3.4 The Unscented Kalman Filter

The Taylor series expansion applied by the EKF is only one way to linearize the transformation of a Gaussian. Two other approaches have often been found to yield superior results. One is known as *moments matching* (and the resulting filter is known as *assumed density filter*, or *ADF*), in which the linearization is calculated in a way that preserves the true mean and the true covariance of the posterior distribution (which is not the case for EKFs). Another linearization method is applied by the *unscented Kalman filter*, or *UKF*, which performs a stochastic linearization through the use of a weighted statistical linear regression process. We now discuss the UKF algorithm without mathematical derivation. The reader is encouraged to read more details in the literature referenced in the bibliographical remarks.

UNSCENTED KALMAN FILTER

3.4.1 Linearization Via the Unscented Transform

Figure 3.7 illustrates the linearization applied by the UKF, called the *unscented transform*. Instead of approximating the function g by a Taylor series expansion, the UKF deterministically extracts so-called *sigma points* from the Gaussian and passes these through g. In the general case, these sigma points are located at the mean and symmetrically along the main axes of the covariance (two per dimension). For an n-dimensional Gaussian with mean μ and covariance Σ, the resulting $2n+1$ sigma points $\mathcal{X}^{[i]}$ are chosen according to the following rule:

SIGMA POINT

(3.66)
$$\begin{aligned} \mathcal{X}^{[0]} &= \mu \\ \mathcal{X}^{[i]} &= \mu + \left(\sqrt{(n+\lambda)\,\Sigma}\right)_i \qquad \text{for } i = 1, \ldots, n \\ \mathcal{X}^{[i]} &= \mu - \left(\sqrt{(n+\lambda)\,\Sigma}\right)_{i-n} \qquad \text{for } i = n+1, \ldots, 2n \end{aligned}$$

Here $\lambda = \alpha^2(n+\kappa) - n$, with α and κ being scaling parameters that determine how far the sigma points are spread from the mean. Each sigma point $\mathcal{X}^{[i]}$ has two weights associated with it. One weight, $w_m^{[i]}$, is used when computing the mean, the other weight, $w_c^{[i]}$, is used when recovering the covariance of the Gaussian.

(3.67)
$$\begin{aligned} w_m^{[0]} &= \frac{\lambda}{n+\lambda} \\ w_c^{[0]} &= \frac{\lambda}{n+\lambda} + (1 - \alpha^2 + \beta) \end{aligned}$$

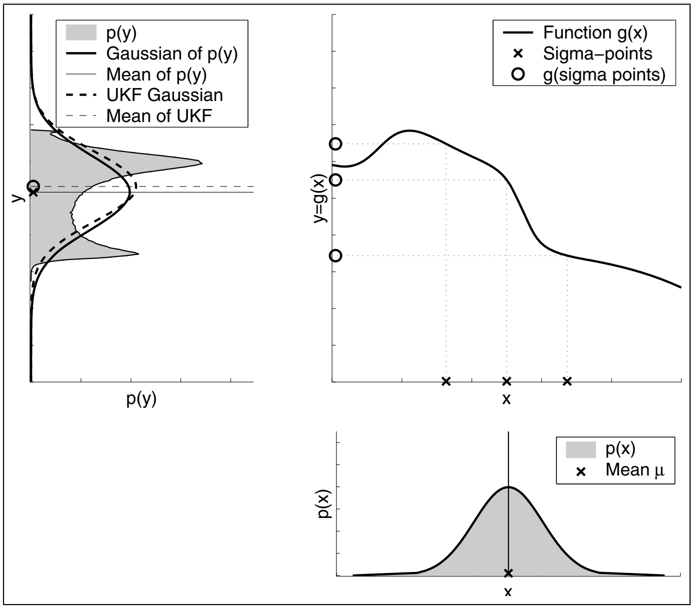

Figure 3.7 Illustration of linearization applied by the UKF. The filter first extracts $2n+1$ weighted sigma points from the n-dimensional Gaussian ($n = 1$ in this example). These sigma points are passed through the nonlinear function g. The linearized Gaussian is then extracted from the mapped sigma points (small circles in the upper right plot). As for the EKF, the linearization incurs an approximation error, indicated by the mismatch between the linearized Gaussian (dashed) and the Gaussian computed from the highly accurate Monte-Carlo estimate (solid).

$$w_m^{[i]} = w_c^{[i]} = \frac{1}{2(n+\lambda)} \quad \text{for } i = 1, \ldots, 2n.$$

The parameter β can be chosen to encode additional (higher order) knowledge about the distribution underlying the Gaussian representation. If the distribution is an exact Gaussian, then $\beta = 2$ is the optimal choice.

The sigma points are then passed through the function g, thereby probing how g changes the shape of the Gaussian.

$$\mathcal{Y}^{[i]} = g(\mathcal{X}^{[i]}) \tag{3.68}$$

The parameters $(\mu' \quad \Sigma')$ of the resulting Gaussian are extracted from the mapped sigma points $\mathcal{Y}^{[i]}$ according to

$$(3.69) \qquad \mu' = \sum_{i=0}^{2n} w_m^{[i]} \, \mathcal{Y}^{[i]}$$

$$\Sigma' = \sum_{i=0}^{2n} w_c^{[i]} \, (\mathcal{Y}^{[i]} - \mu')(\mathcal{Y}^{[i]} - \mu')^T.$$

Figure 3.8 illustrates the dependency of the unscented transform on the uncertainty of the original Gaussian. For comparison, the results using the EKF Taylor series expansion are plotted alongside the UKF results.

Figure 3.9 shows an additional comparison between UKF and EKF approximation, here in dependency of the local nonlinearity of the function g. As can be seen, the unscented transform is more accurate than the first order Taylor series expansion applied by the EKF. In fact, it can be shown that the unscented transform is accurate in the first two terms of the Taylor expansion, while the EKF captures only the first order term. (It should be noted, however, that both the EKF and the UKF can be modified to capture higher order terms.)

3.4.2 The UKF Algorithm

The UKF algorithm utilizing the unscented transform is presented in Table 3.4. The input and output are identical to the EKF algorithm. Line 2 determines the sigma points of the previous belief using Equation (3.66), with γ short for $\sqrt{n + \lambda}$. These points are propagated through the noise-free state prediction in line 3. The predicted mean and variance are then computed from the resulting sigma points (lines 4 and 5). R_t in line 5 is added to the sigma point covariance in order to model the additional prediction noise uncertainty (compare line 3 of the EKF algorithm in Table 3.3). The prediction noise R_t is assumed to be additive. Later, in Chapter 7, we present a version of the UKF algorithm that performs more accurate estimation of the prediction and measurement noise terms.

A new set of sigma points is extracted from the predicted Gaussian in line 6. This sigma point set $\bar{\mathcal{X}}_t$ now captures the overall uncertainty after the prediction step. In line 7, a predicted observation is computed for each sigma point. The resulting observation sigma points $\bar{\mathcal{Z}}_t$ are used to compute the predicted observation $\hat{z}_t$ and its uncertainty, S_t. The matrix Q_t is the covariance matrix of the additive measurement noise. Note that S_t represents the same uncertainty as $H_t \; \bar{\Sigma}_t \; H_t^T + Q_t$ in line 4 of the EKF algorithm in