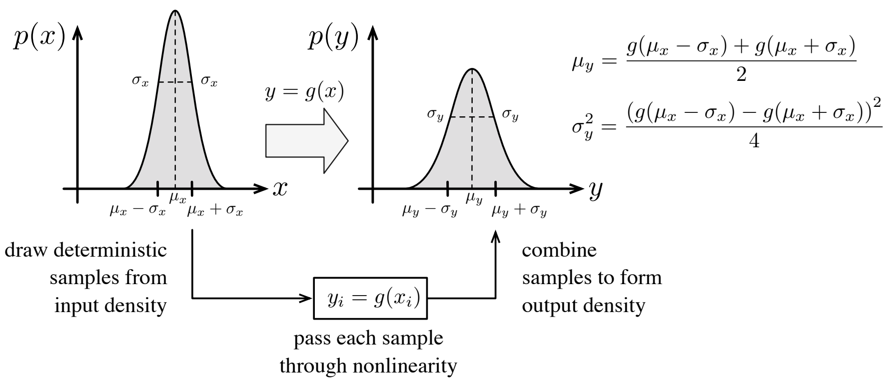

Figure 4.10 One-dimensional Gaussian PDF transformed through a deterministic nonlinear function, $g(\cdot)$. Here the basic sigmapoint transformation is used in which only two deterministic samples (one on either side of the mean) approximate the input density.

by passing the output PDF through the inverse of the nonlinearity (using the same linearization procedure). This is not true for all methods of passing PDFs through nonlinearities since they do not all make the same approximations as linearization. For example, the sigmapoint transformation is not reversible in this way.

Sigmapoint Transformation

In a sense, the *sigmapoint (SP)* or *unscented* transformation (Julier and Uhlmann, 1996) is the compromise between the Monte Carlo and linearization methods when the input density is roughly a Gaussian PDF. It is more accurate than linearization, but for a comparable computational cost to linearization. Monte Carlo is still the most accurate method, but the computational cost is prohibitive in most situations.

It is actually a bit misleading to refer to 'the' sigmapoint transformation, as there is actually a whole family of such transformations. Figure 4.10 depicts the very simplest version in one dimension. In general, a version of the SP transformation is used that includes one additional sample beyond the basic version at the mean of the input density. The steps are as follows:

1. A set of $2L+1$ *sigmapoints* is computed from the input density, $\mathcal{N}(\boldsymbol{\mu}_x, \boldsymbol{\Sigma}_{xx})$, according to

$$\mathbf{L}\mathbf{L}^T = \boldsymbol{\Sigma}_{xx}, \quad \text{(Cholesky decomposition, } \mathbf{L} \text{ lower-triangular)} \tag{4.47a}$$

$$\mathbf{x}_0 = \boldsymbol{\mu}_x, \tag{4.47b}$$

$$\mathbf{x}_i = \boldsymbol{\mu}_x + \sqrt{L+\kappa}\,\mathrm{col}_i\mathbf{L}, \quad i = 1 \ldots L \tag{4.47c}$$

$$\mathbf{x}_{i+L} = \boldsymbol{\mu}_x - \sqrt{L+\kappa}\,\mathrm{col}_i\mathbf{L}, \quad i = 1 \ldots L \tag{4.47d}$$

where $L = \dim(\boldsymbol{\mu}_x)$. We note that

$$\boldsymbol{\mu}_x = \sum_{i=0}^{2L} \alpha_i \, \mathbf{x}_i, \tag{4.48a}$$

$$\boldsymbol{\Sigma}_{xx} = \sum_{i=0}^{2L} \alpha_i \left(\mathbf{x}_i - \boldsymbol{\mu}_x\right)\left(\mathbf{x}_i - \boldsymbol{\mu}_x\right)^T, \tag{4.48b}$$

where

$$\alpha_i = \begin{cases} \frac{\kappa}{L+\kappa} & i = 0 \\ \frac{1}{2}\frac{1}{L+\kappa} & \text{otherwise} \end{cases}, \tag{4.49}$$

which we note sums to 1. The user-definable parameter, κ, will be explained in the next section.

2. Each of the sigmapoints is individually passed through the nonlinearity, $\mathbf{g}(\cdot)$:

$$\mathbf{y}_i = \mathbf{g}\left(\mathbf{x}_i\right), \qquad i = 0 \ldots 2L. \tag{4.50}$$

3. The mean of the output density, $\boldsymbol{\mu}_y$, is computed as

$$\boldsymbol{\mu}_y = \sum_{i=0}^{2L} \alpha_i \, \mathbf{y}_i. \tag{4.51}$$

4. The covariance of the output density, $\boldsymbol{\Sigma}_{yy}$, is computed as

$$\boldsymbol{\Sigma}_{yy} = \sum_{i=0}^{2L} \alpha_i \left(\mathbf{y}_i - \boldsymbol{\mu}_y\right)\left(\mathbf{y}_i - \boldsymbol{\mu}_y\right)^T. \tag{4.52}$$

5. The output density, $\mathcal{N}\left(\boldsymbol{\mu}_y, \boldsymbol{\Sigma}_{yy}\right)$, is returned.

This method of transforming a PDF through a nonlinearity has a number of advantages over linearization:

(i) By approximating the input density instead of linearizing, we avoid the need to compute the Jacobian of the nonlinearity (either in closed form or numerically). Figure 4.11 provides an example of the sigmapoints for a two-dimensional Gaussian.
(ii) We employ only standard linear algebra operations (Cholesky decomposition, outer products, matrix summations).
(iii) The computation cost is similar to linearization (when a numerical Jacobian is used).
(iv) There is no requirement that the nonlinearity be smooth and differentiable.

The next section will furthermore show that the unscented transformation can also more accurately capture the posterior density than linearization (by way of an example).

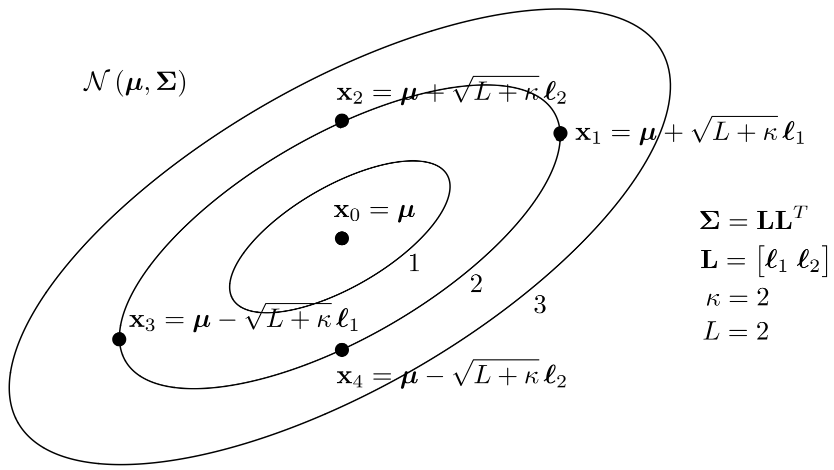

Figure 4.11 Two-dimensional ($L = 2$) Gaussian PDF, whose covariance is displayed using elliptical equiprobable contours of 1, 2, and 3 standard deviations, and the corresponding $2L + 1 = 5$ sigmapoints for $\kappa = 2$.

Example 4.1 We will use a simple one-dimensional nonlinearity, $f(x) = x^2$, as an example and compare the various transformation methods. Let the prior density be $\mathcal{N}(\mu_x, \sigma_x^2)$.

Monte Carlo Method

In fact, for this particularly nonlinearity, we can essentially use the Monte Carlo method in closed form (i.e., we do not actually draw any samples) to get the exact answer for transforming the input density through the nonlinearity. An arbitrary sample (a.k.a., realization) of the input density is given by

$$x_i = \mu_x + \delta x_i, \qquad \delta x_i \leftarrow \mathcal{N}(0, \sigma_x^2). \tag{4.53}$$

Transforming this sample through the nonlinearity, we get

$$y_i = f(x_i) = f(\mu_x + \delta x_i) = (\mu_x + \delta x_i)^2 = \mu_x^2 + 2\mu_x \delta x_i + \delta x_i^2. \tag{4.54}$$

Taking the expectation of both sides, we arrive at the mean of the output:

$$\mu_y = E[y_i] = \mu_x^2 + 2\mu_x \underbrace{E[\delta x_i]}_{0} + \underbrace{E\left[\delta x_i^2\right]}_{\sigma_x^2} = \mu_x^2 + \sigma_x^2. \tag{4.55}$$

We do a similar thing for the variance of the output:

$$\sigma_y^2 = E\left[(y_i - \mu_y)^2\right] \tag{4.56a}$$
$$= E\left[(2\mu_x \delta x_i + \delta x_i^2 - \sigma_x^2)^2\right] \tag{4.56b}$$
$$= \underbrace{E\left[\delta x_i^4\right]}_{3\sigma_x^4} + 4\mu_x \underbrace{E\left[\delta x_i^3\right]}_{0} + (4\mu_x^2 - 2\sigma_x^2) \underbrace{E\left[\delta x_i^2\right]}_{\sigma_x^2} - 4\mu_x \sigma_x^2 \underbrace{E\left[\delta x_i\right]}_{0} + \sigma_x^4 \tag{4.56c}$$
$$= 4\mu_x^2 \sigma_x^2 + 2\sigma_x^4, \tag{4.56d}$$

where $E\left[\delta x_i^3\right] = 0$ and $E\left[\delta x_i^4\right] = 3\sigma_x^4$ are the well-known third and fourth moments for a Gaussian PDF.

In truth, the resulting output density is not Gaussian. We could go on to compute higher moments of the output (and they would not all match a Gaussian). However, if we want to approximate the output as Gaussian by not considering the moments beyond the variance, we can. In this case, the resulting output density is $\mathcal{N}\left(\mu_y, \sigma_y^2\right)$. We have effectively used the Monte Carlo method with an infinite number of samples to carry out the computation of the first two moments of the posterior exactly in closed form. Let us now see how linearization and the sigmapoint transformation perform.

Linearization

Linearizing the nonlinearity about the mean of the input density, we have

$$y_i = f(\mu_x + \delta x_i) \approx \underbrace{f(\mu_x)}_{\mu_x^2} + \underbrace{\left.\frac{\partial f}{\partial x}\right|_{\mu_x}}_{2\mu_x} \delta x_i = \mu_x^2 + 2\mu_x \delta x_i. \tag{4.57}$$

Taking the expectation, we arrive at the mean of the output:

$$\mu_y = E[y_i] = \mu_x^2 + 2\mu_x \underbrace{E[\delta x_i]}_{0} = \mu_x^2, \tag{4.58}$$

which is just the mean of the input passed through the nonlinearity: $\mu_y = f(\mu_x)$. For the variance of the output we have

$$\sigma_y^2 = E\left[(y_i - \mu_y)^2\right] = E\left[(2\mu_x \delta x_i)^2\right] = 4\mu_x^2 \sigma_x^2. \tag{4.59}$$

Comparing (4.55) with (4.58), and (4.56) with (4.59), we see there are some discrepancies. In fact, the linearized mean has a bias and the variance is too small (i.e., overconfident). Let us see what happens with the sigmapoint transformation.

Sigmapoint Transformation

There are $2L + 1 = 3$ sigmapoints in dimension $L = 1$:

$$x_0 = \mu_x, \quad x_1 = \mu_x + \sqrt{1+\kappa}\,\sigma_x, \quad x_2 = \mu_x - \sqrt{1+\kappa}\,\sigma_x, \tag{4.60}$$

where κ is a user-definable parameter that we discuss below. We pass each sigmapoint through the nonlinearity:

$$y_0 = f(x_0) = \mu_x^2, \tag{4.61a}$$

$$y_1 = f(x_1) = \left(\mu_x + \sqrt{1+\kappa}\,\sigma_x\right)^2 = \mu_x^2 + 2\mu_x\sqrt{1+\kappa}\,\sigma_x + (1+\kappa)\sigma_x^2, \tag{4.61b}$$

$$y_2 = f(x_2) = \left(\mu_x - \sqrt{1+\kappa}\,\sigma_x\right)^2 = \mu_x^2 - 2\mu_x\sqrt{1+\kappa}\,\sigma_x + (1+\kappa)\sigma_x^2. \tag{4.61c}$$

The mean of the output is given by

$$\mu_y = \frac{1}{1+\kappa}\left(\kappa y_0 + \frac{1}{2}\sum_{i=1}^{2} y_i\right) \tag{4.62a}$$

$$= \frac{1}{1+\kappa}\Big(\kappa\mu_x^2 + \frac{1}{2}(\mu_x^2 + 2\mu_x\sqrt{1+\kappa}\,\sigma_x + (1+\kappa)\sigma_x^2 + \mu_x^2 - 2\mu_x\sqrt{1+\kappa}\,\sigma_x + (1+\kappa)\sigma_x^2)\Big) \tag{4.62b}$$

$$= \frac{1}{1+\kappa}\left(\kappa\mu_x^2 + \mu_x^2 + (1+\kappa)\sigma_x^2\right) \tag{4.62c}$$

$$= \mu_x^2 + \sigma_x^2, \tag{4.62d}$$

which is independent of κ and exactly the same as (4.55). For the variance we have

$$\sigma_y^2 = \frac{1}{1+\kappa}\left(\kappa\left(y_0 - \mu_y\right)^2 + \frac{1}{2}\sum_{i=1}^{2}\left(y_i - \mu_y\right)^2\right) \tag{4.63a}$$

$$= \frac{1}{1+\kappa}\left(\kappa\sigma_x^4 + \frac{1}{2}\left(\left(2\mu_x\sqrt{1+\kappa}\,\sigma_x + \kappa\sigma_x^2\right)^2 + \left(-2\mu_x\sqrt{1+\kappa}\,\sigma_x + \kappa\sigma_x^2\right)^2\right)\right) \tag{4.63b}$$

$$= \frac{1}{1+\kappa}\left(\kappa\sigma_x^4 + 4(1+\kappa)\mu_x^2\sigma_x^2 + \kappa^2\sigma_x^4\right) \tag{4.63c}$$

$$= 4\mu_x^2\sigma_x^2 + \kappa\sigma_x^4, \tag{4.63d}$$

which can be made to be identical to (4.56) by selecting the user-definable parameter, κ, to be 2. Thus, for this nonlinearity, the unscented transformation can exactly capture the correct mean and variance of the output.

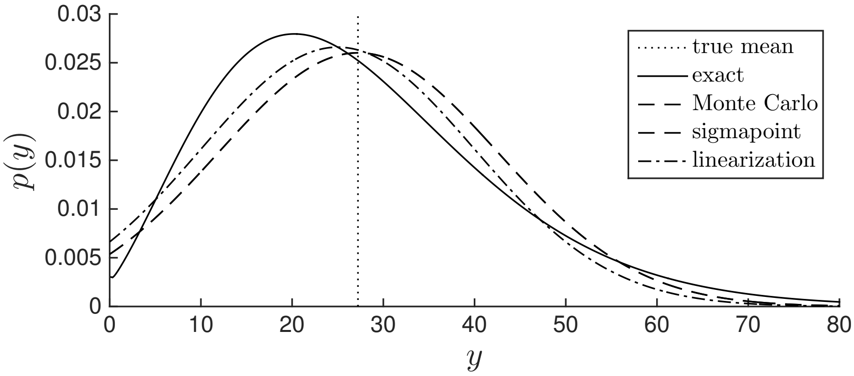

Figure 4.12 Graphical depiction of passing a Gaussian PDF, $p(x) = \mathcal{N}\left(5, (3/2)^2\right)$, through the nonlinearity, $y = x^2$, using various methods. We see that the Monte Carlo and sigmapoint methods match the true mean, while linearization does not. We also show the exact transformed PDF, which is not Gaussian and therefore does not have its mean at its mode.

To understand why we should pick $\kappa = 2$, we need look no further than the input density. The parameter κ scales how far away the sigmapoints are from the mean. This does not affect the first three moments of the sigmapoints (i.e., μ_x, σ_x^2, and the zero *skewness*). However, changing κ does influence the fourth moment, *kurtosis*. We already used the fact that for a Gaussian PDF, the fourth moment is $3\sigma_x^4$. We can chose κ to make the fourth moment of the sigmapoints match the true kurtosis of the Gaussian input density:

$$3\sigma_x^4 = \frac{1}{1+\kappa}\left(\kappa\underbrace{(x_0-\mu_x)^4}_{0} + \frac{1}{2}\sum_{i=1}^{2}(x_i-\mu_x)^4\right) \tag{4.64a}$$

$$= \frac{1}{2(1+\kappa)}\left(\left(\sqrt{1+\kappa}\,\sigma_x\right)^4 + \left(-\sqrt{1+\kappa}\,\sigma_x\right)^4\right) \tag{4.64b}$$

$$= (1+\kappa)\sigma_x^4. \tag{4.64c}$$

Comparing the desired and actual kurtosis, we should pick $\kappa = 2$ to make them match exactly. Not surprisingly, this has a positive effect on accuracy of the transformation.

In summary, this example shows that linearization is an inferior method of transforming a PDF through a nonlinearity if the goal is to capture the true mean of the output. Figure 4.12 provides a graphical depiction of this example.

In the next few sections, we return to the Bayes filter and use our new knowledge about the different methods of passing PDFs through nonlinearities to make some useful improvements to the EKF. We will begin with the particle filter, which makes use of the Monte Carlo method. We will then try to implement a Gaussian filter using the SP transformation.