
Gaussian Elimination

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1 GAUSSIAN ELIMINATION

Gaussian elimination can be used to solve the linear system

$$\mathbf{Ax} = \mathbf{b}$$

where $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{x}, \mathbf{b} \in \mathbb{R}^n$. In Algorithm 1 below, we will be referring to the rows and columns of \mathbf{A} as well as to the elements of \mathbf{x} at different points. Here are the conventions we use:

$$\mathbf{A} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_j \quad \dots \quad \mathbf{a}_{n-1} \quad \mathbf{a}_n] = \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \vdots \\ \bar{a}_i \\ \vdots \\ \bar{a}_{n-1} \\ \bar{a}_n \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Algorithm 1 below implements Gaussian elimination with partial pivoting and is adapted from Algorithm 6.2 in [1, pp. 374–375]. Note that ε is used to denote the machine epsilon.

Algorithm 1: gaussian_elimination

Gaussian elimination with partial pivoting.

Given:

- $\mathbf{A} \in \mathbb{R}^{n \times n}$ - matrix
- $\mathbf{b} \in \mathbb{R}^n$ - vector

Note:

- \mathbf{A} and \mathbf{b} define the linear system $\mathbf{Ax} = \mathbf{b}$.

Procedure:

1. Determine n (where $\mathbf{A} \in \mathbb{R}^{n \times n}$).
2. Redefine \mathbf{A} by augmenting the original \mathbf{A} with \mathbf{b} .

$$\mathbf{A} = [\mathbf{A} \quad \mathbf{b}]$$

3. Initialize a boolean variable to keep track of if the matrix is singular.

$$\text{singular} = \text{false}$$

4. Perform the elimination process.

$$\text{for } i = 1 \text{ to } n - 1$$

- (a) Determine the pivot row as follows:
- Define the vector $\mathbf{p} = (i, i + 1, \dots, n - 1, n)^T$.
 - Remove the elements p_k of \mathbf{p} where $A_{k,i} = 0$.
 - Find the index (p) of the pivot row.

$$p = \min(\mathbf{p})$$

- (b) Exit the loop if all possible pivots in the i^{th} column are zero (to within machine precision) because that would result in a singular matrix.

```

if  $\max |\mathbf{a}_i| \leq \varepsilon$ 
    singular = true
    Exit loop – the matrix is singular and no unique solution exists.
end

```

- (c) Switch pivot row with the i^{th} row if $p \neq i$.

```

if  $p \neq i$ 
    Store the  $i^{\text{th}}$  row of  $\mathbf{A}$  as  $\bar{a}_i$ .
    Store the  $p^{\text{th}}$  row of  $\mathbf{A}$  as  $\bar{a}_p$ .
    Set the  $i^{\text{th}}$  row of  $\mathbf{A}$  to  $\bar{a}_p$ .
    Set the  $p^{\text{th}}$  row of  $\mathbf{A}$  to  $\bar{a}_i$ .
end

```

- (d) Perform the elementary row operation.

```

for  $j = i + 1$  to  $n$ 
     $\bar{a}_j = \bar{a}_j - \left( \frac{A_{j,i}}{A_{i,i}} \right) \bar{a}_i$ 
end

```

end

5. Determine if \mathbf{A} is singular (if the bottom right element of \mathbf{A} is 0 (to within machine precision), then the entire bottom row is 0 and \mathbf{A} is singular).

```

if  $|A_{n,n}| \leq \varepsilon$ 
    singular = true
end

```

6. Preallocate $\mathbf{x} \in \mathbb{R}^n$ to store the solution.
 7. Perform backward substitution to solve $\mathbf{Ax} = \mathbf{b}$ if \mathbf{A} is nonsingular.

if !(singular)

```
      |  $x_n = \frac{A_{n,n+1}}{A_{n,n}}$   
      | for  $i = n - 1$  to 1 by -1  
      |   |  $S = 0$   
      |   | for  $j = i + 1$  to  $n$   
      |   |   |  $S = S + A_{i,j}x_j$   
      |   |   | end  
      |   |  $x_i = \frac{A_{i,n+1} - S}{A_{i,i}}$   
      |   | end  
      | end
```

Return:

- $\mathbf{x} \in \mathbb{R}^n$ - solution of the linear system $\mathbf{Ax} = \mathbf{b}$

REFERENCES

- [1] Richard L. Burden and J. Douglas Faires. *Numerical Analysis*. 9th. Boston, MA: Brooks/Cole, Cengage Learning, 2011.