$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$

for all $x, y f$ and $0 < \lambda < 1$

a) $f(x) = x^2$ is convex. prove.

$f(x) = x^3$ not convex.

$$(\lambda x + (1-\lambda)y)^2 \leq \lambda x^2 + (1-\lambda)y^2$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

let $a = \lambda x$, $b = (1-\lambda)y$

then $(\lambda x + (1-\lambda y)^2 = (\lambda x)^2 + 2(\lambda x)((1-\lambda)y)$
$+ (1-\lambda)^2 y^2$

$$\lambda x + (1-\lambda y)^2 = \lambda^2 x^2 + 2\lambda(1-\lambda)xy + (1-\lambda)^2 y^2$$

$$\lambda^2 x^2 + 2\lambda(1-\lambda)xy + (1-\lambda)^2 y^2 - \lambda x - (1-\lambda y)^2 \geq 0$$

$$\lambda(1-\lambda)x^2 - 2\lambda(1-\lambda)xy + \lambda(1-\lambda)y^2$$

$$= \lambda(1-\lambda)(\underbrace{x^2 - 2xy + y^2}_{(x-y)^2})$$

$$\boxed{\begin{array}{c} 0 \geq \lambda(1-\lambda)(x-y)^2 \\ f(x) = x^2 \text{ is convex} \end{array}}$$

$f(x) = x^3$ not convex

$x = -1,\ y = 1,\ \lambda = 1/2$

$\frac{1}{2} f(-1) + \frac{1}{2} f(1) = \frac{1}{2}(-1) + \frac{1}{2}(1) = 0$

$x = 0,\ y = 2,\ \lambda = 1/2$

$f(1) = 1^3 = 1$

$\frac{1}{2}(f(0)) + \frac{1}{2}(f(2)) = \frac{1}{2}(0) + \frac{1}{2}(8)$

$= 4$

$1 \le 4$

$x = -2, \; y = 1, \; \lambda = 1/2$

$\lambda x + (1-\lambda) y = \frac{1}{2}(-2) + \frac{1}{2}(1) = -1/2$

$f(-\frac{1}{2}) = (-\frac{1}{2})^3 = \cancel{\frac{1}{8}} -\frac{1}{8}$

$\frac{1}{2} f(-2) + \frac{1}{2} f(1) = \frac{1}{2}(-8) + \frac{1}{2}(1)$

$= 4 + 0.5 = -3.5$

$-1/8 \not\le -3.5$

fails not convex.

b) $n \times n$ matrix A $x^T A x \ge 0$

$x \in \mathbb{R}^n \; ; \; x \ne 0$

prove $f(x) = x^T A x$ is convex if

A is a pos. semi definite matrix

for $x \in \mathbb{R}^n$

$f(x) = x^T A x$ if convex A is psd.

$$f(\lambda x + (1-\lambda) y \leq \lambda f(x) + (1-\lambda) f(y)$$

sub $f(x) = x^T A x$

$$(\lambda x + (1-\lambda) y)^T A (\lambda x + (1-\lambda) y \leq \lambda (x^T A x) + (1-\lambda)(y^T A y)$$

$u = \lambda x + (1-\lambda) y$

$f(u) = u^T A u$

$$(\lambda x + (1-\lambda) y^T A (\lambda x + (1-\lambda) y) = \lambda^2 x^T A x + 2\lambda (1-\lambda) x^T A y + (1-\lambda)^2 y^T A y$$

$\lambda - \lambda^2 = \lambda(1-\lambda)$

$\lambda x^2 A x - \lambda^2 x^T A x = \lambda (1-\lambda) x^T A x$

$(1-\lambda) - (1-\lambda)^2 = \lambda(1-\lambda)$

$= \lambda(1-\lambda) x^T A x - \lambda(1-\lambda) y^T A y - 2\lambda(1-\lambda) x^T A y$

$= \lambda(1-\lambda)(x^2 A x + y^T A y - 2 x^T A y)$

$x^TAx + y^TAy - 2x^TAy = (x-y)^TA(x-y)$

$(x-y)^TA(x-y) \geq 0$

$\lambda(1-\lambda) > 0$

$x(1-\lambda)(x-y)^TA(x-y) \geq 0$

$\lambda(f(x)) + (1-\lambda)(f(y)) \geq$

$f(\lambda x + (1-\lambda)y)$

$f(x) = x^TAx$ is convex