

Gaussian Elimination

MATLAB Implementation

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gaussian_elimination

Solves the linear system $\mathbf{Ax} = \mathbf{b}$ for \mathbf{x} using Gaussian elimination with partial pivoting.

Syntax

`x = gaussian_elimination(A,b)`

Description

`x = gaussian_elimination(A,b)` solves the linear system $\mathbf{Ax} = \mathbf{b}$ for $\mathbf{x} \in \mathbb{R}^n$, where $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{b} \in \mathbb{R}^n$.

Examples

Example 1

Solve the linear system $\mathbf{Ax} = \mathbf{b}$ for \mathbf{x} , where

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 5 \\ 1 & 1 & -3 \\ 2 & 4 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 10 \\ -2 \\ 1 \end{bmatrix}$$

■ SOLUTION

Entering \mathbf{A} and \mathbf{b} into MATLAB,

```
% defines matrix A
A = [2,-1, 5;
     1, 1,-3;
     2, 4, 1];

% defines vector b
b = [10;
     -2;
     1];
```

To solve the linear system for \mathbf{x} ,

```
x = gaussian_elimination(A,b)
```

This yields the result

```
x =
     2
    -1
     1
```

Links

MATLAB® Central's File Exchange:

https://www.mathworks.com/matlabcentral/fileexchange/89306-gaussian-elimination-gaussian_elimination

GitHub®:

https://github.com/tamaskis/gaussian_elimination-MATLAB

Gaussian Elimination

Gaussian elimination can be used to solve the linear system

$$\mathbf{Ax} = \mathbf{b}$$

where $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{x}, \mathbf{b} \in \mathbb{R}^{n \times 1}$. In Algorithms 1 and 2 below, we will be referring to the rows and columns of \mathbf{A} as well as to the elements of \mathbf{x} at different points. Here are the conventions we use:

$$\mathbf{A} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_j \quad \dots \quad \mathbf{a}_{n-1} \quad \mathbf{a}_n] = \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \vdots \\ \bar{a}_i \\ \vdots \\ \bar{a}_{n-1} \\ \bar{a}_n \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Algorithms 1 and 2¹ below implement Gaussian elimination with partial pivoting and are adapted from Algorithm 6.2 in [1, pp. 374–375]. Note that ε is used to denote the machine epsilon.

¹ Note that together, these algorithms represent a single algorithm. However, this single algorithm is split into two due to its length and the limitations of typesetting algorithms in \LaTeX .

Algorithm 1: Gaussian elimination with partial pivoting (part 1).

1 Given: \mathbf{A} , \mathbf{x} , \mathbf{b}

// determines n (where $\mathbf{A} \in \mathbb{R}^{n \times n}$)

2 $n = \text{size}(\mathbf{A}, 1)$

// redefines A by augmenting the original A with \mathbf{b}

3 $\mathbf{A} = [\mathbf{A} \ \mathbf{b}]$

// initializes boolean variable to keep track if matrix is singular

4 singular = false

// elimination process

5 for $i = 1$ **to** $n - 1$ **do**

// determines pivot row

6 Define the vector $\mathbf{p} = (i, i + 1, \dots, n - 1, n)^T$.

7 Remove the elements p_k of \mathbf{p} where $A_{k,i} = 0$.

8 $p = \min(\mathbf{p})$

// exit the loop if all possible pivots in the i^{th} column are zero (to machine precision) because that would result in a singular matrix

9 if $\max |\mathbf{a}_i| \leq \varepsilon$ **then**

10 | singular = true

11 | Exit loop – the matrix is singular and no unique solution exists.

12 end

// switches pivot row with the i^{th} row if $p \neq i$

13 if $p \neq i$ **then**

14 | Store the i^{th} row of \mathbf{A} as \bar{a}_i .

15 | Store the p^{th} row of \mathbf{A} as \bar{a}_p .

16 | Set the i^{th} row of \mathbf{A} to \bar{a}_p .

17 | Set the p^{th} row of \mathbf{A} to \bar{a}_i .

18 end

// elementary row operation

19 for $j = i + 1$ **to** n **do**

20 | $\bar{a}_j = \bar{a}_j - \left(\frac{A_{j,i}}{A_{i,i}} \right) \bar{a}_i$

21 end

22 end

Algorithm 2: Gaussian elimination with partial pivoting (part 2).

```
// determines if  $\mathbf{A}$  is singular (if the bottom right element of  $\mathbf{A}$  is 0 (to within machine precision), then the entire bottom row is 0 and  $\mathbf{A}$  is singular)
1 if  $|A_{n,n}| \leq \varepsilon$  then
2   | singular = true
3 end

4 Preallocate/initialize the solution vector  $\mathbf{x} \in \mathbb{R}^{n \times 1}$ .

// perform backward substitution to solve  $\mathbf{Ax} = \mathbf{b}$  if  $\mathbf{A}$  is nonsingular
5 if !(singular) then
6   |  $x_n = \frac{A_{n,n+1}}{A_{n,n}}$ 
7   | for  $i = n - 1$  to 1 by  $-1$  do
8     |  $S = 0$  for  $i + 1$  to  $n$  do
9       |  $S = S + A_{i,j}x_j$ 
10    | end
11    |  $x_i = \frac{A_{i,n+1} - S}{A_{i,i}}$ 
12  | end
13 end
```

References

- [1] Richard L. Burden and J. Douglas Faires. *Numerical Analysis*. 9th. Boston, MA: Brooks/Cole, Cengage Learning, 2011.