Bisection Method

Tamas Kis | tamas.a.kis@outlook.com | https://tamaskis.github.io

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1 BISECTION METHOD

The **bisection method** can be used to find the root of a univariate function $f(x)$, with no restrictions on the differentiability of f. The basic idea behind the bisection is starting off with some interval $[a, b]$ containing a root, iteratively "shrinking" this interval until it is below some tolerance threshold, and then taking the root to be the midpoint of this interval. The general procedure is as follows:

1. Make an initial guess for the interval $[a, b]$ containing the root.
2. Assume the root, c, is the midpoint of this interval: $c = (a + b)/2$.
3. Evaluate $f(a)$ and $f(c)$.
   (a) If $f(a) < 0$ and $f(c) > 0$ (i.e. they have different sign), we know the true root is contained in the interval $[a, c]$. Therefore, we update our interval so that a remains the same, but b is updated to be c.
   (b) If $f(a)$ and $f(c)$ have the same sign (either both negative or both positive), we know the true root must be contained in the interval $[c, b]$. Therefore, we update our interval so that b remains the same, but a is updated to be c.
4. Repeating steps 2 and 3, the interval $[a, b]$ will keep shrinking. Once the difference $(b - a)$ is small enough, we say that the estimate of the root has **converged** to the true root, within some **tolerance** (which we denote as TOL). Therefore, if we predetermine that, at most, the root must be restricted to an interval of length TOL, we will keep repeating steps 2 and 3 until $(b - a) < \mathrm{TOL}$.

In some cases, the difference $(b - a)$ may never decrease below TOL, or take too long to decrease to below TOL. Therefore, we also define the **maximum number of iterations** ($i_{\max}$) so that the algorithm does not keep iterating forever, or for too long of a time [1, 2].

There are two basic algorithms for implementing the bisection method. The first implementation, shown in Algorithm 1 below, does *not* store the result of each iteration. On the other hand, the second implementation, shown in Algorithm 2, *does* store the result of each iteration. `bisection_method` implements both of these algorithms.

Since Algorithm 2 first needs to preallocate a potentially huge array to store all of the intermediate solutions, Algorithm 1 is significantly faster. Even if $i_{\max}$ (determines size of the preallocated array) is set to be a small number (for example, 10), Algorithm 1 is still faster. The reason we still consider and implement Algorithm 2 is so that convergence studies may be performed.

Algorithm 1:
Bisection method ("fast" implementation).

Given:
- $f(x)$ - function
- a - initial guess for lower bound of interval with root
- b - initial guess for upper bound of interval with root
- TOL - tolerance
- $i_{\max}$ - maximum number of iterations

Procedure:
1. Initial guess for root.

$$c = \frac{a+b}{2}$$

2. Find the root using the bisection method.

$$i = 1$$

while $((b-a) > \mathrm{TOL})$ **and** $(i < i_{\max})$

(a) Update interval.

if $f(c) = 0$

  Stop

else if $\mathrm{sgn}\,[f(c)] = \mathrm{sgn}\,[f(a)]$

  $a = c$

else

  $b = c$

end

(b) Update root estimate.

$$c = \frac{a+b}{2}$$

(c) Increment loop index.

$$i = i + 1$$

end

Return:

- root $= c$ - converged root

Algorithm 2:

Bisection method ("return all" implementation).

Given:

- $f(x)$ - function
- a - initial guess for lower bound of interval with root
- b - initial guess for upper bound of interval with root
- TOL - tolerance
- $i_{\max}$ - maximum number of iterations

Procedure:

1. Preallocate $\mathbf{x} \in \mathbb{R}^{i_{\max}}$ to store the estimates of the root at each iteration.
2. Store the initial guess for the root in the first element of $\mathbf{x}$.

$$x_1 = \frac{a+b}{2}$$

3. Find the root using the bisection method.

$$i = 1$$

while $((b-a) > \text{TOL})$ **and** $(i < i_{\max})$

(a) Update interval.

if $f(x_i) = 0$

Stop

else if $\text{sgn}\left[f(x_i)\right] = \text{sgn}\left[f(a)\right]$

$a = x_i$

else

$b = x_i$

end

(b) Update root estimate.

$$x_{i+1} = \frac{a+b}{2}$$

(c) Increment loop index.

$$i = i + 1$$

end

Return:

- $\mathbf{x}$ - vector where the first element is the initial guess for the root, the subsequent elements are the intermediate root estimates, and the final element is the converged root

REFERENCES

[1] *Bisection method*. Wikipedia. Accessed: February 7, 2021. URL: https://en.wikipedia.org/wiki/Bisection_method.

[2] Richard L. Burden and J. Douglas Faires. "The Bisection Method". In: *Numerical Analysis*. 9th ed. Boston, MA: Brooks/Cole, Cengage Learning, 2011. Chap. 2.1, pp. 48–55.