Gaussian Elimination

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CONTENTS

1 GAUSSIAN ELIMINATION

Gaussian elimination can be used to solve the linear system

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

where $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{x}, \mathbf{b} \in \mathbb{R}^n$. In Algorithm 1 below, we will be referring to the rows and columns of $\mathbf{A}$ as well as to the elements of $\mathbf{x}$ at different points. Here are the conventions we use:

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_j & \dots & \mathbf{a}_{n-1} & \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \vdots \\ \bar{a}_i \\ \vdots \\ \bar{a}_{n-1} \\ \bar{a}_n \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Algorithm 1 below implements Gaussian elimination with partial pivoting and is adapted from Algorithm 6.2 in [1, pp. 374–375]. Note that ε is used to denote the machine epsilon.

Algorithm 1: `gaussian_elimination`
Gaussian elimination with partial pivoting.

Given:
- $\mathbf{A} \in \mathbb{R}^{n \times n}$ - matrix
- $\mathbf{b} \in \mathbb{R}^n$ - vector

Note:
- $\mathbf{A}$ and $\mathbf{b}$ define the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$.

Procedure:
1. Determine n (where $\mathbf{A} \in \mathbb{R}^{n \times n}$).
2. Redefine $\mathbf{A}$ by augmenting the original $\mathbf{A}$ with $\mathbf{b}$.

$$\mathbf{A} = [\mathbf{A}\ \mathbf{b}]$$

3. Initialize a boolean variable to keep track of if the matrix is singular.

$$\text{singular} = \text{false}$$

4. Perform the elimination process.

$$\textbf{for } i = 1 \textbf{ to } n-1$$

(a) Determine the pivot row as follows:
   i. Define the vector $\mathbf{p} = (i, i+1, ..., n-1, n)^T$.
   ii. Remove the elements p_k of $\mathbf{p}$ where $A_{k,i} = 0$.
   iii. Find the index (p) of the pivot row.

$$p = \min(\mathbf{p})$$

(b) Exit the loop if all possible pivots in the i^{th} column are zero (to within machine precision) because that would result in a singular matrix.

if $\max |\mathbf{a}_i| \leq \varepsilon$
   $\text{singular} = \text{true}$
   Exit loop – the matrix is singular and no unique solution exists.
end

(c) Switch pivot row with the i^{th} row if $p \neq i$.

if $p \neq i$
   Store the i^{th} row of $\mathbf{A}$ as $\bar{a}_i$.
   Store the p^{th} row of $\mathbf{A}$ as $\bar{a}_p$.
   Set the i^{th} row of $\mathbf{A}$ to $\bar{a}_p$.
   Set the p^{th} row of $\mathbf{A}$ to $\bar{a}_i$.
end

(d) Perform the elementary row operation.

for $j = i+1$ **to** n

$$\bar{a}_j = \bar{a}_j - \left(\frac{A_{j,i}}{A_{i,i}}\right)\bar{a}_i$$

end

end

5. Determine if $\mathbf{A}$ is singular (if the bottom right element of $\mathbf{A}$ is 0 (to within machine precision), then the entire bottom row is 0 and $\mathbf{A}$ is singular).

if $|A_{n,n}| \leq \varepsilon$
   $\text{singular} = \text{true}$
end

6. Preallocate $\mathbf{x} \in \mathbb{R}^n$ to store the solution.
7. Perform backward substitution to solve $\mathbf{A}\mathbf{x} = \mathbf{b}$ if $\mathbf{A}$ is nonsingular.

if !(singular)

$x_n = \frac{A_{n,n+1}}{A_{n,n}}$
for $i = n-1$ **to** 1 **by** -1
    $S = 0$
    for $j = i+1$ **to** n
        $S = S + A_{i,j}x_j$
    end
    $x_i = \frac{A_{i,n+1} - S}{A_{i,i}}$
end
end

Return:

- $\mathbf{x} \in \mathbb{R}^n$ - solution of the linear system $\mathbf{Ax} = \mathbf{b}$

REFERENCES

[1] Richard L. Burden and J. Douglas Faires. *Numerical Analysis*. 9th. Boston, MA: Brooks/Cole, Cengage Learning, 2011.