problem 2. zero mean

eigen analysis of covariance matrix and PCA

$$K = \begin{bmatrix} 4 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

var   covar of X_1 and X_3 covar betwn X_2 and X_3

find

a) eigen values λ_k + orthonormal eigenvectors e_k of K.

eigenvalues tell you how much data is spread in each direction.

eigenvalues (λ) → determinant $(K - \lambda I) = 0$

I = identity matrix = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\lambda I = \lambda \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$K - \lambda I = \begin{bmatrix} 4 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$K - \lambda I = \begin{bmatrix} 4-\lambda & -1 & 2 \\ -1 & 5-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix}$$

$$\det(K - \lambda I) = \det(\uparrow)$$

ex. $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ $\det = a(ei - fh) - b(di - fg) + c(dh - eg)$

$$\det = (4-\lambda)\left[[(5-\lambda)(3-\lambda)] - 1\right]$$
$$+ 1\left[-(3-\lambda) + 2\right]$$
$$+ 2\left[1 - [(5-\lambda)\cdot 2]\right]$$

term A

$$= 15 - 5\lambda + -3\lambda + \lambda^2 - 1$$
$$= 15 - 8\lambda + \lambda^2 - 1$$
$$= (4-\lambda)(\lambda^2 - 8\lambda + 14)$$
$$= 4\lambda^2 - 32\lambda + 56 - \lambda^3 + 8\lambda^2 - 14\lambda$$

$= 12\lambda^2 - 46\lambda + 56 - \lambda^3$

term A $= -\lambda^3 + 12\lambda^2 - 46\lambda + 56$

term B $= -(3-\lambda) + 2$

$= -3 + \lambda + 2$

$= -1 + \lambda$

$= \lambda - 1$

term C $= 2\left[1-(5-\lambda)2\right]$

$= 2\left[1 - 10 - 2\lambda\right]$

$= 18 - 4\lambda$

$\det(K - \lambda I) = (-\lambda^3 + 2\lambda^2 - 46\lambda + 56)$
$+ (\lambda - 1) + (4\lambda - 18)$

$0 = -\lambda^3 + 12\lambda^2 - 41\lambda + 37$

$\lambda_1 = 6.7 \quad \lambda_2 = 3.9 \quad \lambda_3 = 1.4$

finding eigenvectors

$$(K - \lambda I)e = 0$$

$$K - 6.7I = \begin{bmatrix} 4-6.7 & -1 & 2 \\ -1 & 5-6.7 & -1 \\ 2 & 3-6.7 & 3-6.7 \end{bmatrix}$$

$$= \begin{bmatrix} -2.7 & -1 & 2 \\ -1 & -1.7 & -1 \\ 2 & -1 & -3.7 \end{bmatrix}$$

$$e_1 = \begin{bmatrix} -.6 \\ 0.6 \\ -.5 \end{bmatrix}$$

$$e_2 = \begin{bmatrix} .5 \\ .8 \\ .3 \end{bmatrix}$$

$$e_3 = \begin{bmatrix} 0.6 \\ -0.06 \\ -0.8 \end{bmatrix}$$

2b) spectral decomposition

$$k = \lambda_1 e_1 e_1^T + \lambda_2 e_2 e_2^T + \lambda_3 e_3 e_3^T$$

$$\text{term 1} = \begin{bmatrix} 2.4 & -2.5 & 1.9 \\ -2.5 & 2.7 & -2.1 \\ 1.96 & -2.1 & 1.6 \end{bmatrix}$$

$$\text{term 2} = \begin{bmatrix} 1.1 & 1.6 & 0.7 \\ 1.6 & 2.3 & 1.0 \\ 0.7 & 1.0 & 0.47 \end{bmatrix}$$

$$\text{term 3} = \begin{bmatrix} 0.5 & -.05 & -.7 \\ -.05 & .004 & 0.66 \\ -.7 & .06 & 0.9 \end{bmatrix}$$

$$k = \begin{bmatrix} 4 & -1 & 2 \\ -1 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

c) KL expansion

$$x = Z_1 e_1 + Z_2 e_2 + Z_3 e_3$$

$$= Z_1 \begin{bmatrix} -.6 \\ 0.6 \\ -.5 \end{bmatrix} + Z_2 \begin{bmatrix} .5 \\ .8 \\ .3 \end{bmatrix} + Z_3 \begin{bmatrix} 0.6 \\ -0.06 \\ -0.8 \end{bmatrix}$$

d) $\tilde{x} = Z_1 e_1 + Z_2 e_2$

$$= Z_1 \begin{bmatrix} -.6 \\ 0.6 \\ -.5 \end{bmatrix} + Z_2 \begin{bmatrix} .5 \\ .8 \\ .3 \end{bmatrix}$$

total var = 12

captured var = 10.57 12

10.57/12 = 88.09%

by using 2 pc we captured 88% of total var while reducing dimensionality by a third.

e) $MSE = E[||x - \hat{x}||^2]$?

$$x = z_1 e_1 + z_2 e_2 + z_3 e_3$$

$$\hat{x} = z_1 e_1 + z_2 e_2$$

$$x - \hat{x} = (z_1 e_1 + z_2 e_2 + z_3 e_3) - (z_1 e_1 + z_2 e_2)$$

$$= z_3 e_3$$

$$MSE = E[||x - \hat{x}||^2]$$

$$= E||z_3 e_3||^2$$

$=$

z_3 hase zero meanso $E(z_3)=0$

$= E[z_3^2]-0$

$= var(z_3)$

$$\boxed{MSE = \lambda_3 = 1.428}$$

sum of discarded eigenvaues. represents variance lost going from 3D to 2D.