2\mathsf{Re}\left\{\tilde{f}(\omega_0)e^{\mathrm{i}\vartheta(\omega_0)\varrho}\sqrt{\frac{\pi k_0^3}{\varrho\omega_*^2}}(1-\mathrm{i})\right\}\,. \end{aligned} \tag{50}$$

In order to bring this result into the form of Eqs. (53), (56)-(58) of our PRD [?], we can write

$$\begin{aligned} 1-\mathrm{i} &= \sqrt{2}e^{-\mathrm{i}\pi/4}\,, \\ \vartheta(\omega_0)\varrho &= \left(k_0 - \omega_0\frac{\eta}{\varrho}\right)\varrho = \sqrt{\omega_0^2-\omega_*^2}\,\varrho - \omega_0\eta\,, \\ 2\frac{1}{2\pi}\sqrt{\frac{\pi k_0^3}{\omega_*^2\varrho}} &= \sqrt{\frac{1}{\pi}}\frac{k_0^{3/2}}{\omega_*\sqrt{\varrho}} = \frac{1}{\sqrt{\pi}}\frac{(\omega_0^2-\omega_*^2)^{3/4}}{\omega_*\sqrt{\varrho}} \\ \tilde{f}(\omega_0) &= |\tilde{f}(\omega_0)|\,e^{\mathrm{i}\arg[\tilde{f}(\omega_0)]}\,. \end{aligned} \tag{51}$$

Equation (50) then becomes

$$\sigma(\eta,r) = A(\eta,r)e^{\mathrm{i}\phi(\eta,r)} \quad \text{with} \tag{52}$$

$$\phi(\eta,r) = \sqrt{\Omega^2-\omega_*^2}\,(r-r_\mathrm{e}) - \Omega\eta + \arg[\tilde{\sigma}(\omega,r_\mathrm{e})] - \frac{\pi}{4}\,, \tag{53}$$

$$A(\eta,r) = \sqrt{\frac{2}{\pi}}\,\frac{(\Omega^2-\omega_*^2)^{3/4}}{\omega_*\sqrt{r-r_\mathrm{e}}}\,|\tilde{\sigma}(\omega,r_\mathrm{e})|\,, \tag{54}$$

$$\Omega(\eta,r) = \omega_0 = \frac{\omega_*\eta}{\sqrt{\eta^2-(r-r_\mathrm{e})^2}}\,. \tag{55}$$

Here, we have used the fact that for an outgoing signal at positive frequencies, $\tilde{f}(\omega)=\tilde{\sigma}(\omega,r_\mathrm{e})$. Note also that $\eta^2-(r-r_\mathrm{e})^2>0$ for velocities $0\leq v<1$.

Case 4: $\boldsymbol{-1<v<0}$

For completeness, we also consider the case of ingoing modes. The calculation proceeds almost identically to that of outgoing modes, but we need to be careful with the signs.

Again, we have two contributions, but now $\mathrm{sig}(v)=-1$ in Eq. (43) and the contributions come from $\vartheta'(-\omega_0)=0$ and $\theta'(\omega_0)=0$. Equation (40) with the SPA (38) becomes

$$2\pi\sigma(t,r) = \tilde{f}(-\omega_0)e^{\mathrm{i}\vartheta(-\omega_0)\varrho}I_f + \tilde{g}(\omega_0)e^{\mathrm{i}\theta(\omega_0)\varrho}I_g\,, \tag{56}$$

with

$$I_f = \int_{-\infty}^{\infty} e^{\mathrm{i}(\omega+\omega_0)^2/\varepsilon_f}\mathrm{d}\omega = \int_{-\infty}^{\infty} e^{\mathrm{i}\omega^2/\varepsilon_f}\mathrm{d}\omega \quad \text{with} \quad \varepsilon_f = \frac{2}{\vartheta''(-\omega_0)\varrho}\,,$$

$$I_g = \int_{-\infty}^{\infty} e^{\mathrm{i}(\omega-\omega_0)^2/\varepsilon_g}\mathrm{d}\omega = \int_{-\infty}^{\infty} e^{\mathrm{i}\omega^2/\varepsilon_g}\mathrm{d}\omega \quad \text{with} \quad \varepsilon_g = \frac{2}{\theta''(\omega_0)\varrho}\,. \tag{57}$$

Note that $\varrho<0$ now which changes the signs of ε_f and ε_g relative to the outgoing case. Using

$$\vartheta''(-\omega_0) = \frac{-\omega_*^2}{k_0^3}\,, \qquad \theta''(\omega_0) = \frac{\omega_*^2}{k_0^3}\,, \qquad k_0 := k(\omega_0) = \sqrt{\omega_0^2-\omega_*^2}\,, \tag{58}$$

$\varepsilon_f=-\varepsilon_g>0$ and, with the Fresnel integral (39),

$$I_f = \sqrt{\frac{\pi}{2}}\sqrt{\frac{2}{|\vartheta''(-\omega_0)|\,|\varrho|}}(1+\mathrm{i}) = \sqrt{\frac{\pi k_0^3}{-\varrho\omega_*^2}}\,(1+\mathrm{i})\,,$$

$$I_g = \sqrt{\frac{\pi}{2}}\sqrt{\frac{2}{|\theta''(\omega_0)|\,|\varrho|}}(1-\mathrm{i}) = \sqrt{\frac{\pi k_0^3}{-\varrho\omega_*^2}}\,(1-\mathrm{i})\,. \tag{59}$$

Combining Eqs. (56)-(59), our wave signal in the time domain becomes, using $z + z^* = 2\mathsf{Re}(z)$ for $z \in \mathbb{C}$,

$$
\begin{aligned}
2\pi\sigma(\eta, r) &= \underbrace{\tilde{f}(-\omega_0)}_{=\tilde{g}^*(\omega_0)} \underbrace{e^{\mathrm{i}\vartheta(-\omega_0)\varrho}}_{=e^{-\mathrm{i}\theta(\omega_0)\varrho}} \sqrt{\frac{\pi k_0^3}{-\varrho\omega_*^2}}(1+\mathrm{i}) + \tilde{g}(\omega_0)e^{\mathrm{i}\theta(\omega_0)\varrho}\sqrt{\frac{\pi k_0^3}{-\varrho\omega_*^2}}(1-\mathrm{i}) \\
&= 2\mathsf{Re}\left\{\tilde{g}(\omega_0)e^{\mathrm{i}\theta(\omega_0)\varrho}\sqrt{\frac{\pi k_0^3}{-\varrho\omega_*^2}}(1-\mathrm{i})\right\}. \qquad (60)
\end{aligned}
$$

In order to bring this result into the form of Eqs. (53), (56)-(58) of our PRD [?], we can write

$$
\begin{aligned}
1-\mathrm{i} &= \sqrt{2}e^{-\mathrm{i}\pi/4}, \\
\theta(\omega_0)\varrho &= \left(-k_0 - \omega_0\frac{\eta}{\varrho}\right)\varrho \;=\; -\sqrt{\omega_0^2-\omega_*^2}\,\varrho - \omega_0\eta, \\
2\frac{1}{2\pi}\sqrt{\frac{\pi k_0^3}{-\omega_*^2\varrho}} &= \sqrt{\frac{1}{\pi}}\frac{k_0^{3/2}}{\omega_*\sqrt{-\varrho}} \;=\; \sqrt{\frac{1}{\pi}}\frac{(\omega_0^2-\omega_*^2)^{3/4}}{\omega_*\sqrt{-\varrho}} \\
\tilde{g}(\omega_0) &= |\tilde{g}(\omega_0)|\,e^{\mathrm{i}\arg[\tilde{g}(\omega_0)]}. \qquad (61)
\end{aligned}
$$

Equation (60) then becomes

$$
\boxed{
\begin{aligned}
\sigma(\eta, r) &= A(\eta, r)e^{\mathrm{i}\phi(\eta,r)} \qquad \text{with} && (62) \\
\phi(\eta, r) &= \sqrt{\Omega^2-\omega_*^2}\,(r_\mathrm{e}-r) - \Omega\eta + \arg[\tilde{\sigma}(\Omega, r_\mathrm{e})] - \frac{\pi}{4}, && (63) \\
A(\eta, r) &= \sqrt{\frac{2}{\pi}}\frac{(\Omega^2-\omega_*^2)^{3/4}}{\omega_*\sqrt{r_\mathrm{e}-r}}\,|\tilde{\sigma}(\omega, r_\mathrm{e})|, && (64) \\
\Omega(\eta, r) &= \omega_0 \;=\; \frac{\omega_*\eta}{\sqrt{\eta^2-(r-r_\mathrm{e})^2}}. && (65)
\end{aligned}
}
$$

It appears plausible that the final expressions for the outgoing and ingoing signals are identical up to the reversion of $r - r_\mathrm{e} \;\to\; r_\mathrm{e} - r$; after all, the one-dimensional wave equation is democratic about the direction of propagation. The only difference arises in the reconstruction of the original signal ϕ which acquires a rescaling by r_e/r in the outgoing and by r/r_e in the ingoing case.

We conclude this discussion by drawing the attention, in so far as necessary for observant readers, to our rather sloppy use of the time variable, either t or η. Whereas t and η are measured relative to the big bang in the cosmological line element (12), they are measured relative to the time of emission in the propagation of the wave signal. This does not affect our results, since the former time coordinate only enters our calculations through the cosmological scale factor $a(t)$. In the next section, we will replace these occurrences of the scale factor with a redshift factor $1+z$, so that the cosmological no longer appears in our expressions in explicit form. Every t or η present in our final expressions is then measured relative to the emission of the signal.

1.5 Interpretation of the wave signal

From now on, we will focus on the case of an outgoing wave signal with $0 < v < 1$. For ease of discussion, we repeat here the result of our calculation in the SPA under the assumption that signals are short lived compared to the timescale of cosmological expansion,

$$\sigma(\eta, r) = A(\eta, r) e^{\mathrm{i}\phi(\eta, r)} \quad \text{with} \tag{66}$$

$$\phi(\eta, r) = \sqrt{\Omega^2 - \omega_*^2}\,(r - r_\mathrm{e}) - \Omega\eta + \arg[\tilde{\sigma}(\Omega, r_\mathrm{e})] - \frac{\pi}{4}\,, \tag{67}$$

$$A(\eta, r) = \sqrt{\frac{2}{\pi}}\,\frac{(\Omega^2 - \omega_*^2)^{3/4}}{\omega_*\sqrt{r - r_\mathrm{e}}}\,|\tilde{\sigma}(\Omega, r_\mathrm{e})|\,, \tag{68}$$

$$\Omega(\eta, r) = \omega_0 = \frac{\omega_*\eta}{\sqrt{\eta^2 - (r - r_\mathrm{e})^2}}\,, \tag{69}$$

$$\omega_* = \sqrt{a^2 m^2 - B} = a\sqrt{m^2 - H^2(1 - q)}\,, \qquad B = \frac{\partial_\eta^2 a}{a} = a^2 H^2 (1 - q)\,. \tag{70}$$

The use of conformal time is not entirely intuitive for the human mind (it might not have been sufficiently important in escaping sabretooth tigers...). Let us therefore consider first a prototypical Universe that is spatially flat, matter dominated and has a zero cosmological constant.

1.5.1 The Einstein-de Sitter Universe

In a matter dominated Universe with energy density $\rho(t)$,

$$C := \frac{8\pi}{3} a^3 \rho = \text{const}\,, \tag{71}$$

and the Friedmann equations have the comparatively simple solution

$$a^3 = \frac{9C}{4} t^2\,, \qquad H = \frac{2}{3t}\,, \qquad q = \frac{1}{2}\,. \tag{72}$$

From this solution, we compute the conformal time, defining $\tilde{C} = (9C/4)^{1/3}$,

$$\mathrm{d}\eta = \frac{1}{a}\mathrm{d}t = \tilde{C}^{-1}t^{-2/3}\mathrm{d}t \qquad \Rightarrow \qquad \eta = 3\tilde{C}^{-1}t^{1/3} \qquad \Rightarrow \qquad t^{2/3} = \frac{\tilde{C}^2}{9}\eta^2 \,. \tag{73}$$

so that

$$a(t) = \tilde{C}t^{2/3} \,, \qquad a(\eta) = \frac{\tilde{C}^3}{9}\eta^2 \,. \tag{74}$$

Since $\tilde{C} > 0$, we clearly have $\partial_t^2 a < 0$ and $\partial_\eta^2 > 0$. This is compatible with Eq. (19), with $q = 1/2$. In short, the term $(\partial_t a)^2$ dominates over $a\partial_t^2 a$ in Eq. (19) which gives us a positive B for a decelerating Universe. This also explains at least in part the counter intuitive observation that our positive B leads to an effective reduction in the scalar mass. B is only positive because in conformal time, the expansion of the Universe is *not* decelerated, but accelerated. And from the wave equation (17), we see that it is the acceleration/deceleration in conformal time that matters for the effective scalar mass.

Let us pursue this train of thought a bit further and consider Universes with more general scale factor

$$a(t) \propto t^n \,. \tag{75}$$

We then straightforwardly find

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = -\frac{n-1}{n} \,. \tag{76}$$

For Einstein-de Sitter, we have $n = 2/3$ and $q = 1/2$ as expected. But we also see that the interval $n \in (0,1)$ maps to $\mathbb{R}^+$, i.e. any degree of deceleration we may dream of. A particularly intriguing case is the radiation dominated analog of Einstein-de Sitter (i.e. $k = \Lambda = 0$), where

$$\begin{aligned} a &= \sqrt{2}\, D^{1/4}\sqrt{t} \,, \qquad \text{with} \qquad D := \frac{8\pi}{3}a^4\rho = \text{const} \\ \Rightarrow \quad q &= 1 \,, \end{aligned} \tag{77}$$

and our correction B to the scalar mass completely vanishes. This is in agreement with setting $a = \lambda t^{1/2}$, $\lambda = \text{const}$ in Eq. (19) which straightforwardly gives us $(\partial_t a)^2 + a\partial_t^2 a = 0$. At this point, I do not have an explanation why a radiation dominated Universe would leave the threshold between radiative and trapped wave modes unchanged. Is it just a mathematical result or does the entirely lightlike matter content give us a deeper physical reason why the Universe's expansion is linear in conformal time?

1.5.2 Quantifying the cosmological expansion

We now address the question which of our cosmological modifications to the wave propagation are relevant and which we may discard as negligible. For this purpose, we first express the

Hubble parameter as a frequency. This is achieved by setting $\hbar = c = 1$ which directly gives us

$$1 = \frac{\hbar}{c^2} = 1.17 \times 10^{-51}\ \mathrm{kg\,s} \qquad \Rightarrow \qquad 1\,\mathrm{kg} = 8.5223 \times 10^{50}\,\mathrm{s}^{-1}\,. \tag{78}$$

We have to be careful with the interpretation of units s^{-1} here, since this could either denote frequency or angular frequency. In our case, we set $\hbar = 1$ (and not h), so our frequencies are by construction angular unless we apply additional factors of 2π. Using furthermore

$$1\,\mathrm{pc} = 3.085678 \times 10^{16}\,\mathrm{m}\,, \tag{79}$$

and setting $H = 75\,(\mathrm{km/s})/\mathrm{Mpc}$, we find

$$H = 0.243058 \times 10^{-17}\,\mathrm{s}^{-1}\,. \tag{80}$$

Of course, this is approximately the inverse of the age of the Universe, modified by 2π for an angular frequency. Finally, we need

$$1\,\mathrm{eV} = 1.602177 \times 10^{-19}\,\mathrm{J} = 1.78269 \times 10^{-36}\,\mathrm{kg}\,, \tag{81}$$

to find that

$$1\,\mathrm{eV} = 0.151926 \times 10^{16}\,\mathrm{s}^{-1}\,, \tag{82}$$

and, eventually,

$$\boxed{H = 1.600 \times 10^{-32}\,\mathrm{eV}\,.} \tag{83}$$

This value holds for a Hubble parameter of 75, but all that matters is an order-of-magnitude estimate; in the local Universe, we have $H \approx 10^{-32}\,\mathrm{eV}$ which is very small even by the standards of our small scalar masses $\mathcal{O}(10^{-14})\,\mathrm{eV}$. We won't be far off the mark, if we approximate

$$\omega_* = a\sqrt{m^2 - H^2(1-q)} \approx am\,. \tag{84}$$

Note that one factor of a remains.

1.5.3 Space and time coordinates

The expressions for the observed wave signal in Eqs. (66)-(70) are given in terms of the conformal time η and radius r. These are particularly convenient coordinates for our calculations, but we need to relate them to the time and radius that an observer of a supernova works with.

Let us imagine, for this purpose, what exactly happens in the case of a supernova event that we observe and whose gravitational-wave signal we seek to detect.

(1) First, a supernova explosion occurs somewhere far away in the Universe. Our coordinate system is centered on the collapsing star's origin which corresponds to $r = 0$ and the event happens at some cosmological time $\eta_{\rm e}$; 'e' stands for emission. In our GR1D simulations, we extract the signal at some distance from the star, so our wave propagation starts at radius $r_{\rm e}$. We thus have an event with all wave propagation (GWs or electromagnetic) starting at spacetime location $(\eta_{\rm e}, r_{\rm e})$. At this time, the Universe's scale factor is $a(\eta_{\rm e})$ and we will eventually use the scale freedom to set $a(\eta_{\rm e}) = 1$. From this point in spacetime on, the Universe will keep expanding, light propagates with speed c and the massive GW modes propagate according to their dispersion relation.

(2) The next thing that happens is that an observer on Earth sees the supernova in the electromagnetic spectrum, say in optical light. This happens at conformal time $\eta_{\rm o}$ and at radial position $r_{\rm o}$; 'o' stands for observation.

(3) Let us first consider the distance of the supernova. Astronomers typically measure the location of events in terms of the luminosity distance which, in more mathematical terms, is the areal radius. Our line element, centered on the emitting supernova, is

$$\mathrm{d}s^2 = a^2(-\mathrm{d}\eta^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega)\,,$$

and the proper area of a sphere $\mathrm{d}\eta = \mathrm{d}r = 0$ is clearly given by $A = 4\pi a^2 r^2$. The areal radius R is therefore related to the coordinate radius by

$$R^2 = \frac{A}{4\pi} \qquad \Rightarrow \qquad R = ar\,. \tag{85}$$

Note that the areal radius of a sphere of constant r is time dependent; it increases as the universe is expanding. We therefore need to carefully specify the time whenever we evaluate an areal radius. For now, we consider the time of observation of the supernova event; the areal radius at that moment is $R_{\rm o} = a(\eta_{\rm o})r_{\rm o}$.

(4) It is now time to finally address a question we have in rather cavalier fashion swept a bit under the rug in our calculation hitherto: the reference point of the conformal time coordinate η. With our present choice, $\eta = 0$ marks the starting time of the propagation of our wave signal σ. In other words, $\eta = 0$ is the time of emission of the gravitational-wave signal (which, we recall, is treated as instantaneous on the timescale of cosmological expansion). Ultimately, however, we wish to relate η to the time measured in the observers frame.

Let us calculate for this purpose the amount of conformal time that has elapsed between the supernova event and the observation on Earth. Clearly, the light rays that inform us about the SN event propagate on null curves; for radial null curves, our line element gives us

$$\mathrm{d}s^2 = a^2(-\mathrm{d}\eta^2 + \mathrm{d}r^2) = 0\,, \tag{86}$$

and therefore (note that our light ray is outgoing) $\mathrm{d}\eta = \mathrm{d}r$. This is easy to integrate and we obtain

$$\eta = r - r_{\rm e} + \text{const}\,. \tag{87}$$

The moment of photon emission at $r = r_\mathrm{e}$ corresponds to our supernova explosion where by definition $\eta = 0$, so that our additive constant in this equation vanishes.

We can find the same result by inspecting Eq. (69) for the frequency. The GW modes with infinite frequency arrive together with the photons. Rewriting Eq. (69) as

$$\Omega = \frac{\omega_*}{\sqrt{1 - \left(\frac{r-r_\mathrm{e}}{\eta}\right)^2}}\,, \tag{88}$$

we see that $\Omega = \infty$ corresponds to

$$\eta = r - r_\mathrm{e} =: \eta_\mathrm{o}\,, \tag{89}$$

in agreement with setting the constant to zero in Eq. (87).

So, we have identified the moment we see the supernova in light as $\eta = r - r_\mathrm{e}$. At this moment, our observation clock starts ticking. Naturally, our observer will measure her progression of age in terms of proper time. Being at rest at r, this proper time is simply given by

$$\mathrm{d}\tau = a\mathrm{d}\eta\,. \tag{90}$$

Assuming once again that the observation programme is much shorter than the expansion time scale of the Universe, we can approximate $a \approx a(\eta_\mathrm{o})$ in this equation and we obtain

$$\tau = a(\eta - \eta_\mathrm{o}) = a[\eta - (r - r_\mathrm{e})]\,. \tag{91}$$

To summarize, our convention is that $\eta = r - r_\mathrm{e}$ corresponds to the moment we see the supernova in light, at which $\tau = 0$ and our observer clock starts ticking.

This discussion was probably more lengthy than it needed to be, but the whole issue has twisted my mind in a rather agonizing way and I wanted to make sure I still understand what the heck is going on here even years from now. Fortunately, the key summary is very short. In order to apply Eqs. (66)-(70) to an actual observation, we need to convert the observer's proper time τ (measured since the optical detection of the supernova event) and their estimate of the luminosity distance R into our conformal time and radius coordinates η and r. These expressions are given by inverting Eqs. (85) and (91),

$$r = \frac{R}{a_\mathrm{o}}\,, \qquad \eta = \frac{\tau}{a_\mathrm{o}} + r - r_\mathrm{e}\,. \tag{92}$$

We need one final approximation to arrive at rather simple expressions for the wave signal. While we have centered our coordinate system on the supernova source, we extract the wave signal from our GR1D simulations at finite radius r_e. The value of this radius depends on the scalar field mass, but is always on the time scale of light seconds and, therefore, much smaller than r (if you wonder about the physical units of r, recall that we set $a_\mathrm{e} = 1$ without loss of generality). Clearly, $r - r_\mathrm{e} \approx r$ with high precision, so that

$$r = \frac{R}{a_\mathrm{o}}\,, \qquad \eta = \frac{\tau + R}{a_\mathrm{o}}\,. \tag{93}$$

The rest is easy. Inserting Eqs. (93) and (84) into Eqs. (66)-(70), we obtain, setting $a_\mathrm{e} = 1$, $a_\mathrm{o} = 1 + z$, and using

$$\Omega^2 - \omega_*^2 = (1+z)^2 m^2 \frac{R^2}{(\tau+R)^2 - R^2}\,, \tag{94}$$

we obtain after some algebra

$$\sigma(\tau, R) = A(\tau, R) e^{\mathrm{i}\phi(\tau,R)} \qquad \text{with} \tag{95}$$

$$\phi(\tau, R) = -m\sqrt{(\tau+R)^2 - R^2} + \arg[\tilde{\sigma}(\Omega, r_\mathrm{e})] - \frac{\pi}{4}\,, \tag{96}$$

$$A(\tau, R) = \sqrt{\frac{2}{\pi}}(1+z)\sqrt{m}\frac{R}{[(\tau+R)^2 - R^2]^{3/4}} \left|\tilde{\sigma}(\Omega, r_\mathrm{e})\right|\,, \tag{97}$$

$$\Omega(\tau, R) = (1+z)\frac{m}{\sqrt{1 - \left(\frac{R}{\tau+R}\right)^2}}\,, \tag{98}$$

$$\omega_* = (1+z)m\,. \tag{99}$$

Here, m is the scalar mass parameter in the form the angular frequency $m/\hbar$ in the rest frame of the core collapse event. Finally, we reconstruct the scalar field φ according to

$$\varphi(\tau, R) = \frac{\sigma(\tau, R)}{R}\,. \tag{100}$$

This result needs some comments to lift any possible fog of surprise. Let us start with the frequency $\Omega(\tau, R)$. Equation (98) may give the impression of a blue shift due to the factor (1+z). This, however, is not the correct way to interpret our result. Let us start with the supernova which goes off at a distance R. Without cosmological expansion, we would see a mode of some given frequency with a delay of τ relative to the photons as predicted by the mode's group velocity. With cosmological expansion, we would instead see a higher frequency mode at τ; the lower frequency modes "just haven't made it" because they got slowed down by the cosmic expansion. Note furthermore that it is this increased value Ω that enters in our initial data $\tilde{\sigma}(\Omega, r_\mathrm{e})$ on the right-hand sides of Eqs. (96 and (97).

Next, we consider the phase $\phi(\tau, R)$. Here, the only effect arises from the argument of $\tilde{\sigma}(\Omega, r_\mathrm{e})$ which now has to be evaluated (in the source frame) at the enlarged frequency $\Omega(\tau, R)$. Likewise, the amplitude $A(\tau, R)$ is now sourced by the amplitude of $\tilde{\sigma}(\Omega, R)$, i.e. the source's power at higher frequency. Typically, the power spectrum of our collapse events at r_e rapidly drops with frequency, and we correspondingly expect a drop in the signal amplitude as compared to a non-expanding Universe. The factor $(1 + z)$ on the right-hand side of Eq. (97) mitigates this reduction in the signal, but I would expect it to be subdominant. Nonetheless, it might be worth while developing an intuitive explanation for this blueshift factor. At the moment, I do not have one.