Fundamentals of Linear Time-Varying Systems

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3.1 Introduction

In this chapter, various fundamental elements of the theory of linear time-varying systems are studied in both the continuous-time and discrete-time cases. The chapter is a revised and updated version of Chapter 25 that appeared in the first edition of *The Control Handbook*. The revision includes material on the existence of coordinate transformations that transform time-varying system matrices into diagonal forms, and the connection to the concept of "dynamic" eigenvalues and eigenvectors. Also included in the revision is the use of a simplified expression derived in [11] for the feedback gain in the design of a state feedback controller in the single-input case based on the control canonical form. Generalizations to the multi-input multi-output case can be carried out using the results in [12]. The theory begins in Section 3.2 after the following comments.

The distinguishing characteristic of a time-varying system is that the values of the output response depend on when the input is applied to the system. Time variation is often a result of system parameters changing as a function of time, such as aerodynamic coefficients in high-speed aircraft, circuit parameters in electronic circuits, mechanical parameters in machinery, and diffusion coefficients in chemical processes. Time variation may also be a result of linearizing a nonlinear system about a family of operating points and/or about a time-varying operating point.

3-2 Control System Advanced Methods

The values of the time-varying parameters in a system are often not known *a priori*, that is, before the system is put into operation, but can be measured or estimated during system operation. Systems whose parameters are not known *a priori* are often referred to as parameter varying systems, and the control of such systems is referred to as gain scheduling. Parameter varying systems and gain scheduling are considered in another chapter of this handbook, and are not pursued here. In some applications, time variations in the coefficients of the system model are known *a priori*. For example, this may be the case if the system model is a linearization of a nonlinear system about a known time-varying nominal trajectory. In such cases, the theory developed in this chapter can be directly applied.

In this chapter, the study of linear time-varying systems is carried out in terms of input/output equations and the state model. The focus is on the relationships between input/output and state models, the construction of canonical forms, the study of the system properties of stability, controllability, and observability, and the design of controllers. Both the continuous-time and the discrete-time cases are considered. In the last part of the chapter, an example is given on the application to state observers and state feedback control, and examples are given on checking for stability. Additional references on the theory of linear time-varying systems are given in Further Reading.

3.2 Analysis of Continuous-Time Causal Linear Time-Varying Systems

Consider the single-input single-output continuous-time system given the input/output relationship

$$y(t) = \int_{-\infty}^{t} h(t,\tau)u(\tau)\,d\tau \tag{3.1}$$

where t is the continuous-time variable, $y(t)$ is the output response resulting from input $u(t)$, and $h(t,\tau)$ is a real-valued continuous function of t and τ. It is assumed that there are conditions on $h(t,\tau)$ and/or $u(t)$, which insure that the integral in Equation 3.1 exists. The system given by Equation 3.1 is causal since the output $y(t)$ at time t depends only on the input $u(\tau)$ for $\tau \leq t$. The system is also linear since integration is a linear operation. Linearity means that if $y_1(t)$ is the output response resulting from input $u_1(t)$, and $y_2(t)$ is the output response resulting from input $u_2(t)$, then for any real numbers a and b, the output response resulting from input $au_1(t) + bu_2(t)$ is equal to $ay_1(t) + by_2(t)$.

Let $\delta(t)$ denote the unit impulse defined by $\delta(t) = 0$, $t \neq 0$ and $\int_{-\varepsilon}^{\varepsilon} \delta(\lambda)\,d\lambda = 1$ for any real number $\varepsilon > 0$. For any real number t_1, the time-shifted impulse $\delta(t - t_1)$ is the unit impulse located at time $t = t_1$. Then from Equation 3.1 and the sifting property of the impulse, the output response $y(t)$ resulting from input $u(t) = \delta(t - t_1)$ is given by

$$y(t) = \int_{-\infty}^{t} h(t,\tau)u(\tau)\,d\tau = \int_{-\infty}^{t} h(t,\tau)\delta(\tau - t_1)\,d\tau = h(t,t_1)$$

Hence, the function $h(t,\tau)$ in Equation 3.1 is the *impulse response function* of the system, that is, $h(t,\tau)$ is the output response resulting from the impulse $\delta(t-\tau)$ applied to the system at time τ.

The linear system given by Equation 3.1 is *time invariant* (or *constant*) if and only if

$$h(t+\gamma, \tau+\gamma) = h(t,\tau), \quad \text{for all real numbers } t, \tau, \gamma \tag{3.2}$$

Time invariance means that if $y(t)$ is the response to $u(t)$, then for any real number t_1, the time-shifted output $y(t - t_1)$ is the response to the time-shifted input $u(t - t_1)$. Setting $\gamma = -\tau$ in Equation 3.2 gives

$$h(t-\tau, 0) = h(t,\tau), \quad \text{for all real numbers } t, \tau \tag{3.3}$$

Hence, the system defined by Equation 3.1 is time invariant if and only if the impulse response function $h(t,\tau)$ is a function only of the difference $t-\tau$. In the time-invariant case, Equation 3.1 reduces to the convolution relationship

$$y(t) = h(t) * u(t) = \int_{-\infty}^{t} h(t-\tau)u(\tau)\, d\tau \tag{3.4}$$

where $h(t) = h(t,0)$ is the impulse response (i.e., the output response resulting from the impulse $\delta(t)$ applied to the system at time 0).

The linear system defined by Equation 3.1 is *finite-dimensional* or *lumped* if the input $u(t)$ and the output $y(t)$ are related by the nth-order input/output differential equation

$$y^{(n)}(t) + \sum_{i=0}^{n-1} a_i(t)y^{(i)}(t) = \sum_{i=0}^{m} b_i(t)u^{(i)}(t) \tag{3.5}$$

where $y^{(i)}(t)$ is the ith derivative of $y(t)$, $u^{(i)}(t)$ is the ith derivative of $u(t)$, and $a_i(t)$ and $b_i(t)$ are real-valued functions of t. In Equation 3.5, it is assumed that $m \leq n$. The linear system given by Equation 3.5 is time invariant if and only if all coefficients in Equation 3.5 are constants, that is, $a_i(t) = a_i$ and $b_i(t) = b_i$ for all i, where a_i and b_i are real constants.

3.2.1 State Model Realizations

A state model for the system given by Equation 3.5 can be constructed as follows. First, suppose that $m = 0$ so that Equation 3.5 becomes

$$y^{(n)}(t) + \sum_{i=0}^{n-1} a_i(t)y^{(i)}(t) = b_0(t)u(t). \tag{3.6}$$

Then defining the state variables

$$x_i(t) = y^{(i-1)}(t), \quad i = 1, 2, \ldots, n, \tag{3.7}$$

the system defined by Equation 3.6 has the state model

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{3.8}$$

$$y(t) = Cx(t) \tag{3.9}$$

where the coefficient matrices $A(t)$, $B(t)$, and C are given by

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \cdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0(t) & -a_1(t) & -a_2(t) & \cdots & -a_{n-2}(t) & -a_{n-1}(t) \end{bmatrix} \tag{3.10}$$

$$B(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b_0(t) \end{bmatrix} \tag{3.11}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{3.12}$$

and $x(t)$ is the n-dimensional state vector given by

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix}$$

When $m \geq 1$ in Equation 3.5, the definition in Equation 3.7 of the state variables will not yield a state model of the form given in Equations 3.8 and 3.9. If $m < n$, a state model can be generated by first rewriting Equation 3.5 in the form

$$D^n y(t) + \sum_{i=0}^{n-1} D^i \left[\alpha_i(t) y(t)\right] = \sum_{i=0}^{m} D^i \left[\beta_i(t) u(t)\right] \tag{3.13}$$

where D is the derivative operator and $\alpha_i(t)$ and $\beta_i(t)$ are real-valued functions of t. The form of the input/output differential equation given by Equation 3.13 exists if $a_i(t)$ and $b_i(t)$ are differentiable a suitable number of times. If $a_i(t)$ are constants so that $a_i(t) = a_i$ for all t, then $\alpha_i(t)$ are constants and $\alpha_i(t) = a_i$, $i = 0, 1, \ldots, n-1$. If $b_i(t)$ are constants so that $b_i(t) = b_i$ for all t, then $\beta_i(t)$ are constants and $\beta_i(t) = b_i$, $i = 0, 1, \ldots, m$.

When $a_i(t)$ and $b_i(t)$ are not constants, $\alpha_i(t)$ is a linear combination of $a_i(t)$ and the derivatives of $a_j(t)$ for $j = n-i-1$, $n-i-2, \ldots 1$, and $\beta_i(t)$ is a linear combination of $b_i(t)$ and the derivatives of $b_j(t)$ for $j = m-i-1, m-i-2, \ldots 1$. For example, when $n = 2, \alpha_0(t)$ and $\alpha_1(t)$ are given by

$$\alpha_0(t) = a_0(t) - \dot{a}_1(t) \tag{3.14}$$

$$\alpha_1(t) = a_1(t) \tag{3.15}$$

When $n = 3, \alpha_i(t)$ are given by

$$\alpha_0(t) = a_0(t) - \dot{a}_1(t) + \ddot{a}_2(t) \tag{3.16}$$

$$\alpha_1(t) = a_1(t) - 2\dot{a}_2(t) \tag{3.17}$$

$$\alpha_2(t) = a_2(t) \tag{3.18}$$

In the general case (n arbitrary), there is a one-to-one and onto correspondence between the coefficients $a_i(t)$ in the left-hand side of Equation 3.5 and the coefficients $\alpha_i(t)$ in the left-hand side of Equation 3.13. Similarly, there is a one-to-one and onto correspondence between the coefficients in the right-hand side of Equation 3.5 and the right-hand side of Equation 3.13.

Now given the system defined by Equation 3.5 written in the form of Equation 3.13, define the state variables

$$\begin{aligned}
x_n(t) &= y(t) \\
x_{n-1}(t) &= Dx_n(t) + \alpha_{n-1}(t)x_n(t) - \beta_{n-1}(t)u(t) \\
x_{n-2}(t) &= Dx_{n-1}(t) + \alpha_{n-2}(t)x_n(t) - \beta_{n-2}(t)u(t) \\
&\vdots \\
x_1(t) &= Dx_2(t) + \alpha_1(t)x_n(t) - \beta_1(t)u(t)
\end{aligned} \tag{3.19}$$

where $\beta_i(t) = 0$ for $i > m$. Then with the state variables defined by Equations 3.19, the system given by Equation 3.5 has the state model

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{3.20}$$

$$y(t) = Cx(t) \tag{3.21}$$

where the coefficient matrices $A(t)$, $B(t)$, and C are given by

$$A(t) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -\alpha_0(t) \\ 1 & 0 & 0 & \cdots & 0 & -\alpha_1(t) \\ 0 & 1 & 0 & \cdots & 0 & -\alpha_2(t) \\ \vdots & & & \cdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -\alpha_{n-2}(t) \\ 0 & 0 & 0 & \cdots & 1 & -\alpha_{n-1}(t) \end{bmatrix} \tag{3.22}$$

$$B(t) = \begin{bmatrix} \beta_0(t) \\ \beta_1(t) \\ \beta_2(t) \\ \vdots \\ \beta_{n-2}(t) \\ \beta_{n-1}(t) \end{bmatrix} \tag{3.23}$$

and

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \tag{3.24}$$

The state model with $A(t)$ and C specified by Equations 3.22 and 3.24 is said to be in *observer canonical form*, which is observable as discussed below. Note that for this particular state realization, the row vector C is constant (independent of t).

In addition to the state model defined by Equations 3.22 through 3.24, there are other possible state models for the system given by Equation 3.5. For example, another state model can be constructed in the case when the left-hand side of Equation 3.5 can be expressed in a Floquet factorization form (see [1]), so that Equation 3.5 becomes

$$(D - p_1(t))(D - p_2(t)) \ldots (D - p_n(t))y(t) = \sum_{i=0}^{m} b_i(t)u^{(i)}(t) \tag{3.25}$$

where again D is the derivative operator and the $p_i(t)$ are real-valued or complex-valued functions of the time variable t. For example, consider the $n = 2$ case for which

$$\begin{aligned} (D - p_1(t))(D - p_2(t))y(t) &= (D - p_1(t))[\dot{y}(t) - p_2(t)y(t)] \\ (D - p_1(t))(D - p_2(t))y(t) &= \ddot{y}(t) - [p_1(t) + p_2(t)]\dot{y}(t) + [p_1(t)p_2(t) - \dot{p}_2(t)]y(t) \end{aligned} \tag{3.26}$$

With $n = 2$ and $m = 1$, the state variables may be defined by

$$x_1(t) = \dot{y}(t) - p_2(t)y(t) - b_1(t)u(t) \tag{3.27}$$

$$x_2(t) = y(t) \tag{3.28}$$

which results in the state model given by Equations 3.20 and 3.21 with

$$A(t) = \begin{bmatrix} p_1(t) & 0 \\ 1 & p_2(t) \end{bmatrix} \tag{3.29}$$

$$B(t) = \begin{bmatrix} b_0(t) - \dot{b}_1(t) + p_1(t)b_1(t) \\ b_1(t) \end{bmatrix} \tag{3.30}$$

and

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix} \tag{3.31}$$

In the general case given by Equation 3.25, there is a state model in the form of Equations 3.20 and 3.21 with $A(t)$ and C given by

$$A(t) = \begin{bmatrix} p_1(t) & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & p_2(t) & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & p_3(t) & 0 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & p_{n-1}(t) & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & p_n(t) \end{bmatrix} \tag{3.32}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{3.33}$$

A very useful feature of the state model with $A(t)$ given by Equation 3.32 is the lower triangular form of $A(t)$. In particular, as discussed below, stability conditions can be specified in terms of $p_i(t)$, which can be viewed as "time-varying poles" of the system. However, in the time-varying case the computation of $p_i(t)$ based on the Floquet factorization given in the left-hand side of Equation 3.25 requires that nonlinear Riccati-type differential equations must be solved (see, e.g., [2,13]). The focus of [2] is on the computation of poles and zeros and the application to stability and transmission blocking, with the emphasis on time-varying difference equations. In [4], the authors generate a Floquet factorization given by the left-hand side of Equation 3.25 by utilizing successive Riccati transformations in a state-space setting.

A significant complicating factor in the theory of factoring polynomial operators with time-varying coefficients is that in general there are an infinite number of different "pole sets"$\{p_1(t), p_2(t), \ldots, p_n(t)\}$. This raises the question as to whether one pole set may be "better" in some sense than another pole set. In the case of linear difference equations with time-varying coefficients, it is shown in [2] that poles can be computed using a nonlinear recursion and that uniqueness of pole sets can be achieved by specifying initial values for the poles. For recent work on the factorization of time-varying differential equations by using ground field extensions, see [6]. As discussed below, in [7] the definition and computation of time-varying poles is pursued by using Lyapunov coordinate transformations in a state-space setting.

3.2.2 The State Model

For an m-input p-output linear n-dimensional time-varying continuous-time system, the general form of the state model is

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{3.34}$$

$$y(t) = C(t)x(t) + D(t)u(t) \tag{3.35}$$

where Equation 3.34 is the *state equation* and Equation 3.35 is the *output equation*. In Equations 3.34 and 3.35, $A(t)$ is the $n \times n$ *system matrix*, $B(t)$ is the $n \times m$ *input matrix*, $C(t)$ is the $p \times n$ *output matrix*, $D(t)$ is the $p \times m$ *direct feed matrix*, $u(t)$ is the m-dimensional input vector, $x(t)$ is the n-dimensional state vector, and $y(t)$ is the p-dimensional output vector. The term $D(t)u(t)$ in Equation 3.35 is of little significance in the theory, and thus $D(t)u(t)$ is usually omitted from Equation 3.35, which will be done here.

To solve Equation 3.34, first consider the homogeneous equation

$$\dot{x}(t) = A(t)x(t), \quad t > t_0 \tag{3.36}$$

with the initial condition $x(t_0)$ at initial time t_0. For any $A(t)$ whose entries are continuous functions of t, it is well known (see, e.g., [8]) that for any initial condition $x(t_0)$, there is a unique continuously differentiable solution of Equation 3.36 given by

$$x(t) = \Phi(t, t_0)x(t_0), \quad t > t_0 \tag{3.37}$$

where $\Phi(t, t_0)$ is an $n \times n$ matrix function of t and t_0, called the *state-transition matrix*. The state-transition matrix has the following fundamental properties:

$$\Phi(t, t) = I = n \times n \text{ identity matrix}, \quad \text{for all } t \tag{3.38}$$

$$\Phi(t, \tau) = \Phi(t, t_1)\Phi(t_1, \tau), \quad \text{for all } t_1, t, \tau \tag{3.39}$$

$$\Phi^{-1}(t, \tau) = \Phi(\tau, t), \quad \text{for all } t, \tau \tag{3.40}$$

$$\frac{\partial}{\partial t}\Phi(t, \tau) = A(t)\Phi(t, \tau), \quad \text{for all } t, \tau \tag{3.41}$$

$$\frac{\partial}{\partial \tau}\Phi(t, \tau) = -\Phi(t, \tau)A(\tau), \quad \text{for all } t, \tau \tag{3.42}$$

$$\det\Phi(t, \tau) = \exp\left[\int_\tau^t \operatorname{tr}[A(\sigma)]\, d\sigma\right] \tag{3.43}$$

In Equation 3.43, "det" denotes the determinant and "tr" denotes the trace. Equation 3.39 is called the *composition property*. It follows from this property that $\Phi(t, \tau)$ can be written in the factored form

$$\Phi(t, \tau) = \Phi(t, 0)\Phi(0, \tau), \quad \text{for all } t, \tau \tag{3.44}$$

Another important property of the state-transition matrix $\Phi(\tau, t)$ is that it is a continuously differentiable matrix function of t and τ.

It follows from Equation 3.42 that the *adjoint equation*

$$\dot{\gamma}(t) = -A^T(t)\gamma(t) \tag{3.45}$$

has state-transition matrix equal to $\Phi^T(\tau, t)$, where again $\Phi(t, \tau)$ is the state-transition matrix for Equation 3.36 and superscript "T" denotes the transpose operation.

If the system matrix $A(t)$ is constant over the interval $[t_1, t_2]$, that is, $A(t) = A$, for all $t \in [t_1, t_2]$, then the state-transition matrix is equal to the matrix exponential over $[t_1, t_2]$:

$$\Phi(t, \tau) = e^{A(t-\tau)} \quad \text{for all } t, \tau \in [t_1, t_2] \tag{3.46}$$

If $A(t)$ is time varying and $A(t)$ commutes with its integral over the interval $[t_1, t_2]$, that is,

$$A(t)\left[\int_\tau^t A(\sigma)\, d\sigma\right] = \left[\int_\tau^t A(\sigma)\, d\sigma\right]A(t), \quad \text{for all } t, \tau \in [t_1, t_2] \tag{3.47}$$

then $\Phi(t, \tau)$ is given by

$$\Phi(t, \tau) = \exp\left[\int_\tau^t A(\tau)\, d\tau\right], \quad \text{for all } t, \tau \in [t_1, t_2] \tag{3.48}$$

Note that the commutativity condition in Equation 3.47 is always satisfied in the time-invariant case. It is also always satisfied in the one-dimensional ($n = 1$) time-varying case since scalars commute. Thus, $\Phi(t, \tau)$ is given by the exponential form in Equation 3.48 when $n = 1$. Unfortunately, the exponential form for $\Phi(t, \tau)$ does not hold for an arbitrary time-varying matrix $A(t)$ when $n > 1$, and, in general, there is no known closed-form expression for $\Phi(t, \tau)$ when $n > 1$. However, approximations to $\Phi(t, \tau)$ can be readily computed from $A(t)$ by numerical techniques, such as the method of successive approximations (see, e.g., [8]). Approximations to $\Phi(t, \tau)$ can also be determined by discretizing the time variable t as shown next.

Let k be an integer-valued variable, let T be a positive number, and let $a_{ij}(t)$ denote the i,j entry of the matrix $A(t)$. Suppose that by choosing T to be sufficiently small, the absolute values $|a_{ij}(t) - a_{ij}(kT)|$ can be made as small as desired over the interval $t \in [kT, kT+T]$ and for all integer values of k. Then for a suitably small $T, A(t)$ is approximately equal to $A(kT)$ for $t \in [kT, kT+T]$, and from Equation 3.46 the state-transition matrix $\Phi(t, kT)$ is given approximately by

$$\Phi(t, kT) = e^{A(kT)(t-kT)} \quad \text{for all } t \in [kT,\ kT+T] \quad \text{and} \quad \text{all } k \tag{3.49}$$

Setting $t = kT + T$ in Equation 3.49 yields

$$\Phi(kT+T, kT) = e^{A(kT)T} \quad \text{for all } k \tag{3.50}$$

The state-transition matrix $\Phi(kT+T, kT)$ given by Equation 3.50 can be computed by using a parameterized Laplace transform. To show this, first define the $n \times n$ matrix

$$q(k,t) = e^{A(kT)t} \tag{3.51}$$

Then $q(k,t)$ is equal to the inverse Laplace transform of

$$Q(k,s) = [sI - A(kT)]^{-1} \tag{3.52}$$

Note that the transform $Q(k,s)$ is parameterized by the integer-valued variable k. A closed-form expression for $q(k,t)$ as a function of t can be obtained by expanding the entries of $Q(k,s)$ into partial fraction expansions and then using standard Laplace transform pairs to determine the inverse transform of the terms in the partial faction expansions. Then from Equation 3.51, setting $t = T$ in $q(k,t)$ yields $q(k,T) = \Phi(kT+T, kT)$.

Again consider the general case with the state equation given by Equation 3.34. Given the state-transition matrix $\Phi(t,\tau)$, for any given initial state $x(t_0)$ and input $u(t)$ applied for $t \geq t_0$, the complete solution to Equation 3.34 is

$$x(t) = \Phi(t,t_0)x(t_0) + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)\,d\tau, \quad t > t_0 \tag{3.53}$$

Then, when $y(t) = C(t)x(t)$, the output response $y(t)$ is given by

$$y(t) = C(t)\Phi(t,t_0)x(t_0) + \int_{t_0}^{t} C(t)\Phi(t,\tau)B(\tau)u(\tau)\,d\tau, \quad t > t_0 \tag{3.54}$$

If the initial time t_0 is taken to be $-\infty$ and the initial state is zero, Equation 3.54 becomes

$$y(t) = \int_{-\infty}^{t} C(t)\Phi(t,\tau)B(\tau)u(\tau)\,d\tau \tag{3.55}$$

Comparing Equation 3.55 with the m-input, p-output version of Equation 3.1 reveals that

$$H(t,\tau) = \begin{cases} C(t)\Phi(t,\tau)B(\tau) & \text{for } t \geq \tau \\ 0 & \text{for } t < \tau \end{cases} \tag{3.56}$$

where $H(t,\tau)$ is the $p \times m$ impulse response function matrix. Inserting Equation 3.44 into Equation 3.56 reveals that $H(t,\tau)$ can be expressed in the factored form,

$$H(t,\tau) = H_1(t)H_2(\tau), \quad t \geq \tau \tag{3.57}$$

where

$$H_1(t) = C(t)\Phi(t,0) \quad \text{and} \quad H_2(\tau) = \Phi(0,\tau)B(\tau) \tag{3.58}$$

It turns out [8] that a linear time-varying system with impulse response matrix $H(t,\tau)$ has a state realization given by Equations 3.34 and 3.35 with $D(t) = 0$ if and only if $H(t,\tau)$ can be expressed in the factored form given in Equation 3.57.

3.2.3 Change of State Variables

Suppose that the system under study has the n-dimensional state model

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{3.59}$$

$$y(t) = C(t)x(t) \tag{3.60}$$

In the following development, the system given by Equations 3.59 and 3.60 will be denoted by the triple $[A(t), B(t), C(t)]$.

Given any $n \times n$ invertible continuously differentiable matrix $P(t)$, which is real or complex valued, another state model can be generated by defining the new state vector $z(t) = P^{-1}(t)x(t)$, where $P^{-1}(t)$ is the inverse of $P(t)$. The matrix $P^{-1}(t)$ (or the matrix $P(t)$) is called the transformation matrix and the process of going from $x(t)$ to $z(t)$ is referred to as a coordinate transformation. The state variables for the new state model (i.e., the $z_i(t)$) are linear combinations with time-varying coefficients of the state variables of the given state model. To determine the state equations for the new state model, first note that $P(t)z(t) = x(t)$. Then taking the derivative of both sides of $P(t)z(t) = x(t)$ gives

$$P(t)\dot{z}(t) + \dot{P}(t)z(t) = \dot{x}(t) \tag{3.61}$$

Inserting the expression for $\dot{x}(t)$ given by Equation 3.59 yields

$$P(t)\dot{z}(t) + \dot{P}(t)z(t) = A(t)x(t) + B(t)u(t) \tag{3.62}$$

and replacing $x(t)$ by $P(t)z(t)$ in Equation 3.62 and rearranging terms results in

$$P(t)\dot{z}(t) = [A(t)P(t) - \dot{P}(t)]z(t) + B(t)u(t) \tag{3.63}$$

Finally, multiplying both sides of Equation 3.63 on the left by $P^{-1}(t)$ gives

$$\dot{z}(t) = [P^{-1}(t)A(t)P(t) - P^{-1}(t)\dot{P}(t)]z(t) + P^{-1}(t)B(t)u(t) \tag{3.64}$$

and replacing $x(t)$ by $P(t)z(t)$ in Equation 3.60 yields

$$y(t) = C(t)P(t)z(t) \tag{3.65}$$

The state equations for the new state model are given by Equations 3.64 and 3.65. This new model will be denoted by the triple $[\bar{A}(t), \bar{B}(t), \bar{C}(t)]$ where

$$\bar{A}(t) = P^{-1}(t)A(t)P(t) - P^{-1}(t)\dot{P}(t) \tag{3.66}$$

$$\bar{B}(t) = P^{-1}(t)B(t) \tag{3.67}$$

$$\bar{C}(t) = C(t)P(t) \tag{3.68}$$

Multiplying both sides of Equation 3.66 on the left by $P(t)$ and rearranging the terms yield the result that the transformation matrix $P(t)$ satisfies the differential equation

$$\dot{P}(t) = A(t)P(t) - P(t)\bar{A}(t) \tag{3.69}$$

The state models $[A(t), B(t), C(t)]$ and $[\overline{A}(t), \overline{B}(t), \overline{C}(t)]$ with $\bar{A}(t), \bar{B}(t), \bar{C}(t)$ given by Equations 3.66 through 3.68 are said to be *algebraically equivalent*. The state-transition matrix $\bar{\Phi}(t, \tau)$ for $[\bar{A}(t), \bar{B}(t), \bar{C}(t)]$ is given by

$$\bar{\Phi}(t, \tau) = P^{-1}(t)\Phi(t, \tau)P(\tau) \tag{3.70}$$

where $\Phi(t, \tau)$ is the state-transition matrix for $[A(t), B(t), C(t)]$.

Given an n-dimensional state model $[A(t), B(t), C(t)]$ and any $n \times n$ continuous matrix function $\Gamma(t)$, there is a transformation matrix $P(t)$ that transforms $A(t)$ into $\Gamma(t)$, that is, $\bar{A}(t) = \Gamma(t)$. To show this, define $P(t)$ by

$$P(t) = \Phi(t, 0)\bar{\Phi}(0, t) \tag{3.71}$$

where $\bar{\Phi}(t, \tau)$ is the state-transition matrix for $\dot{z}(t) = \Gamma(t)z(t)$. Note that $P(t)$ is continuously differentiable since $\Phi(t, 0)$ and $\bar{\Phi}(0, t)$ are continuously differentiable, and $P(t)$ is invertible since state-transition matrices are invertible. Then taking the derivative of both sides of Equation 3.71 gives

$$\dot{P}(t) = \Phi(t, 0)\dot{\bar{\Phi}}(0, t) + \dot{\Phi}(t, 0)\bar{\Phi}(0, t) \tag{3.72}$$

Using Equations 3.41 and 3.42 in Equation 3.72 yields

$$\dot{P}(t) = -\Phi(t, 0)\bar{\Phi}(0, t)\Gamma(t) + A(t)\Phi(t, 0)\bar{\Phi}(0, t) \tag{3.73}$$

Finally, inserting Equation 3.71 into Equation 3.73 results in Equation 3.69 with $\bar{A}(t) = \Gamma(t)$, and thus, $P(t)$ transforms $A(t)$ into $\Gamma(t)$.

This result shows that via a change of state, $A(t)$ can be transformed to any desired continuous matrix $\Gamma(t)$. The fact that any continuous system matrix $A(t)$ can be transformed to any other continuous matrix raises some interesting issues. For example, suppose that the transformation $z(t) = P^{-1}(t)x(t)$ is defined so that the new system matrix $\bar{A}(t)$ is equal to a diagonal matrix $\Lambda(t)$ with real or complex-valued functions $\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)$ on the diagonal. Then from Equation 3.66,

$$\Lambda(t) = P^{-1}(t)A(t)P(t) - P^{-1}(t)\dot{P}(t) \tag{3.74}$$

and multiplying both sides of Equation 3.74 on the left by $P(t)$ gives

$$P(t)\Lambda(t) = A(t)P(t) - \dot{P}(t) \tag{3.75}$$

Now for $i = 1, 2, \ldots, n$, let $\gamma_i(t)$ denote the ith column of the transformation matrix $P(t)$. Then from Equation 3.75, it follows that

$$\lambda_i(t)\gamma_i(t) = A(t)\gamma_i(t) - \dot{\gamma}_i(t), \quad i = 1, 2, \ldots, n \tag{3.76}$$

Rearranging terms in Equation 3.76 gives

$$[A(t) - \lambda_i(t)I]\gamma_i(t) = \dot{\gamma}_i(t), \quad i = 1, 2, \ldots, n \tag{3.77}$$

where I is the $n \times n$ identity matrix.

Equation 3.77 has appeared in the literature on linear time-varying systems, and has sometimes been used as the justification for referring to the $\lambda_i(t)$ as "dynamic" eigenvalues or poles, and the corresponding $\gamma_i(t)$ as eigenvectors, of the time-varying system having system matrix $A(t)$. But since Equation 3.77 holds for any continuous scalar-valued functions $\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)$, or for any constants $\lambda_1, \lambda_2, \ldots, \lambda_n$, the existence of $\lambda_i(t)$ and $\gamma_i(t)$ satisfying Equation 3.77 is not useful unless further conditions are placed on the transformation matrix $P(t)$. For example, one can consider stronger notions of equivalence such as *topological equivalence*. This means that, in addition to being algebraically equivalent, the transformation matrix $P(t)$ has the properties

$$|\det P(t)| \geq c_0 \quad \text{for all } t \tag{3.78}$$

$$|p_{ij}(t)| \leq c_1 \quad \text{for all } t \quad \text{and} \quad i, j = 1, 2, \ldots, n \tag{3.79}$$

where $p_{ij}(t)$ is the i, j entry of $P(t)$, and c_0 and c_1 are finite positive constants. The conditions given in Equations 3.78 and 3.79 are equivalent to requiring that $P(t)$ and its inverse $P^{-1}(t)$ be bounded matrix functions of t. A transformation $z(t) = P^{-1}(t)x(t)$ with $P(t)$ satisfying Equations 3.78 and 3.79 is called a *Lyapunov transformation*. As noted in Section 3.2.4, stability is preserved under a Lyapunov transformation.

If the transformation $z(t) = P^{-1}(t)x(t)$ is a Lyapunov transformation and puts the system matrix $A(t)$ into the diagonal form $\Lambda(t) = \text{diag}(\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t))$, then the $\lambda_i(t)$ are of interest since they determine the stability of the system. However, in the general linear time-varying case, the existence of a Lyapunov transformation that puts $A(t)$ into a diagonal form is a strong condition. A much weaker condition is the existence of a Lyapunov transformation that puts $A(t)$ into an upper or lower triangular form (see, e.g., [7]). This is briefly discussed in Section 3.2.4.

3.2.4 Stability

Given a system with n-dimensional state model $[A(t), B(t), C(t)]$, again consider the homogeneous equation

$$\dot{x}(t) = A(t)x(t), \quad t > t_0 \tag{3.80}$$

with solution

$$x(t) = \Phi(t, t_0)x(t_0), \quad t > t_0 \tag{3.81}$$

The system is said to be *asymptotically stable* if for some initial time t_0, the solution $x(t)$ satisfies the condition $\|x(t)\| \to 0$ as $t \to \infty$ for any initial state $x(t_0)$ at initial time t_0. Here $\|x(t)\|$ denotes the *Euclidean norm* of the state $x(t)$ given by

$$\|x(t)\| = \sqrt{x^T(t)x(t)} = \sqrt{x_1^2(t) + x_2^2(t) + \cdots + x_n^2(t)} \tag{3.82}$$

where $x(t) = [x_1(t)\ x_2(t) \ldots x_n(t)]^T$. A system is asymptotically stable if and only if

$$\|\Phi(t, t_0)\| \to 0 \quad \text{as } t \to \infty \tag{3.83}$$

where $\|\Phi(t, t_0)\|$ is the matrix norm equal to the square root of the largest eigenvalue of $\Phi^T(t, t_0)\Phi(t, t_0)$, where t is viewed as a parameter.

A stronger notion of stability is *exponential stability*, which requires that for some initial time t_0, there exist finite positive constants c and λ such that for any $x(t_0)$, the solution $x(t)$ to Equation 3.80 satisfies

$$\|x(t)\| \leq ce^{-\lambda(t-t_0)}\,\|x(t_0)\|\,, \quad t \geq t_0 \tag{3.84}$$

If the condition in Equation 3.84 holds for all t_0 with the constants c and λ fixed, the system is said to be *uniformly exponential stable*. Uniform exponential stability is equivalent to requiring that there exists finite positive constants γ and λ such that

$$\|\Phi(t, \tau)\| \leq \gamma e^{-\lambda(t-\tau)} \quad \text{for all } t, \tau \text{ such that } t \geq \tau \tag{3.85}$$

Uniform exponential stability is also equivalent to requiring that, given any positive constant δ, there exists a positive constant T such that for any t_0 and $x(t_0)$, the solution $x(t)$ to Equation 3.80 satisfies

$$\|x(t)\| \leq \delta\,\|x(t_0)\| \quad \text{for } t \geq t_0 + T \tag{3.86}$$

It follows from Equation 3.85 that uniform exponential stability is preserved under a Lyapunov transformation $z(t) = P^{-1}(t)x(t)$. To see this, let $\bar{\Phi}(t, \tau)$ denote the state-transition matrix for $\dot{z}(t) = \bar{A}(t)z(t)$, where

$$\bar{A}(t) = P^{-1}(t)A(t)P(t) - P^{-1}(t)\dot{P}(t) \tag{3.87}$$

Then using Equation 3.70 gives

$$\left\|\bar{\Phi}(t,\tau)\right\| \le \left\|P^{-1}(t)\right\| \, \|\Phi(t,\tau)\| \, \|P(\tau)\| \tag{3.88}$$

But Equations 3.78 and 3.79 imply that

$$\|P(\tau)\| \le M_1 \quad \text{and} \quad \left\|P^{-1}(t)\right\| \le M_2 \tag{3.89}$$

for some finite constants M_1 and M_2. Then inserting Equations 3.85 and 3.89 into Equation 3.88 yields

$$\left\|\bar{\Phi}(t,\tau)\right\| \le \gamma M_1 M_2 e^{-\lambda(t-\tau)} \quad \text{for all } t, \tau \text{ such that } t \ge \tau \tag{3.90}$$

which verifies that $\dot{z}(t) = \bar{A}(t)z(t)$ is uniformly exponentially stable.

If $P(t)$ and $P^{-1}(t)$ are bounded only for $t \ge t_1$ for some finite t_1, the coordinate transformation $z(t) = P^{-1}(t)x(t)$ preserves exponential stability of the given system. This can be proved using constructions similar to those given above. The details are omitted.

When $n = 1$ so that $A(t) = a(t)$ is a scalar-valued function of t, as seen from Equation 3.48,

$$\Phi(t,t_0) = \exp\left[\int_{t_0}^{t} a(\tau)\, d\tau\right]. \tag{3.91}$$

It follows that the system is asymptotically stable if and only if

$$\int_{t_0}^{t} a(\sigma)\, d\sigma \to -\infty \text{ as } t \to \infty \tag{3.92}$$

and the system is uniformly exponentially stable if a positive constant λ exists so that

$$\int_{\tau}^{t} a(\sigma)\, d\sigma \le -\lambda(t-\tau) \quad \text{for all } t, \tau \text{ such that } t \ge \tau \tag{3.93}$$

When $n > 1$, if $A(t)$ commutes with its integral for $t > t_1$ for some $t_1 \ge t_0$ (see Equation 3.47), then

$$\Phi(t,t_1) = \exp\left[\int_{t_1}^{t} A(\sigma)\, d\sigma\right], \quad t > t_1 \tag{3.94}$$

and a sufficient condition for exponential stability (not uniform in general) is that the matrix function

$$\frac{1}{t}\int_{t_2}^{t} A(\sigma)\, d\sigma \tag{3.95}$$

be a bounded function of t and its *pointwise eigenvalues* have real parts $\le -\beta$ for $t > t_2$ for some $t_2 \ge t_1$, where β is a positive constant. The pointwise eigenvalues of a time-varying matrix $M(t)$ are the eigenvalues of the constant matrix $M(\tau)$ for each fixed value of τ viewed as a time-independent parameter. If $A(t)$ is upper or lower triangular with $p_1(t), p_2(t), \ldots, p_n(t)$ on the diagonal, then a sufficient condition for uniform exponential stability is that the off-diagonal entries of $A(t)$ be bounded and the scalar systems

$$\dot{x}_i(t) = p_i(t)x_i(t) \tag{3.96}$$

be uniformly exponentially stable for $i = 1, 2, \ldots, n$. Note that the system matrix $A(t)$ given by Equation 3.32 is lower triangular, and thus, in this case, the system with this particular system matrix is uniformly exponentially stable if the poles $p_i(t)$ in the Floquet factorization given in the left-hand side of

Equation 3.25 are stable in the sense that the scalar systems in Equation 3.96 are uniformly exponentially stable.

If there exists a Lyapunov transformation $z(t) = P^{-1}(t)x(t)$ that puts the system matrix $A(t)$ into an upper or lower triangular form with $p_1(t), p_2(t), \ldots, p_n(t)$ on the diagonal, then sufficient conditions for uniform exponential stability are that the off-diagonal entries of the triangular form be bounded and the scalar systems defined by Equation 3.96 be uniformly exponentially stable. The construction of a Lyapunov transformation that puts $A(t)$ into an upper triangular form is given in [7]. In that paper, the authors define the set $\{p_1(t), p_2(t), \ldots, p_n(t)\}$ of elements on the diagonal of the upper triangular form to be a pole set of the linear time-varying system with system matrix $A(t)$.

Another condition for uniform exponential stability is that a symmetric positive-definite matrix $Q(t)$ exists with $c_1 I \leq Q(t) \leq c_2 I$ for some positive constants c_1, c_2, such that

$$A^T(t)Q(t) + Q(t)A(t) + \dot{Q}(t) \leq -c_3 I \tag{3.97}$$

for some positive constant c_3. Here $F \leq G$ means that F and G are symmetric matrices and $G - F$ is positive semidefinite (all pointwise eigenvalues are ≥ 0). This stability test is referred to as the Lyapunov criterion. For more details, see [8].

Finally, it is noted that if the entries of $B(t)$ and $C(t)$ are bounded, then uniform exponential stability implies that the system is *bounded-input, bounded-output (BIBO) stable*, that is, a bounded input $u(t)$ always results in a bounded output response $y(t)$.

3.2.5 Controllability and Observability

Given a system with the n-dimensional state model $[A(t), B(t), C(t)]$, it is now assumed that the entries of $A(t), B(t)$, and $C(t)$ are at least continuous functions of t. The system is said to be *controllable* on the interval $[t_0, t_1]$, where $t_1 > t_0$, if for any states x_0 and x_1, a continuous input $u(t)$ exists that drives the system to the state $x(t_1) = x_1$ at time $t = t_1$ starting from the state $x(t_0) = x_0$ at time $t = t_0$.

Define the *controllability Gramian* which is the $n \times n$ matrix given by

$$W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, t)B(t)B^T(t)\Phi^T(t_0, t)\, dt \tag{3.98}$$

The controllability Gramian $W(t_0, t_1)$ is symmetric positive semidefinite and is the solution to the matrix differential equation

$$\begin{aligned} &\frac{d}{dt}W(t, t_1) = A(t)W(t, t_1) + W(t, t_1)A^T(t) - B(t)B^T(t), \\ &W(t_1, t_1) = 0 \end{aligned} \tag{3.99}$$

Then the system is controllable on $[t_0, t_1]$ if and only if $W(t_0, t_1)$ is invertible, in which case a continuous input $u(t)$ that drives the system from $x(t_0) = x_0$ to $x(t_1) = x_1$ is

$$u(t) = -B^T(t)\Phi^T(t_0, t)W^{-1}(t_0, t_1)\left[x_0 - \Phi(t_0, t_1)x_1\right], \quad t_0 \leq t \leq t_1 \tag{3.100}$$

There is a sufficient condition for controllability that does not require that the controllability Gramian be computed: Given a positive integer q, suppose that the entries of $B(t)$ are $q - 1$ times continuously differentiable and the entries of $A(t)$ are $q - 2$ times continuously differentiable, and define the $n \times m$ matrices

$$K_0(t) = B(t) \tag{3.101}$$

$$K_i(t) = -A(t)K_{i-1}(t) + \dot{K}_{i-1}(t), \quad i = 1, 2, \ldots, q - 1 \tag{3.102}$$

Finally, let $K(t)$ denotes the $n \times mq$ matrix whose ith block column is equal to $K_{i-1}(t)$, that is

$$K(t) = [K_0(t)\, K_1(t) \ldots K_{q-1}(t)] \tag{3.103}$$

Then a sufficient condition for the system $[A(t), B(t), C(t)]$ to be controllable on the interval $[t_0, t_1]$ is that the $n \times mq$ matrix $K(t)$ defined by Equation 3.103 has rank n for at least one value of $t \in [t_0, t_1]$. This condition was first derived in [9].

The rank condition on the matrix $K(t)$ is preserved under a change of state variables. To show this, suppose the coordinate transformation $z(t) = P^{-1}(t)x(t)$ results in the state model $[\overline{A}(t), \overline{B}(t), \overline{C}(t)]$. For this model, define

$$\bar{K}(t) = [\bar{K}_0(t)\, \bar{K}_1(t)\, \ldots\, \bar{K}_{q-1}(t)] \tag{3.104}$$

where the $\bar{K}_i(t)$ are given by Equations 3.101 and 3.102 with $A(t)$ and $B(t)$ replaced by $\bar{A}(t)$ and $\bar{B}(t)$, respectively. It is assumed that $\bar{A}(t)$ and $\bar{B}(t)$ satisfy the same differentiability requirements as given above for $A(t)$ and $B(t)$ so that $\bar{K}(t)$ is well defined. Then it follows that

$$P^{-1}(t)K(t) = \overline{K}(t) \tag{3.105}$$

Now since $P^{-1}(t)$ is invertible for all $t, P^{-1}(t)$ has rank n for all $t \in [t_0, t_1]$, and thus $\bar{K}(t)$ has rank n for some $t_c \in [t_0, t_1]$ if and only if $K(t_c)$ has rank n. Therefore, the system defined by $[A(t), B(t), C(t)]$ is controllable on the interval $[t_0, t_1]$ if and only if the transformed system $[\overline{A}(t), \overline{B}(t), \overline{C}(t)]$ is controllable on $[t_0, t_1]$.

If the matrix $K(t)$ has rank n for all t, the $n \times n$ matrix $K(t)K^T(t)$ is invertible for all t, and in this case Equation 3.105 can be solved for $P^{-1}(t)$. This gives

$$P^{-1}(t) = \overline{K}(t)K^T(t)\left[K(t)K^T(t)\right]^{-1} \tag{3.106}$$

Note that Equation 3.106 can be used to compute the transformation matrix $P^{-1}(t)$ directly from the coefficient matrices of the state models $[A(t), B(t), C(t)]$ and $[\overline{A}(t), \overline{B}(t), \overline{C}(t)]$.

Now suppose that the system input $u(t)$ is zero, so that the state model is given by

$$\dot{x}(t) = A(t)x(t) \tag{3.107}$$

$$y(t) = C(t)x(t) \tag{3.108}$$

Inserting Equation 3.108 into the solution of Equation 3.107 results in the output response $y(t)$ resulting from initial state $x(t_0)$:

$$y(t) = C(t)\Phi(t, t_0)x(t_0), \quad t > t_0 \tag{3.109}$$

The system is said to be observable on the interval $[t_0, t_1]$ if any initial state $x(t_0)$ can be determined from the output response $y(t)$ given by Equation 3.109 for $t \in [t_0, t_1]$. Define the *observability Gramian* which is the $n \times n$ matrix given by

$$M(t_0, t_1) = \int_{t_0}^{t_1} \Phi^T(t, t_0)C^T(t)C(t)\Phi(t, t_0)\, dt \tag{3.110}$$

The observability Gramian $M(t_0, t_1)$ is symmetric positive semidefinite and is the solution to the matrix differential equation

$$\begin{aligned} &\frac{d}{dt}M(t, t_1) = -A^T(t)M(t, t_1) - M(t, t_1)A(t) - C^T(t)C(t), \\ &M(t_1, t_1) = 0 \end{aligned} \tag{3.111}$$

Then the system is *observable* on $[t_0, t_1]$ if and only if $M(t_0, t_1)$ is invertible, in which case the initial state $x(t_0)$ is given by

$$x(t_0) = M^{-1}(t_0, t_1) \int_{t_0}^{t_1} \Phi^T(t, t_0) C^T(t) y(t)\, dt \tag{3.112}$$

Again given an m-input p-output n-dimensional system with state model $[A(t), B(t), C(t)]$, the *adjoint system* is the p-input m-output n-dimensional system with state model $[-A^T(t), C^T(t), B^T(t)]$. The adjoint system is given by the state equations

$$\dot{\gamma}(t) = -A^T(t)\gamma(t) + C^T(t)\nu(t) \tag{3.113}$$

$$\eta(t) = B^T(t)\gamma(t) \tag{3.114}$$

where $\gamma(t), \nu(t)$, and $\eta(t)$ are the state, input, and output, respectively, of the adjoint system. As noted above in Section 3.2.2, the state-transition matrix of the adjoint system $[-A^T(t), C^T(t), B^T(t)]$ is equal to $\Phi^T(\tau, t)$, where $\Phi(t, \tau)$ is the state-transition matrix for the given system $[A(t), B(t), C(t)]$. Using this fact and the definition of the input and output coefficient matrices of the adjoint system given by Equations 3.113 and 3.114, it is clear that the controllability Gramiam of the adjoint system is identical to the observability Gramiam of the given system. Thus, the given system is observable on $[t_0, t_1]$ if and only if the adjoint system is controllable on $[t_0, t_1]$. In addition, the observability Gramiam of the adjoint system is identical to the controllability Gramiam of the given system. Hence, the given system is controllable on $[t_0, t_1]$ if and only if the adjoint system is observable on $[t_0, t_1]$.

A sufficient condition for observability is given next in terms of the system matrix $A(t)$ and the output matrix $C(t)$: Consider the system with state model $[A(t), B(t), C(t)]$, and define the $p \times n$ matrices

$$L_0(t) = C(t) \tag{3.115}$$

$$L_i(t) = L_{i-1}(t)A(t) + \dot{L}_{i-1}(t), \quad i = 1, 2, \ldots, q-1 \tag{3.116}$$

where q is a positive integer. Consider the $pq \times n$ matrix $L(t)$ whose ith block row is equal to $L_{i-1}(t)$, that is,

$$L(t) = \begin{bmatrix} L_0(t) \\ L_1(t) \\ \vdots \\ L_{q-1}(t) \end{bmatrix} \tag{3.117}$$

It is assumed that the entries of $C(t)$ are $q-1$ times differentiable and the entries of $A(t)$ are $q-2$ times differentiable, so that $L(t)$ is well defined. Then as first shown in [9], the system is observable on $[t_0, t_1]$ if the $pq \times n$ matrix $L(t)$ defined by Equations 3.115 through 3.117 has rank n for at least one value of $t \in [t_0, t_1]$.

Now consider the $n \times pq$ matrix $U(t) = [U_0(t)\ U_1(t) \ldots U_{q-1}(t)]$ generated from the adjoint system $[-A^T(t), C^T(t), B^T(t)]$, where

$$U_0(t) = C^T(t) \tag{3.118}$$

$$U_i(t) = A^T(t)U_{i-1}(t) + \dot{U}_{i-1}(t), \quad i = 1, 2, \ldots, q-1 \tag{3.119}$$

Then from Equations 3.115 through 3.119, it is seen that the transpose $U^T(t)$ of the $n \times pq$ matrix $U(t)$ is equal to the $pq \times n$ matrix $L(t)$ of the given system with state model $[A(t), B(t), C(t)]$. Since the transpose operation does not affect the rank of a matrix, the sufficient condition rank $L(t) = n$ for observability of the system $[A(t), B(t), C(t)]$ implies that the adjoint system $[-A^T(t), C^T(t), B^T(t)]$ is controllable. In addition, the rank condition for observability of the adjoint system implies that the given system is controllable. It also follows from the above constructions that the rank condition for observabiliy of the system $[A(t), B(t), C(t)]$ is preserved under a coordinate transformation.

3.2.6 Control Canonical Form and Controller Design

Now suppose that the system with state model $[A(t), B(t), C(t)]$ has a single input ($m = 1$) so that the input matrix $B(t)$ is an n-element column vector. Assuming that $B(t)$ and $A(t)$ can be differentiated an appropriate number of times, let $R(t)$ denote the $n \times n$ matrix whose columns $r_i(t)$ are defined by

$$r_1(t) = B(t) \tag{3.120}$$

$$r_{i+1}(t) = A(t)r_i(t) - \dot{r}_i(t), \quad i = 1, 2, \ldots, n-1 \tag{3.121}$$

Note that $R(t)$ is a minor variation of the $n \times n$ matrix $K(t)$ defined by Equations 3.101 through 3.103 with $q = n$. In fact, since $R(t)$ is equal to $K(t)$ with a sign change in the columns, $R(t)$ has rank n for all t if and only if $K(t)$ has rank n for all t. Thus, $R(t)$ is invertible for all t if and only if the matrix $K(t)$ has rank n for all t.

In some textbooks, such as [1], the matrix $R(t)$ is called the *controllability matrix* of the system with state model $[A(t), B(t), C(t)]$.

Assuming that $R(t)$ is invertible for all t, define the n-element column vector

$$\eta(t) = -R^{-1}(t)r_{n+1}(t) \tag{3.122}$$

where $r_{n+1}(t)$ is defined by Equation 3.121 with $i = n$. The vector $\eta(t)$ is invariant under any change of state $z(t) = P^{-1}(t)x(t)$. In other words, if Equations 3.120 through 3.122 are evaluated for the new state model $[\overline{A}(t), \overline{B}(t), \overline{C}(t)]$ resulting from the transformation $z(t) = P^{-1}(t)x(t)$, Equation 3.122 will yield the same result for $\eta(t)$. In addition, if $A(t) = A$ and $B(t) = B$ where A and B are constant matrices, the vector η is constant and is given by

$$\eta = [a_0 \; a_1 \; \ldots \; a_{n-1}]^T \tag{3.123}$$

where a_i are the coefficients of the characteristic polynomial of A, that is,

$$\det(sI - A) = s^n + \sum_{i=0}^{n-1} a_i s^i \tag{3.124}$$

Given the analogy with the time-invariant case, the vector $\eta(t)$ given by Equation 3.122 can be viewed as a time-varying version of the *characteristic vector* of the system.

If the $n \times n$ matrix $R(t)$ with columns defined by Equations 3.120 and 3.121 is invertible for all t, there is a coordinate transformation $z(t) = P^{-1}(t)x(t)$, which converts $[A(t), B(t), C(t)]$ into the *control canonical form* $[\bar{A}(t), \bar{B}, \bar{C}(t)]$ with $\bar{A}(t)$ and $\bar{B}$ given by

$$\bar{A}(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \cdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -\psi_0(t) & -\psi_1(t) & -\psi_2(t) & \cdots & -\psi_{n-2}(t) & -\psi_{n-1}(t) \end{bmatrix} \tag{3.125}$$

$$\overline{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{3.126}$$

This form was first derived in [10].

A recursive procedure for computing the rows $c_i(t)$ of the matrix $P^{-1}(t)$ in the transformation $z(t) = P^{-1}(t)x(t)$ to the control canonical form was derived in [11], and is given by

$$c_1(t) = (e_n)^T R^{-1}(t) \tag{3.127}$$

$$c_{i+1}(t) = c_i(t)A(t) + \dot{c}_i(t), \quad i = 1, 2, \ldots, n-1 \tag{3.128}$$

where

$$(e_n)^T = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \tag{3.129}$$

As shown in [11], the entries $\psi_i(t)$ in the bottom row of $\bar{A}(t)$ are given by

$$\psi(t) = [\psi_0(t)\ \psi_1(t)\ \ldots\ \psi_{n-1}(t)] = -(c_n(t)A(t) + \dot{c}_n(t))P(t) \tag{3.130}$$

Now consider the system in the control canonical form given by the state equation

$$\dot{z}(t) = \bar{A}(t)z(t) + \bar{B}u(t) \tag{3.131}$$

where $\bar{A}(t)$ and $\bar{B}$ are defined by Equations 3.125 and 3.126. Then with the *state feedback control* $u(t) = -\bar{g}(t)z(t)$, where $\bar{g}(t)$ is an n-element row vector, from Equation 3.131 the state equation for the resulting closed-loop system is

$$\dot{z}(t) = [\bar{A}(t) - \bar{B}\bar{g}(t)]z(t) \tag{3.132}$$

Let $d = [d_0\ d_1\ \ldots\ d_{n-1}]$ be an n-element row vector with any desired constants $d_0, d_1, \ldots, d_{n-1}$. Then from the form of $\bar{A}(t)$ and $\bar{B}$, it is clear that if $\bar{g}(t)$ is taken to be $\bar{g}(t) = d - \psi(t)$, then the resulting closed-loop system matrix $\bar{A}(t) - \bar{B}\bar{g}(t)$ is equal to the matrix in the right-hand side of Equation 3.125 with the elements $-d_0, -d_1, \ldots, -d_{n-1}$ in the bottom row. Thus, the matrix $\bar{A}(t) - \bar{B}\bar{g}(t)$ is constant and its characteristic polynomial is equal to $s^n + d_{n-1}s^{n-1} + \cdots d_1 s + d_0$. The coefficients of the characteristic polynomial can be assigned to have any desired values $d_0, d_1, \ldots, d_{n-1}$.

The state feedback $u(t) = -\bar{g}(t)z(t)$ in the control canonical form can be expressed in terms of the state $x(t)$ of the given system $[A(t), B(t), C(t)]$ by using the coordinate transformation $z(t) = P^{-1}(t)x(t)$. This results in

$$u(t) = -\bar{g}(t)z(t) = -\bar{g}(t)P^{-1}(t)x(t) = -[d - \psi(t)]P^{-1}(t)x(t) \tag{3.133}$$

Inserting the feedback $u(t)$ given by Equation 3.133 into the state equation $\dot{x}(t) = A(t)x(t) + B(t)u(t)$ for the system $[A(t), B(t), C(t)]$ yields the following equation for the resulting closed-loop system:

$$\dot{x}(t) = (A(t) - B(t)g(t))x(t) \tag{3.134}$$

where the feedback gain vector $g(t)$ is given by

$$g(t) = \bar{g}(t)P^{-1}(t) = [d - \psi(t)]P^{-1}(t) \tag{3.135}$$

By the above constructions, the coordinate transformation $z(t) = P^{-1}(t)x(t)$ transforms the system matrix $A(t) - B(t)g(t)$ of the closed-loop system defined by Equation 3.134 into the system matrix $\bar{A}(t) - \bar{B}\bar{g}(t)$ of the closed-loop system given by Equation 3.132.

As shown in [11], the gain vector $g(t)$ defined by Equation 3.135 can be computed without having to determine $\bar{A}(t)$ by using the following formula:

$$g(t) = -\left[c_{n+1}(t) + \sum_{i=0}^{n-1} d_i c_{i+1}\right] \tag{3.136}$$

where the $c_i(t)$ are given by Equations 3.127 and 3.128 with $i = 1, 2, \ldots, n$. If the zeros of the characteristic polynomial of $\bar{A}(t) - \bar{B}\bar{g}(t)$ are chosen so that they are located in the open left half of the complex plane and if the coordinate transformation $z(t) = P^{-1}(t)x(t)$ is a Lyapunov transformation, then the closed-loop system given by Equation 3.134 is uniformly exponentially stable. If $P(t)$ and $P^{-1}(t)$ are bounded only for $t \geq t_1$ for some finite t_1, then the closed-loop system is exponentially stable. In Section 3.4, an example is given which illustrates the computation of the feedback gain $g(t)$.

3.3 Discrete-Time Linear Time-Varying Systems

A discrete-time causal linear time-varying system with single-input $u(k)$ and single-output $y(k)$ can be modeled by the input/output relationship

$$y(k) = \sum_{j=-\infty}^{k} h(k, j)u(j) \tag{3.137}$$

where k is an integer-valued variable (the discrete-time index) and $h(k,j)$ is the output response resulting from the unit pulse $\delta(k-j)$ (where $\delta(k-j) = 1$ for $k = j$ and $= 0$ for $k \neq j$) applied at time j. It is assumed that $u(k)$ and/or $h(k,j)$ is constrained so that the summation in Equation 3.137 is well defined. The system defined by Equation 3.137 is time invariant if and only if $h(k,j)$ is a function of only the difference $k-j$, in which case Equation 3.137 reduces to the convolution relationship

$$y(k) = h(k) * u(k) = \sum_{j=-\infty}^{k} h(k-j)u(j) \tag{3.138}$$

where $h(k-j) = h(k-j, 0)$.

The system defined by Equation 3.138 is finite dimensional if the input $u(k)$ and the output $y(k)$ are related by the nth-order difference equation

$$y(k+n) + \sum_{i=0}^{n-1} a_i(k)y(k+i) = \sum_{i=0}^{m} b_i(k)u(k+i) \tag{3.139}$$

where $m \leq n$ and the $a_i(k)$ and the $b_i(k)$ are real-valued functions of the discrete-time variable k. The system given by Equation 3.139 is time invariant if and only if all coefficients in Equation 3.139 are constants, that is, $a_i(k) = a_i$ and $b_i(k) = b_i$ for all i, where a_i and b_i are constants.

When $m < n$ the system defined by Equation 3.139 has the n-dimensional state model

$$x(k+1) = A(k)x(k) + B(k)u(k) \tag{3.140}$$

$$y(k) = Cx(k) \tag{3.141}$$

where

$$A(k) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0(k) \\ 1 & 0 & 0 & \cdots & 0 & -a_1(k-1) \\ 0 & 1 & 0 & \cdots & 0 & -a_2(k-2) \\ \vdots & & & \cdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -a_{n-2}(k-n+2) \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1}(k-n+1) \end{bmatrix} \tag{3.142}$$

$$B(k) = \begin{bmatrix} b_0(k) \\ b_1(k-1) \\ b_2(k-2) \\ \vdots \\ b_{n-2}(k-n+2) \\ b_{n-1}(k-n+1) \end{bmatrix} \tag{3.143}$$

and

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

where $b_i(k) = 0$ for $i > m$. This particular state model is referred to as the observer canonical form. As in the continuous-time case, there are other possible state realizations of Equation 3.139, but these will not be considered here. It is interesting to note that the entries of $A(k)$ and $B(k)$ in the observer canonical form are simply time shifts of the coefficients of the input/output difference Equation 3.139, whereas as shown above, in the continuous-time case this relationship is rather complicated.

3.3.1 State Model in the General Case

For an m-input p-output linear n-dimensional time-varying discrete-time system, the general form of the state model is

$$x(k+1) = A(k)x(k) + B(k)u(k) \tag{3.144}$$

$$y(k) = C(k)x(k) + D(k)u(k) \tag{3.145}$$

where the system matrix $A(k)$ is $n \times n$, the input matrix $B(k)$ is $n \times m$, the output matrix $C(k)$ is $p \times n$, and the direct feed matrix $D(k)$ is $p \times m$. The state model given by Equations 3.144 and 3.145 may arise as a result of sampling a continuous-time system given by the state model

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{3.146}$$

$$y(t) = C(t)x(t) + D(t)u(t) \tag{3.147}$$

If the sampling interval is equal to T, then setting $t = kT$ in Equation 3.147 yields an output equation of the form in Equation 3.145, where $C(k) = C(t)|_{t=kT}$ and $D(k) = D(t)|_{t=kT}$. To "discretize" Equation 3.146, first recall (see Equation 3.53) that the solution to Equation 3.146 is

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \phi(t, \tau)B(\tau)u(\tau)\, d\tau, \quad t > t_0 \tag{3.148}$$

Then setting $t = kT + T$ and $t_0 = kT$ in Equation 3.148 yields

$$x(kT+T) = \Phi(kT+T, kT)x(kT) + \int_{kT}^{kT+T} \Phi(kT+T, \tau)B(\tau)u(\tau)\, d\tau \tag{3.149}$$

The second term on the right-hand side of Equation 3.149 can be approximated by

$$\left[\int_{kT}^{kT+T} \Phi(kT+T, \tau)B(\tau)\, d\tau\right] u(kT)$$

and thus Equation 3.149 is in the form of Equation 3.144 with

$$A(k) = \Phi(kT+T, kT) \tag{3.150}$$

$$B(k) = \int_{kT}^{kT+T} \Phi(kT+T, \tau)B(\tau)\, d\tau \tag{3.151}$$

Note that the matrix $A(k)$ given by Equation 3.150 is always invertible since $\Phi(kT+T, kT)$ is always invertible (see the property given by Equation 3.40). As discussed below, this implies that discretized or sampled data systems are "reversible."

From Equations 3.150 and 3.151 it is seen that the computation of $A(k)$ and $B(k)$ requires knowledge of the state-transition matrix $\Phi(t, \tau)$ for $t = kT + T$ and $\tau \in [kT, kT+T)$. As discussed in Section 3.3, if $A(t)$ in Equation 3.146 is a continuous function of t and the variation of $A(t)$ over the intervals

$[kT, kT+T]$ is sufficiently small for all k, then $\Phi(kT+T, \tau)$ can be approximated by

$$\Phi(kT+T,\tau) = e^{A(kT)(kT+T-\tau)} \quad \text{for } \tau \in [kT, kT+T) \tag{3.152}$$

and hence, $A(k)$ and $B(k)$ can be determined using

$$A(k) = e^{A(kT)T} \tag{3.153}$$

$$B(k) = \int_{kT}^{kT+T} e^{A(kT)(kT+T-\tau)} B(\tau)\, d\tau \tag{3.154}$$

Given the discrete-time system defined by Equations 3.144 and 3.145, the solution to Equation 3.144 is

$$x(k) = \Phi(k,k_0)x(k_0) + \sum_{j=k_0}^{k-1} \Phi(k, j+1)B(j)u(j), \quad k > k_0 \tag{3.155}$$

where the $n \times n$ state-transition matrix $\Phi(k, j)$ is given by

$$\Phi(k,k_0) = \begin{cases} \text{not defined for} & k < k_0 \\ I, & k = k_0 \\ A(k-1)A(k-2)\ldots A(k_0), & k > k_0 \end{cases} \tag{3.156}$$

It follows directly from Equation 3.156 that $\Phi(k,k_0)$ is invertible for $k > k_0$ only if $A(k)$ is invertible for $k \geq k_0$. Thus, in general, the initial state $x(k_0)$ cannot be determined from the relationship $x(k) = \Phi(k,k_0)x(k_0)$. In other words, a discrete-time system is not necessarily *reversible*, although any continuous-time system given by Equations 3.146 and 3.147 is reversible since $\Phi(t,t_0)$ is always invertible. However, as noted above, any sampled data system is reversible.

The state-transition matrix $\Phi(k,k_0)$ satisfies the composition property:

$$\Phi(k,k_0) = \Phi(k,k_1)\Phi(k_1,k_0), \quad \text{where } k_0 \leq k_1 \leq k \tag{3.157}$$

and in addition,

$$\Phi(k+1,k_0) = A(k)\Phi(k,k_0), \quad k \geq k_0 \tag{3.158}$$

If $A(k)$ is invertible for all k, $\Phi(k,k_0)$ can be written in the factored form

$$\Phi(k,k_0) = \Phi_1(k)\Phi_2(k_0), \quad k \geq k_0 \tag{3.159}$$

where

$$\Phi_1(k) = \begin{cases} A(k-1)A(k-2)\ldots A(0), & k \geq 1 \\ I, & k = 0 \\ A^{-1}(k-2)A^{-1}(k-3)\ldots A^{-1}(-1), & k < 0 \end{cases} \tag{3.160}$$

$$\Phi_2(k_0) = \begin{cases} A^{-1}(0)A^{-1}(1)\ldots A^{-1}(k_0-1), & k_0 > 0 \\ I, & k_0 = 0 \\ A(-1)A(-2)\ldots A(k_0), & k_0 < 0 \end{cases} \tag{3.161}$$

When the direct feed matrix $D(k)$ in Equation 3.145 is zero, so that $y(k) = C(k)x(k)$, the output response $y(k)$ is given by

$$y(k) = C(k)\Phi(k,k_0)x(k_0) + \sum_{j=k_0}^{k-1} C(k)\Phi(k,j+1)B(j)u(j), \quad k > k_0 \tag{3.162}$$

If the initial time k_0 is set equal to $-\infty$ and the initial state is zero, Equation 3.162 becomes

$$y(k) = \sum_{j=-\infty}^{k-1} C(k)\Phi(k, j+1)B(j)u(j) \tag{3.163}$$

Comparing Equation 3.163 with the m-input p-output version of the input/output Equation 3.137 reveals that

$$H(k,j) = \begin{cases} C(k)\Phi(k,j+1)B(j), & k > j \\ 0, & k \leq j \end{cases} \tag{3.164}$$

where $H(k,j)$ is the $p \times m$ unit-pulse response function matrix. Note that if $A(k)$ is invertible so that $\Phi(k, k_0)$ has the factorization given in Equation 3.159, then $H(k,j)$ can be expressed in the factored form as

$$H(k,j) = [C(k)\Phi_1(k)][\Phi_2(j+1)B(j)] \quad \text{for } k > j. \tag{3.165}$$

As in the continuous-time case, this factorization is a fundamental property of unit-pulse response matrices $H(k,j)$ that are realizable by a state model (with invertible $A(k)$).

3.3.2 Stability

Given an n-dimensional discrete-time system defined by Equations 3.144 and 3.145, consider the homogeneous equation

$$x(k+1) = A(k)x(k), \quad k \geq k_0. \tag{3.166}$$

The solution is

$$x(k) = \Phi(k, k_0)x(k_0), \quad k > k_0 \tag{3.167}$$

where $\Phi(k, k_0)$ is the state-transition matrix defined by Equation 3.156.

The system is said to be asymptotically stable if for some initial time k_0, the solution $x(k)$ satisfies the condition $\|x(k)\| \to 0$ as $k \to 0$ for any initial state $x(k_0)$ at time k_0. This is equivalent to requiring that

$$\|\Phi(k, k_0)\| \to 0, \quad \text{as } k \to \infty \tag{3.168}$$

The system is exponentially stable if for some initial time k_0, there exist finite positive constants c and ρ with $\rho < 1$, such that for any $x(k_0)$ the solution $x(k)$ satisfies

$$\|x(k)\| \leq c\rho^{k-k_0} \|x(k_0)\|, \quad k > k_0 \tag{3.169}$$

If Equation 3.169 holds for all k_0 with the constants c and ρ fixed, the system is said to be uniformly exponentially stable. This is equivalent to requiring that there exist a finite positive constant γ and a nonnegative constant ρ with $\rho < 1$ such that

$$\|\Phi(k,j)\| \leq \gamma\rho^{k-j}, \quad \text{for all } k, j \text{ such that } k \geq j \tag{3.170}$$

Uniform exponential stability is also equivalent to requiring that given any positive constant δ, there exists a positive integer q such that for any k_0 and $x(k_0)$, the solution to Equation 3.166 satisfies

$$\|x(k)\| \leq \delta \|x(k_0)\|, \quad k \geq k_0 + q \tag{3.171}$$

Uniform exponential stability is also equivalent to the existence of a finite positive constant β such that

$$\sum_{i=j+1}^{k} \|\Phi(k,i)\| \leq \beta \quad \text{for all } k, j \text{ such that } k \geq j+1 \tag{3.172}$$

Another necessary and sufficient condition for uniform exponential stability is that a symmetric positive-definite matrix $Q(k)$ exists with $c_1 I \leq Q(k) \leq c_2 I$ for some positive constants c_1 and c_2 so that

$$A^T(k)Q(k+1)A(k) - Q(k) \leq -c_3 I \tag{3.173}$$

for some positive constant c_3.

3.3.3 Controllability and Observability

The discrete-time system defined by Equations 3.144 and 3.145 with $D(k) = 0$ will be denoted by the triple $[A(k), B(k), C(k)]$. The system is said to be controllable on the interval $[k_0, k_1]$ with $k_1 > k_0$ if, for any states x_0 and x_1, an input $u(k)$ exists that drives the system to the state $x(k_1) = x_1$ at time $k = k_1$ starting from the state $x(k_0) = x_0$ at time $k = k_0$. To determine a necessary and sufficient condition for controllability, first solve Equation 3.144 to find the state $x(k_1)$ at time $k = k_1$ resulting from state $x(k_0)$ at time $k = k_0$ and the input sequence $u(k_0), u(k_0 + 1), \ldots, u(k_1 - 1)$. The solution is

$$\begin{aligned} x(k_1) &= \Phi(k_1, k_0)x(k_0) + \Phi(k_1, k_0 + 1)B(k_0)u(k_0) + \Phi(k_1, k_0 + 2)B(k_0 + 1)u(k_0 + 1) \\ &\quad + \cdots \Phi(k_1, k_1 - 1)B(k_1 - 2)u(k_1 - 2) + B(k_1 - 1)u(k_1 - 1) \end{aligned} \tag{3.174}$$

Let $R(k_1, k_0)$ be the *controllability (or reachability) matrix* with n rows and $(k_1 - k_0)m$ columns defined by

$$R(k_0, k_1) = [B(k_1 - 1)\ \Phi(k_1, k_1 - 1)B(k_1 - 2) \ldots \Phi(k_1, k_0 + 2)B(k_0 + 1)\ \Phi(k_1, k_0 + 1)B(k_0)] \tag{3.175}$$

Then Equation 3.174 can be written the form

$$x(k_1) = \Phi(k_1, k_0)x(k_0) + R(k_0, k_1)U(k_0, k_1) \tag{3.176}$$

where $U(k_0, k_1)$ is the $(k_1 - k_0)m$-element column vector of inputs given by

$$U(k_0, k_1) = \left[u^T(k_1 - 1)\, u^T(k_1 - 2) \cdots u^T(k_0 + 1)\, u^T(k_0)\right]^T \tag{3.177}$$

Now for any states $x(k_0) = x_0$ and $x(k_1) = x_1$, from Equation 3.176, there is a sequence of inputs given by $U(k_0, k_1)$ that drives the system from x_0 to x_1 if and only if the matrix $R(k_0, k_1)$ has rank n. If this is the case, Equation 3.176 can be solved for $U(k_0, k_1)$, giving

$$U(k_0, k_1) = R^T(k_0, k_1)\left[R(k_0, k_1)R^T(k_0, k_1)\right]^{-1}[x_1 - \Phi(k_1, k_0)x_0] \tag{3.178}$$

Hence, rank $R(k_0, k_1) = n$ is a necessary and sufficient condition for controllability over the interval $[k_0, k_1]$.

Given a fixed positive integer N, set $k_0 = k - N + 1$ and $k_1 = k + 1$ in $R(k_0, k_1)$, which results in the matrix $R(k - N + 1, k + 1)$. Note that $R(k - N + 1, k + 1)$ is a function of only the integer variable k and that the size of the matrix $R(k - N + 1, k + 1)$ is equal to $n \times Nm$ since $k_1 - k_0 = N$. The $n \times Nm$ matrix $R(k - N + 1, k + 1)$ will be denoted by $R(k)$. By definition of the state-transition matrix $\Phi(k, k_0)$, $R(k)$ can be written in the form

$$R(k) = [R_0(k)\, R_1(k) \ldots R_{N-1}(k)] \tag{3.179}$$

where the block columns $R_i(k)$ of $R(k)$ are given by

$$R_0(k) = B(k) \tag{3.180}$$

$$R_i(k) = A(k)R_{i-1}(k - 1), \quad i = 1, 2, \ldots, N - 1 \tag{3.181}$$

The system is said to be *uniformly N-step controllable* if rank $R(k) = n$ for all k. Uniformly N-step controllable means that the system is controllable on the interval $[k - N + 1,\ k + 1]$ for all k.

Now suppose that the system input $u(k)$ is zero, so that the state model is given by

$$x(k + 1) = A(k)x(k) \tag{3.182}$$

$$y(k) = C(k)x(k) \tag{3.183}$$

From Equations 3.182 and 3.183, the output response $y(k)$ resulting from initial state $x(k_0)$ is given by

$$y(k) = C(k)\Phi(k, k_0)x(k_0), \quad k \geq k_0 \tag{3.184}$$

Then the system is said to be observable on the interval $[k_0, k_1]$ if any initial state $x(k_0) = x_0$ can be determined from the output response $y(k)$ given by Equation 3.184 for $k = k_0, k_0 + 1, \ldots, k_1 - 1$. Using

Equation 3.184 for $k = k_0$ to $k = k_1 - 1$ yields

$$\begin{bmatrix} y(k_0) \\ y(k_0+1) \\ \vdots \\ y(k_1-2) \\ y(k_1-1) \end{bmatrix} = \begin{bmatrix} C(k_0)x_0 \\ C(k_0+1)\Phi(k_0+1,k_0)x_0 \\ \vdots \\ C(k_1-2)\Phi(k_1-2,k_0)x_0 \\ C(k_1-1)\Phi(k_1-1,k_0)x_0 \end{bmatrix} \tag{3.185}$$

The right-hand side of Equation 3.185 can be written in the form $O(k_0, k_1)x_0$ where $O(k_0, k_1)$ is the $(k_1 - k_0)p \times n$ *observability matrix* defined by

$$O(k_0,k_1) = \begin{bmatrix} C(k_0) \\ C(k_0+1)\Phi(k_0+1,k_0) \\ \vdots \\ C(k_1-2)\Phi(k_1-2,k_0) \\ C(k_1-1)\Phi(k_1-1,k_0) \end{bmatrix} \tag{3.186}$$

Equation 3.185 can be solved for any initial state x_0 if and only if rank $O(k_0, k_1) = n$, which is a necessary and sufficient condition for observability on $[k_0, k_1]$. If the rank condition holds, the solution of Equation 3.185 for x_0 is

$$x_0 = \left[O^T(k_0,k_1)O(k_0,k_1)\right]^{-1} O^T(k_0,k_1)Y(k_0,k_1) \tag{3.187}$$

where $Y(k_0, k_1)$ is the $(k_1 - k_0)p$-element column vector of outputs given by

$$Y(k_0,k_1) = \left[y^T(k_0)\, y^T(k_0+1) \cdots y^T(k_1-2)\, y^T(k_1-1)\right]^T \tag{3.188}$$

Given a positive integer N, setting $k_0 = k$ and $k_1 = k + N$ in $O(k_0,\ k_1)$ yields the $Np \times n$ matrix $O(k, k+N)$, which will be denoted by $O(k)$. By definition of the state-transition matrix $\Phi(k, k_0)$, $O(k)$ can be written in the form

$$O(k) = \begin{bmatrix} O_0(k) \\ O_1(k) \\ \vdots \\ O_{N-1}(k) \end{bmatrix} \tag{3.189}$$

where the block rows $O_i(k)$ of $O(k)$ are given by

$$O_0(k) = C(k) \tag{3.190}$$

$$O_i(k) = O_{i-1}(k+1)A(k), \quad i = 1, 2, \ldots, N-1 \tag{3.191}$$

The system is said to be *uniformly N-step observable* if rank $O(k) = n$ for all k. Uniformly N-step observable means that the system is observable on the interval $[k, k+N]$ for all k.

3.3.4 Change of State Variables and Canonical Forms

Again, consider the discrete-time system with state model $[A(k), B(k), C(k)]$. For any $n \times n$ invertible matrix $P(k)$, another state model can be generated by defining the new state vector $z(k) = P^{-1}(k)x(k)$.

The new state model is given by

$$z(k+1) = \bar{A}(k)z(k) + \bar{B}(k)u(k) \tag{3.192}$$

$$y(k) = \bar{C}(k)z(k) \tag{3.193}$$

where

$$\bar{A}(k) = P^{-1}(k+1)A(k)P(k) \tag{3.194}$$

$$\bar{B}(k) = P^{-1}(k+1)B(k) \tag{3.195}$$

$$\bar{C}(k) = C(k)P(k) \tag{3.196}$$

The state-transition matrix $\bar{\Phi}(k, k_0)$ for the new state model is given by

$$\bar{\Phi}(k, k_0) = P^{-1}(k)\Phi(k, k_0)P(k_0) \tag{3.197}$$

where $\Phi(k, k_0)$ is the state-transition matrix for $[A(k), B(k), C(k)]$. The new state model, which will be denoted by $[\bar{A}(k), \bar{B}(k), \bar{C}(k)]$, and the given state model $[A(k), B(k), C(k)]$ are said to be algebraically equivalent.

Given an n-dimensional state model $[A(k), B(k), C(k)]$ and any $n \times n$ invertible matrix function $\Gamma(k)$, if $A(k)$ is invertible, there is an invertible coordinate transformation matrix $P(k)$ for $k \geq k_0$, which transforms $A(k)$ into $\Gamma(k)$ for $k \geq k_0$, that is, $\bar{A}(k) = \Gamma(k)$, $k \geq k_0$. To show this, define $P(k)$ by the matrix difference equation

$$P(k+1) = A(k)P(k)\Gamma^{-1}(k), \quad k \geq k_0 \tag{3.198}$$

with initial condition $P(k_0) = I = n \times n$ identity matrix. Then multiplying both sides of Equation 3.198 on the right by $\Gamma(k)$ and multiplying the resulting equation on the left by $P^{-1}(k+1)$ yields Equation 3.194 with $\bar{A}(k) = \Gamma(k)$ for $k \geq k_0$. This result shows that an invertible matrix $A(k)$ can be put into a diagonal form via a coordinate transformation with any desired nonzero functions or constants on the diagonal. Thus, as in the continuous-time case, there is no useful generalization of the notion of eigenvalues and eigenvectors in the time-varying case, unless additional conditions are placed on the coordinate transformation. For example, one can require that the transformation be a Lyapunov transformation, which means that both $P(k)$ and its inverse $P^{-1}(k)$ are bounded matrix functions of the integer variable k. It follows from Equation 3.197 that uniform exponential stability is preserved under a Lyapunov transformation.

Suppose that the systems $[A(k), B(k), C(k)]$ and $[\bar{A}(k), \bar{B}(k), \bar{C}(k)]$ are algebraically equivalent and let $R(k)$ denote the $n \times Nm$ controllability matrix for the system $[A(k), B(k), C(k)]$, where $R(k)$ is defined by Equations 3.179 through 3.181. Similarly, for the system given by $[\bar{A}(k), \bar{B}(k), \bar{C}(k)]$ define

$$\overline{R}(k) = [\bar{R}_0(k)\ \bar{R}_1(k)\ \ldots\ \bar{R}_{n-1}(k)] \tag{3.199}$$

where the $\bar{R}_i(k)$ are given by Equations 3.180 and 3.181 with $A(k)$ and $B(k)$ replaced by $\bar{A}(k)$ and $\bar{B}(k)$, respectively. Then the coordinate transformation $P^{-1}(k)$ is given by

$$P^{-1}(k+1)R(k) = \bar{R}(k) \tag{3.200}$$

If the system $[A(k), B(k), C(k)]$ is uniformly N-step controllable, $R(k)$ has rank n for all k, and the $n \times n$ matrix $R(k)R^T(k)$ is invertible. Thus, Equation 3.200 can be solved for $P^{-1}(k+1)$, which gives

$$P^{-1}(k+1) = \bar{R}(k)R^T(k)[R(k)R^T(k)]^{-1} \tag{3.201}$$

It follows from Equation 3.200 that uniform N-step controllability is preserved under a change of state variables.

Now suppose that the n-dimensional system with state model $[A(k), B(k), C(k)]$ is uniformly N-step controllable with $N = n$ and that the system has a single input ($m = 1$) so that $B(k)$ is an n-element column vector and $R(k)$ is a $n \times n$ invertible matrix. Define

$$R_n(k) = A(k)R_{n-1}(k-1) \tag{3.202}$$

$$\eta(k) = -R^{-1}(k)R_n(k) \tag{3.203}$$

where $R_{n-1}(k)$ is the column vector defined by Equation 3.181 with $i = n-1$. The n-element column vector $\eta(k)$ defined by Equation 3.203 is invariant under any change of state $z(k) = P^{-1}(k)x(k)$, and in the time-invariant case, η is constant and is given by

$$\eta = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \end{bmatrix}^T \tag{3.204}$$

where the a_i are the coefficients of the characteristic polynomial of A.

Given $\eta(k)$ defined by Equation 3.203, write $\eta(k)$ in the form

$$\eta(k) = \begin{bmatrix} \eta_0(k) & \eta_1(k) & \cdots & \eta_{n-1}(k) \end{bmatrix} \tag{3.205}$$

Then as proved in [3], there is a transformation $P(k)$ which converts $[A(k), B(k), C(k)]$ into the control canonical form $[\bar{A}(k), \bar{B}, \bar{C}(k)]$, with $\bar{A}(k)$ and $\bar{B}$ given by

$$\bar{A}(k) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \cdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -\eta_0(k) & -\eta_1(k+1) & -\eta_2(k+2) & \cdots & -\eta_{n-2}(k+n-2) & -\eta_{n-1}(k+n-1) \end{bmatrix} \tag{3.206}$$

$$\bar{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{3.207}$$

The transformation matrix $P^{-1}(k)$ that yields the control canonical form can be determined using Equation 3.201. As in the continuous-time case, the control canonical form can be used to design a state feedback control which results in a uniformly exponentially stable closed-loop system if the coordinate transformation defined by $P^{-1}(k)$ is a Lyapunov transformation. If $P(k)$ and $P^{-1}(k)$ are bounded only for $k \geq k_1$ for some finite k_1, the resulting closed-loop system is exponentially stable. The details are a straightforward modification of the continuous-time case, and thus are not pursued here.

3.4 Applications and Examples

In Section 3.4.1, an example is given on the construction of canonical forms and the design of observers and controllers.

3.4.1 Observer and Controller Design

Consider the single-input single-output linear time-varying continuous-time system given by the input/output differential equation

$$\ddot{y}(t) + e^{-t}\dot{y}(t) + y(t) = \dot{u}(t) \tag{3.208}$$

To determine the state model, which is in observer canonical form, write Equation 3.208 in the form

$$\ddot{y}(t) + D[\alpha_1(t)y(t)] + \alpha_0(t)y(t) = \dot{u}(t) \tag{3.209}$$

In this case,

$$\alpha_1(t) = e^{-t} \quad \text{and} \quad \alpha_0(t) = 1 + e^{-t} \tag{3.210}$$

Then from Equations 3.22 through 3.24, the observer canonical form of the state model is

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -1-e^{-t} \\ 1 & -e^{-t} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{3.211}$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{3.212}$$

The state variables $x_1(t)$ and $x_2(t)$ in this state model are given by

$$x_1(t) = \dot{y}(t) + e^{-t}y(t) - u(t) \tag{3.213}$$

$$x_2(t) = y(t) \tag{3.214}$$

If the output $y(t)$ of the system can be differentiated, then $x_1(t)$ and $x_2(t)$ can be directly determined from the input $u(t)$ and the output $y(t)$ by using Equations 3.213 and 3.214. In practice, however, differentiation of signals should be avoided, and thus directly determining $x_1(t)$ using Equation 3.213 is usually not viable. As discussed next, by using a state observer, the state $x(t)$ can be estimated without having to differentiate signals. Actually, in this particular example it is necessary to estimate only $x_1(t)$ since the output $y(t) = x_2(t)$ is known, and thus a reduced-order observer could be used, but this is not considered here.

An observer for the state $x(t)$ is given by

$$\frac{d}{dt}\hat{x}(t) = A(t)\hat{x}(t) + H(t)[y(t) - C(t)\hat{x}(t)] + B(t)u(t) \tag{3.215}$$

where $H(t)$ is the n-element observer gain vector and $\hat{x}(t)$ is the estimate of $x(t)$. With the estimation error $e(t)$ defined by $e(t) = x(t) - \hat{x}(t)$, the error is given by the differential equation

$$\dot{e}(t) = [A(t) - H(t)C(t)]\, e(t), \quad t > t_0 \tag{3.216}$$

with initial error $e(t_0)$ at initial time t_0. The objective is to choose the gain vector $H(t)$ so that, for any initial error $e(t_0)$, $\|e(t)\| \to 0$ as $t \to \infty$, with some desired rate of convergence.

For the system given by Equations 3.211 and 3.212, the error Equation 3.216 is

$$\dot{e}(t) = \begin{bmatrix} 0 & -1-e^{-t}-h_1(t) \\ 1 & -e^{-t}-h_2(t) \end{bmatrix} e(t) \tag{3.217}$$

where $H(t) = [h_1(t) \quad h_2(t)]^T$. From Equation 3.217, it is obvious that by setting

$$h_1(t) = m_0 - 1 - e^{-t} \tag{3.218}$$

$$h_2(t) = m_1 - e^{-t} \tag{3.219}$$

where m_0 and m_1 are constants, the coefficient matrix on the right-hand side of Equation 3.217 is constant, and its eigenvalues can be assigned by choosing m_0 and m_1. Hence, any desired rate of convergence to zero can be achieved for the error $e(t)$.

The estimate $\hat{x}(t)$ of $x(t)$ can then be used to realize a feedback control law of the form

$$u(t) = -g(t)\hat{x}(t) \tag{3.220}$$

where $g(t)$ is the feedback gain vector. The first step in pursuing this is to consider the extent to which the system can be controlled by state feedback of the form given in Equation 3.220 with $\hat{x}(t)$ replaced by $x(t)$; in other words, the true system state $x(t)$ is assumed to be available. In particular, we can ask whether or not there is a gain vector $g(t)$ so that with $u(t) = -g(t)x(t)$, the state of the resulting closed-loop system decays to zero exponentially with some desired rate of convergence. This can be answered by attempting to transform the state model given by Equations 3.211 and 3.212 to control canonical form. Following the procedure given in Section 3.2, the steps are as follows.

Let $R(t)$ denote the 2×2 matrix whose columns $r_i(t)$ are defined by Equations 3.120 and 3.121 with $n = 2$. This yields

$$r_1(t) = B(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{3.221}$$

$$r_2(t) = A(t)r_1(t) + \dot{r}_1(t) = \begin{bmatrix} -1-e^{-t} \\ -e^{-t} \end{bmatrix} \tag{3.222}$$

Then

$$R(t) = [r_1(t) \quad r_2(t)] = \begin{bmatrix} 0 & -1-e^{-t} \\ 1 & -e^{-t} \end{bmatrix} \tag{3.223}$$

and

$$\det[R(t)] = 1 + e^{-t} \quad \text{for all } t \tag{3.224}$$

where "det" denotes the determinant. Since $\det[R(t)] \neq 0$ for all $t, R(t)$ has rank 2 for all t, and thus the control canonical form exists for the system given by Equations 3.211 and 3.212.

The rows $c_i(t)$ of the matrix $P^{-1}(t)$ in the transformation $z(t) = P^{-1}(t)x(t)$ to the control canonical form are computed using Equations 3.127 and 3.128. This yields

$$c_1(t) = [-(1+e^{-t})^{-1} \quad 0] \quad \text{and} \quad c_2(t) = [-e^{-t}(1+e^{-t})^{-2} \quad 1] \tag{3.225}$$

and thus

$$P^{-1}(t) = \begin{bmatrix} -(1+e^{-t})^{-1} & 0 \\ -e^{-t}(1+e^{-t})^{-2} & 1 \end{bmatrix} \quad \text{and} \quad P(t) = \begin{bmatrix} -(1+e^{-t}) & 0 \\ -e^{-t}(1+e^{-t})^{-1} & 1 \end{bmatrix} \tag{3.226}$$

Now choose the coefficients of the characteristic polynomial of the closed-loop system matrix in the control canonical form to be d_0 and d_1. Then from Equation 3.136, the feedback gain vector $g(t)$ back in the original state coordinates is given by

$$g(t) = -\left(c_3(t) + d_0c_1(t) + d_1c_2(t)\right) \tag{3.227}$$

where

$$c_3(t) = c_2(t)A(t) + \dot{c}_2(t) \tag{3.228}$$

$$c_3(t) = [1 - 2e^{-2t}(1+e^{-t})^{-3} + e^{-t}(1+e^{-t})^{-2} \quad e^{-t}(1+e^{-t})^{-1} - e^{-t}] \tag{3.229}$$

Inserting the expressions for $c_1(t), c_2(t)$, and $c_3(t)$ given by Equations 3.225 and 3.229 into Equation 3.227 results in the feedback gain vector $g(t)$.

Since $P(t)$ and $P^{-1}(t)$ given by Equation 3.226 are bounded for $t \geq t_1$ for any finite t_1, by choosing appropriate values for d_0 and d_1 it follows that via the state feedback control given by $u(t) = -g(t)x(t)$, the resulting closed-loop system is exponentially stable with any desired rate λ of convergence to zero. It also follows that with the state feedback control $u(t) = -g(t)\hat{x}(t)$, where $\hat{x}(t)$ is the estimated state, the resulting closed-loop system is also exponentially stable with any desired rate of convergence to zero.

3.4.2 Exponential Systems

A system with n-dimensional state model $[A(t), B(t), C(t)]$ is said to be an *exponential system* if its state-transition matrix $\Phi(t,\tau)$ can be written in the matrix exponential form

$$\Phi(t,\tau) = e^{\Gamma(t,\tau)} \tag{3.230}$$

where $\Gamma(t,\tau)$ is a $n \times n$ matrix function of t and τ. The form given in Equation 3.230 is valid (at least locally, that is, when t is close to τ) for a large class of time-varying systems. In fact, as noted above, for $t \in [\tau, \tau + T]$, $\Phi(t,\tau)$ can be approximated by the matrix exponential $e^{A(kT)(t-\tau)}$ if $A(t)$ is approximately equal to $A(\tau)$ for $t \in [\tau, \tau + T]$. For a mathematical development of the matrix exponential form, see [5].

As noted previously, the exponential form in Equation 3.230 is valid for any system where $A(t)$ commutes with its integral (see Equation 3.47), in which case

$$\Gamma(t,\tau) = \int_\tau^t A(\sigma)\, d\sigma \tag{3.231}$$

The class of systems for which $A(t)$ commutes with its integral is actually fairly large; in particular, this is the case for any $A(t)$ given by

$$A(t) = \sum_{i=1}^{r} f_i(t) A_i \tag{3.232}$$

where $f_i(t)$ are arbitrary real-valued functions of t and the A_i are arbitrary constant $n \times n$ matrices that satisfy the commutativity conditions

$$A_i A_j = A_j A_i, \quad \text{for all integers } 1 \le i, j \le r \tag{3.233}$$

For example, suppose that

$$A(t) = \begin{bmatrix} f_1(t) & c_1 f_2(t) \\ c_2 f_2(t) & f_1(t) \end{bmatrix} \tag{3.234}$$

where $f_1(t)$ and $f_2(t)$ are arbitrary real-valued functions of t and c_1 and c_2 are arbitrary constants. Then

$$A(t) = f_1(t)A_1 + f_2(t)A_2 \tag{3.235}$$

where $A_1 = I$ and

$$A_2 = \begin{bmatrix} 0 & c_1 \\ c_2 & 0 \end{bmatrix} \tag{3.236}$$

Obviously, A_1 and A_2 commute, and thus $\Phi(t,\tau)$ is given by Equations 3.230 and 3.231. In this case, $\Phi(t,\tau)$ can be written in the form

$$\Phi(t,\tau) = \exp\left[\left(\int_\tau^t f_1(\sigma)\, d\sigma\right) I\right] \exp\left[\left(\int_\tau^t f_2(\sigma)\, d\sigma\right) A_2\right] \tag{3.237}$$

Given an n-dimensional system with exponential state-transition matrix $\Phi(t,\tau) = e^{\Gamma(t,\tau)}$, $\Phi(t,\tau)$ can be expressed in terms of scalar functions using the Laplace transform as in the time-invariant case.

In particular, let

$$\Phi(t,\beta,\tau) = \text{inverse transform of } [sI - (1/\beta)\Gamma(\beta,\tau)]^{-1} \tag{3.238}$$

where $\Gamma(\beta,\tau) = \Gamma(t,\tau)|_{t=\beta}$ and β is viewed as a parameter. Then

$$\Phi(t,\tau) = \Phi(t,\beta,\tau)|_{\beta=t} \tag{3.239}$$

For example, suppose that

$$A(t) = \begin{bmatrix} f_1(t) & f_2(t) \\ -f_2(t) & f_1(t) \end{bmatrix} \tag{3.240}$$

where $f_1(t)$ and $f_2(t)$ are arbitrary functions of t with the constraint that $f_2(t) \geq 0$ for all t. Then

$$\Gamma(t,\tau) = \int_\tau^t A(\sigma)\,d\sigma \tag{3.241}$$

and $\Phi(t,\beta,\tau)$ is equal to the inverse transform of

$$\Phi(s,\beta,\tau) = \begin{bmatrix} s-\gamma_1(\beta,\tau) & -\gamma_2(\beta,\tau) \\ \gamma_2(\beta,\tau) & s-\gamma_1(\beta,\tau) \end{bmatrix}^{-1} \tag{3.242}$$

where

$$\gamma_1(\beta,\tau) = (1/\beta)\int_\tau^\beta f_1(\sigma)\,d\sigma \tag{3.243}$$

$$\gamma_2(\beta,\tau) = (1/\beta)\int_\tau^\beta f_2(\sigma)\,d\sigma \tag{3.244}$$

Computing the inverse Laplace transform of $\Phi(s,\beta,\tau)$ and using Equation 3.239 give (for $t > 0$)

$$\Phi(t,\tau) = \begin{bmatrix} e^{\gamma_1(t,\tau)t}\cos[\gamma_2(t,\tau)t] & e^{\gamma_1(t,\tau)t}\sin[\gamma_2(t,\tau)t] \\ -e^{\gamma_1(t,\tau)t}\sin[\gamma_2(t,\tau)t] & e^{\gamma_1(t,\tau)t}\cos[\gamma_2(t,\tau)t] \end{bmatrix} \tag{3.245}$$

3.4.3 Stability

Again consider an n-dimensional exponential system $[A(t), B(t), C(t)]$ with state-transition matrix $\Phi(t,\tau) = e^{\Gamma(t,\tau)}$. A sufficient condition for exponential stability of the differential equation $\dot{x}(t) = A(t)x(t)$is that the $n \times n$ matrix $(1/t)\Gamma(t,\tau)$ be bounded as a function of t and its pointwise eigenvalues have real parts $\leq -\nu$ for some $\nu > 0$ and all $t > \tau$ for some finite τ. For example, suppose that $A(t)$ is given by Equation 3.234 so that

$$(1/t)\Gamma(t,\tau) = \begin{bmatrix} \gamma_1(t,\tau) & c_1\gamma_2(t,\tau) \\ c_2\gamma_2(t,\tau) & \gamma_1(t,\tau) \end{bmatrix} \tag{3.246}$$

where $\gamma_1(t,\tau)$ and $\gamma_2(t,\tau)$ are given by Equations 3.243 and 3.244 with $\beta = t$. Then

$$\det[sI - (1/t)\Gamma(t,\tau)] = s^2 - 2\gamma_1(t,\tau)s + \gamma_1^2(t,\tau) - c_1c_2\gamma_2^2(t,\tau) \tag{3.247}$$

and the pointwise eigenvalues of $(1/t)\Gamma(t,\tau)$ have real parts $\leq -\nu$ for all $t > \tau$ for some $\nu > 0$ and τ if

$$\gamma_1(t,\tau) \leq \nu_1, \quad \text{for all } t > \tau \quad \text{and} \quad \text{some } \nu_1 < 0, \tag{3.248}$$

$$\gamma_1^2(t,\tau) - c_1c_2\gamma_2^2(t,\tau) \geq \nu_2 \quad \text{for all } t > \tau \quad \text{and} \quad \text{some } \nu_2 > 0 \tag{3.249}$$

Therefore, if Equations 3.248 and 3.249 are satisfied and $\gamma_1(t,\tau)$ and $\gamma_2(t,\tau)$ are bounded functions of t, the solutions to $\dot{x}(t) = A(t)x(t)$ with $A(t)$ given by Equation 3.234 decay to zero exponentially.

It is well known that in general there is no pointwise eigenvalue condition on the system matrix $A(t)$ that insures exponential stability, or even asymptotic stability. For an example (taken from [8]), suppose that

$$A(t) = \begin{bmatrix} -1 + \alpha(\cos^2 t) & 1 - \alpha(\sin t)(\cos t) \\ -1 - \alpha(\sin t)(\cos t) & -1 + \alpha(\sin^2 t) \end{bmatrix} \tag{3.250}$$

where α is a real parameter. The pointwise eigenvalues of $A(t)$ are equal to

$$\frac{\alpha - 2 \pm \sqrt{\alpha^2 - 4}}{2} \tag{3.251}$$

which are strictly negative if $0 < \alpha < 2$. But

$$\Phi(t,0) = \begin{bmatrix} e^{(\alpha-1)t}(\cos t) & e^{-t}(\sin t) \\ -e^{(\alpha-1)t}(\sin t) & e^{-t}(\cos t) \end{bmatrix} \tag{3.252}$$

and thus, the system is obviously not asymptotically stable if $\alpha > 1$.

3.4.4 The Lyapunov Criterion

By using the Lyapunov criterion (see Equation 3.97), it is possible to derive sufficient conditions for uniform exponential stability without computing the state-transition matrix. For example, suppose that

$$A(t) = \begin{bmatrix} 0 & 1 \\ -1 & -a(t) \end{bmatrix} \tag{3.253}$$

where $a(t)$ is a real-valued function of t with $a(t) \geq c$ for all $t > t_1$, for some t_1 and some constant $c > 0$. Now in Equation 3.97, choose

$$Q(t) = \begin{bmatrix} a(t) + \dfrac{2}{a(t)} & 1 \\ 1 & \dfrac{2}{a(t)} \end{bmatrix} \tag{3.254}$$

Then, $c_1 I \leq Q(t) \leq c_2$, for all $t > t_1$ for some constants $c_1 > 0$ and $c_2 > 0$. Now

$$Q(t)A(t) + A^T(t)Q(t) + \dot{Q}(t) = \begin{bmatrix} -2 + \dot{a}(t) - \dfrac{\dot{a}(t)}{a^2(t)} & 0 \\ 0 & -1 - \dfrac{\dot{a}(t)}{a^2(t)} \end{bmatrix} \tag{3.255}$$

Hence, if

$$-2 + \dot{a}(t) - \frac{\dot{a}(t)}{a^2(t)} \leq -c_3 \quad \text{for } t > t_1 \quad \text{for some } c_3 > 0 \tag{3.256}$$

and

$$-1 - \frac{\dot{a}(t)}{a^2(t)} \leq -c_4 \quad \text{for } t > t_1 \quad \text{for some } c_4 > 0 \tag{3.257}$$

the system is uniformly exponentially stable. For instance, if $a(t) = b - \cos t$, then Equations 3.256 and 3.257 are satisfied if $b > 2$, in which case the system is uniformly exponentially stable.

Now suppose that

$$A(t) = \begin{bmatrix} 0 & 1 \\ -a_1(t) & -a_2(t) \end{bmatrix} \tag{3.258}$$

As suggested in [8, p. 109], sufficient conditions for uniform exponential stability can be derived by taking

$$Q(t) = \begin{bmatrix} a_1(t) + a_2(t) + \dfrac{a_1(t)}{a_2(t)} & 1 \\ 1 & 1 + \dfrac{1}{a_2(t)} \end{bmatrix} \tag{3.259}$$

.5 Defining Terms

tate model: For linear time-varying systems, this is a mathematical representation of the system in terms of state equations of the form $\dot{x}(t) = A(t)x(t) + B(t)u(t), y(t) = C(t)x(t) + D(t)u(t)$.

tate-transition matrix: The matrix $\Phi(t, t_0)$ where $\Phi(t, t_0)x(t_0)$ is the state at time t starting with state $x(t_0)$ at time t_0 and with no input applied for $t \geq t_0$.

xponential system: A system whose state-transition matrix $\Phi(t, \tau)$ can be written in the exponential form $e^{\Gamma(t,\tau)}$ for some $n \times n$ matrix function $\Gamma(t, \tau)$.

leversible system: A system whose state-transition matrix is invertible.

ampled data system: A discrete-time system generated by sampling the inputs and outputs of a continuous-time system.

hange of state: A transformation $z(t) = P^{-1}(t)x(t)$ from the state vector $x(t)$ to the new state vector $z(t)$.

lgebraic equivalence: Refers to two state models of the same system related by a change of state.

yapunov transformation: A change of state $z(t) = P^{-1}(t)x(t)$ where $P(t)$ and its inverse $P^{-1}(t)$ are both bounded functions of t.

anonical form: A state model $[A(t), B(t), C(t)]$ with one or more of the coefficient matrices $A(t), B(t), C(t)$ in a special form.

ontrol canonical form: In the single-input case, a canonical form for $A(t)$ and $B(t)$ that facilitates the study of state feedback control.

bserver canonical form: In the single-output case, a canonical form for $A(t)$ and $C(t)$ that facilitates the design of a state observer.

haracteristic vector: A time-varying generalization corresponding to the vector of coefficients of the characteristic polynomial in the time-invariant case.

symptotic stability: Convergence of the solutions of $\dot{x}(t) = A(t)x(t)$ to zero for any initial state $x(t_0)$.

xponential stability: Convergence of the solutions of $\dot{x}(t) = A(t)x(t)$ to zero at an exponential rate.

niform exponential stability: Convergence of the solutions of $\dot{x}(t) = A(t)x(t)$ to zero at an exponential rate uniformly with respect to the initial time.

ointwise eigenvalues: The eigenvalues of a $n \times n$ time-varying matrix $M(t)$ with t replaced by τ, where τ is viewed as a time-independent parameter.

ontrollability: The existence of inputs that drive a system from any initial state to any desired state.

bservability: The ability to compute the initial state $x(t_0)$ from knowledge of the output response $y(t)$ for $t \geq t_0$.

tate feedback control: A control signal of the form $u(t) = -F(t)x(t)$ where $F(t)$ is the feedback gain matrix and $x(t)$ is the system state.

bserver: A system which provides an estimate $\hat{x}(t)$ of the state $x(t)$ of a system.

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eferences

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Further Reading

There are a large number of research papers and textbooks that contain results on the theory of linear time-varying systems. Only a small portion of the existing work is mentioned here, with the emphasis on textbooks: In addition to [1] and [8] listed in the references above, textbooks that contain material on the fundamentals of linear time-varying systems include the following:

14. Zadeh, L.A. and Desoer, C.A., *Linear System Theory*, McGraw-Hill, New York, 1963.
15. Brockett, R.W., *Finite Dimensional Linear Systems*, Wiley & Sons, New York, 1970.
16. D'Angelo, H., *Linear Time-Varying Systems*, Allyn and Bacon, Boston, MA, 1970.
17. Kailath, T., *Linear Systems*, Prentice-Hall, Englewood Cliffs, NJ, 1980.
18. Sontag, E.D., *Mathematical Control Theory*, Springer-Verlag, New York, 1990.
19. Antsaklis, P.J. and Michel, A.N., *Linear Systems*, Birkhauser, Boston, MA, 2006.

For textbooks on H-infinity and $H2$ control of time-varying systems, see

20. Peters, M.A. and Iglesias, P.A., *Minimum Entropy Control for Time-Varying Systems*, Birkhauser, Boston, MA, 1997.
21. Ichikawa, A. and Katayama, H., *Linear Time-Varying and Sampled-Data Systems, Lecture Notes in Control and Information Science*, 265, Springer-Verlag, London, 2001.

An approach to linear time-varying systems given in terms of matrix algebra and analytic function theory is developed in the following textbook:

22. Dewilde, P. and Veen, A. van der, *Time-Varying Systems and Computations*, Kluwer, Boston, MA, 1998.

For textbooks on the adaptive control of time-varying systems and observers for time-varying systems, see

23. Tsakalis, K.S. and Ioannou, P.A., *Linear Time-Varying Plants: Control and Adaptation,* Prentice-Hall, Englewood Cliffs, NJ, 1993.
24. O'Reilly, J., *Observers for Linear* Systems, Academic Press, New York, 1983.

Many textbooks exist on the stability of time-varying differential equations and systems. Examples are

25. Bellman, R., *Stability of Differential Equations*, McGraw-Hill, New York, 1953.
26. Hahn, W., *Stability of Motion*, Springer-Verlag, New York, 1967.
27. Harris, C.J. and Miles, J.F., *Stability of Linear* Systems, Academic Press, New York, 1980.
28. Lukes, D.L., *Differential Equations: Classical to Controlled*, Academic Press, New York, 1982.
29. Miller, R.K. and Michel, A.N., *Ordinary Differential Equations,* Academic Press, New York, 1982.

30. Michel, A.N., Liu, D., and Hou, L., *Stability of Dynamical Systems: Continuous, Discontinuous, and Discrete Systems*, Birkhauser Verlag, Boston, MA, 2008.

For an in-depth treatment of the stability of second-order linear time-varying differential equations, see

31. Duc, L.H., Ilchmann, A., Siegmund, S., and Taraba, P., On stability of linear time-varying second-order differential equations, *Quart. Appl. Math.* 64, 137–151, 2006.

There are a number of papers on the study of time-varying systems given in terms of rings of differential or difference polynomials. In addition to [2], [3], and [6] in the references above, examples of this work are as follows:

32. Kamen, E.W., Khargonekar, P.P., and Poolla, K.R., A transfer function approach to linear time-varying discrete-time systems, *SIAM J. Control Optim.*, 23, 550–565, 1985.
33. Poolla, K. R. and Khargonekar, P.P., Stabilizability and stable proper factorizations for linear time-varying systems, *SIAM J. Control Optim.*, 25, 723–736, 1987.
34. Fliess, M., Some basic structural properties of generalized linear systems, *Systems Control Lett.*, 15, 391–396, 1990.

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where t is the continuous-time variable, $y(t)$ is the output response resulting from input $u(t)$, and $h(t,\tau)$ is a real-valued continuous function of t and τ. It is assumed that there are conditions on $h(t,\tau)$ and/or $u(t)$, which insure that the integral in Equation 3.1 exists. The system given by Equation 3.1 is causal since the output $y(t)$ at time t depends only on the input $u(\tau)$ for $\tau \le t$. The system is also linear since integration is a linear operation. Linearity means that if $y_1(t)$ is the output response resulting from input $u_1(t)$, and $y_2(t)$ is the output response resulting from input $u_2(t)$, then for any real numbers a and b, the output response resulting from input $au_1(t) + bu_2(t)$ is equal to $ay_1(t) + by_2(t)$.

Let $\delta(t)$ denote the unit impulse defined by $\delta(t) = 0$, $t \neq 0$ and $\int_{-\varepsilon}^{\varepsilon} \delta(\lambda)\, d\lambda = 1$ for any real number $\varepsilon > 0$. For any real number t_1, the time-shifted impulse $\delta(t - t_1)$ is the unit impulse located at time $t = t_1$. Then from Equation 3.1 and the sifting property of the impulse, the output response $y(t)$ resulting from input $u(t) = \delta(t - t_1)$ is given by

$$y(t) = \int_{-\infty}^{t} h(t,\tau)u(\tau)\, d\tau = \int_{-\infty}^{t} h(t,\tau)\delta(\tau - t_1)\, d\tau = h(t, t_1)$$

Hence, the function $h(t,\tau)$ in Equation 3.1 is the *impulse response function* of the system, that is, $h(t,\tau)$ is the output response resulting from the impulse $\delta(t-\tau)$ applied to the system at time τ.

The linear system given by Equation 3.1 is *time invariant* (or *constant*) if and only if

$$h(t+\gamma, \tau+\gamma) = h(t,\tau), \quad \text{for all real numbers } t, \tau, \gamma \tag{3.2}$$

Time invariance means that if $y(t)$ is the response to $u(t)$, then for any real number t_1, the time-shifted output $y(t - t_1)$ is the response to the time-shifted input $u(t - t_1)$. Setting $\gamma = -\tau$ in Equation 3.2 gives

$$h(t-\tau, 0) = h(t,\tau), \quad \text{for all real numbers } t, \tau \tag{3.3}$$

Hence, the system defined by Equation 3.1 is time invariant if and only if the impulse response function $h(t,\tau)$ is a function only of the difference $t - \tau$. In the time-invariant case, Equation 3.1 reduces to the