Chapter 11

Continuous-Time Linear State-Space Models

11.1 Introduction

In this chapter, we focus on the solution of CT state-space models. The development here follow the previous chapter.

11.2 The Time-Varying Case

Consider the nth-order continuous-time linear state-space description

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t) \ . \end{aligned} \tag{11.1}$$

We shall always assume that the coefficient matrices in the above model are sufficiently well behaved for there to *exist* a *unique* solution to the state-space model for any specified initial condition $x(t_0)$ and any integrable input $u(t)$. For instance, if these coefficient matrices are piecewise continuous, with a finite number of discontinuities in any finite interval, then the desired existence and uniqueness properties hold.

We can describe the solution of (11.1) in terms of a matrix function $\Phi(t, \tau)$ that has the following two properties:

$$\dot{\Phi}(t, \tau) = A(t)\Phi(t, \tau) \ , \tag{11.2}$$

$$\Phi(\tau, \tau) = I \ . \tag{11.3}$$

This matrix function is referred to as the **state transition matrix**, and under our assumption on the nature of $A(t)$ it turns out that the state transition matrix *exists* and is *unique.*

We will show that, given $x(t_0)$ and $u(t)$,

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau \ . \tag{11.4}$$

Observe again that, as in the DT case, the terms corresponding to the zero-input and zero-state responses are evident in (11.4). In order to verify (11.4), we differentiate it with respect to t:

$$\dot{x}(t) = \dot{\Phi}(t, t_0)x(t_0) + \int_{t_0}^{t} \dot{\Phi}(t, \tau)B(\tau)u(\tau)d\tau + \Phi(t, t)B(t)u(t) \ . \tag{11.5}$$

Using (11.2) and (11.3),

$$\dot{x}(t) = A(t)\Phi(t, t_0)x(t_0) + \int_{t_0}^{t} A(t)\Phi(t, \tau)B(\tau)u(\tau)d\tau + B(t)u(t) \ . \tag{11.6}$$

Now, since the integral is taken with respect to τ, $A(t)$ can be factored out:

$$\begin{aligned} \dot{x}(t) &= A(t)\left[\Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau\right] + B(t)u(t) \ , && (11.7) \\ &= A(t)x(t) + B(t)u(t) \ , && (11.8) \end{aligned}$$

so the expression in (11.4) does indeed satisfy the state evolution equation. To verify that it also matches the specified initial condition, note that

$$x(t_0) = \Phi(t_0, t_0)x(t_0) = x(t_0). \tag{11.9}$$

We have now shown that the matrix function $\Phi(t, \tau)$ satisfying (11.2) and (11.3) yields the solution to the continuous-time system equation (11.1).

Exercise: Show that $\Phi(t, \tau)$ must be nonsingular. (Hint: Invoke our claim about uniqueness of solutions.)

The question that remains is how to find the state transition matrix. For a general linear time-varying system, there is no analytical expression that expresses $\Phi(t, \tau)$ analytically as a function of $A(t)$. Instead, we are essentially limited to numerical solution of the equation (11.2) with the boundary condition (11.3). This equation may be solved one column at a time, as follows. We numerically compute the respective solutions $x^i(t)$ of the homogeneous equation

$$\dot{x}(t) = A(t)x(t) \tag{11.10}$$

for each of the n initial conditions below:

$$x^1(t_0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad x^2(t_0) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad x^n(t_0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} .$$

Then

$$\Phi(t, t_0) = \left[\begin{array}{ccc} x^1(t) & \ldots & x^n(t) \end{array} \right]. \tag{11.11}$$

In summary, knowing n solutions of the homogeneous system for n independent initial conditions, we are able to construct the general solution of this linear time varying system. The underlying reason this construction works is that solutions of a linear system may be superposed, and our system is of order n.

Example 11.1 A Special Case

Consider the following time-varying system

$$\frac{d}{dt}\left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array} \right] = \left[\begin{array}{cc} \alpha(t) & \beta(t) \\ -\beta(t) & \alpha(t) \end{array} \right] \left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array} \right],$$

where $\alpha(t)$ and $\beta(t)$ are continuous functions of t. It turns out that the special structure of the matrix $A(t)$ here permits an analytical solution. Specifically, verify that the state transition matrix of the system is

$$\Phi(t, t_0) = \left[\begin{array}{cc} \exp(\int_{t_0}^t \alpha(\tau)d\tau)\cos(\int_{t_0}^t \beta(\tau)d\tau) & \exp(\int_{t_0}^t \alpha(\tau)d\tau)\sin(\int_{t_0}^t \beta(\tau)d\tau) \\ -\exp(\int_{t_0}^t \alpha(\tau)d\tau)\sin(\int_{t_0}^t \beta(\tau)d\tau) & \exp(\int_{t_0}^t \alpha(\tau)d\tau)\cos(\int_{t_0}^t \beta(\tau)d\tau) \end{array} \right]$$

The secret to solving the above system — or equivalently, to obtaining its state transition matrix — is to transform it to polar co-ordinates via the definitions

$$\begin{aligned} r^2(t) &= (x_1)^2(t) + (x_2)^2(t) \\ \theta(t) &= \tan^{-1}\left(\frac{x_2}{x_1}\right) . \end{aligned}$$

We leave you to deduce now that

$$\begin{aligned} \frac{d}{dt}r^2 &= 2\alpha r^2 \\ \frac{d}{dt}\theta &= -\beta . \end{aligned}$$

The solution of this system of equations is then given by

$$r^2(t) = \exp\left(2\int_{t_0}^t \alpha(\tau)d\tau\right) r^2(t_0)$$

and

$$\theta(t) = \theta(t_0) - \int_{t_0}^t \beta(\tau)d\tau$$