MATHEMATICS

Affine transformation applied to a multivariate Gaussian random variable - what is the mean vector and covariance matrix of the new variable?

Asked 9 years ago Modified 4 years, 5 months ago Viewed 34k times

40

29

Given a random vector $\mathbf{x} \sim N(\bar{\mathbf{x}}, \mathbf{C_x})$ with normal distribution. $\bar{\mathbf{x}}$ is the mean value vector and $\mathbf{C_x}$ is the covariance matrix of $\mathbf{x}$.

An affine transformation is applied to the $\mathbf{x}$ vector to create a new random $\mathbf{y}$ vector:

$$\mathbf{y} = \mathbf{Ax} + \mathbf{b}$$

Can we find mean value $\bar{\mathbf{y}}$ and covariance matrix $\mathbf{C_y}$ of this new vector $\mathbf{y}$ in terms of already given parameters ($\bar{\mathbf{x}}$, $\mathbf{C_x}$, $\mathbf{A}$ and $\mathbf{b}$)?

Can you please show the steps. Once I learn the method, I will use it on several other distributions myself.

linear-algebra multivariable-calculus normal-distribution random-variables

Share Cite Follow

edited Apr 21, 2016 at 18:20
Cong Ma
115 6

asked Mar 17, 2013 at 2:19
hkBattousai
4,095 7 36 53

2 Hint: $\bar{\mathbf{y}} = E[\mathbf{y}] = E[\mathbf{Ax} + \mathbf{b}]$. Now apply *linearity of expectation*. $\mathbf{C_y} = E[(\mathbf{y} - \bar{\mathbf{y}})(\mathbf{y} - \bar{\mathbf{y}})^T] = E[\mathbf{y}\mathbf{y}^T] - E[\bar{\mathbf{y}}\bar{\mathbf{y}}^T]$. – Dilip Sarwate Mar 17, 2013 at 2:46

1 Answer

Sorted by: Highest score (default)

49

We find the mean of $\mathbf{y}$ by using the fact that $\mathbb{E}\{\}$ is a linear operator.

$$\bar{\mathbf{y}} = \mathbb{E}\{\mathbf{y}\} = \mathbb{E}\{\mathbf{Ax} + \mathbf{b}\} = \mathbf{A}\mathbb{E}\{\mathbf{x}\} + \mathbf{b} = \mathbf{A}\bar{\mathbf{x}} + \mathbf{b}$$

Then we find covariance of

$$
\begin{aligned}
\mathbf{C}_{\mathbf{y}} &\triangleq \mathbb{E}\{(\mathbf{y}-\bar{\mathbf{y}})(\mathbf{y}-\bar{\mathbf{y}})^\top\} \\
&= \mathbb{E}\left\{\left[(\mathbf{A}\mathbf{x}+\mathbf{b})-(\mathbf{A}\bar{\mathbf{x}}+\mathbf{b})\right]\left[(\mathbf{A}\mathbf{x}+\mathbf{b})-(\mathbf{A}\bar{\mathbf{x}}+\mathbf{b})\right]^\top\right\} \\
&= \mathbb{E}\left\{\left[\mathbf{A}(\mathbf{x}-\bar{\mathbf{x}})\right]\left[\mathbf{A}(\mathbf{x}-\bar{\mathbf{x}})\right]^\top\right\} \\
&= \mathbb{E}\left\{\mathbf{A}(\mathbf{x}-\bar{\mathbf{x}})(\mathbf{x}-\bar{\mathbf{x}})^\top\mathbf{A}^\top\right\} \\
&= \mathbf{A}\mathbb{E}\left\{(\mathbf{x}-\bar{\mathbf{x}})(\mathbf{x}-\bar{\mathbf{x}})^\top\right\}\mathbf{A}^\top \\
&= \mathbf{A}\mathbf{C}_{\mathbf{x}}\mathbf{A}^\top
\end{aligned}
$$

Then, $\mathbf{y}$ is defined as,

$$\mathbf{y} \sim \mathcal{N}(\mathbf{A}\bar{\mathbf{x}}+\mathbf{b}, \mathbf{A}\mathbf{C}_{\mathbf{x}}\mathbf{A}^\top)$$

That is,

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{\sqrt{|2\pi\mathbf{A}\mathbf{C}_{\mathbf{x}}\mathbf{A}^\top|}}\exp\left(-\frac{1}{2}\left[\mathbf{y}-(\mathbf{A}\bar{\mathbf{x}}+\mathbf{b})\right]^\top(\mathbf{A}\mathbf{C}_{\mathbf{x}}\mathbf{A}^\top)^{-1}\left[\mathbf{y}-(\mathbf{A}\bar{\mathbf{x}}+\mathbf{b})\right]\right)$$

Share Cite Follow

edited Oct 16, 2017 at 20:44
Atcold
213 2 10

answered Mar 17, 2013 at 12:07
hkBattousai
4,095 7 36 53

Are there restrictions on A? I'm thinking, for example, about A={{1,1},{0,0}} (Ones are top row, 0s are bottom row) and sigma={{1,0},{0,1}}. Then rank A.sigma.Transpose(A) is 1 and not invertible any more. – BenB Jun 1, 2015 at 2:20

@BenB If the rank of A is not full, then the transformation is not linear. And that is not the case mentioned in the question statement. – hkBattousai Feb 6, 2016 at 13:24

4 Not all linear transformations have full rank. If the rank isn't full, the formula $y \sim N(A\bar{x}+b, AC_xA^T)$ still works. The pdf also works if interpreted to use pseudoinverses and pseudodeterminants. – Geoffrey Irving May 8, 2016 at 4:58

To be even more complete this answer could include the cross covariance! nice answer btw – Marco Aguiar Feb 26, 2018 at 13:20