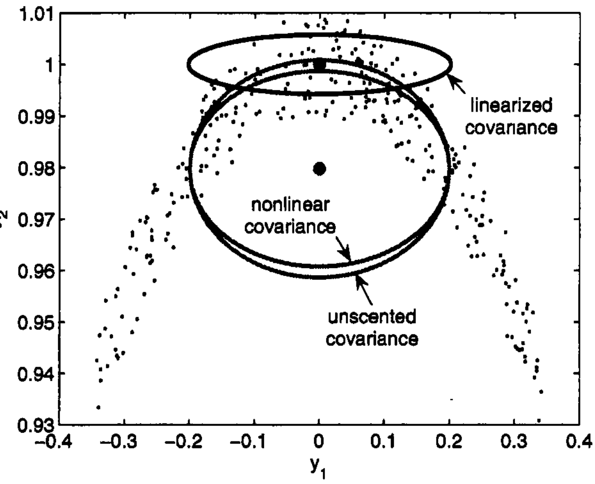

Figure 14.3 Results of Example 14.1. A comparison of the exact, linearized, and unscented mean and covariance of 300 randomly generated points with $\tilde{r}$ uniformly distributed between ± 0.01 and $\tilde{\theta}$ uniformly distributed between ± 0.35 radians.

to propagate the mean and covariance of a system using a time-update and a measurement update. If the system is linear, then the mean and covariance can be exactly updated with the Kalman filter (Chapter 5). If the system is nonlinear, then the mean and covariance can be approximately updated with the extended Kalman filter (Section 13.2). However, the EKF is based on linearization, and the previous section showed that unscented transformations are more accurate than linearization for propagating means and covariances. Therefore, we simply replace the EKF equations with unscented transformations to obtain the UKF algorithm. The UKF algorithm can be summarized as follows.

The unscented Kalman filter

1. We have an n-state discrete-time nonlinear system given by

$$\begin{aligned} x_{k+1} &= f(x_k, u_k, t_k) + w_k \\ y_k &= h(x_k, t_k) + v_k \\ w_k &\sim (0, Q_k) \\ v_k &\sim (0, R_k) \end{aligned} \tag{14.56}$$

2. The UKF is initialized as follows.

$$\begin{aligned} \hat{x}_0^+ &= E(x_0) \\ P_0^+ &= E\left[(x_0 - \hat{x}_0^+)(x_0 - \hat{x}_0^+)^T\right] \end{aligned} \tag{14.57}$$

3. The following time update equations are used to propagate the state estimate and covariance from one measurement time to the next.

(a) To propagate from time step $(k-1)$ to k, first choose sigma points $x_{k-1}^{(i)}$ as specified in Equation (14.29), with appropriate changes since the current best guess for the mean and covariance of x_k are $\hat{x}_{k-1}^{+}$ and P_{k-1}^{+}:

$$\begin{aligned} \hat{x}_{k-1}^{(i)} &= \hat{x}_{k-1}^{+} + \tilde{x}^{(i)} \quad i = 1, \cdots, 2n \\ \tilde{x}^{(i)} &= \left(\sqrt{nP_{k-1}^{+}}\right)_i^T \quad i = 1, \cdots, n \\ \tilde{x}^{(n+i)} &= -\left(\sqrt{nP_{k-1}^{+}}\right)_i^T \quad i = 1, \cdots, n \end{aligned} \tag{14.58}$$

(b) Use the known nonlinear system equation $f(\cdot)$ to transform the sigma points into $\hat{x}_k^{(i)}$ vectors as shown in Equation (14.30), with appropriate changes since our nonlinear transformation is $f(\cdot)$ rather than $h(\cdot)$:

$$\hat{x}_k^{(i)} = f(\hat{x}_{k-1}^{(i)}, u_k, t_k) \tag{14.59}$$

(c) Combine the $\hat{x}_k^{(i)}$ vectors to obtain the *a priori* state estimate at time k. This is based on Equation (14.33):

$$\hat{x}_k^{-} = \frac{1}{2n} \sum_{i=1}^{2n} \hat{x}_k^{(i)} \tag{14.60}$$

(d) Estimate the *a priori* error covariance as shown in Equation (14.43). However, we should add Q_{k-1} to the end of the equation to take the process noise into account:

$$P_k^{-} = \frac{1}{2n} \sum_{i=1}^{2n} \left(\hat{x}_k^{(i)} - \hat{x}_k^{-}\right) \left(\hat{x}_k^{(i)} - \hat{x}_k^{-}\right)^T + Q_{k-1} \tag{14.61}$$

4. Now that the time update equations are done, we implement the measurement-update equations.

(a) Choose sigma points $x_k^{(i)}$ as specified in Equation (14.29), with appropriate changes since the current best guess for the mean and covariance of x_k are $\hat{x}_k^{-}$ and P_k^{-}:

$$\begin{aligned} \hat{x}_k^{(i)} &= \hat{x}_k^{-} + \tilde{x}^{(i)} \quad i = 1, \cdots, 2n \\ \tilde{x}^{(i)} &= \left(\sqrt{nP_k^{-}}\right)_i^T \quad i = 1, \cdots, n \\ \tilde{x}^{(n+i)} &= -\left(\sqrt{nP_k^{-}}\right)_i^T \quad i = 1, \cdots, n \end{aligned} \tag{14.62}$$

This step can be omitted if desired. That is, instead of generating new sigma points we can reuse the sigma points that were obtained from the time update. This will save computational effort if we are willing to sacrifice performance.

(b) Use the known nonlinear measurement equation $h(\cdot)$ to transform the sigma points into $\hat{y}_k^{(i)}$ vectors (predicted measurements) as shown in Equation (14.30):

$$\hat{y}_k^{(i)} = h(\hat{x}_k^{(i)}, t_k) \tag{14.63}$$

(c) Combine the $\hat{y}_k^{(i)}$ vectors to obtain the predicted measurement at time k. This is based on Equation (14.33):

$$\hat{y}_k = \frac{1}{2n} \sum_{i=1}^{2n} \hat{y}_k^{(i)} \tag{14.64}$$

(d) Estimate the covariance of the predicted measurement as shown in Equation (14.43). However, we should add R_k to the end of the equation to take the measurement noise into account:

$$P_y = \frac{1}{2n} \sum_{i=1}^{2n} \left(\hat{y}_k^{(i)} - \hat{y}_k\right)\left(\hat{y}_k^{(i)} - \hat{y}_k\right)^T + R_k \tag{14.65}$$

(e) Estimate the cross covariance between $\hat{x}_k^-$ and $\hat{y}_k$ based on Equation (14.43):

$$P_{xy} = \frac{1}{2n} \sum_{i=1}^{2n} \left(\hat{x}_k^{(i)} - \hat{x}_k^-\right)\left(\hat{y}_k^{(i)} - \hat{y}_k\right)^T \tag{14.66}$$

(f) The measurement update of the state estimate can be performed using the normal Kalman filter equations as shown in Equation (10.100):

$$\begin{aligned} K_k &= P_{xy} P_y^{-1} \\ \hat{x}_k^+ &= \hat{x}_k^- + K_k(y_k - \hat{y}_k) \\ P_k^+ &= P_k^- - K_k P_y K_k^T \end{aligned} \tag{14.67}$$

The algorithm above assumes that the process and measurement equations are linear with respect to the noise, as shown in Equation (14.56). In general, the process and measurement equations may have noise that enters the process and measurement equations nonlinearly. That is,

$$\begin{aligned} x_{k+1} &= f(x_k, u_k, w_k, t_k) \\ y_k &= h(x_k, v_k, t_k) \end{aligned} \tag{14.68}$$

In this case, the UKF algorithm presented above is not rigorous because it treats the noise as additive, as seen in Equations (14.61) and (14.65). To handle this situation, we can augment the noise onto the state vector as shown in [Jul04, Wan01]:

$$x_k^{(a)} = \begin{bmatrix} x_k \\ w_k \\ v_k \end{bmatrix} \tag{14.69}$$

Then we can use the UKF to estimate the augmented state $x_k^{(a)}$. The UKF is initialized as

$$\begin{aligned} \hat{x}_0^{a+} &= \begin{bmatrix} E(x_0) \\ 0 \\ 0 \end{bmatrix} \\ P_0^{a+} &= \begin{bmatrix} E\left[(x_0-\hat{x}_0)(x_0-\hat{x}_0)^T\right] & 0 & 0 \\ 0 & Q_0 & 0 \\ 0 & 0 & R_0 \end{bmatrix} \end{aligned} \tag{14.70}$$

Then we use the UKF algorithm presented above, except that we are estimating the augmented mean and covariance, so we remove Q_{k-1} and R_k from Equations (14.61) and (14.65).

■ EXAMPLE 14.2

Suppose we are trying to estimate the altitude x_1, velocity x_2, and constant ballistic coefficient x_3 of a body as it falls toward earth. A range measuring device is located at an altitude a and the horizontal range between the measuring device and the body is M. This system is the same as the one in Example 13.3. The equations for this system are

$$\begin{aligned} \dot{x}_1 &= x_2 + w_1 \\ \dot{x}_2 &= \rho_0 \exp(-x_1/k) x_2^2 x_3/2 - g + w_2 \\ \dot{x}_3 &= w_3 \\ y(t_k) &= \sqrt{M^2 + (x_1(t_k) - a)^2} + v_k \end{aligned} \tag{14.71}$$

As usual, w_i is the noise that affects the ith process equation, and v is the measurement noise. ρ_0 is the air density at sea level, k is a constant that defines the relationship between air density and altitude, and g is the acceleration due to gravity. We will use the continuous-time system equations to simulate the system, and suppose that we obtain range measurements every 0.5 seconds. The constants that we will use are given as

$$\begin{aligned} \rho_0 &= 2 \text{ lb-sec}^2/\text{ft}^4 \\ g &= 32.2 \text{ ft/sec}^2 \\ k &= 20,000 \text{ ft} \\ E[v_k^2] &= 10,000 \text{ ft}^2 \\ E[w_i^2(t)] &= 0 \quad i = 1,2,3 \\ M &= 100,000 \text{ ft} \\ a &= 100,000 \text{ ft} \end{aligned} \tag{14.72}$$

The initial conditions of the system and the estimator are given as

$$\begin{aligned} x_0 &= \begin{bmatrix} 300,000 & -20,000 & 0.001 \end{bmatrix}^T \\ \hat{x}_0^+ &= x_0 \\ P_0^+ &= \begin{bmatrix} 1,000,000 & 0 & 0 \\ 0 & 4,000,000 & 0 \\ 0 & 0 & 10 \end{bmatrix} \end{aligned} \tag{14.73}$$