problem 1 : linear MMSE estimation

estimating scalar Y

from vector $X \in \mathbb{R}^n$

And linear MMSE estimator $\hat{Y} = w^T X$

that minimizes MSE, $\mathbb{E}[(Y - \hat{Y})^2]$

a) $K = \begin{bmatrix} 5 & 2 \\ 2 & 4 \end{bmatrix}$ find optimal w^*

$\hat{Y} = w \times X$

variance = how spread out values are from the mean.

variance = avg of (value - mean)^2

covariance = if variables move together

$$K = \begin{bmatrix} Var(X) & cov(X,Y) \\ cov(X,Y) & var(Y) \end{bmatrix}$$

$var(X) = 5$

$var(Y) = 4$

$cov(X, Y) = 2 \rightarrow$ move together positively

1a) finding optimal weight

$\hat{Y} = w * X$

$\hat{Y}$ = our prediction of Y

w = ? weight

X = value we observe

good weight → close to true values → mean squared error.

$MSE = Avg\ of\ (Y - \hat{Y})^2$

$$\text{optimal } w^* = \frac{cov(X, Y)}{var(X)}$$

$$\boxed{w^* = \frac{2}{5} = 0.4}$$

1b) Calculate min. MMSE

$$MMSE = Var(Y) - \frac{[cov(X, Y)]^2}{var(X)}$$

Var(Y) = total uncertaintity in Y

$$\frac{[\mathrm{cov}(X,Y)]^2}{\mathrm{var}(x)}$$ = how much uncertaintity we remove by using X

the difference = leftover uncertainty

$$MMSE = 4 - \left(\frac{(2)^2}{5}\right)$$

$$= 4 - \frac{4}{5} = \frac{16}{5} = 3.2$$

$$\boxed{MMSE = 3.2}$$

1c) $\underbrace{W^* X}_{\text{best predictor of gaussian random variables}} = \underbrace{E[Y|X]}_{\text{conditional expectation}}$

best predictor of gaussian random variables

conditional expectation
average value of Y, given we know x

$$E[Y|X] = \left(\frac{\mathrm{cov}(X,Y)}{\mathrm{var}(X)}\right) \times X$$

$$E[Y|X] = E[Y] + \frac{cov(X,Y)}{var(X)} * (X - E[X])$$

zero mean.

$$E[Y|X] = \frac{2}{5} * X \qquad w^{*} = 2/5$$

$$E[Y|X] = w^{*} * X \checkmark$$

1d) non gaussian

$$\hat{Y} = \tilde{w} X$$

linear mmse only uses the covariance matrix
doesnt care abt the actual distribution.
therefore w stays the same and mmse
stays the same. it doesnt matter
where they come from (uniform, gaussian, etc)
since the matrix is the same.