Example 5.7-2. Test the system described by Eq. 5.7-2 for complete controllability.
In matrix form, the system becomes

$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)$$

$$y(k) = [1 \quad 0]\,\mathbf{x}(k)$$

and the controllability test matrix is

$$\mathbf{\Theta}_c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which has rank equal to 2, meaning that the system is completely controllable.

Complete Observability

If a system has this property, then all modes of the system can be detected at the output. This does not mean that all states are present in the output equation, $\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k)$. The modes are contained in $\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) = \mathbf{C}\,\mathbf{M}\mathbf{q}(k)$ and $\mathbf{M}$ was found using $\mathbf{A}$. This also points out that complete observability is determined by the matrices $(\mathbf{A}, \mathbf{C})$.

Definition 5.7-3. A system is completely observable if and only if the output $y(k)$ during $0 \leq k \leq K$ can be used to determine $\mathbf{x}(0)$. The input, $\mathbf{u}(k)$, in the interval is assumed known.

This definition is the starting point for the development of a test for complete observability. The general single-input, single-output system is

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}u(k)$$

$$y(k) = \mathbf{C}\mathbf{x}(k)$$

Previous arguments indicate that $u(k)$ has no effect on complete observability. This suggests eliminating the input and working with

$$y(0) = \mathbf{C}\mathbf{x}(0)$$

$$y(1) = \mathbf{C}\mathbf{x}(1) = \mathbf{C}\mathbf{A}\mathbf{x}(0)$$

$$y(2) = \mathbf{C}\mathbf{x}(2) = \mathbf{C}\mathbf{A}^2\mathbf{x}(0)$$

$$\vdots$$

$$y(k) = \mathbf{C}\mathbf{A}^k\mathbf{x}(0)$$

or

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \cdot \\ \cdot \\ \cdot \\ y(k) \end{bmatrix} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{CA}^k \end{bmatrix} \mathbf{x}(0)$$

According to the definition of complete observability, it must be possible to solve this equation for $\mathbf{x}(0)$. This will be possible if and only if

$$\text{rank} \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{CA}^k \end{bmatrix} = n$$

This is the observability test matrix and is more commonly written as

$$\mathbf{\Theta}_o = [\mathbf{C}^T \quad \mathbf{A}^T\mathbf{C}^T \quad (\mathbf{A}^2)^T\mathbf{C}^T \quad \cdots \quad (\mathbf{A}^{n-1})^T\mathbf{C}^T]$$

$(\mathbf{A}^n)^T\mathbf{C}^T$ is not needed since $\mathbf{A}^n$ is linearly dependent on $\mathbf{A}^{n-1}, \ldots, \mathbf{A}$.

Example 5.7-3. Test the system in Ex. 5.7-1 for complete observability.
The observability test matrix is

$$\mathbf{\Theta}_o = [\mathbf{C}^T \quad \mathbf{A}^T\mathbf{C}^T] = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

which has rank <2, implying that the system is not completely observable.

Example 5.7-4. Test the system in Ex. 5.7-2 to see whether it is completely observable.
Here, the observability test matrix becomes,

$$\mathbf{\Theta}_o = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which has rank $n = 2$ and so the system is completely observable.