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# Gaussian Elimination

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# 1 GAUSSIAN ELIMINATION

Gaussian elimination can be used to solve the linear system

$$\mathbf{Ax} = \mathbf{b}$$

where  $\mathbf{A} \in \mathbb{R}^{n \times n}$  and  $\mathbf{x}, \mathbf{b} \in \mathbb{R}^n$ . In Algorithm 1 below, we will be referring to the rows and columns of  $\mathbf{A}$  as well as to the elements of  $\mathbf{x}$  at different points. Here are the conventions we use:

$$\mathbf{A} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_j \quad \dots \quad \mathbf{a}_{n-1} \quad \mathbf{a}_n] = \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \vdots \\ \bar{a}_i \\ \vdots \\ \bar{a}_{n-1} \\ \bar{a}_n \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Algorithm 1 below implements Gaussian elimination with partial pivoting and is adapted from Algorithm 6.2 in [1, pp. 374–375]. Note that  $\varepsilon$  is used to denote the machine epsilon.

## Algorithm 1: gaussian\_elimination

Gaussian elimination with partial pivoting.

### Given:

- $\mathbf{A} \in \mathbb{R}^{n \times n}$  - matrix
- $\mathbf{b} \in \mathbb{R}^n$  - vector

### Note:

- $\mathbf{A}$  and  $\mathbf{b}$  define the linear system  $\mathbf{Ax} = \mathbf{b}$ .

### Procedure:

1. Determine  $n$  (where  $\mathbf{A} \in \mathbb{R}^{n \times n}$ ).
2. Redefine  $\mathbf{A}$  by augmenting the original  $\mathbf{A}$  with  $\mathbf{b}$ .

$$\mathbf{A} = [\mathbf{A} \quad \mathbf{b}]$$

3. Initialize a boolean variable to keep track of if the matrix is singular.

$$\text{singular} = \text{false}$$

4. Perform the elimination process.

$$\text{for } i = 1 \text{ to } n - 1$$

- (a) Determine the pivot row as follows:
- Define the vector  $\mathbf{p} = (i, i + 1, \dots, n - 1, n)^T$ .
  - Remove the elements  $p_k$  of  $\mathbf{p}$  where  $A_{k,i} = 0$ .
  - Find the index ( $p$ ) of the pivot row.

$$p = \min(\mathbf{p})$$

- (b) Exit the loop if all possible pivots in the  $i^{\text{th}}$  column are zero (to within machine precision) because that would result in a singular matrix.

```

if  $\max |\mathbf{a}_i| \leq \varepsilon$ 
    singular = true
    Exit loop – the matrix is singular and no unique solution exists.
end

```

- (c) Switch pivot row with the  $i^{\text{th}}$  row if  $p \neq i$ .

```

if  $p \neq i$ 
    Store the  $i^{\text{th}}$  row of  $\mathbf{A}$  as  $\bar{a}_i$ .
    Store the  $p^{\text{th}}$  row of  $\mathbf{A}$  as  $\bar{a}_p$ .
    Set the  $i^{\text{th}}$  row of  $\mathbf{A}$  to  $\bar{a}_p$ .
    Set the  $p^{\text{th}}$  row of  $\mathbf{A}$  to  $\bar{a}_i$ .
end

```

- (d) Perform the elementary row operation.

```

for  $j = i + 1$  to  $n$ 
     $\bar{a}_j = \bar{a}_j - \left( \frac{A_{j,i}}{A_{i,i}} \right) \bar{a}_i$ 
end

```

**end**

5. Determine if  $\mathbf{A}$  is singular (if the bottom right element of  $\mathbf{A}$  is 0 (to within machine precision), then the entire bottom row is 0 and  $\mathbf{A}$  is singular).

```

if  $|A_{n,n}| \leq \varepsilon$ 
    singular = true
end

```

6. Preallocate  $\mathbf{x} \in \mathbb{R}^n$  to store the solution.  
 7. Perform backward substitution to solve  $\mathbf{A}\mathbf{x} = \mathbf{b}$  if  $\mathbf{A}$  is nonsingular.

**if** !(singular)

```
      |  
      | $x_n = \frac{A_{n,n+1}}{A_{n,n}}$   
      |for  $i = n - 1$  to 1 by -1  
      |  
      | $S = 0$   
      |for  $j = i + 1$  to  $n$   
      | $S = S + A_{i,j}x_j$   
      |end  
      | $x_i = \frac{A_{i,n+1} - S}{A_{i,i}}$   
      |end  
end
```

**Return:**

- $\mathbf{x} \in \mathbb{R}^n$  - solution of the linear system  $\mathbf{Ax} = \mathbf{b}$

## REFERENCES

- [1] Richard L. Burden and J. Douglas Faires. *Numerical Analysis*. 9<sup>th</sup>. Boston, MA: Brooks/Cole, Cengage Learning, 2011.