1 Preliminary comments

Purpose of this document is to summarize issues arising from the matter terms in the Geroch-decomposed implementation of higher-dimensional black-hole evolutions. The starting point are the notes from Miguel and Carlos as of Sep 22, 2009. The main difficulty is that the norm of the Killing vector λ behaves like $\sim y^2$ on the y axis and we have various places where we divide by λ. Things appear to get even worse on the z axis, i. e. when $x = y = 0$. There are some apparently neat ways of handling these issues, but I have not yet come up with a completely satisfactory solution.
For now I will focus on the matter source term labeled $E + S$ by both, Miguel and Carlos. This source term is given in Eq. (5.11) of Miguel's notes and Eq. (2.46a) of Carlos' draft (note that the two differ by a factor of 2 in the definition of K_λ), and Carlos' expansion of the covariant derivatives. I will follow Carlos' choice of K_λ and Miguel's shorter notation in keeping covriant devivatives. Either choice is immaterial for highlighting the problems we are facing. The source terms are given by

$$\begin{aligned}\frac{4\pi(E+S)}{D-4} &= -(D-5)\lambda^{-1} + \frac{1}{2}\lambda^{-1}\gamma^{ij}D_j\partial_i\lambda + \frac{D-6}{4}\lambda^{-2}\gamma^{ij}(\partial_i\lambda)(\partial_j\lambda) \\ &\quad -\frac{1}{2}\lambda^{-1}KK_\lambda - \frac{D-5}{4}\lambda^{-2}K_\lambda^2. \qquad (1.1)\end{aligned}$$

For the Tangherlini solution we know for sure that

$$\lambda = y^2\psi^2, \qquad (1.2)$$

where ψ is the conformal factor, as given, for example, in Eq. (3.18) of Carlos' draft. The same will hold for Brill-Lindquist data and I will assume for now that nothing worse happens for more general data. This may or may not be the case, but we'll be happy for now if we can handle evolutions of Brill-Lindquist data. In Eq. (1.1) we have various divisions by λ. Because we know the Tangherlini solution to be regular at $y = 0$, the overall coefficients of $1/\lambda$ terms must vanish in this limit at least as fast as y^2. Numerically, we cannot implement the equations this way, however, because we'd be dividing by zero and numerically even an expression like $0/0$ will generate non-assigned number ("nan") and immediately screw the evolution.
¿From experience with polar and spherical coordinates in simpler models involving, for example, neutron stars, I conclude that it is not a good idea to work with singular variables such as λ but instead use regular functions. In our case, this is simply achieve by introducing

$$\kappa \equiv \frac{1}{y^2}\lambda. \qquad (1.3)$$

Note that for now I do not factor out the conformal factor. This can be done for κ just as well as it has been done already by Miguel for λ, but it would result in more complicated expressions and is independent of the issues resulting from the y^2 behavior.
In terms of the new variable κ, the source term becomes

$$\begin{aligned}\frac{4\pi(E+S)}{D-4} &= \frac{1}{2\kappa}\gamma^{ij}\partial_i\partial_j\kappa + \frac{D-6}{4}\gamma^{ij}(\partial_i\kappa)(\partial_j\kappa) - \frac{1}{2\kappa}\gamma^{ij}\Gamma^m_{ij}\partial_m\kappa + \frac{1}{y^2\kappa}(D-5)(\kappa\gamma^{yy}-1) \\ &\quad +\frac{1}{y\kappa}\left[(D-4)\gamma^{yj}\partial_j\kappa - \kappa\gamma^{ij}\Gamma^y_{ij}\right] \\ &\quad -\frac{1}{\kappa}KK_\kappa - \frac{1}{\alpha y}\beta^y - (D-5)\left[\left(\frac{K_\kappa}{\kappa}\right)^2 + \frac{2}{y\alpha\kappa}\beta^y + \frac{1}{y^2\alpha^2}(\beta^y)^2\right]. \qquad (1.4)\end{aligned}$$

Because κ is regular on the axis, we can now focus our attention on those terms which involve explicit divisions by y. We will discuss these in turn, but in reverse order as the last term on the first line of Eq. (1.4) is by far the worst. Starting from the back we first encounter terms of the form β^y/y, either in linear or quadratic form. Naturally, these have to be regular at $y = 0$. This is indeed guaranteed by the symmetry condition of β^y which behaves like a vector and therefore satisfies

$$\beta^y(-y) = -\beta^y(y) \tag{1.5}$$

in the neighborhood of $y = 0$. In consequence, an expansion around $y = 0$ yields

$$\beta^y(\Delta y) = b_1 y + \mathcal{O}(y^2). \tag{1.6}$$

In consequence, β^y/y will behave nicely on the y axis, but there still remains the question of how to calculate it without dividing by zero. This can be achieved nicely by taking the derivative

$$\partial_y \beta^y = b_1 = \frac{\beta^y}{y}. \tag{1.7}$$

Numerically, we have no problems at all calculating the left hand side of Eq. (1.7) and, thus the potentially trouble some right hand side. The only remaining danger is that at gridpoints Δy away from the xz plane, the standard evaluation using β^y/y might be inaccurate and generate some numerical noise. From experience, however, this should not happen for variables linear in y. The solution for these terms is therefore to evaluate them straightforwardly off the $y = 0$ plane and use `if` statements to employ the derivatives instead on the plane.

The same idea holds for the terms in the middle row of Eq. (1.4). In these terms, β^y is replaced by γ^{yj} and Γ^y_{ij}, respectively. Both have the same behavior $\sim y$ near the xz plane.

Finally, we consider the last term in the first row of Eq. (1.4). Note that the case $d = 5$ appears to be somewhat benevolent in that this more complicated term vanishes identically. Still, we want to be able to evolve arbitrary dimensions and would like to have a viable solution for this term, too. What we need to verify is that $\kappa\gamma^{yy} = 1 + \mathcal{O}(y^2)$.

For this purpose we first note that $\gamma_{xy} = 0 = \gamma_{yz}$ at $y = 0$, so that

$$\gamma_{ij} = \begin{pmatrix} \gamma_{xx} & 0 & \gamma_{xz} \\ 0 & \gamma_{yy} & 0 \\ \gamma_{zx} & 0 & \gamma_{zz} \end{pmatrix} \tag{1.8}$$

The determinant thus simplifies to

$$\det \gamma_{ij} = \gamma_{yy}(\gamma_{xx}\gamma_{zz} - \gamma_{xz}^2), \tag{1.9}$$

and for the inverse metric we find

$$\gamma^{yy} = \frac{1}{\det \gamma_{ij}}(\gamma_{xx}\gamma_{zz} - \gamma_{xz}^2) = \frac{1}{\gamma_{yy}}. \tag{1.10}$$

On the xz plane we also obtain a simplification for the derivatives of $\tilde{\gamma}^{ij}$. Specifically, we have

$$\tilde{\gamma}^{xy} = \frac{1}{\det \tilde{\gamma}_{ij}}(\tilde{\gamma}_{xz}\tilde{\gamma}_{yz} - \tilde{\gamma}_{xy}\tilde{\gamma}_{zz}) = \tilde{\gamma}_{xz}\tilde{\gamma}_{yz} - \tilde{\gamma}_{xy}\tilde{\gamma}_{zz}, \tag{1.11}$$

where we have used the fact that $\det \tilde{\gamma}_{ij} = 1$ by construction. We thus have

$$\begin{aligned}
\partial_y \tilde{\gamma}^{xy} &= \underbrace{(\partial_y \tilde{\gamma}_{xz})}_{=\mathcal{O}(y)} \underbrace{\tilde{\gamma}_{yz}}_{=\mathcal{O}(y)} + \tilde{\gamma}_{xz}(\partial \tilde{\gamma}_{yz}) - (\partial_y \tilde{\gamma}_{xy})\tilde{\gamma}_{zz} - \underbrace{\tilde{\gamma}_{xy}}_{=\mathcal{O}(y)} \underbrace{(\partial_y \tilde{\gamma}_{zz})}_{=\mathcal{O}(y)}, \\
&= \tilde{\gamma}_{xz}\partial_y \tilde{\gamma}_{yz} - \tilde{\gamma}_{zz}\partial_y \tilde{\gamma}_{xy} + \mathcal{O}(y^2). \qquad (1.12)
\end{aligned}$$

We likewise obtain

$$\begin{aligned}
\partial_y \tilde{\gamma}^{yz} &= \tilde{\gamma}_{xz}\partial_y \tilde{\gamma}_{xy} - \tilde{\gamma}_{xx}\partial_y \tilde{\gamma}_{yz} + \mathcal{O}(y^2), \qquad (1.13) \\
\partial_y \tilde{\gamma}^{yy} &= \tilde{\gamma}_z z \partial_y \tilde{\gamma}_{xx} + \tilde{\gamma}_{xx}\partial_y \tilde{\gamma}_{zz} - 2\tilde{\gamma}_{xz}\partial_y \tilde{\gamma}_{xz}, \qquad (1.14) \\
\partial_y \partial_y \tilde{\gamma}^{yy} &= \tilde{\gamma}_{zz}\partial_y \partial_y \tilde{\gamma}_{xx} + \tilde{\gamma}_{xx}\partial_y \partial_y \tilde{\gamma}_{zz} - 2\tilde{\gamma}_{xz}\partial_y \partial_y \tilde{\gamma}_{xz} + \mathcal{O}(y^2). \qquad (1.15)
\end{aligned}$$

The benefit of these relations is that we do not need to store the contravariant version of the conformally rescaled metric in grid functions to evaluate their derivatives. The covariant metric is already stored in grid functions and their derivatives are evaluated already in the four-dimensional implementation of BSSN and are thus readily available. We need to monitor, however, the performance of this simplification because the condition $\det \tilde{\gamma}_{ij} = 1$ is satisfied up to numerical accuracy only. We will bear this in mind when performing the simulations.

Let us now discuss a few of these relations in the special case of the Tangherlini solution. There, we have

$$\kappa \gamma^{yy} = \frac{\kappa}{\gamma_{yy}} = 1 \qquad (1.16)$$

from Eq. (3.20) of Carlos' draft and our definition of κ. Because κ is a scalar and γ^{yy} a tensor, the lowest correciton terms off the axis are of the order of y^2 and we are safe at least in the case of the Tangherlini solution.

But how about the more general case and its numerical implementation? This is where I am still uncertain. So let us summarize what results we have so far. Assuming axial symmetry, we have Miguel's relation (4.5) which translates into

$$\kappa = \frac{1}{x^2 + y^2}(y^2 \gamma_{xx} + x^2 \gamma_{yy} - 2xy\gamma_{xy}). \qquad (1.17)$$

It is a consequence of the transformation from spherical to Cartesian coordinates and the result

$$\lambda = \sin^2\theta g_{\theta\theta} = \frac{y^2}{\rho^2} g_{\theta\theta}. \qquad (1.18)$$

Our critical expression thus becomes

$$\frac{1}{y^2}(\kappa\gamma^{yy} - 1) = \frac{1}{y^2}\left(\frac{\kappa}{\gamma_{yy}} - 1\right) = \frac{1}{\gamma_{yy}} \frac{\gamma_{xx} - 2\frac{x}{y}\gamma_{xy} - \gamma_{yy}}{x^2 + y^2}. \qquad (1.19)$$

I am prepared to believe that this is regular even on the z axis where $x = 0 = y$. But the limit as $\rho \to 0$ should be independent of the direction in which we take this limit. Is that criterion met?

It appears to me that we somehow should view this expression as a term

$$\frac{f}{\rho^2}; \qquad f(\rho) = \mathcal{O}(\rho^2) + \ldots, \qquad (1.20)$$

and should be able to use a second derivative

$$\frac{f}{\rho^2} = \frac{1}{2}\partial_\rho \partial_\rho f(\rho). \tag{1.21}$$

We can convert the second derivative with respect to ρ into Cartesian coordinates and obtain

$$\partial_\rho \partial_\rho = \frac{x^2}{\rho^2}\partial_x \partial_x + 2\frac{xy}{\rho^2}\partial_x \partial_y + \frac{y^2}{\rho^2}\partial_y \partial_y, \tag{1.22}$$

but there remains the question what combination of the metric components we should use for f and how the derivative exactly looks like. I currently do not have the solution for this...
A more promising approach is as follows. We use Eq. (1.10) and rewrite the source term as

$$\begin{aligned} \frac{1}{y^2 \kappa}(D-5)(\kappa\gamma^{yy} - 1) &= \frac{D-5}{y^2 \kappa \gamma_{yy}}(\kappa - \gamma_{yy}) \\ &= \frac{D-5}{2\kappa\gamma_{yy}}\partial_y \partial_y (\kappa - \gamma_{yy}). \end{aligned} \tag{1.23}$$

The second derivatives can be evaluated using standard stencils at $y = 0$ and we should obtain regular expressions.
We next consider the other source terms to see what other potentially troublesome terms we encounter. The source term for the extrinsic curvature is given in Eq. (2.46b) in Carlos' draft or Eq. (5.12) in Miguel's notes. As before, we replace λ in terms of the regularized variable κ and obtain

$$\frac{8\pi}{D-4}(S_{ij} - \frac{1}{3}S\gamma_{ij}) = \frac{1}{2}\left(\delta_j{}^y \frac{\partial_i \kappa}{y\kappa} + \delta_i{}^y \frac{\partial_j \kappa}{y\kappa} + \frac{1}{\kappa}\partial_i \partial_j \kappa - \frac{1}{2}\frac{\partial_i \kappa}{\kappa}\frac{\partial_j \kappa}{\kappa} - \frac{K_\kappa}{\kappa}K_{ij} - \frac{1}{\alpha}\frac{\beta^y}{y}K_{ij}\right)^{\mathrm{TF}} \tag{1.24}$$

where "TF" denotes the trace free part. Note that a factor of χ cancels on both sides of the equation. As far as I can see, no difficulties further to those in the above case appear in these source terms. Taking derivatives should enable us to obtan regular expressions for all terms involved.
Finally, we need to consider the source vector j_i which is given by

$$\begin{aligned} \frac{16\pi j_i}{D-4} &= \frac{2}{y\kappa\alpha}\left(\alpha K_\kappa \delta_i{}^y + \kappa \partial_i \beta^y\right) + \frac{2}{\kappa}\partial_i K_\kappa + \frac{\beta^y}{y\kappa\alpha}\partial_i \kappa \\ &\quad -2\frac{\beta^y}{y\alpha^2}\partial_i \alpha - \frac{1}{\kappa^2}K_\kappa \partial_i \kappa - \frac{K_i{}^y}{y\kappa} - \frac{1}{2\kappa}K_i{}^j \partial_j \kappa. \end{aligned} \tag{1.25}$$

Here I am somewhat puzzled by the first term on the right hand side. The individual terms enclosed by the parentheses do not vanish on the y axis but the combination has to vanish $\sim y$. Barring bugs in the equations, of course. It appears far from obvious to me that the sum vanish.
Let us consider this term further. First, we look at the case $i \neq y$. We then obtain

$$\alpha K_\kappa \delta_i{}^y + \kappa \partial_i \beta^y = 0 + \kappa \partial_i \beta^y. \tag{1.26}$$

For example, this term becomes $\kappa \partial_z \beta^y$ for the z component. I can only assume that the condition $\beta^y \sim y$ for $y \to 0$ must be preserved and guarantees that the overall term is regular. Let us assume that for $y \to 0$

$$\beta^y = b_1 y + \mathcal{O}(y^2). \tag{1.27}$$

We then obtain

$$\partial_z \beta^y = y \partial_z b_1 + \mathcal{O}(y^2), \tag{1.28}$$

(note that b_1 may depend on x and z but not on y). We are now able to use our above trick with derivatives, so that

$$\partial_z \partial_y \beta^y = \partial_z [b_1 + \mathcal{O}(y)] = \partial_z b_1 = \frac{1}{y} \partial_z \beta^y. \tag{1.29}$$

The same argument should hold for $i = x$. We are thus left with one difficult term, namely the case $i = y$. The tricky term is then given by

$$\alpha K_\kappa + \kappa \partial_y \beta^y. \tag{1.30}$$

This term must involve nontrivial cancelations because

$$\partial_y \beta^y = b_1, \tag{1.31}$$

which does not vanish $\sim y$ on its own, but instead must be canceled somehow by K_{kappa}. While this is not obvious to me, the regularity requirements appear to me to make this term much easier than I had originally thought. For this purpose we recall that we are dealing with the y component of j_i here. But j_i is a vectorial quantity and it's y component has therefore to vanish in the xz plane, i. e. at $y = 0$. There remains the question how well this is actually satisfied numerically at grid points adjacent to $y = 0$, but *at* $y = 0$ we should simply be allowed to set

$$j_y = 0. \tag{1.32}$$

We have already found a method to obtain j_x and j_z, so we should be fine here. Assuming the method with derivatives works, of course.

We next turn our attention to the sourceterms of the evolution of κ and K_κ themselves. To be continued...

2 The final equations

2.1 Definitions and relations

Before we summarize the final expressions of the source terms, we will give a comprehensive list of the definitions and useful relations that have been and will be used throughout the derivation of our formalism.

The BSSN equations work with the conformally rescaled metric and extrinsic curvature

$$\tilde{\gamma}_{ij} = W^2 \gamma_{ij} \quad \Leftrightarrow \quad \gamma_{ij} = \frac{1}{W^2} \tilde{\gamma}_{ij}, \tag{2.1}$$

$$\tilde{\gamma}^{ij} = \frac{1}{W^2} \gamma^{ij} \quad \Leftrightarrow \quad \gamma^{ij} = W^2 \tilde{\gamma}^{ij}, \tag{2.2}$$

$$\tilde{A}_{ij} = W^2 A_{ij} = W^2 \left(K_{ij} - \frac{1}{3} \gamma_{ij} K \right) \quad \Leftrightarrow \quad K_{ij} = \frac{1}{W^2} \left(\tilde{A}_{ij} + \frac{1}{3} \tilde{\gamma}_{ij} K \right), \tag{2.3}$$

where the conformal factor is given by

$$W = (\det \gamma_{ij})^{-1/6}. \tag{2.4}$$

In consequence, the conformally rescaled metric has unit determinant

$$\det \tilde{\gamma}_{ij} = 1. \tag{2.5}$$

The BSSN system further promotes the contracted Christoffel symbol to an independent evolution variable

$$\Gamma^k_{ij} = \tilde{\Gamma}^k_{ij} - \frac{1}{W}\left(\delta_i{}^k \partial_j W + \delta_j{}^k \partial_i W - \tilde{\gamma}_{ij}\tilde{\gamma}^{kl}\partial_l W\right), \tag{2.6}$$

$$\Rightarrow \quad \Gamma^k = W^2 \tilde{\Gamma}^k + W\tilde{\gamma}^{kl}\partial_l W. \tag{2.7}$$

The additional field that we will need to evolve or express in terms of the metric components is the norm of the Killing field λ. In order to formulate its time evolution in terms of a first order system, we also introduce the variable

$$K_\lambda := -\frac{1}{2\alpha}\left(\partial_t \lambda - \beta^m \partial_m \lambda\right). \tag{2.8}$$

We expect, however, that the norm of the Killing vector will have a y^2 fall off as $y \to 0$, that is, on the xz plane. This leads to divisions by zero on the right hand side of the BSSN evolution equations. We expect all variables to remain regular on the xz plane and therefore all divisions by y need to be cancelled by a corresponding fall off behavior of the numerators. At $y = 0$, however, we will not be able to implement this numerically unless we isolate the irregular terms in question and explicitly use cancelations or find other methods to evaluate expressions such as

$$\lim_{y\to 0} \frac{f}{y}, \tag{2.9}$$

where f is some example function which behaves like y^n with $n \geq 1$ near the xz plane. It is necessary, for this purpose, to formulate the equations in terms of variables which are manifestly regular at $y = 0$. We also prefer to apply a conformal rescaling of the norm of the Killing vector and thus use the evolution variable

$$\tilde{\kappa} := \frac{W^2}{y^2}\lambda. \tag{2.10}$$

As before, we would like to obtain a first order evolution system in time and therefore introduce the auxiliary variable

$$K_{\tilde{\kappa}} := -\frac{1}{2\alpha}\left(\partial_t \tilde{\kappa} - \beta^m \partial_m \tilde{\kappa} + \frac{2}{3}\tilde{\kappa}\partial_m \beta^m\right). \tag{2.11}$$

Here, the last term on the right hand side arises from the fact that $\tilde{\kappa}$ is not a scalar, but a scalar density of weight $-2/3$. It is not entirely clear at this moment whether the inclusion of this term is really necessary for a stable numerical implementation or not. For consistency with the rest of the BSSN variables, however, we decide to keep this form of $K_{\tilde{\kappa}}$. A straightforward calculation shows that

$$K_\lambda = \frac{y^2}{W^2}K_{\tilde{\kappa}} + \frac{1}{3}\frac{y^2\tilde{\kappa}}{W^2}K + \frac{\beta^y}{\alpha}\frac{y\tilde{\kappa}}{W^2}. \tag{2.12}$$

In the following we will discuss in detail the individual source terms as well as the evolution equation for $K_{\tilde{\kappa}}$. We will also give explicit recipes for handling those terms which involve divisions by y or higher powers thereof. We will see, how all such terms result in regular expressions which can be evaluated with little additional coding on the xz plane.

2.2 The scalar source term

The evolution of the trace of the extrinsic curvature as well as that of $K_{\tilde{\kappa}}$ involve the source term $E+S$. Starting with Eq. (2.46a) of Carlos' draft, we obtain for this source term

$$\begin{aligned}
\frac{4\pi(E+S)}{D-4} &= (D-5)\frac{\chi}{\tilde{\kappa}}\frac{\tilde{\gamma}^{yy}\tilde{\kappa}-1}{y^2} - \frac{2D-7}{4\tilde{\kappa}}\tilde{\gamma}^{mn}(\partial_m\tilde{\kappa})(\partial_n\chi) + \frac{D-1}{4}\frac{1}{\chi}\tilde{\gamma}^{mn}(\partial_m\chi)(\partial_n\chi) \\
&\quad + \frac{D-6}{4}\frac{\chi}{\tilde{\kappa}^2}\tilde{\gamma}^{mn}(\partial_m\tilde{\kappa})(\partial_n\tilde{\kappa}) + \frac{1}{2\tilde{\kappa}}\tilde{\gamma}^{mn}(\chi\partial_m\partial_n\tilde{\kappa} - \tilde{\kappa}\partial_m\partial_n\chi) - \frac{KK_{\tilde{\kappa}}}{\tilde{\kappa}} - \frac{1}{3}K^2 - \frac{\beta^y}{y\alpha}K \\
&\quad + \left[(D-4)\frac{\tilde{\gamma}^{ym}}{y} - \frac{1}{2}\tilde{\Gamma}^m\right]\left(\frac{\chi}{\tilde{\kappa}}\partial_m\tilde{\kappa} - \partial_m\chi\right) - \frac{1}{2}\frac{\tilde{\gamma}^{ym}}{y}\partial_m\chi - \chi\frac{\tilde{\Gamma}^y}{y} \\
&\quad - (D-5)\left[\left(\frac{K_{\tilde{\kappa}}}{\tilde{\kappa}}\right)^2 + \frac{2}{3}\frac{K_{\tilde{\kappa}}}{\tilde{\kappa}}K + \frac{K^2}{9} + 2\frac{\beta^y}{y\alpha}\frac{K_{\tilde{\kappa}}}{\tilde{\kappa}} + \frac{2}{3}\frac{\beta^y}{y\alpha}K + \left(\frac{\beta^y}{y\alpha}\right)^2\right] \\
&= (D-5)\frac{\chi}{\tilde{\kappa}}\frac{\tilde{\gamma}^{yy}\tilde{\kappa}-1}{y^2} - \frac{2D-7}{4\tilde{\kappa}}\tilde{\gamma}^{mn}(\partial_m\tilde{\kappa})(\partial_n\chi) + \frac{D-1}{4}\frac{1}{\chi}\tilde{\gamma}^{mn}(\partial_m\chi)(\partial_n\chi) \\
&\quad + \frac{D-6}{4}\frac{\chi}{\tilde{\kappa}^2}\tilde{\gamma}^{mn}(\partial_m\tilde{\kappa})(\partial_n\tilde{\kappa}) + \frac{1}{2\tilde{\kappa}}\tilde{\gamma}^{mn}(\chi\tilde{D}_m\partial_n\tilde{\kappa} - \tilde{\kappa}\tilde{D}_m\partial_n\chi) - \frac{KK_{\tilde{\kappa}}}{\tilde{\kappa}} - \frac{1}{3}K^2 - \frac{\beta^y}{y\alpha}K \\
&\quad + (D-4)\frac{\tilde{\gamma}^{ym}}{y}\left(\frac{\chi}{\tilde{\kappa}}\partial_m\tilde{\kappa} - \partial_m\chi\right) - \frac{1}{2}\frac{\tilde{\gamma}^{ym}}{y}\partial_m\chi - \chi\frac{\tilde{\Gamma}^y}{y} \\
&\quad - (D-5)\left[\left(\frac{K_{\tilde{\kappa}}}{\tilde{\kappa}}\right)^2 + \frac{2}{3}\frac{K_{\tilde{\kappa}}}{\tilde{\kappa}}K + \frac{K^2}{9} + 2\frac{\beta^y}{y\alpha}\frac{K_{\tilde{\kappa}}}{\tilde{\kappa}} + \frac{2}{3}\frac{\beta^y}{y\alpha}K + \left(\frac{\beta^y}{y\alpha}\right)^2\right],
\end{aligned} \tag{2.13}$$

where we have used the covariant derivative $\tilde{D}_m$ of the conformally rescaled three-metric $\tilde{\gamma}_{ij}$. The following expressions need special treatment at $y=0$. For clarity, we will introduce indices a, b,... which denote either x or z but not y. The function f stands for either of the variables χ, $\tilde{\kappa}$ or α.

$$\begin{aligned}
\frac{\tilde{\gamma}^{yy}\tilde{\kappa}-1}{y^2} &\to \frac{1}{2}(\tilde{\kappa}\partial_y\partial_y\tilde{\gamma}^{yy} + \tilde{\gamma}^{yy}\partial_y\partial_y\tilde{\kappa}), \\
\frac{\tilde{\gamma}^{ya}}{y}\partial_a f &\to (\partial_y\tilde{\gamma}^{ya})(\partial_a f), \\
\frac{\tilde{\gamma}^{yy}}{y}\partial_y f &\to \tilde{\gamma}^{yy}\partial_y\partial_y f, \\
\Rightarrow \frac{\tilde{\gamma}^{ym}\partial_m f}{y} &\to (\partial_y\tilde{\gamma}^{ya})(\partial_a f) + \tilde{\gamma}^{yy}\partial_y\partial_y f, \\
\frac{\tilde{\Gamma}^y}{y} &\to \partial_y\tilde{\Gamma}^y, \\
\frac{\beta^y}{y} &\to \partial_y\beta^y,
\end{aligned} \tag{2.14}$$

with $\partial_y\partial_y\tilde{\gamma}^{yy}$ given by Eq. (1.15). Note that the first line of (2.14) implies that

$$\tilde{\gamma}^{yy}\tilde{\kappa} - 1 = \mathcal{O}(y^2) \quad \Rightarrow \quad \tilde{\kappa} - \tilde{\gamma}_{yy} = \mathcal{O}(y^2). \tag{2.15}$$

While we cannot use the latter expression to regularize the term on the first line of (2.14) as has erroneously been claimed in an earlier version of these notes, we shall be able to use it in the discussion of the vectorial source term in Secs. 2.3 and 3.4

2.3 The vectorial source term

The evolution of the BSSN variable $\tilde{\Gamma}^i$ involves a source term j_i. Using the same fundamental variables as in the previous section we obtain for this term (note that the contravariant version follows from $j^i = \chi\tilde{\gamma}^{im} j_m$)

$$\begin{aligned} \frac{16\pi j_i}{D-4} &= \frac{2}{y}\left[\delta_i{}^y \frac{K_{\tilde{\kappa}}}{\tilde{\kappa}} + \frac{1}{\alpha}\partial_i\beta^y - \tilde{\gamma}^{ym}\tilde{A}_{mi}\right] + 2\frac{1}{\tilde{\kappa}}\partial_i K_{\tilde{\kappa}} - \frac{K_{\tilde{\kappa}}}{\tilde{\kappa}}\left(\frac{1}{\chi}\partial_i\chi + \frac{1}{\tilde{\kappa}}\partial_i\tilde{\kappa}\right) \\ &\quad + \frac{2}{3}\partial_i K - 2\frac{\beta^y}{y\alpha^2}\partial_i\alpha + \frac{\beta^y}{y\alpha}\left(\frac{1}{\tilde{\kappa}}\partial_i\tilde{\kappa} - \frac{1}{\chi}\partial_i\chi\right) \\ &\quad - \tilde{\gamma}^{nm}\tilde{A}_{mi}\left(\frac{1}{\tilde{\kappa}}\partial_n\tilde{\kappa} - \frac{1}{\chi}\partial_n\chi\right). \end{aligned} \tag{2.16}$$

Here we need to discuss three terms which need special attention on the xz plane. The first we are already familiar with:

$$\frac{\beta^y}{y} \to \partial_y\beta^y. \tag{2.17}$$

For the other terms we need to distinguish between y components on the one hand and x, z components on the other. As before we denote the latter by early latin indices a, b... . The second special term is

$$\frac{1}{y}\tilde{\gamma}^{ym}\tilde{A}_{mi} \tag{2.18}$$

and becomes

$$\frac{1}{y}\tilde{\gamma}^{ym}\tilde{A}_{ma} = \frac{1}{y}\tilde{\gamma}^{yc}\tilde{A}_{ca} + \frac{1}{y}\tilde{\gamma}^{yy}\tilde{A}_{ya} \to \tilde{A}_{ca}\partial_y\tilde{\gamma}^{yc} + \tilde{\gamma}^{yy}\partial_y\tilde{A}_{ya}, \tag{2.19}$$

$$\frac{1}{y}\tilde{\gamma}^{ym}\tilde{A}_{my} = \frac{1}{y}\tilde{\gamma}^{yc}\tilde{A}_{cy} + \frac{1}{y}\tilde{\gamma}^{yy}\tilde{A}_{yy} \to \frac{\mathcal{O}(y^2)}{y} + \frac{1}{y}\tilde{\gamma}^{yy}\tilde{A}_{yy} = \frac{1}{y}\tilde{\gamma}^{yy}\tilde{A}_{yy}. \tag{2.20}$$

Both terms on the right hand side of the first line are manifestly regular and can be calculated straightforwardly at $y = 0$. The y component leaves us with a term, however, which we need to combine with the next set of terms. This final term we need to discuss is

$$\frac{2}{y}\left[\delta_i{}^y\frac{K_{\tilde{\kappa}}}{\tilde{\kappa}} + \frac{1}{\alpha}\partial_i\beta^y\right]. \tag{2.21}$$

This term is trickier and we still would like to understand analytically why the terms in brackets must vanish at $y = 0$. Assuming, though, that this indeed be the case, we arrive at a remarkably simple solution for this term. Let us first discuss the case $i = y$ which requires us to also include the term from Eq. (2.20) and leads to

$$\frac{2}{y}\left(\frac{K_{\tilde{\kappa}}}{\tilde{\kappa}} + \frac{1}{\alpha}\partial_y\beta^y - \tilde{\gamma}^{yy}\tilde{A}_{yy}\right). \tag{2.22}$$

In trying to understand why this term must vanish at $y = 0$, we consider the time derivative of the above condition (2.14), first line. For this purpose we recall that

$$
\begin{aligned}
\tilde{\kappa} &= \tilde{\gamma}_{yy} + \mathcal{O}(y^2), && (2.23)\\
\partial_t \chi &= \beta^m \partial_m \chi - \frac{2}{3}\chi \partial_m \beta^m + \frac{2}{3}\alpha \chi K, && (2.24)\\
\partial_t \tilde{\kappa} &= \beta^m \partial_m \tilde{\kappa} - \frac{2}{3}\tilde{\kappa}\partial_m \beta^m - 2\alpha K_{\tilde{\kappa}}, && (2.25)\\
\partial_t \tilde{\gamma}_{ij} &= \beta^m \partial_m \tilde{\gamma}_{ij} + \tilde{\gamma}_{mj}\partial_i \beta^m + \tilde{\gamma}_{im}\partial_j \beta^m - \frac{2}{3}\tilde{\gamma}_{ij}\partial_m \beta^m - 2\alpha \tilde{A}_{ij}, && (2.26)\\
\Rightarrow \partial_t \tilde{\gamma}_{yy} &= \beta^m \partial_m \tilde{\gamma}_{yy} + 2\tilde{\gamma}_{my}\partial_y \beta^m - \frac{2}{3}\tilde{\gamma}_{yy}\partial_m \beta^m - 2\alpha \tilde{A}_{yy}, && (2.27)
\end{aligned}
$$

In order to avoid a conical singularity at $y = 0$ we therefore obtain the condition

$$
\begin{aligned}
\partial_t(\tilde{\kappa} - \tilde{\gamma}_{yy}) &= \beta^m \partial_m \tilde{\kappa} - \frac{2}{3}\tilde{\kappa}\partial_m \beta^m - 2\alpha K_{\tilde{\kappa}} - \beta^m \partial_m \tilde{\gamma}_{yy} - 2\tilde{\gamma}_{my}\partial_y \beta^m + \frac{2}{3}\tilde{\gamma}_{yy}\partial_m \beta^m + 2\alpha \tilde{A}_{yy}\\
&= -2\alpha K_{\tilde{\kappa}} - 2\tilde{\gamma}_{my}\partial_y \beta^m + 2\alpha \tilde{A}_{yy} + \mathcal{O}(y^2)\\
&= -2\alpha K_{\tilde{\kappa}} - 2\tilde{\gamma}_{yy}\partial_y \beta^y - \underbrace{2\tilde{\gamma}_{cy}\partial_y \beta^c}_{=\mathcal{O}(y^2)} + 2\alpha \tilde{A}_{yy} + \mathcal{O}(y^2)\\
&= -2\alpha K_{\tilde{\kappa}} - 2\tilde{\kappa}\partial_y \beta^y + 2\tilde{\kappa}\tilde{\gamma}^{yy}\alpha \tilde{A}_{yy} + \mathcal{O}(y^2)\\
&= -2\alpha\tilde{\kappa}\left(\frac{K_{\tilde{\kappa}}}{\tilde{\kappa}} + \frac{1}{\alpha}\partial_y \beta^y - \tilde{\gamma}^{yy}\tilde{A}_{yy}\right) + \mathcal{O}(y^2) && (2.28)
\end{aligned}
$$

Up to the regular factor $-2\alpha\tilde{\kappa}$ this is identical to the term (2.22) which we require to vanish. Even more, we know that the left hand side is the time derivative of Eq. (2.15) which we know to vanish $\sim y^2$. Put another way, all terms on the right hand side of (2.28) are even in y and therefore

$$
\text{rhs} = \frac{1}{y}\left(f_0 + f_2 y^2\right). \tag{2.29}
$$

Regularity requires that $f_0 = 0$, so that we are left with $\mathcal{O}(y^2)/y = 0$. The case $i = a \neq y$ is simpler and directly leads to

$$
\frac{1}{y}\left[0 + \frac{2}{\alpha}\partial_a \beta^y\right] \to \frac{2}{\alpha}\partial_y \partial_a \beta^y. \tag{2.30}
$$

Again, we can straightforwardly calculate second derivatives of β^y at $y = 0$ and will obtain a regular expression. In summary, we have

$$
\begin{aligned}
\frac{2}{y}\left[\delta_a{}^y \frac{K_{\tilde{\kappa}}}{\tilde{\kappa}} + \frac{1}{\alpha}\partial_a \beta^y - \tilde{\gamma}^{ym}\tilde{A}_{ma}\right] &\to \frac{2}{\alpha}\partial_y \partial_a \beta^y - 2\tilde{A}_{ca}\partial_y \tilde{\gamma}^{yc} - 2\tilde{\gamma}^{yy}\partial_y \tilde{A}_{ya},\\
\frac{2}{y}\left[\delta_y{}^y \frac{K_{\tilde{\kappa}}}{\tilde{\kappa}} + \frac{1}{\alpha}\partial_y \beta^y - \tilde{\gamma}^{ym}\tilde{A}_{my}\right] &\to 0. && (2.31)
\end{aligned}
$$

2.4 The tensorial source term

We next discuss the source term of the traceless part of the extrinsic curvature, or, rather, the conformally rescaled version thereof. Starting from Eq. (2.46b) of Carlos draft, we obtain

$$\begin{aligned}
\frac{8\pi\chi\left(S_{ij}-\frac{1}{3}\gamma_{ij}S\right)}{D-4} &= \frac{1}{2}\left[\frac{\chi}{y\tilde{\kappa}}\left(\delta_j{}^y\partial_i\tilde{\kappa}+\delta_i{}^y\partial_j\tilde{\kappa}-2\tilde{\kappa}\tilde{\Gamma}^y_{ij}\right)+\frac{1}{2\chi}(\partial_i\chi)(\partial_j\chi)+\frac{\chi}{\tilde{\kappa}}\partial_i\partial_j\tilde{\kappa}\right.\\
&\quad -\partial_i\partial_j\chi+\left(\tilde{\Gamma}^m_{ij}+\frac{1}{2\chi}\tilde{\gamma}_{ij}\tilde{\gamma}^{mn}\partial_n\chi\right)\left(\partial_m\chi-\frac{\chi}{\tilde{\kappa}}\partial_m\tilde{\kappa}\right)-\tilde{\gamma}_{ij}\frac{\gamma^{\tilde{y}m}}{y}\partial_m\chi\\
&\quad \left.-\frac{\chi}{2\tilde{\kappa}^2}(\partial_i\tilde{\kappa})(\partial_j\tilde{\kappa})\right]^{\mathrm{TF}}-\left(\frac{K_{\tilde{\kappa}}}{\tilde{\kappa}}+\frac{1}{3}K+\frac{\beta^y}{y\alpha}\right)\tilde{A}_{ij},\\
&= \frac{1}{2}\left[\frac{\chi}{y\tilde{\kappa}}\left(\delta_j{}^y\partial_i\tilde{\kappa}+\delta_i{}^y\partial_j\tilde{\kappa}-2\tilde{\kappa}\tilde{\Gamma}^y_{ij}\right)+\frac{1}{2\chi}(\partial_i\chi)(\partial_j\chi)+\frac{\chi}{\tilde{\kappa}}\tilde{D}_i\partial_j\tilde{\kappa}\right.\\
&\quad -\tilde{D}_i\partial_j\chi+\frac{1}{2\chi}\tilde{\gamma}_{ij}\tilde{\gamma}^{mn}\partial_n\chi\left(\partial_m\chi-\frac{\chi}{\tilde{\kappa}}\partial_m\tilde{\kappa}\right)-\tilde{\gamma}_{ij}\frac{\gamma^{\tilde{y}m}}{y}\partial_m\chi\\
&\quad \left.-\frac{\chi}{2\tilde{\kappa}^2}(\partial_i\tilde{\kappa})(\partial_j\tilde{\kappa})\right]^{\mathrm{TF}}-\left(\frac{K_{\tilde{\kappa}}}{\tilde{\kappa}}+\frac{1}{3}K+\frac{\beta^y}{y\alpha}\right)\tilde{A}_{ij}.
\end{aligned} \tag{2.32}$$

We again have our familiar term

$$\frac{\beta^y}{y}\to\partial_y\beta^y, \tag{2.33}$$

The term

$$\frac{\tilde{\gamma}^{ym}}{y}\partial_m\chi \tag{2.34}$$

needs the ususal distinction of $m=y$ and $m=c\neq y$ and becomes

$$\frac{\tilde{\gamma}^{yc}}{y}\partial_c\chi \quad\to\quad \left(\partial_y\tilde{\gamma}^{yc}\right)\partial_c\chi, \tag{2.35}$$

$$\frac{\tilde{\gamma}^{yy}}{y}\partial_y\chi \quad=\quad \tilde{\gamma}^{yy}\frac{\partial_y\chi}{y}\to\tilde{\gamma}^{yy}\partial_y\partial_y\chi, \tag{2.36}$$

so that

$$\frac{1}{y}\tilde{\gamma}^{my}\partial_m\chi\to(\partial_y\tilde{\gamma}^{yc})\partial_c\chi+\tilde{\gamma}^{yy}\partial_y\partial_y\chi. \tag{2.37}$$

The first term on the right hand side of Eq. (2.32), however, requires a little more attention. We distinguish between 4 different cases. First, $i=y$, $j=y$ leads to

$$\begin{aligned}
&\frac{\chi}{y\tilde{\kappa}}\left[\partial_y\tilde{\kappa}+\partial_y\tilde{\kappa}-\tilde{\kappa}\tilde{\gamma}^{ym}(\partial_y\tilde{\gamma}_{ym}+\partial_y\tilde{\gamma}_{ym}-\partial_m\tilde{\gamma}_{yy})\right]\\
=\;&\frac{\chi}{y\tilde{\kappa}}\left[2\partial_y\tilde{\kappa}-\tilde{\kappa}\tilde{\gamma}^{ym}(2\partial_y\tilde{\gamma}_{ym}-\partial_m\tilde{\gamma}_{yy})\right]\\
=\;&\frac{\chi}{y\tilde{\kappa}}\left[2\partial_y\tilde{\kappa}-\tilde{\kappa}\tilde{\gamma}^{yy}\partial_y\tilde{\gamma}_{yy}-\tilde{\kappa}\tilde{\gamma}^{yc}(2\partial_y\tilde{\gamma}_{yc}-\partial_c\tilde{\gamma}_{yy})\right]\\
\to\;&\frac{\chi}{\tilde{\kappa}}\left[2\partial_y\partial_y\tilde{\kappa}-\tilde{\kappa}\tilde{\gamma}^{yy}\partial_y\partial_y\tilde{\gamma}_{yy}-\tilde{\kappa}(\partial_y\tilde{\gamma}^{yc})(2\partial_y\tilde{\gamma}_{yc}-\partial_c\tilde{\gamma}_{yy})\right]
\end{aligned} \tag{2.38}$$

Second, we consider $i = a \neq y$, $j = y$ and obtain

$$\begin{aligned}
& \frac{\chi}{y\tilde{\kappa}}\left[\partial_a\tilde{\kappa} - \tilde{\kappa}\tilde{\gamma}^{ym}(\partial_a\tilde{\gamma}_{ym} + \partial_y\tilde{\gamma}_{ma} - \partial_m\tilde{\gamma}_{ay})\right] \\
= \; & \frac{\chi}{y\tilde{\kappa}}\left[\partial_a\tilde{\kappa} - \tilde{\kappa}\tilde{\gamma}^{yy}(\partial_a\tilde{\gamma}_{yy} + \partial_y\tilde{\gamma}_{ya} - \partial_y\tilde{\gamma}_{ay}) - \tilde{\kappa}\tilde{\gamma}^{yc}(\partial_a\tilde{\gamma}_{yc} + \partial_y\tilde{\gamma}_{ca} - \partial_c\tilde{\gamma}_{ay})\right] \\
\to \; & \frac{\chi}{y\tilde{\kappa}}\left[\partial_a\tilde{\kappa} - \tilde{\kappa}\tilde{\gamma}^{yy}\partial_a\tilde{\gamma}_{yy} - \mathcal{O}(y)\times\mathcal{O}(y)\right] \\
\to \; & \frac{\chi}{y\tilde{\kappa}}\left[\partial_a\tilde{\kappa} + \tilde{\kappa}\frac{1}{\tilde{\gamma}^{yy}}\partial_a\tilde{\gamma}^{yy}\right] = \frac{\chi}{y\tilde{\kappa}}\left[\frac{1}{\tilde{\gamma}^{yy}}\partial_a(\tilde{\kappa}\tilde{\gamma}^{yy})\right] = \frac{\chi}{y\tilde{\kappa}}\tilde{\gamma}_{yy}\partial_a(\tilde{\kappa}\tilde{\gamma}^{yy} - 1) \\
\to \; & \frac{\chi}{y\tilde{\kappa}}\tilde{\gamma}_{yy}\mathcal{O}(y^2),
\end{aligned} \tag{2.39}$$

where we have used $\tilde{\gamma}^{yy} = 1/\tilde{\gamma}_{yy}$ at $y = 0$ and the first line of Eq. (2.14) which implies that $\tilde{\kappa}\tilde{\gamma}^{yy} - 1 = \mathcal{O}(y^2)$. From symmetry we deduce that the third case, $i = y$, $j = b$ vanishes, too.
Finally, we consider case 4, i. e. $i = a$, $j = b$ which results in

$$\begin{aligned}
& \frac{\chi}{y\tilde{\kappa}}\left[-\tilde{\kappa}\tilde{\gamma}^{yy}(\partial_a\tilde{\gamma}_{by} + \partial_b\tilde{\gamma}_{ya} - \partial_y\tilde{\gamma}_{ab}) - \tilde{\kappa}\tilde{\gamma}^{yc}(\partial_a\tilde{\gamma}_{bc} + \partial_b\tilde{\gamma}_{ac} - \partial_c\tilde{\gamma}_{ab})\right] \\
= \; & -\chi\left[\tilde{\gamma}^{yy}\frac{1}{y}(\partial_a\tilde{\gamma}_{by} + \partial_b\tilde{\gamma}_{ya} - \partial_y\tilde{\gamma}_{ab}) + \frac{\tilde{\gamma}^{yc}}{y}(\partial_a\tilde{\gamma}_{bc} + \partial_b\tilde{\gamma}_{ac} - \partial_c\tilde{\gamma}_{ab})\right] \\
\to \; & -\chi\left[\tilde{\gamma}^{yy}(\partial_y\partial_a\tilde{\gamma}_{by} + \partial_y\partial_b\tilde{\gamma}_{ya} - \partial_y\partial_y\tilde{\gamma}_{ab}) + (\partial_y\tilde{\gamma}^{yc})(\partial_a\tilde{\gamma}_{bc} + \partial_b\tilde{\gamma}_{ac} - \partial_c\tilde{\gamma}_{ab})\right]
\end{aligned} \tag{2.40}$$

2.5 The evolution of $\tilde{\kappa}$

The evolution of $\tilde{\kappa}$ is formulated in terms of a first order in time system of equations. for this purpose we have introduced in Eq. (2.11) the variable $K_{\tilde{\kappa}}$. The time evolution of $\tilde{\kappa}$ is thus given by

$$\partial_t\tilde{\kappa} = -2\alpha K_{\tilde{\kappa}} + \beta^m\partial_m\tilde{\kappa} - \frac{2}{3}\tilde{\kappa}\partial_m\beta^m. \tag{2.41}$$

The evolution of K_λ is given by Eq. (2.37) of Carlos' draft and we wish to rewrite this equation in terms of the rescaled variable $\tilde{\kappa}$. For this purpose we recall Eq. (2.12) and apply the derivative operator

$$\partial_0 := \partial_t - \beta^m\partial_m$$

to both sides. The resulting expression

$$\begin{aligned}
\partial_0 K_{\tilde{\kappa}} = \; & \frac{\chi}{y^2}\partial_0 K_\lambda + 4\frac{\beta^y}{y}K_{\tilde{\kappa}} + \frac{4}{3}\alpha K K_{\tilde{\kappa}} + \frac{4}{3}\tilde{\kappa}\frac{\beta^y}{y}K + \frac{2}{9}\tilde{\kappa}\alpha K^2 \\
& -\frac{\tilde{\kappa}}{3}\partial_0 K - \frac{\tilde{\kappa}}{\alpha}\frac{\partial_0\beta^y}{y} + \tilde{\kappa}\frac{\beta^y}{y\alpha^2}\partial_0\alpha + \frac{\tilde{\kappa}}{\alpha}\left(\frac{\beta^y}{y}\right)^2 - \frac{2}{3}K_{\tilde{\kappa}}\partial_m\beta^m
\end{aligned} \tag{2.42}$$

enables us to use Eq. (2.37) of Carlos' draft. We further need the time derivatives

$$\partial_0\tilde{\kappa} = -2\alpha K_{\tilde{\kappa}} - \frac{2}{3}\tilde{\kappa}\partial_m\beta^m, \tag{2.43}$$

$$\partial_0\chi = \frac{2}{3}\chi(\alpha K - \partial_m\beta^m), \tag{2.44}$$

$$\partial_0 y = -\beta^m\partial_m y = -\beta^y, \tag{2.45}$$

$$\partial_0 K = -D^m\partial_m\alpha + \alpha\left(\tilde{A}^{mn}\tilde{A}_{mn} + \frac{1}{3}K^2\right) + 4\pi\alpha(E + S). \tag{2.46}$$

Finally we need the gauge conditions because the source terms also contain time derivatives of lapse and shift. We will have to be careful to adjust these expressions in case we change the gauge conditions. At the moment, however, we work with the following coordinate choices

$$\partial_0\alpha = -2\alpha K, \tag{2.47}$$

$$\partial_0\beta^i = \frac{3}{4}\tilde{\Gamma}^i - \eta\beta^i, \tag{2.48}$$

where η is a free parameter or function. We have experimented both with constants and functional expressions of the form

$$\eta = \eta_0 \frac{r_a + r_b}{2(m_a r_b + m_b r_a)}, \tag{2.49}$$

where m_a is the (constant) mass parameter and r_a the coordinate distance of the point in question from the puncture position of hole a and likewise for hole b. η_0 is a constant which determines the overall scaling of this gauge term. In particular, we see from Eqs. (2.47), (2.48) that we will be able to use the symmetry of these expressions in imposing regularity on the xz plane.
We have now assembled all the tools necessary to obtain the time evolution of $K_{\tilde{\kappa}}$. A straightforward calculation converts Eq. (2.37) of Carlos' draft into

$$\begin{aligned}
\partial_t K_{\tilde{\kappa}} &= \beta^m\partial_m K_{\tilde{\kappa}} - \frac{2}{3}K_{\tilde{\kappa}}\partial_m\beta^m + \alpha\Bigg[(5-D)\frac{\chi}{y^2}(\tilde{\kappa}\tilde{\gamma}^{yy}-1) + (4-D)\frac{\chi}{y}\tilde{\gamma}^{ym}\partial_m\tilde{\kappa} \\
&\quad + \frac{2D-7}{2}\frac{\tilde{\kappa}}{y}\tilde{\gamma}^{ym}\partial_m\chi + \frac{6-D}{4}\frac{\chi}{\tilde{\kappa}}\tilde{\gamma}^{mn}(\partial_m\tilde{\kappa})(\partial_n\tilde{\kappa}) + \frac{2D-7}{4}\tilde{\gamma}^{mn}(\partial_m\tilde{\kappa})(\partial_n\chi) \\
&\quad + \frac{1-D}{4}\frac{\tilde{\kappa}}{\chi}\tilde{\gamma}^{mn}(\partial_m\chi)(\partial_n\chi) + (D-6)\frac{1}{\tilde{\kappa}}K_{\tilde{\kappa}}^2 + \frac{2D-5}{3}\left(K_{\tilde{\kappa}} + \tilde{\kappa}\frac{\beta^y}{y\alpha}\right)K \\
&\quad + 2(D-4)\frac{\beta^y}{y\alpha}K_{\tilde{\kappa}} + \frac{D-1}{9}\tilde{\kappa}K^2 + (D-5)\tilde{\kappa}\left(\frac{\beta^y}{y\alpha}\right)^2 + \frac{1}{2}\tilde{\gamma}^{mn}\left(\tilde{\kappa}\partial_m\partial_n\chi - \chi\partial_m\partial_n\tilde{\kappa}\right) \\
&\quad + \chi\tilde{\kappa}\frac{\tilde{\Gamma}^y}{y} + \frac{1}{2}\tilde{\Gamma}^m(\chi\partial_m\tilde{\kappa} - \tilde{\kappa}\partial_m\chi)\Bigg] - \frac{\chi\tilde{\kappa}}{y}\tilde{\gamma}^{ym}\partial_m\alpha - \frac{1}{2}\tilde{\gamma}^{mn}(\partial_m\alpha)(\chi\partial_n\tilde{\kappa} - \tilde{\kappa}\partial_n\chi) \\
&\quad - \frac{1}{3}\tilde{\kappa}\partial_0 K - \tilde{\kappa}\frac{\partial_0\beta^y}{y\alpha} + \tilde{\kappa}\frac{\beta^y}{y\alpha^2}\partial_0\alpha \\
&= \beta^m\partial_m K_{\tilde{\kappa}} - \frac{2}{3}K_{\tilde{\kappa}}\partial_m\beta^m + \alpha\Bigg[(5-D)\frac{\chi}{y^2}(\tilde{\kappa}\tilde{\gamma}^{yy}-1) + (4-D)\frac{\chi}{y}\tilde{\gamma}^{ym}\partial_m\tilde{\kappa} \\
&\quad + \frac{2D-7}{2}\frac{\tilde{\kappa}}{y}\tilde{\gamma}^{ym}\partial_m\chi + \frac{6-D}{4}\frac{\chi}{\tilde{\kappa}}\tilde{\gamma}^{mn}(\partial_m\tilde{\kappa})(\partial_n\tilde{\kappa}) + \frac{2D-7}{4}\tilde{\gamma}^{mn}(\partial_m\tilde{\kappa})(\partial_n\chi) \\
&\quad + \frac{1-D}{4}\frac{\tilde{\kappa}}{\chi}\tilde{\gamma}^{mn}(\partial_m\chi)(\partial_n\chi) + (D-6)\frac{1}{\tilde{\kappa}}K_{\tilde{\kappa}}^2 + \frac{2D-5}{3}\left(K_{\tilde{\kappa}} + \tilde{\kappa}\frac{\beta^y}{y\alpha}\right)K + 2(D-4)\frac{\beta^y}{y\alpha}K_{\tilde{\kappa}} \\
&\quad + \frac{D-1}{9}\tilde{\kappa}K^2 + (D-5)\tilde{\kappa}\left(\frac{\beta^y}{y\alpha}\right)^2 + \frac{1}{2}\tilde{\gamma}^{mn}\left(\tilde{\kappa}\tilde{D}_m\partial_n\chi - \chi\tilde{D}_m\partial_n\tilde{\kappa}\right) + \chi\tilde{\kappa}\frac{\tilde{\Gamma}^y}{y}\Bigg] \\
&\quad - \frac{\chi\tilde{\kappa}}{y}\tilde{\gamma}^{ym}\partial_m\alpha - \frac{1}{2}\tilde{\gamma}^{mn}(\partial_m\alpha)(\chi\partial_n\tilde{\kappa} - \tilde{\kappa}\partial_n\chi) - \frac{1}{3}\tilde{\kappa}\partial_0 K - \tilde{\kappa}\frac{\partial_0\beta^y}{y\alpha} + \tilde{\kappa}\frac{\beta^y}{y\alpha^2}\partial_0\alpha
\end{aligned} \tag{2.50}$$

Divisions by y occur on a few occassions in these source terms, but we are already familiar with all these cases from Eq. (2.14) and need no further treatment at this stage.

3 An alternative formulation

3.1 Motivation

With the formulation of the previous section we encountered problems in the case of a collapsed lapse. In retrospect, this is not surprising if we consider the final two terms in Eq. (2.50). As the lapse $\alpha \to 0$, these terms will diverge for standard moving puncture gauge conditions (2.47) and (2.48). The same applies to the term involving $\beta^{y2}/(y^2\alpha^2)$ in the second last line of Eq. (2.50). Even if these terms miraculously cancel, it is highly unlikely that this is reflected accurately in a numerical treatment. We do not even have reason to believe that such a miraculous cancelation exists. After all, the variable κ evolved according to Eq. (2.41) where $K_{\tilde{\kappa}}$ is multiplied by a factor of α on the right hand side. There is nothing intrinsically wrong with a diverging $K_{\tilde{\kappa}}$, therefore, as it involves only a finite and regular change of $\tilde{\kappa}$ at locations where the lapse has collapsed. We could attempt to cure this behavior in three ways: (i) we start with $\alpha = 1$ everywhere. Indeed, such evolutions start without any diverging terms, but a collapse of the lapse would still result in all the problems mentioned above. A collapse of the lapse, on the other hand, is what we would expect in a stable evolution, so this possibility is unlikely to work and we find numerical experiments to go unstable after about 50 iterations uding this approach[1]. (ii) We can use different gauge conditions by introducing an extra factor of α on the right hand sides of both Eqs. (2.47) and (2.48). We have tested corresponding simulations of traditional "3+1" Schwarzschild holes using such gauge conditions. These were running stably for about 500 iterations, but showed stronger variations of the variables that standard moving puncture gauge. Moreover, we would loose the key ingredient of the moving puncture trick, namely the non-vanishing shift β^i at the puncture which allows the holes to move across the grid. (iii) The same as before, but we apply the extra factor of α in the evolution of the shift to β^y, only. As before, however, this introduces more variation and less smooth profiles into variables such as Γ^y as compared with the standard moving puncture evolutions.

3.2 The evolution of $\tilde{\kappa}$

It seems, in consequence, that the variable $K_{\tilde{\kappa}}$ as defined in Eq. (2.11) is not an ideal choice for converting the evolution of $\tilde{\kappa}$ into a first order system. In the spirit of the comment in the previous paragraph, it appears that we are piping a finite variation of $\tilde{\kappa}$ as $\alpha \to 0$ through an infinite variation in $K_{\tilde{\kappa}}$ only to regularize it in Eq. (2.41) with an additional factor of α. These potential irregularities in $K_{\tilde{\kappa}}$ can be traced back to the appearance of the term β^y/α in Eq. (2.12) which relates the regular K_λ to $K_{\tilde{\kappa}}$. In this section, we will prescribe an alternative evolution of the D-dimensional Einstein equations using the variable

$$K_{\bar{\kappa}} := K_{\tilde{\kappa}} + \tilde{\kappa}\frac{\beta^y}{y\alpha} = -\frac{1}{2\alpha}\left(\partial_t\tilde{\kappa} - \beta^m\partial_m\tilde{\kappa} + \frac{2}{3}\tilde{\kappa}\partial_m\beta^m - 2\tilde{\kappa}\frac{\beta^y}{y}\right), \tag{3.1}$$

$$K_\lambda = \frac{y^2}{W^2}K_{\bar{\kappa}} + \frac{1}{3}\frac{y^2\tilde{\kappa}}{W^2}K. \tag{3.2}$$

so that

$$\partial_t\tilde{\kappa} = -2\alpha K_{\bar{\kappa}} + \beta^m\partial_m\tilde{\kappa} - \frac{2}{3}\tilde{\kappa}\partial_m\beta^m + 2\tilde{\kappa}\frac{\beta^y}{y}, \tag{3.3}$$

$$\tag{3.4}$$

[1]We cannot rule out at this stage, of course, that such instabilities are the result of erroneous implementation of the equations, well checked though the code is.

is our evolution equation of $\tilde{\kappa}$. The evolution of the new auxiliary variable $K_{\bar{\kappa}}$ now simplifies to

$$
\begin{aligned}
\partial_t K_{\bar{\kappa}} \;=\;& \beta^m \partial_m K_{\bar{\kappa}} - \frac{2}{3} K_{\bar{\kappa}} \partial_m \beta^m + 2\frac{\beta^y}{y} K_{\bar{\kappa}} - \frac{1}{3}\tilde{\kappa}\partial_0 K - \frac{W^2\tilde{\kappa}}{y}\tilde{\gamma}^{ym}\partial_m\alpha - \frac{W}{2}\tilde{\gamma}^{mn}(\partial_m\alpha)(W\partial_n\tilde{\kappa} - 2\tilde{\kappa}\partial_n W) \\
& +\alpha\left[(5-D)\frac{W^2}{y^2}(\tilde{\kappa}\tilde{\gamma}^{yy}-1) + (4-D)\frac{W^2}{y}\tilde{\gamma}^{ym}\partial_m\tilde{\kappa} + (2D-7)\frac{W\tilde{\kappa}}{y}\tilde{\gamma}^{ym}\partial_m W\right. \\
& +\frac{6-D}{4}\frac{W^2}{\tilde{\kappa}}\tilde{\gamma}^{mn}(\partial_m\tilde{\kappa})(\partial_n\tilde{\kappa}) + \frac{2D-7}{2}\tilde{\gamma}^{mn}W(\partial_m\tilde{\kappa})(\partial_n W) + (2-D)\tilde{\kappa}\tilde{\gamma}^{mn}(\partial_m W)(\partial_n W) \\
& \left.+(D-6)\frac{K_{\bar{\kappa}}^2}{\tilde{\kappa}} + \frac{2D-5}{3}KK_{\bar{\kappa}} + \frac{D-1}{9}\tilde{\kappa}K^2 + \frac{1}{2}\tilde{\gamma}^{mn}W\left(2\tilde{\kappa}\tilde{D}_m\partial_n W - W\tilde{D}_m\partial_n\tilde{\kappa}\right) + W^2\tilde{\kappa}\frac{\tilde{\Gamma}^y}{y}\right]
\end{aligned}
\tag{3.5}
$$

In the remainder of this section we discuss in turn the three source terms and how they are modified by the introduction of $K_{\bar{\kappa}}$ in place of $K_{\tilde{\kappa}}$.

3.3 The scalar source term

The source term $E+S$ can either be calculated from scratch starting with Eq. (2.46a) of Carlos' draft or by using the above equation (2.13) where we substitute $K_{\tilde{\kappa}}$ with $K_{\bar{\kappa}}$. The result is

$$
\begin{aligned}
\frac{4\pi(E+S)}{D-4} \;=\;& (D-5)\frac{W^2}{\tilde{\kappa}}\frac{\tilde{\gamma}^{yy}\tilde{\kappa}-1}{y^2} - \frac{2D-7}{2}\frac{W}{\tilde{\kappa}}\tilde{\gamma}^{mn}(\partial_m\tilde{\kappa})(\partial_n W) + (D-2)\tilde{\gamma}^{mn}(\partial_m W)(\partial_n W) \\
& +\frac{D-6}{4}\frac{W^2}{\tilde{\kappa}^2}\tilde{\gamma}^{mn}(\partial_m\tilde{\kappa})(\partial_n\tilde{\kappa}) + \frac{W}{2\tilde{\kappa}}\tilde{\gamma}^{mn}(W\tilde{D}_m\partial_n\tilde{\kappa} - 2\tilde{\kappa}\tilde{D}_m\partial_n W) - \frac{KK_{\bar{\kappa}}}{\tilde{\kappa}} - \frac{1}{3}K^2 \\
& +(D-4)W\frac{\tilde{\gamma}^{ym}}{y}\left(\frac{W}{\tilde{\kappa}}\partial_m\tilde{\kappa} - 2\partial_m W\right) - W\frac{\tilde{\gamma}^{ym}}{y}\partial_m W - W^2\frac{\tilde{\Gamma}^y}{y} - (D-5)\left(\frac{K_{\bar{\kappa}}}{\tilde{\kappa}} + \frac{K}{3}\right)^2 .
\end{aligned}
\tag{3.6}
$$

3.4 The vectorial source term

For the vectorial source term we similarly obtain

$$
\begin{aligned}
\frac{16\pi j_i}{D-4} \;=\;& \frac{2}{y}\left[\delta_i{}^y\frac{K_{\bar{\kappa}}}{\tilde{\kappa}} - \tilde{\gamma}^{ym}\tilde{A}_{mi}\right] + 2\frac{1}{\tilde{\kappa}}\partial_i K_{\bar{\kappa}} - \frac{K_{\bar{\kappa}}}{\tilde{\kappa}}\left(\frac{2}{W}\partial_i W + \frac{1}{\tilde{\kappa}}\partial_i\tilde{\kappa}\right) \\
& +\frac{2}{3}\partial_i K - \tilde{\gamma}^{nm}\tilde{A}_{mi}\left(\frac{1}{\tilde{\kappa}}\partial_n\tilde{\kappa} - \frac{2}{W}\partial_n W\right).
\end{aligned}
\tag{3.7}
$$

In this case, we also have a small modification of the regularization procedure at $y=0$. The discussion of the term

$$\frac{1}{y}\tilde{\gamma}^{ym}\tilde{A}_{mi}$$

in Sec. 2.3 is valid as before. The combined term for $i=y$, however, is now given by

$$\frac{2}{y}\left(\frac{K_{\bar{\kappa}}}{\tilde{\kappa}} - \tilde{\gamma}^{yy}\tilde{A}_{yy}\right). \tag{3.8}$$

In trying to understand why this term must vanish at $y = 0$, we again consider the time derivative of the above condition (2.14), first line. For this purpose we recall that now

$$\begin{aligned}
\tilde{\kappa} &= \tilde{\gamma}_{yy} + \mathcal{O}(y^2), && (3.9)\\
\partial_t \chi &= \beta^m \partial_m \chi - \frac{2}{3}\chi \partial_m \beta^m + \frac{2}{3}\alpha \chi K, && (3.10)\\
\partial_t \tilde{\kappa} &= \beta^m \partial_m \tilde{\kappa} - \frac{2}{3}\tilde{\kappa}\partial_m \beta^m + 2\tilde{\kappa}\frac{\beta^y}{y} - 2\alpha K_{\tilde{\kappa}}, && (3.11)\\
\partial_t \tilde{\gamma}_{ij} &= \beta^m \partial_m \tilde{\gamma}_{ij} + \tilde{\gamma}_{mj}\partial_i \beta^m + \tilde{\gamma}_{im}\partial_j \beta^m - \frac{2}{3}\tilde{\gamma}_{ij}\partial_m \beta^m - 2\alpha \tilde{A}_{ij}, && (3.12)\\
\Rightarrow \partial_t \tilde{\gamma}_{yy} &= \beta^m \partial_m \tilde{\gamma}_{yy} + 2\tilde{\gamma}_{my}\partial_y \beta^m - \frac{2}{3}\tilde{\gamma}_{yy}\partial_m \beta^m - 2\alpha \tilde{A}_{yy}, && (3.13)
\end{aligned}$$

In order to avoid a conical singularity at $y = 0$ we therefore obtain the condition

$$\begin{aligned}
\partial_t(\tilde{\kappa} - \tilde{\gamma}_{yy}) &= \beta^m \partial_m \tilde{\kappa} - \frac{2}{3}\tilde{\kappa}\partial_m \beta^m + 2\tilde{\kappa}\frac{\beta^y}{y} - 2\alpha K_{\tilde{\kappa}} - \beta^m \partial_m \tilde{\gamma}_{yy} - 2\tilde{\gamma}_{my}\partial_y \beta^m + \frac{2}{3}\tilde{\gamma}_{yy}\partial_m \beta^m + 2\alpha \tilde{A}_{yy}\\
&= -2\alpha K_{\tilde{\kappa}} + 2\tilde{\kappa}\frac{\beta^y}{y} - 2\tilde{\gamma}_{my}\partial_y \beta^m + 2\alpha \tilde{A}_{yy} + \mathcal{O}(y^2)\\
&= -2\alpha K_{\tilde{\kappa}} + 2\tilde{\kappa}\frac{\beta^y}{y} - 2\tilde{\gamma}_{yy}\partial_y \beta^y - \underbrace{2\tilde{\gamma}_{cy}\partial_y \beta^c}_{=\mathcal{O}(y^2)} + 2\alpha \tilde{A}_{yy} + \mathcal{O}(y^2)\\
&= -2\alpha K_{\tilde{\kappa}} + 2\tilde{\kappa}\frac{\beta^y}{y} - 2\tilde{\kappa}\partial_y \beta^y + 2\tilde{\kappa}\tilde{\gamma}^{yy}\alpha \tilde{A}_{yy} + \mathcal{O}(y^2)\\
&= -2\alpha \tilde{\kappa}\left(\frac{K_{\tilde{\kappa}}}{\tilde{\kappa}} - \frac{1}{\alpha}\frac{\beta^y}{y} + \frac{1}{\alpha}\partial_y \beta^y - \tilde{\gamma}^{yy}\tilde{A}_{yy}\right) + \mathcal{O}(y^2)\\
&= -2\alpha \tilde{\kappa}\left(\frac{K_{\tilde{\kappa}}}{\tilde{\kappa}} - \tilde{\gamma}^{yy}\tilde{A}_{yy}\right) + \mathcal{O}(y^2), && (3.14)
\end{aligned}$$

where we have used that $\frac{\beta^y}{y} = \partial_y \beta^y + \mathcal{O}(y^2)$ as $y \to 0$. Up to the regular factor $-2\alpha\tilde{\kappa}$ this is identical to the term (3.8) which we require to vanish. Even more, we know that the left hand side is the time derivative of Eq. (2.15) which we know to vanish $\sim y^2$. Put another way, all terms on the right hand side of (3.14) are even in y and therefore

$$\text{rhs} = \frac{1}{y}\left(f_0 + f_2 y^2\right). \tag{3.15}$$

Regularity requires that $f_0 = 0$, so that we are left with $\mathcal{O}(y^2)/y = 0$. The case $i = a \neq y$ directly yields zero because of the Kronecker symbol. In summary, we have

$$\begin{aligned}
\frac{2}{y}\left[\delta_a{}^y \frac{K_{\tilde{\kappa}}}{\tilde{\kappa}} - \tilde{\gamma}^{ym}\tilde{A}_{ma}\right] &\to -2\tilde{A}_{ca}\partial_y \tilde{\gamma}^{yc} - 2\tilde{\gamma}^{yy}\partial_y \tilde{A}_{ya},\\
\frac{2}{y}\left[\delta_y{}^y \frac{K_{\tilde{\kappa}}}{\tilde{\kappa}} - \tilde{\gamma}^{ym}\tilde{A}_{my}\right] &\to 0. && (3.16)
\end{aligned}$$

3.5 The tensorial source term

The tensorial term is only mildly affected by our modified variable $K_{\tilde\kappa}$ and becomes

$$\begin{aligned}\frac{8\pi W^2\left(S_{ij}-\frac{1}{3}\gamma_{ij}S\right)}{D-4} \quad = \quad & \frac{1}{2}\left[\frac{W^2}{y\tilde\kappa}\left(\delta_j{}^y\partial_i\tilde\kappa+\delta_i{}^y\partial_j\tilde\kappa-2\tilde\kappa\tilde\Gamma^y_{ij}\right)+\frac{W^2}{\tilde\kappa}\tilde D_i\partial_j\tilde\kappa\right.\\ & -2W\tilde D_i\partial_j W+\tilde\gamma_{ij}\tilde\gamma^{mn}\partial_n W\left(2\partial_m W-\frac{W}{\tilde\kappa}\partial_m\tilde\kappa\right)-2W\tilde\gamma_{ij}\frac{\gamma^{\tilde y m}}{y}\partial_m W\\ & \left.-\frac{W^2}{2\tilde\kappa^2}(\partial_i\tilde\kappa)(\partial_j\tilde\kappa)\right]^{\mathrm{TF}}-\left(\frac{K_{\tilde\kappa}}{\tilde\kappa}+\frac{1}{3}K\right)\tilde A_{ij}. \end{aligned} \tag{3.17}$$

The regularization from Sec. 2.4 remains valid.

4 Constraints

The constraints including matter terms are now given by

$$\mathcal{H} := R+\frac{2}{3}K^2-\tilde\gamma^{mn}\tilde\gamma^{kl}\tilde A_{mk}\tilde A_{nl}-16\pi E, \tag{4.1}$$

$$\mathcal{M}_i := \tilde\gamma^{mn}\left(\tilde D_n\tilde A_{im}-\frac{3}{2}\tilde A_{mi}\frac{\partial_n\chi}{\chi}\right)-\frac{2}{3}\partial_i K-8\pi j_i. \tag{4.2}$$

In addition to the above calculated source terms we also need to express E in terms of our fundamental variables which becomes

$$\begin{aligned}\frac{16\pi E}{D-4} \quad = \quad & (D-5)\frac{W^2}{\tilde\kappa}\frac{\tilde\gamma^{yy}\tilde\kappa-1}{y^2}+(D-3)\frac{W^2}{y\tilde\kappa}\tilde\gamma^{ym}\partial_m\tilde\kappa-2(D-2)\frac{W}{y}\tilde\gamma^{ym}\partial_m W+\frac{D-7}{4}\frac{W^2}{\tilde\kappa^2}\tilde\gamma^{mn}(\partial_m\tilde\kappa)(\partial_n\tilde\kappa)\\ & -(D-2)\frac{W}{\tilde\kappa}\tilde\gamma^{mn}(\partial_m\tilde\kappa)(\partial_n W)+(D+1)\tilde\gamma^{mn}(\partial_m W)(\partial_n W)-(D-5)\frac{K^2_{\tilde\kappa}}{\tilde\kappa^2}-\frac{2D-4}{3}K\frac{K_{\tilde\kappa}}{\tilde\kappa}\\ & -\frac{D+1}{9}K^2+\frac{W^2}{\tilde\kappa}\tilde\gamma^{mn}\tilde D_m\partial_n\tilde\kappa-2W\tilde\gamma^{mn}\tilde D_m\partial_n W-2W^2\frac{\tilde\Gamma^y}{y}. \end{aligned} \tag{4.3}$$

In order to complete our list of expressions, we will also give S_{ij} (not the tracefree version) and S in terms of our fields

$$\begin{aligned}\frac{8\pi S_{ij}}{D-4} \quad = \quad & \frac{1}{2y\tilde\kappa}\delta_j{}^y\partial_i\tilde\kappa+\frac{1}{2y\tilde\kappa}\delta_i{}^y\partial_j\tilde\kappa+\frac{1}{2\tilde\kappa}\tilde D_i\partial_j\tilde\kappa-\frac{1}{W}\tilde D_i\partial_j\tilde W-\frac{1}{y}\tilde\Gamma^y_{ij}-\frac{1}{yW}\tilde\gamma_{ij}\tilde\gamma^{ym}\partial_m W-\frac{1}{\tilde\kappa^2}(\partial_i\tilde\kappa)(\partial_j\tilde\kappa)\\ & +\frac{1}{2}\tilde\gamma_{ij}\tilde\gamma^{mn}\left[\frac{2}{W^2}(\partial_m W)(\partial_n W)-\frac{1}{W\tilde\kappa}(\partial_m W)(\partial_n\tilde\kappa)\right]-\frac{1}{W^2}\left(\frac{K_{\tilde\kappa}}{\tilde\kappa}+\frac{1}{3}K\right)\left(\tilde A_{ij}+\frac{1}{3}\tilde\gamma_{ij}K\right)\\ & -\frac{D-5}{2}\left\{-\frac{\tilde\gamma^{yy}\tilde\kappa-1}{y^2\tilde\kappa}-\frac{1}{y\tilde\kappa}\tilde\gamma^{ym}\partial_m\tilde\kappa+\frac{2}{yW}\tilde\gamma^{ym}\partial_m W-\tilde\gamma^{mn}\left[\frac{1}{4\tilde\kappa^2}(\partial_m\tilde\kappa)(\partial_n\tilde\kappa)-\frac{1}{W\tilde\kappa}(\partial_m W)(\partial_n\tilde\kappa)\right.\right.\\ & \left.\left.+\frac{1}{W^2}(\partial_m W)(\partial_n W)\right]+\frac{1}{W^2}\left(\frac{K_{\tilde\kappa}}{\tilde\kappa}+\frac{1}{3}K\right)^2\right\}\tilde\gamma_{ij} \end{aligned} \tag{4.4}$$

$$
\begin{aligned}
\frac{8\pi S}{D-4} &= \frac{W^2}{y\tilde{\kappa}}\tilde{\gamma}^{ym}\partial_m\tilde{\kappa} - \frac{3}{y}W\tilde{\gamma}^{ym}\partial_m W + \tilde{\gamma}^{mn}\left[3(\partial_m W)(\partial_n W) - \frac{3W}{2\tilde{\kappa}}(\partial_m W)(\partial_n\tilde{\kappa}) - \frac{W^2}{4\tilde{\kappa}^2}(\partial_m\tilde{\kappa})(\partial_n\tilde{\kappa})\right. \\
&\quad \left. + \frac{W^2}{2\tilde{\kappa}}\tilde{D}_m\partial_n\tilde{\kappa} - W\tilde{D}_m\partial_n W\right] - \frac{W^2}{y}\tilde{\Gamma}^y - K\left(\frac{K_{\tilde{\kappa}}}{\tilde{\kappa}} + \frac{1}{3}K\right) + \frac{3(D-5)}{2}\left\{\frac{W^2}{\tilde{\kappa}}\frac{\tilde{\gamma}^{yy}\tilde{\kappa}-1}{y^2}\right. \\
&\quad + \frac{W^2}{y\tilde{\kappa}}\tilde{\gamma}^{ym}\partial_m\tilde{\kappa} - \frac{2W}{y}\tilde{\gamma}^{ym}\partial_m W + \tilde{\gamma}^{mn}\left[\frac{W^2}{4\tilde{\kappa}^2}(\partial_m\tilde{\kappa})(\partial_n\tilde{\kappa}) - \frac{W}{\tilde{\kappa}}(\partial_m W)(\partial_n\tilde{\kappa}) + (\partial_m W)(\partial_n W)\right] \\
&\quad \left. - \left(\frac{K_{\tilde{\kappa}}}{\tilde{\kappa}} + \frac{K}{3}\right)^2\right\} \qquad (4.5)
\end{aligned}
$$

We conclude this formulation with the modifications of the standard BSSN equations when χ is replaced with $W = \sqrt{\chi}$. Here we merely list those terms which actually do change. Specifically, we obtain

$$
\begin{aligned}
\partial_t\tilde{\Gamma}^i &= -\tilde{A}^{im}3\alpha\frac{\partial_m\chi}{\chi} + \ldots\,, \\
&= -6\alpha\tilde{A}^{im}\frac{\partial_m W}{W} + \ldots\,, && (4.6) \\
\partial_t\tilde{A}_{ij} &= \chi\left(\alpha R_{ij} - D_i\partial_j\alpha\right)^{\mathrm{TF}} + \ldots\,, \\
&= W^2\left(\alpha R_{ij} - D_i\partial_j\alpha\right)^{\mathrm{TF}} + \ldots\,, && (4.7) \\
\partial_t\chi &= \beta^m\partial_m\chi + \frac{2}{3}\chi\left(\alpha K - \partial_m\beta^m\right), && (4.8) \\
\Rightarrow \partial_t W &= \beta^m\partial_m W + \frac{W}{3}\left(\alpha K - \partial_m\beta^m\right), && (4.9) \\
R^\phi_{ij} &= \frac{1}{2\chi}\tilde{D}_i\partial_j\chi - \frac{1}{4\chi^2}(\partial_i\chi)(\partial_j\chi) + \frac{1}{2\chi}\tilde{\gamma}_{ij}\tilde{\gamma}^{mn}\tilde{D}_m\partial_n\chi - \frac{3}{4\chi^2}\tilde{\gamma}_{ij}\tilde{\gamma}^{mn}(\partial_m\chi)(\partial_n\chi), \\
&= \frac{1}{W}\tilde{D}_i\partial_j W + \tilde{\gamma}_{ij}\tilde{\gamma}^{mn}\left[\frac{1}{W}\tilde{D}_m\partial_n W - \frac{2}{W^2}(\partial_m W)(\partial_n W)\right], && (4.10) \\
\chi D_i D_j\alpha &= \frac{1}{2}\left[(\partial_i\chi)(\partial_j\alpha) + (\partial_i\alpha)(\partial_j\chi)\right] + \chi\tilde{D}_i\partial_j\alpha - \frac{1}{2}\tilde{\gamma}_{ij}\tilde{\gamma}^{mn}(\partial_m\chi)(\partial_n\alpha), \\
&= W^2\tilde{D}_i\partial_j\alpha + W\left[(\partial_i W)(\partial_j\alpha) + (\partial_i\alpha)(\partial_j W)\right] - W\tilde{\gamma}_{ij}\tilde{\gamma}^{mn}(\partial_m W)(\partial_n\alpha). && (4.11)
\end{aligned}
$$