and the homogeneous response to initial conditions $x_1(0)$ and $x_2(0)$ is

$$\mathbf{x}_h(t) = \mathbf{\Phi}(t)\mathbf{x}(0) \tag{iii}$$

or

$$x_1(t) = x_1(0)e^{-2t} \tag{iv}$$
$$x_2(t) = x_1(0)\left(e^{-t} - e^{-2t}\right) + x_2(0)e^{-t}. \tag{v}$$

With the given initial conditions the response is

$$x_1(t) = 2e^{-2t} \tag{vi}$$
$$\begin{aligned} x_2(t) &= 2\left(e^{-t} - e^{-2t}\right) + 3e^{-t} \\ &= 5e^{-t} - 2e^{-2t}. \end{aligned} \tag{vii}$$

In general the recognition of the exponential components from the series for each element is difficult and is not normally used for finding a closed form for the state transition matrix.

Although the sum expressed in Eq. (9) converges for all $\mathbf{A}$, in many cases the series converges slowly, and is rarely used for the direct computation of $\mathbf{\Phi}(t)$. There are many methods for computing the elements of $\Phi(t)$, including one presented in Section 4.3, that are much more convenient than the direct series definition. [1,5,6]

2.2 The Forced State Response of Linear Systems

We now consider the complete response of a linear system to an input $\mathbf{u}(t)$. Consider first a first-order system with a state equation $\dot{x} = ax + bu$ written in the form

$$\dot{x}(t) - ax(t) = bu(t). \tag{12}$$

If both sides are multiplied by an integrating factor e^{-at}, the left-hand side becomes a perfect differential

$$e^{-at}\dot{x} - e^{-at}ax = \frac{d}{dt}\left(e^{-at}x(t)\right) = e^{-at}bu \tag{13}$$

which may be integrated directly to give

$$\int_0^t \frac{d}{d\tau}\left(e^{-a\tau}x(\tau)\right) d\tau = e^{-at}x(t) - x(0) = \int_0^t e^{-a\tau}bu(\tau)\, d\tau \tag{14}$$

and rearranged to give the state variable response explicitly:

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)\, d\tau. \tag{15}$$

The development of the expression for the response of higher order systems may be performed in a similar manner using the matrix exponential $e^{-\mathbf{A}t}$ as an integrating factor. Matrix differentiation and integration are defined to be element by element operations, therefore if the state equations $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ are rearranged, and all terms pre-multiplied by the square matrix $e^{-\mathbf{A}t}$:

$$e^{-\mathbf{A}t}\dot{\mathbf{x}}(t) - e^{-\mathbf{A}t}\mathbf{A}\mathbf{x}(t) = \frac{d}{dt}\left(e^{-\mathbf{A}t}\mathbf{x}(t)\right) = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t). \tag{16}$$

Integration of Eq. (16) gives

$$\int_0^t \frac{d}{d\tau}\left(e^{-\mathbf{A}\tau}\mathbf{x}(\tau)\right) d\tau = e^{-\mathbf{A}t}\mathbf{x}(t) - e^{-\mathbf{A}0}\mathbf{x}(0) = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau \tag{17}$$

and because $e^{-\mathbf{A}0} = \mathbf{I}$ and $[e^{-\mathbf{A}t}]^{-1} = e^{\mathbf{A}t}$ the complete state vector response may be written in two similar forms

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau \tag{18}$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau. \tag{19}$$

The full state response, described by Eq. (18) or Eq. (19) consists of two components: the first is a term similar to the system homogeneous response $\mathbf{x}_h(t) = e^{\mathbf{A}t}\mathbf{x}(0)$ that is dependent only on the system initial conditions $\mathbf{x}(0)$. The second term is in the form of a *convolution integral*, and is the particular solution for the input $\mathbf{u}(t)$, with zero initial conditions.

Evaluation of the integral in Eq. (19) involves matrix integration. For a system of order n and with r inputs, the matrix $e^{\mathbf{A}t}$ is $n \times n$, $\mathbf{B}$ is $n \times r$ and $\mathbf{u}(t)$ is an $r \times 1$ column vector. The product $e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)$ is therefore an $n \times 1$ column vector, and solution for each of the state equations involves a single scalar integration.

■ Example 2

Find the response of the two state variables of the system

$$\begin{aligned} \dot{x}_1 &= -2x_1 + u \\ \dot{x}_2 &= x_1 - x_2. \end{aligned}$$

to a constant input $u(t) = 5$ for $t > 0$, if $x_1(0) = 0$, and $x_2 = 0$.

Solution: This is the same system described in Example 1. The state transition matrix was shown to be

$$\mathbf{\Phi}(t) = \begin{bmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t} \end{bmatrix}$$