Equation 3.184 for $k = k_0$ to $k = k_1 - 1$ yields

$$\begin{bmatrix} y(k_0) \\ y(k_0+1) \\ \vdots \\ y(k_1-2) \\ y(k_1-1) \end{bmatrix} = \begin{bmatrix} C(k_0)x_0 \\ C(k_0+1)\Phi(k_0+1,k_0)x_0 \\ \vdots \\ C(k_1-2)\Phi(k_1-2,k_0)x_0 \\ C(k_1-1)\Phi(k_1-1,k_0)x_0 \end{bmatrix} \tag{3.185}$$

The right-hand side of Equation 3.185 can be written in the form $O(k_0, k_1)x_0$ where $O(k_0, k_1)$ is the $(k_1 - k_0)p \times n$ *observability matrix* defined by

$$O(k_0,k_1) = \begin{bmatrix} C(k_0) \\ C(k_0+1)\Phi(k_0+1,k_0) \\ \vdots \\ C(k_1-2)\Phi(k_1-2,k_0) \\ C(k_1-1)\Phi(k_1-1,k_0) \end{bmatrix} \tag{3.186}$$

Equation 3.185 can be solved for any initial state x_0 if and only if rank $O(k_0, k_1) = n$, which is a necessary and sufficient condition for observability on $[k_0, k_1]$. If the rank condition holds, the solution of Equation 3.185 for x_0 is

$$x_0 = \left[O^T(k_0,k_1)O(k_0,k_1)\right]^{-1} O^T(k_0,k_1)Y(k_0,k_1) \tag{3.187}$$

where $Y(k_0, k_1)$ is the $(k_1 - k_0)p$-element column vector of outputs given by

$$Y(k_0,k_1) = \left[y^T(k_0)\; y^T(k_0+1) \cdots y^T(k_1-2)\; y^T(k_1-1)\right]^T \tag{3.188}$$

Given a positive integer N, setting $k_0 = k$ and $k_1 = k + N$ in $O(k_0,\, k_1)$ yields the $Np \times n$ matrix $O(k, k+N)$, which will be denoted by $O(k)$. By definition of the state-transition matrix $\Phi(k, k_0)$, $O(k)$ can be written in the form

$$O(k) = \begin{bmatrix} O_0(k) \\ O_1(k) \\ \vdots \\ O_{N-1}(k) \end{bmatrix} \tag{3.189}$$

where the block rows $O_i(k)$ of $O(k)$ are given by

$$O_0(k) = C(k) \tag{3.190}$$

$$O_i(k) = O_{i-1}(k+1)A(k), \quad i = 1, 2, \ldots, N-1 \tag{3.191}$$

The system is said to be *uniformly N-step observable* if rank $O(k) = n$ for all k. Uniformly N-step observable means that the system is observable on the interval $[k, k+N]$ for all k.

3.3.4 Change of State Variables and Canonical Forms

Again, consider the discrete-time system with state model $[A(k), B(k), C(k)]$. For any $n \times n$ invertible matrix $P(k)$, another state model can be generated by defining the new state vector $z(k) = P^{-1}(k)x(k)$.