
Bisection Method

Tamas Kis | kis@stanford.edu | <https://github.com/tamaskis>

Contents

1 Bisection Method	2
References	4

Copyright © 2021 Tamas Kis

Permission is hereby granted, free of charge, to any person obtaining a copy of this software and associated documentation files (the "Software"), to deal in the Software without restriction, including without limitation the rights to use, copy, modify, merge, publish, distribute, sublicense, and/or sell copies of the Software, and to permit persons to whom the Software is furnished to do so, subject to the following conditions:

The above copyright notice and this permission notice shall be included in all copies or substantial portions of the Software.

THE SOFTWARE IS PROVIDED "AS IS", WITHOUT WARRANTY OF ANY KIND, EXPRESS OR IMPLIED, INCLUDING BUT NOT LIMITED TO THE WARRANTIES OF MERCHANTABILITY, FITNESS FOR A PARTICULAR PURPOSE AND NONINFRINGEMENT. IN NO EVENT SHALL THE AUTHORS OR COPYRIGHT HOLDERS BE LIABLE FOR ANY CLAIM, DAMAGES OR OTHER LIABILITY, WHETHER IN AN ACTION OF CONTRACT, TORT OR OTHERWISE, ARISING FROM, OUT OF OR IN CONNECTION WITH THE SOFTWARE OR THE USE OR OTHER DEALINGS IN THE SOFTWARE.



1 BISECTION METHOD

The **bisection method** can be used to find the root of a univariate function $f(x)$, with no restrictions on the differentiability of f . The basic idea behind the bisection is starting off with some interval $[a, b]$ containing a root, iteratively “shrinking” this interval until it is below some tolerance threshold, and then taking the root to be the midpoint of this interval. The general procedure is as follows:

1. Make an initial guess for the interval $[a, b]$ containing the root.
2. Assume the root, c , is the midpoint of this interval: $c = (a + b)/2$.
3. Evaluate $f(a)$ and $f(c)$.
 - (a) If $f(a) < 0$ and $f(c) > 0$ (i.e. they have different sign), we know the true root is contained in the interval $[a, c]$. Therefore, we update our interval so that a remains the same, but b is updated to be c .
 - (b) If $f(a)$ and $f(c)$ have the same sign (either both negative or both positive), we know the true root must be contained in the interval $[c, b]$. Therefore, we update our interval so that b remains the same, but a is updated to be c .
4. Repeating steps 2 and 3, the interval $[a, b]$ will keep shrinking. Once the difference $(b - a)$ is small enough, we say that the estimate of the root has **converged** to the true root, within some **tolerance** (which we denote as TOL). Therefore, if we predetermine that, at most, the root must be restricted to an interval of length TOL, we will keep repeating steps 2 and 3 until $(b - a) < \text{TOL}$.

In some cases, the difference $(b - a)$ may never decrease below TOL, or take too long to decrease below TOL. Therefore, we also define the **maximum number of iterations** (i_{\max}) so that the algorithm does not keep iterating forever, or for too long of a time [1, 2].

There are two basic algorithms for implementing the bisection method. The first implementation, shown in Algorithm 1 below, does *not* store the result of each iteration. On the other hand, the second implementation, shown in Algorithm 2, *does* store the result of each iteration. `bisection_method` implements both of these algorithms.

Since Algorithm 2 first needs to preallocate a potentially huge array to store all of the intermediate solutions, Algorithm 1 is significantly faster. Even if i_{\max} (determines size of the preallocated array) is set to be a small number (for example, 10), Algorithm 1 is still faster. The reason we still consider and implement Algorithm 2 is so that convergence studies may be performed.

Algorithm 1:

Bisection method [fast implementation].

Given:

- $f(x)$ - function
- a - lower bound for initial guess of interval with root
- b - upper bound for initial guess of interval with root
- TOL - tolerance
- i_{\max} - maximum number of iterations

Procedure:

1. Initial guess for root.

$$c = \frac{a + b}{2}$$

2. Initialize x_{new} so its scope will not be limited to within the while loop.

$$x_{\text{new}} = 0$$

3. Initialize the loop index.

$$i = 1$$

4. Find the root using the bisection method.

```

while  $((b - a) > \text{TOL})$  and  $(i < i_{\max})$ 
    (a) Update interval.

        if  $f(c) = 0$ 
            | Stop
        else if  $\text{sgn}[f(c)] = \text{sgn}[f(a)]$ 
            |  $a = c$ 
        else
            |  $b = c$ 
        end

    (b) Update root estimate.

        
$$c = \frac{a + b}{2}$$


    (c) Increment loop index.

         $i = i + 1$ 
end

```

Return:

- $\text{root} = c$ - converged root

Algorithm 2:

Bisection method [storing intermediate root estimates].

Given:

- $f(x)$ - function
- a - lower bound for initial guess of interval with root
- b - upper bound for initial guess of interval with root
- TOL - tolerance
- i_{\max} - maximum number of iterations

Procedure:

1. Preallocate $\mathbf{x} \in \mathbb{R}^{i_{\max}}$ to store the estimates of the root at each iteration.
2. Store the initial guess for the root in the first element of \mathbf{x} .

$$x_1 = \frac{a + b}{2}$$

3. Initialize the loop index.

$$i = 1$$

4. Find the root using the bisection method.

```

while  $((b - a) > \text{TOL})$  and  $(i < i_{\max})$ 

```

```

    (a) Update interval.

        if  $f(x_i) = 0$ 
        |   Stop
        else if  $\text{sgn}[f(x_i)] = \text{sgn}[f(a)]$ 
        |    $a = x_i$ 
        else
        |    $b = x_i$ 
        end

    (b) Update root estimate.

        
$$x_{i+1} = \frac{a+b}{2}$$


    (c) Increment loop index.

        
$$i = i + 1$$


end
```

Return:

- **x** - vector where the first element is the initial guess for the root, the subsequent elements are the intermediate root estimates, and the final element is the converged root

REFERENCES

- [1] *Bisection method*. Wikipedia. https://en.wikipedia.org/wiki/Bisection_method (accessed: February 7, 2020).
- [2] James Hateley. *Nonlinear Equations*. MATH 3620 Course Reader (Vanderbilt University). 2019.