Magnitude and Phase of a Linear System

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1 MAGNITUDE AND PHASE OF A LINEAR SYSTEM

1.1 Complex Numbers

A complex number, z, can be written as

$$z = x + jy = \mathrm{Re}\,(z) + j\,\mathrm{Im}\,(z)$$

In the complex plane, this can be represented as a vector $z = (x, y)^T$, where this vector has magnitude $|z|$ and phase[1] (i.e. direction) $\angle z$.

$$|z| = \sqrt{x^2 + y^2}$$

$$\angle z = \mathrm{arctan2}\,(y, x)$$

1.2 Continuous-Time Transfer Functions

A continuous-time transfer function, $H(s)$, evaluates to a complex number.

$$H(s) = \mathrm{Re}\,[H(s)] + j\,\mathrm{Im}\,[H(s)]$$

Thus,

$$|H(s)| = \sqrt{\mathrm{Re}\,[H(s)]^2 + \mathrm{Im}\,[H(s)]^2}$$

$$\angle H(s) = \mathrm{arctan2}\,(\mathrm{Im}\,[H(s)], \mathrm{Re}\,[H(s)])$$

1.3 Discrete-Time Transfer Functions

Finding the magnitude and phase of a discrete-time transfer function can be done in an identical matter as continuous-time transfer functions:

$$|G(z)| = \sqrt{\mathrm{Re}\,[G(z)]^2 + \mathrm{Im}\,[G(z)]^2}$$

$$\angle G(z) = \mathrm{arctan2}\,(\mathrm{Im}\,[G(z)], \mathrm{Re}\,[G(z)])$$

1.4 State-Space Representation

Consider the case where we have a linear system expressed in the state-space form

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x} + Du\end{aligned}$$

where $u, y, D \in \mathbb{R}$. This can readily be converted to a continuous-time transfer function $G(s)$ in MATLAB using the `tf2ss` function. However, we can simply define the linear system using

```
sys = ss(A,B,C,D)
```

and then use the `evalfr` function to evaluate the system at a specific point s in the frequency domain.

[1] Note that the phase is often defined as

$$\angle z = \arctan\left(\frac{y}{x}\right)$$

This will not always return the correct phase since the more general definition is

$$\tan\,(\angle z) = \frac{y}{x}$$