problem 1 (Pareto MLE and MOME in astronomy) In astronomy, the distribution of asteroid sizes is of great interest (ever watched "Armageddon"?). Denoting the size of asteroid i X_i , a common model is:

$$f_x(x) = \begin{cases} \alpha\theta^{\alpha}x^{-\alpha-1} & \textbf{if } x \geqslant \theta \\ \\ 0 & \textbf{else} \end{cases}$$

Here, θ is the minimum size of an asteroid; let's call it 1 meter for now. Suppose we observe a sample of n asteroid sizes $X_1, ..., X_n$.

1. Holding θ fixed, what is the likelihood function for α?

If we assume that θ is fixed the likelihood function for α is:
$f_x(x) = \alpha\theta^{\alpha}x^{-\alpha-1}$
$\theta = 1$
$f_x(x) = \alpha \cdot 1^{\alpha}x^{-\alpha-1} = \frac{\alpha}{x^{\alpha+1}}$
$L(\alpha \mid x) = \prod_{i=1}^{n} \frac{\alpha}{x_i^{\alpha+1}}$

2. What is the MLE of α?

$\log(L(\alpha \mid x)) = \ell(\alpha) = n \cdot \log(\alpha) - (\alpha + 1) \cdot \sum_{i=1}^{n} \log(x_i)$
$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \log(x_i)$
To find the MLE for α we set that the first derivative of α is equal to 0:
$\frac{\partial \ell}{\partial \alpha} = 0 \Rightarrow \frac{n}{\alpha} - \sum_{i=1}^{n} \log(x_i) = 0 \Rightarrow \frac{n}{\alpha} = \sum_{i=1}^{n} \log(x_i) = 0$
$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} \log(x_i)}$

3. Now, assume α is fixed. What is the likelihood function for θ? What is its MLE?

If we assume that α is fixed the likelihood function for θ is:
$\alpha = c$ where "c" presents some constant
$f_x(x) = c \cdot \theta^{c}x^{-c-1} = \frac{c \cdot \theta^{c}}{x^{c+1}}$
$L(\theta \mid x) = \prod_{i=1}^{n} \frac{c \cdot \theta^{c}}{x_i^{c+1}}$

MLE for θ is:
$\log(L(\theta \mid x)) = \ell(\theta) = n \cdot \log(c) + c \cdot n \cdot \log(\theta) - (c + 1) \cdot \sum_{i=1}^{n} \log(x_i)$
$\frac{\partial \ell}{\partial \theta} = \frac{c \cdot n}{\theta}$
To find the MLE for θ we set that the first derivative of θ is equal to 0: $\frac{\partial \ell}{\partial \theta} = 0 \Rightarrow \frac{c \cdot n}{\theta} = 0$
Since $\theta \leq x_i$ for all i we maximize the likelihood by setting $\hat{\theta}$ equal to the smallest x_i in the sample.

4. Now assume that $\theta = 3$ meters. Find a MOM estimator for α (The mean of a pareto $\frac{\alpha\theta}{\alpha-1}$ for $\alpha > 1$). It is ok to assume α is at least 1.

$\mu_1 = \frac{\alpha\theta}{\alpha-1} = \frac{3\alpha}{\alpha-1} \Rightarrow (\alpha-1)\cdot\mu_1 = 3\alpha \Rightarrow \alpha\mu_1 - \mu_1 = 3\alpha \Rightarrow \alpha\cdot(\mu_1 - 3) = \mu_1 \Rightarrow \alpha = \frac{\mu_1}{\mu_1-3}$
As first moment is mean we set: $\hat{\mu_1} = \bar{x} \Rightarrow \hat{\alpha} = \frac{\bar{x}}{\bar{x}-3}$

5. Draw observations from a Pareto with $\alpha = 2$ and $\theta = 3$. Compute the MLE and MOM estimates for α for various sample sizes. Compare the accuracy of the estimates in a plot, as a function of sample size.

In order to compute the MLE for α I used the formula from question 2: $\hat{\alpha} = \frac{n}{\sum_{i=1}^{n}\log(x_i)}$
To compute the MOM estimator of I used the formula from the question 4: $\hat{\alpha} = \frac{\bar{x}}{\bar{x}-3}$
For sample sizes I choose values from 10 till 1000 with step size of 2 (e.g. 10, 12, 14, 16). From the figures below, we could see on the y axis that α values get closer to 2 as we are increasing the sample size.

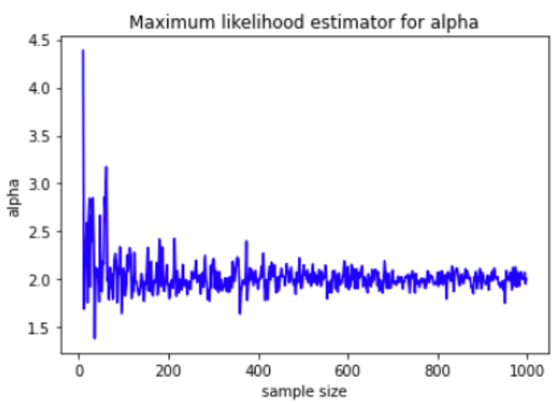

Figure 1: Maximum likelihood estimator for alpha

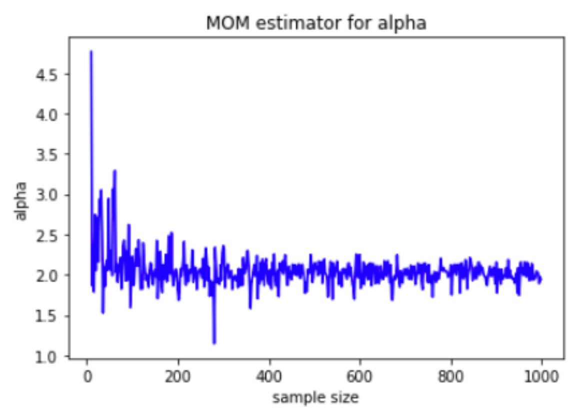

Figure 2: MOM estimator for alpha