A continuous-time, deterministic linear system can be described by the equations

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1.67}$$

where x is the state vector, u is the control vector, and y is the output vector. Matrices A, B, and C are appropriately dimensioned matrices. The A matrix is often called the system matrix, B is often called the input matrix, and C is often called the output matrix. In general, A, B, and C can be time-varying matrices and the system will still be linear. If A, B, and C are constant then the solution to Equation (1.67) is given by

$$\begin{aligned} x(t) &= e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau \\ y(t) &= Cx(t) \end{aligned} \tag{1.68}$$

where t_0 is the initial time of the system and is often taken to be 0. This is easy to verify when all of the quantities in Equation (1.67) are scalar, but it happens to be true in the vector case also. Note that in the zero input case, $x(t)$ is given as

$$x(t) = e^{A(t-t_0)}x(t_0), \qquad \text{zero input case} \tag{1.69}$$

For this reason, e^{At} is called the state-transition matrix of the system.[3] It is the matrix that describes how the state changes from its initial condition in the absence of external inputs. We can evaluate the above equation at $t = t_0$ to see that

$$e^{A0} = I \tag{1.70}$$

in analogy with the scalar exponential of zero.

As stated above, even if x is an n-element vector, then Equation (1.68) still describes the solution of Equation (1.67). However, a fundamental question arises in this case: How can we take the exponential of the matrix A in Equation (1.68)? What does it mean to raise the scalar e to the power of a matrix? There are many different ways to compute this quantity [Mol03]. Three of the most useful are the following:

$$\begin{aligned} e^{At} &= \sum_{j=0}^{\infty} \frac{(At)^j}{j!} \\ &= \mathcal{L}^{-1}[(sI - A)^{-1}] \\ &= Qe^{\hat{A}t}Q^{-1} \end{aligned} \tag{1.71}$$

The first expression above is the definition of e^{At}, and is analogous to the definition of the exponential of a scalar. This definition shows that A must be square in order for e^{At} to exist. From Equation (1.67), we see that a system matrix is always square. The definition of e^{At} can also be used to derive the following properties.

$$\begin{aligned} \frac{d}{dt}e^{At} &= Ae^{At} \\ &= e^{At}A \end{aligned} \tag{1.72}$$

[3]The MATLAB function EXPM computes the matrix exponential. Note that the MATLAB function EXP computes the element-by-element exponential of a matrix, which is generally not the same as the matrix exponential.