A continuous-time, deterministic linear system can be described by the equations

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1.67}$$

where x is the state vector, u is the control vector, and y is the output vector. Matrices A, B, and C are appropriately dimensioned matrices. The A matrix is often called the system matrix, B is often called the input matrix, and C is often called the output matrix. In general, A, B, and C can be time-varying matrices and the system will still be linear. If A, B, and C are constant then the solution to Equation (1.67) is given by

$$\begin{aligned} x(t) &= e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau \\ y(t) &= Cx(t) \end{aligned} \tag{1.68}$$

where t_0 is the initial time of the system and is often taken to be 0. This is easy to verify when all of the quantities in Equation (1.67) are scalar, but it happens to be true in the vector case also. Note that in the zero input case, $x(t)$ is given as

$$x(t) = e^{A(t-t_0)}x(t_0), \qquad \text{zero input case} \tag{1.69}$$

For this reason, e^{At} is called the state-transition matrix of the system.[3] It is the matrix that describes how the state changes from its initial condition in the absence of external inputs. We can evaluate the above equation at $t = t_0$ to see that

$$e^{A0} = I \tag{1.70}$$

in analogy with the scalar exponential of zero.

As stated above, even if x is an n-element vector, then Equation (1.68) still describes the solution of Equation (1.67). However, a fundamental question arises in this case: How can we take the exponential of the matrix A in Equation (1.68)? What does it mean to raise the scalar e to the power of a matrix? There are many different ways to compute this quantity [Mol03]. Three of the most useful are the following:

$$\begin{aligned} e^{At} &= \sum_{j=0}^{\infty} \frac{(At)^j}{j!} \\ &= \mathcal{L}^{-1}[(sI - A)^{-1}] \\ &= Qe^{\hat{A}t}Q^{-1} \end{aligned} \tag{1.71}$$

The first expression above is the definition of e^{At}, and is analogous to the definition of the exponential of a scalar. This definition shows that A must be square in order for e^{At} to exist. From Equation (1.67), we see that a system matrix is always square. The definition of e^{At} can also be used to derive the following properties.

$$\begin{aligned} \frac{d}{dt}e^{At} &= Ae^{At} \\ &= e^{At}A \end{aligned} \tag{1.72}$$

[3]The MATLAB function EXPM computes the matrix exponential. Note that the MATLAB function EXP computes the element-by-element exponential of a matrix, which is generally not the same as the matrix exponential.

In general, matrices do not commute under multiplication but, interestingly, a matrix always commutes with its exponential.

The first expression in Equation (1.71) is not usually practical for computational purposes since it is an infinite sum (although the latter terms in the sum often decrease rapidly in magnitude, and may even become zero). The second expression in Equation (1.71) uses the inverse Laplace transform to compute e^{At}. In the third expression of Equation (1.71), Q is a matrix whose columns comprise the eigenvectors of A, and $\hat{A}$ is the Jordan form[4] of A. Note that Q and $\hat{A}$ are well defined for any square matrix A, so the matrix exponential e^{At} exists for all square matrices A and all finite t. The matrix $\hat{A}$ is often diagonal, in which case $e^{\hat{A}t}$ is easy to compute:

$$\begin{aligned}
\hat{A} &= \begin{bmatrix} \hat{A}_{11} & 0 & \cdots & 0 \\ 0 & \hat{A}_{22} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \hat{A}_{nn} \end{bmatrix} \\
e^{\hat{A}t} &= \begin{bmatrix} e^{\hat{A}_{11}} & 0 & \cdots & 0 \\ 0 & e^{\hat{A}_{22}} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & e^{\hat{A}_{nn}} \end{bmatrix}
\end{aligned} \tag{1.73}$$

This can be computed from the definition of $e^{\hat{A}t}$ in Equation (1.71). Even if the Jordan form matrix $\hat{A}$ is not diagonal, $e^{\hat{A}t}$ is easy to compute [Bay99, Che99, Kai80]. We can also note from the third expression in Equation (1.71) that

$$\begin{aligned}
\left[e^{At}\right]^{-1} &= e^{-At} \\
&= Qe^{-\hat{A}t}Q^{-1}
\end{aligned} \tag{1.74}$$

(Recall that A and $-A$ have the same eigenvectors, and their eigenvalues are negatives of each other. See Problem 1.10.) We see from this that e^{At} is always invertible. This is analogous to the scalar situation in which the exponential of a scalar is always nonzero.

Another interesting fact about the matrix exponential is that all of the individual elements of the matrix exponential e^A are nonnegative if and only if all of the individual elements of A are nonnegative [Bel60, Bel80].

■ EXAMPLE 1.2

As an example of a linear system, suppose that we are controlling the angular acceleration of a motor (for example, with some applied voltage across the motor windings). The derivative of the position is the velocity. A simplified motor model can then be written as

[4]In fact, Equation (1.71) can be used to define the Jordan form of a matrix. That is, if e^{At} can be written as shown in Equation (1.71), where Q is a matrix whose columns comprise the eigenvectors of A, then $\hat{A}$ is the Jordan form of A. More discussion about Jordan forms and their computation can be found in most linear systems books [Kai80, Bay99, Che99].

$$
\begin{aligned}
\dot{\theta} &= \omega \\
\dot{\omega} &= u + w_1
\end{aligned}
\tag{1.75}
$$

The scalar w_1 is the acceleration noise and could consist of such factors as uncertainty in the applied acceleration, motor shaft eccentricity, and load disturbances. If our measurement consists of the angular position of the motor then a state space description of this system can be written as

$$
\begin{aligned}
\begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ w_1 \end{bmatrix} \\
y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x + v
\end{aligned}
\tag{1.76}
$$

The scalar v consists of measurement noise. Comparing with Equation (1.67), we see that the state vector x is a 2×1 vector containing the scalars θ and ω.

$\nabla\nabla\nabla$

■ EXAMPLE 1.3

In this example, we will use the three expressions in Equation (1.71) to compute the state-transition matrix of the system described in Example 1.2. From the first expression in Equation (1.71) we obtain

$$
\begin{aligned}
e^{At} &= \sum_{j=0}^{\infty} \frac{(At)^j}{j!} \\
&= (At)^0 + (At)^1 + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots \\
&= I + At
\end{aligned}
\tag{1.77}
$$

where the last equality comes from the fact that $A^k = 0$ when $k > 1$ for the A matrix given in Example 1.2. We therefore obtain

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}
\end{aligned}
\tag{1.78}
$$

From the second expression in Equation (1.71) we obtain

$$
\begin{aligned}
e^{At} &= \mathcal{L}^{-1}[(sI - A)^{-1}] \\
&= \mathcal{L}^{-1}\left(\begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}^{-1} \right) \\
&= \mathcal{L}^{-1} \begin{bmatrix} 1/s & 1/s^2 \\ 0 & 1/s \end{bmatrix} \\
&= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}
\end{aligned}
\tag{1.79}
$$

In order to use the third expression in Equation (1.71) we first need to obtain the eigendata (i.e., the eigenvalues and eigenvectors) of the A matrix. These are found as

$$\begin{aligned} \lambda(A) &= \{0,0\} \\ v(A) &= \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \end{aligned} \tag{1.80}$$

This shows that

$$\begin{aligned} \hat{A} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ Q &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{1.81}$$

Note that in this simple example A is already in Jordan form, so $\hat{A} = A$ and $Q = I$. The third expression in Equation (1.71) therefore gives

$$\begin{aligned} e^{At} &= Qe^{\hat{A}t}Q^{-1} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{1.82}$$

$\nabla\nabla\nabla$

1.3 NONLINEAR SYSTEMS

The discussion of linear systems in the preceding section is a bit optimistic, because in reality linear systems do not exist. Real systems always have some nonlinearities. Even a simple resistor is ultimately nonlinear if we apply a large enough voltage across it. However, we often model a resistor with the simple linear equation $V = IR$ because this equation accurately describes the operation of the resistor over a wide operating range. So even though linear systems do not exist in the real world, linear systems theory is still a valuable tool for dealing with nonlinear systems.

The general form of a continuous-time nonlinear system can be written as

$$\begin{aligned} \dot{x} &= f(x,u,w) \\ y &= h(x,v) \end{aligned} \tag{1.83}$$

where $f(\cdot)$ and $h(\cdot)$ are arbitrary vector-valued functions. We use w to indicate process noise, and v to indicate measurement noise. If $f(\cdot)$ and $h(\cdot)$ are explicit functions of t then the system is time-varying. Otherwise, the system is time-invariant. If $f(x,u,w) = Ax + Bu + w$, and $h(x,v) = Hx + v$, then the system is linear [compare with Equation (1.67)]. Otherwise, the system is nonlinear.

In order to apply tools from linear systems theory to nonlinear systems, we need to linearize the nonlinear system. In other words, we need to find a linear system