Definíciók

Ha $n, r \geq 1$, akkor

$$\mathrm{H}_n^{(s_1,s_2,\ldots,s_r)} := \sum_{n=n_1>n_2>\cdots>n_r\geq 1} \frac{1}{n_1^{s_1} n_2^{s_2}\cdots n_r^{s_r}} \tag{1}$$

$$\mathrm{H}_n^{*(s_1,s_2,\ldots,s_r)} := \sum_{n=n_1\geq n_2\geq\cdots\geq n_r\geq 1} \frac{1}{n_1^{s_1} n_2^{s_2}\cdots n_r^{s_r}} \tag{2}$$

$$\zeta_n(s_1,s_2,\ldots,s_r) := \sum_{n\geq n_1>n_2>\cdots>n_r\geq 1} \frac{1}{n_1^{s_1} n_2^{s_2}\cdots n_r^{s_r}} \tag{3}$$

$$\zeta_n^*(s_1,s_2,\ldots,s_r) := \sum_{n\geq n_1\geq n_2\geq\cdots\geq n_r\geq 1} \frac{1}{n_1^{s_1} n_2^{s_2}\cdots n_r^{s_r}} \tag{4}$$

$$\zeta(s_1,s_2,\ldots,s_r) := \sum_{n_1>n_2>\cdots>n_r\geq 1} \frac{1}{n_1^{s_1} n_2^{s_2}\cdots n_r^{s_r}} \tag{5}$$

$$\zeta^*(s_1,s_2,\ldots,s_r) := \sum_{n_1\geq n_2\geq\cdots\geq n_r\geq 1} \frac{1}{n_1^{s_1} n_2^{s_2}\cdots n_r^{s_r}} \tag{6}$$

ha $r = 0$, azaz $(s_1, s_2, \ldots, s_r) = (\,) = \emptyset$, akkor

$$\mathrm{H}_n^{()} = \mathrm{H}_n^{*()} = \delta_{n,0} = \begin{cases} 0 & ,\text{ ha } n > 0 \\ 1 & ,\text{ ha } n = 0 \end{cases} \tag{7}$$

$$\zeta_n(\,) = \zeta_n^*(\,) = 1 \tag{8}$$

ha $n = 0$, akkor

$$\mathrm{H}_0^{(s_1,\ldots,s_r)} = \mathrm{H}_0^{*(s_1,,\ldots,s_r)} = \zeta_0(s_1,\ldots,s_r) = \zeta_0^*(s_1,\ldots,s_r) = \delta_{r,0} = \begin{cases} 0 & ,\text{ ha } r > 0 \\ 1 & ,\text{ ha } r = 0 \end{cases} = \begin{cases} 0 & ,\text{ ha } (s_1,s_2,\ldots,s_r) \neq (\,) \\ 1 & ,\text{ ha } (s_1,s_2,\ldots,s_r) = (\,) \end{cases} \tag{9}$$

Rekurziók

$$\mathrm{H}_n^{()} = \delta_{n,0} \quad ; \quad \mathrm{H}_n^{(s_1,s_2,\ldots,s_r)} = \frac{1}{n^{s_1}} \sum_{k=r-1}^{n-1} \mathrm{H}_k^{(s_2,s_3,\ldots,s_r)} \quad (n, r \geq 1) \tag{10}$$

$$\mathrm{H}_n^{*()} = \delta_{n,0} \quad ; \quad \mathrm{H}_n^{*(s_1,s_2,\ldots,s_r)} = \frac{1}{n^{s_1}} \sum_{k=0}^{n} \mathrm{H}_k^{*(s_2,s_3,\ldots,s_r)} \quad (n, r \geq 1) \tag{11}$$

$$\zeta_n(\,) = 1 \quad ; \quad \zeta_n(s_1,s_2,\ldots,s_r) = \sum_{k=\max(1,r-1)}^{n} \frac{\zeta_{k-1}(s_2,s_3,\ldots,s_r)}{k^{s_1}} \quad (n, r \geq 1) \tag{12}$$

$$\zeta_n^*(\,) = 1 \quad ; \quad \zeta_n^*(s_1,s_2,\ldots,s_r) = \sum_{k=1}^{n} \frac{\zeta_k^*(s_2,s_3,\ldots,s_r)}{k^{s_1}} \quad (n, r \geq 1) \tag{13}$$

A rekurziókból (és a definíciókból is) következik, hogy ha $n \geq 1$, akkor $\mathrm{H}_n^{(s)} = \mathrm{H}_n^{*(s)} = \dfrac{1}{n^s}$, és $\zeta_n(s) = \zeta_n^*(s) = \sum_{k=1}^{n} \dfrac{1}{k^s} := \mathscr{H}_n^{(s)}$

Ugyanis, $\mathrm{H}_n^{(s)} = \dfrac{1}{n^s} \sum_{k=1-1}^{n-1} \mathrm{H}_k^{()} = \dfrac{1}{n^s} \sum_{k=0}^{n-1} \delta_{k,0} = \dfrac{\delta_{0,0}}{n^s} = \dfrac{1}{n^s}$. Hasonlóan, $\mathrm{H}_n^{*(s)} = \dfrac{1}{n^s} \sum_{k=0}^{n} \mathrm{H}_k^{*()} = \dfrac{1}{n^s} \sum_{k=0}^{n} \delta_{k,0} = \dfrac{\delta_{0,0}}{n^s} = \dfrac{1}{n^s}$.

Másrészt, $\zeta_n(s) = \sum_{k=\max(1,1-1)}^{n} \dfrac{\zeta_{k-1}()}{k^s} = \sum_{k=\max(1,0)}^{n} \dfrac{1}{k^s} = \sum_{k=1}^{n} \dfrac{1}{k^s}$, és $\zeta_n^*(s) = \sum_{k=1}^{n} \dfrac{\zeta_k^*()}{k^s} = \sum_{k=1}^{n} \dfrac{1}{k^s}$.

$\mathrm{H}_n^{(s_1,s_2,\ldots,s_r)}$ és $\zeta_n^{(s_1,s_2,\ldots,s_r)}$, illetve $\mathrm{H}_n^{*(s_1,s_2,\ldots,s_r)}$ és $\zeta_n^{*(s_1,s_2,\ldots,s_r)}$ kapcsolata

$$\zeta_n(s_1, s_2, \ldots, s_r) = \sum_{k=r}^{n} \mathrm{H}_k^{(s_1,s_2,\ldots,s_r)} \quad (n, r \geq 0) \tag{14}$$

$$\zeta_n^*(s_1, s_2, \ldots, s_r) = \sum_{k=0}^{n} \mathrm{H}_k^{*(s_1,s_2,\ldots,s_r)} \quad (n, r \geq 0) \tag{15}$$

$$\mathrm{H}_n^{(s_1,s_2,\ldots,s_r)} = \frac{\zeta_{n-1}(s_2, s_3, \ldots, s_r)}{n^{s_1}} \quad (n, r \geq 1) \tag{16}$$

$$\mathrm{H}_n^{*(s_1,s_2,\ldots,s_r)} = \frac{\zeta_n^*(s_2, s_3, \ldots, s_r)}{n^{s_1}} \quad (n, r \geq 1) \tag{17}$$

$$\zeta(s_1, s_2, \ldots, s_r) = \lim_{n\to\infty} \zeta_n(s_1, s_2, \ldots, s_r) = \sum_{k=r}^{\infty} \mathrm{H}_k^{(s_1,s_2,\ldots,s_r)} \tag{18}$$

$$\zeta^*(s_1, s_2, \ldots, s_r) = \lim_{n\to\infty} \zeta_n^*(s_1, s_2, \ldots, s_r) = \sum_{k=0}^{\infty} \mathrm{H}_k^{*(s_1,s_2,\ldots,s_r)} \tag{19}$$

$\mathrm{H}_n^{(s_1,s_2,\ldots,s_r)}$ és $\mathrm{H}_n^{*(s_1,s_2,\ldots,s_r)}$, illetve $\zeta_n^{(s_1,s_2,\ldots,s_r)}$ és $\zeta_n^{*(s_1,s_2,\ldots,s_r)}$ kapcsolata

$$\mathrm{H}_n^{*(s_1,s_2,\ldots,s_r)} = \sum_{\bullet\in\{"+",","\}} \mathrm{H}_n^{(s_1\bullet s_2\bullet\cdots\bullet s_r)} \tag{20}$$

$$\zeta_n^*(s_1, s_2, \ldots, s_r) = \sum_{\bullet\in\{"+",","\}} \zeta_n(s_1 \bullet s_2 \bullet \cdots \bullet s_r) \tag{21}$$

$$\zeta^*(s_1, s_2, \ldots, s_r) = \sum_{\bullet\in\{"+",","\}} \zeta(s_1 \bullet s_2 \bullet \cdots \bullet s_r) \tag{22}$$

$$\mathrm{H}_n^{(s_1,s_2,\ldots,s_r)} = (-1)^r \sum_{\bullet\in\{"+",","\}} (-1)^{|(s_1\bullet s_2\bullet\cdots\bullet s_r)|} \cdot \mathrm{H}_n^{*(s_1\bullet s_2\bullet\cdots\bullet s_r)} \tag{23}$$

$$\zeta_n(s_1, s_2, \ldots, s_r) = (-1)^r \sum_{\bullet\in\{"+",","\}} (-1)^{|(s_1\bullet s_2\bullet\cdots\bullet s_r)|} \cdot \zeta_n^*(s_1 \bullet s_2 \bullet \cdots \bullet s_r) \tag{24}$$

$$\zeta(s_1, s_2, \ldots, s_r) = (-1)^r \sum_{\bullet\in\{"+",","\}} (-1)^{|(s_1\bullet s_2\bullet\cdots\bullet s_r)|} \cdot \zeta^*(s_1 \bullet s_2 \bullet \cdots \bullet s_r) \tag{25}$$

$\zeta_n^{(s_1,s_2,\ldots,s_r)}$ és $\zeta^{(s_1,s_2,\ldots,s_r)}$, illetve $\zeta_n^{*(s_1,s_2,\ldots,s_r)}$és $\zeta^{*(s_1,s_2,\ldots,s_r)}$ kapcsolata

$$\zeta(s_1,s_2,\ldots,s_r)=\lim_{n\to\infty}\zeta_n(s_1,s_2,\ldots,s_r)=\sum_{k=r}^{\infty}\mathrm{H}_k^{(s_1,s_2,\ldots,s_r)}=\mathrm{Li}_{\zeta_n(s_1,s_2,\ldots,s_r)}(1)$$

$$\zeta^*(s_1,s_2,\ldots,s_r)=\lim_{n\to\infty}\zeta_n^*(s_1,s_2,\ldots,s_r)=\sum_{k=0}^{\infty}\mathrm{H}_k^{*(s_1,s_2,\ldots,s_r)}=\mathrm{Le}_{\zeta_n(s_1,s_2,\ldots,s_r)}(1)$$

H_n^* dualitási tétele. Ha $(s_1,s_2,\ldots,s_r)^*$ jelöji az $(s_1,s_2,\ldots,s_r)$ vektor konjugáltját, akkor

$$\mathrm{H}_n^{*(s_1,s_2,\ldots,s_r)^*}=-\sum_{k=1}^{n}(-1)^k\binom{n-1}{k-1}\mathrm{H}_n^{*(s_1,s_2,\ldots,s_r)} \tag{26}$$

$$\zeta^*(s_1,s_2,\ldots,s_r)=-\sum_{k=1}^{\infty}(-1)^k\binom{n}{k}\mathrm{H}_n^{*(s_1,s_2,\ldots,s_r)^*} \tag{27}$$