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# State-transition matrix

In control theory, the **state-transition matrix** is a matrix whose product with the state vector $x$ at an initial time $t_0$ gives $x$ at a later time $t$. The state-transition matrix can be used to obtain the general solution of linear dynamical systems.

## Contents



## Linear systems solutions

The state-transition matrix is used to find the solution to a general state-space representation of a linear system in the following form

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \;\; \mathbf{x}(t_0) = \mathbf{x}_0,$$

where $\mathbf{x}(t)$ are the states of the system, $\mathbf{u}(t)$ is the input signal, $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are matrix functions, and $\mathbf{x}_0$ is the initial condition at $t_0$. Using the state-transition matrix $\mathbf{\Phi}(t,\tau)$, the solution is given by:[1][2]

$$\mathbf{x}(t) = \mathbf{\Phi}(t,t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{\Phi}(t,\tau)\mathbf{B}(\tau)\mathbf{u}(\tau)d\tau$$

The first term is known as the **zero-input response** and represents how the system's state would evolve in the absence of any input. The second term is known as the **zero-state response** and defines how the inputs impact the system.

## Peano–Baker series

The most general transition matrix is given by the Peano–Baker series

$$\mathbf{\Phi}(t,\tau) = \mathbf{I} + \int_{\tau}^{t} \mathbf{A}(\sigma_1)\, d\sigma_1 + \int_{\tau}^{t} \mathbf{A}(\sigma_1) \int_{\tau}^{\sigma_1} \mathbf{A}(\sigma_2)\, d\sigma_2\, d\sigma_1 + \int_{\tau}^{t} \mathbf{A}(\sigma_1) \int_{\tau}^{\sigma_1} \mathbf{A}(\sigma_2) \int_{\tau}^{\sigma_2} \mathbf{A}(\sigma_3)\, d\sigma_3\, d\sigma_2\, d\sigma_1 + \ldots$$

where $\mathbf{I}$ is the identity matrix. This matrix converges uniformly and absolutely to a solution that exists and is unique.[2]

## Other properties

The state transition matrix $\mathbf{\Phi}$ satisfies the following relationships:

1. It is continuous and has continuous derivatives.

2, It is never singular; in fact $\mathbf{\Phi}^{-1}(t,\tau) = \mathbf{\Phi}(\tau,t)$ and $\mathbf{\Phi}^{-1}(t,\tau)\mathbf{\Phi}(t,\tau) = I$, where $I$ is the identity matrix.

3. $\mathbf{\Phi}(t,t) = I$ for all $t$ .[3]

4. $\mathbf{\Phi}(t_2,t_1)\mathbf{\Phi}(t_1,t_0) = \mathbf{\Phi}(t_2,t_0)$ for all $t_0 \leq t_1 \leq t_2$.

5. It satisfies the differential equation $\dfrac{\partial \mathbf{\Phi}(t,t_0)}{\partial t} = \mathbf{A}(t)\mathbf{\Phi}(t,t_0)$ with initial conditions $\mathbf{\Phi}(t_0,t_0) = I$.

6. The state-transition matrix $\mathbf{\Phi}(t,\tau)$, given by

$$\mathbf{\Phi}(t,\tau) \equiv \mathbf{U}(t)\mathbf{U}^{-1}(\tau)$$

where the $n \times n$ matrix $\mathbf{U}(t)$ is the fundamental solution matrix that satisfies

$$\dot{\mathbf{U}}(t) = \mathbf{A}(t)\mathbf{U}(t) \text{ with initial condition } \mathbf{U}(t_0) = I.$$

7. Given the state $\mathbf{x}(\tau)$ at any time $\tau$, the state at any other time $t$ is given by the mapping

$$\mathbf{x}(t) = \mathbf{\Phi}(t,\tau)\mathbf{x}(\tau)$$

## Estimation of the state-transition matrix

In the time-invariant case, we can define $\mathbf{\Phi}$, using the matrix exponential, as $\mathbf{\Phi}(t,t_0) = e^{\mathbf{A}(t-t_0)}$. [4]

In the time-variant case, the state-transition matrix $\mathbf{\Phi}(t,t_0)$ can be estimated from the solutions of the differential equation $\dot{\mathbf{u}}(t) = \mathbf{A}(t)\mathbf{u}(t)$ with initial conditions $\mathbf{u}(t_0)$ given by $[1,\ 0,\ \ldots,\ 0]^T$, $[0,\ 1,\ \ldots,\ 0]^T$, ..., $[0,\ 0,\ \ldots,\ 1]^T$. The corresponding solutions provide the $n$ columns of matrix $\mathbf{\Phi}(t,t_0)$. Now, from property 4, $\mathbf{\Phi}(t,\tau) = \mathbf{\Phi}(t,t_0)\mathbf{\Phi}(\tau,t_0)^{-1}$ for all $t_0 \leq \tau \leq t$. The state-transition matrix must be determined before analysis on the time-varying solution can continue.

## See also

- Magnus expansion
- Liouville's formula

## References


1. Baake, Michael; Schlaegel, Ulrike (2011). "The Peano Baker Series". *Proceedings of the Steklov Institute of Mathematics*. **275**: 155–159. doi:10.1134/S0081543811080098 (https://doi.org/10.1134%2FS0081543811080098). S2CID 119133539 (https://api.semanticscholar.org/CorpusID:119133539).
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3. Brockett, Roger W. (1970). *Finite Dimensional Linear Systems*. John Wiley & Sons. ISBN 978-0-471-10585-5.
4. Reyneke, Pieter V. (2012). "Polynomial Filtering: To any degree on irregularly sampled data" (http://hrcak.srce.hr/file/138435). *Automatika*. **53** (4): 382–397. doi:10.7305/automatika.53-4.248 (https://doi.org/10.7305%2Fautomatika.53-4.248). S2CID 40282943 (https://api.semanticscholar.org/CorpusID:40282943).


## Further reading


- Baake, M.; Schlaegel, U. (2011). "The Peano Baker Series". *Proceedings of the Steklov Institute of Mathematics*. **275**: 155–159. doi:10.1134/S0081543811080098 (https://doi.org/10.1134%2FS0081543811080098). S2CID 119133539 (https://api.semanticscholar.org/CorpusID:119133539).
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