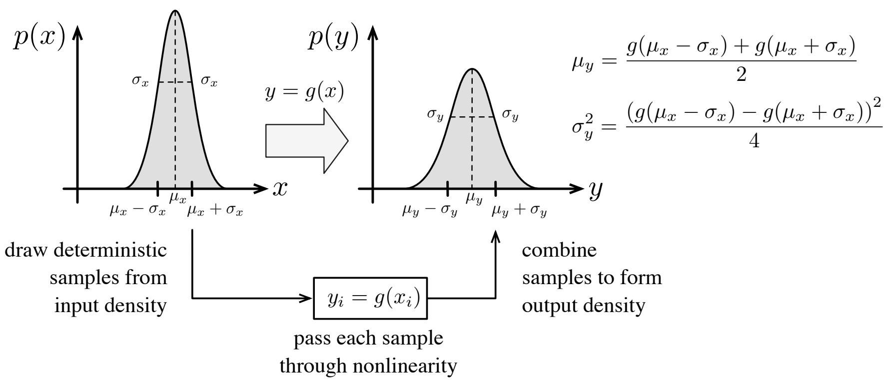


**Figure 4.10** One-dimensional Gaussian PDF transformed through a deterministic nonlinear function, $g(\cdot)$. Here the basic sigmapoint transformation is used in which only two deterministic samples (one on either side of the mean) approximate the input density.

by passing the output PDF through the inverse of the nonlinearity (using the same linearization procedure). This is not true for all methods of passing PDFs through nonlinearities since they do not all make the same approximations as linearization. For example, the sigmapoint transformation is not reversible in this way.

### *Sigmapoint Transformation*

In a sense, the *sigmapoint (SP)* or *unscented* transformation (Julier and Uhlmann, 1996) is the compromise between the Monte Carlo and linearization methods when the input density is roughly a Gaussian PDF. It is more accurate than linearization, but for a comparable computational cost to linearization. Monte Carlo is still the most accurate method, but the computational cost is prohibitive in most situations.

It is actually a bit misleading to refer to 'the' sigmapoint transformation, as there is actually a whole family of such transformations. Figure 4.10 depicts the very simplest version in one dimension. In general, a version of the SP transformation is used that includes one additional sample beyond the basic version at the mean of the input density. The steps are as follows:

1. A set of $2L + 1$ *sigmapoints* is computed from the input density, $\mathcal{N}(\boldsymbol{\mu}_x, \boldsymbol{\Sigma}_{xx})$, according to

$$\mathbf{L}\mathbf{L}^T = \boldsymbol{\Sigma}_{xx}, \quad \text{(Cholesky decomposition, } \mathbf{L} \text{ lower-triangular)} \tag{4.47a}$$
$$\mathbf{x}_0 = \boldsymbol{\mu}_x, \tag{4.47b}$$
$$\mathbf{x}_i = \boldsymbol{\mu}_x + \sqrt{L+\kappa}\,\mathrm{col}_i\mathbf{L}, \qquad i = 1 \ldots L \tag{4.47c}$$
$$\mathbf{x}_{i+L} = \boldsymbol{\mu}_x - \sqrt{L+\kappa}\,\mathrm{col}_i\mathbf{L}, \tag{4.47d}$$

where $L = \dim(\boldsymbol{\mu}_x)$. We note that

$$\boldsymbol{\mu}_x = \sum_{i=0}^{2L} \alpha_i \, \mathbf{x}_i, \tag{4.48a}$$

$$\boldsymbol{\Sigma}_{xx} = \sum_{i=0}^{2L} \alpha_i \left(\mathbf{x}_i - \boldsymbol{\mu}_x\right)\left(\mathbf{x}_i - \boldsymbol{\mu}_x\right)^T, \tag{4.48b}$$

where

$$\alpha_i = \begin{cases} \frac{\kappa}{L+\kappa} & i = 0 \\ \frac{1}{2}\frac{1}{L+\kappa} & \text{otherwise} \end{cases}, \tag{4.49}$$

which we note sums to 1. The user-definable parameter, $\kappa$, will be explained in the next section.

2. Each of the sigmapoints is individually passed through the nonlinearity, $\mathbf{g}(\cdot)$:

$$\mathbf{y}_i = \mathbf{g}\left(\mathbf{x}_i\right), \qquad i = 0 \ldots 2L. \tag{4.50}$$

3. The mean of the output density, $\boldsymbol{\mu}_y$, is computed as

$$\boldsymbol{\mu}_y = \sum_{i=0}^{2L} \alpha_i \, \mathbf{y}_i. \tag{4.51}$$

4. The covariance of the output density, $\boldsymbol{\Sigma}_{yy}$, is computed as

$$\boldsymbol{\Sigma}_{yy} = \sum_{i=0}^{2L} \alpha_i \left(\mathbf{y}_i - \boldsymbol{\mu}_y\right)\left(\mathbf{y}_i - \boldsymbol{\mu}_y\right)^T. \tag{4.52}$$

5. The output density, $\mathcal{N}\left(\boldsymbol{\mu}_y, \boldsymbol{\Sigma}_{yy}\right)$, is returned.

This method of transforming a PDF through a nonlinearity has a number of advantages over linearization:

(i) By approximating the input density instead of linearizing, we avoid the need to compute the Jacobian of the nonlinearity (either in closed form or numerically). Figure 4.11 provides an example of the sigmapoints for a two-dimensional Gaussian.
(ii) We employ only standard linear algebra operations (Cholesky decomposition, outer products, matrix summations).
(iii) The computation cost is similar to linearization (when a numerical Jacobian is used).
(iv) There is no requirement that the nonlinearity be smooth and differentiable.

The next section will furthermore show that the unscented transformation can also more accurately capture the posterior density than linearization (by way of an example).

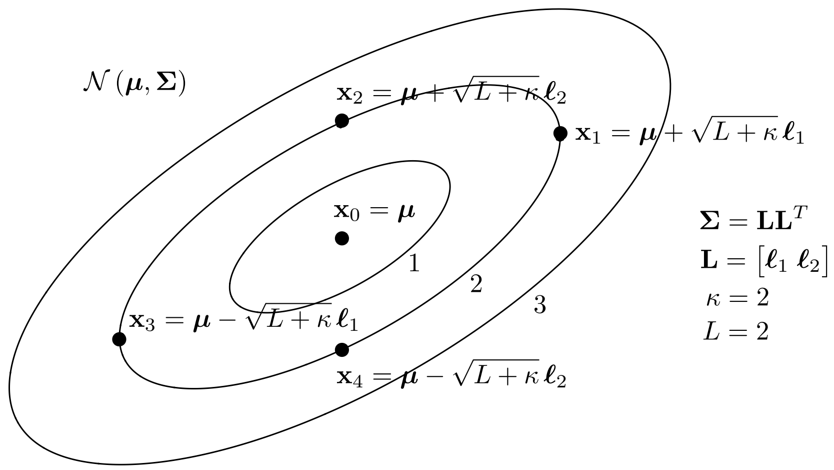


**Figure 4.11** Two-dimensional ($L = 2$) Gaussian PDF, whose covariance is displayed using elliptical equiprobable contours of 1, 2, and 3 standard deviations, and the corresponding $2L + 1 = 5$ sigmapoints for $\kappa = 2$.

**Example 4.1** We will use a simple one-dimensional nonlinearity, $f(x) = x^2$, as an example and compare the various transformation methods. Let the prior density be $\mathcal{N}(\mu_x, \sigma_x^2)$.

*Monte Carlo Method*

In fact, for this particularly nonlinearity, we can essentially use the Monte Carlo method in closed form (i.e., we do not actually draw any samples) to get the exact answer for transforming the input density through the nonlinearity. An arbitrary sample (a.k.a., realization) of the input density is given by

$$x_i = \mu_x + \delta x_i, \qquad \delta x_i \leftarrow \mathcal{N}(0, \sigma_x^2). \tag{4.53}$$

Transforming this sample through the nonlinearity, we get

$$y_i = f(x_i) = f(\mu_x + \delta x_i) = (\mu_x + \delta x_i)^2 = \mu_x^2 + 2\mu_x \delta x_i + \delta x_i^2. \tag{4.54}$$

Taking the expectation of both sides, we arrive at the mean of the output:

$$\mu_y = E[y_i] = \mu_x^2 + 2\mu_x \underbrace{E[\delta x_i]}_{0} + \underbrace{E\left[\delta x_i^2\right]}_{\sigma_x^2} = \mu_x^2 + \sigma_x^2. \tag{4.55}$$