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# Magnitude and Phase of a Linear System

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# 1 MAGNITUDE AND PHASE OF A LINEAR SYSTEM

## 1.1 Complex Numbers

A complex number,  $z$ , can be written as

$$z = x + jy = \operatorname{Re}(z) + j \operatorname{Im}(z)$$

In the complex plane, this can be represented as a vector  $z = (x, y)^T$ , where this vector has magnitude  $|z|$  and phase<sup>1</sup> (i.e. direction)  $\angle z$ .

$$|z| = \sqrt{x^2 + y^2}$$

$$\angle z = \operatorname{arctan2}(y, x)$$

## 1.2 Continuous-Time Transfer Functions

A continuous-time transfer function,  $G(s)$ , evaluates to a complex number.

$$G(s) = \operatorname{Re}[G(s)] + j \operatorname{Im}[G(s)]$$

Thus,

$$|G(s)| = \sqrt{\operatorname{Re}[G(s)]^2 + \operatorname{Im}[G(s)]^2}$$

$$\angle G(s) = \operatorname{arctan2}(\operatorname{Im}[G(s)], \operatorname{Re}[G(s)])$$

## 1.3 Discrete-Time Transfer Functions

Finding the magnitude and phase of a discrete-time transfer function can be done in an identical matter as continuous-time transfer functions:

$$|G(z)| = \sqrt{\operatorname{Re}[G(z)]^2 + \operatorname{Im}[G(z)]^2}$$

$$\angle G(z) = \operatorname{arctan2}(\operatorname{Im}[G(z)], \operatorname{Re}[G(z)])$$

## 1.4 State-Space Representation

Consider the case where we have a linear system expressed in the state-space form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x} + Du$$

where  $u, y, D \in \mathbb{R}$ . This can readily be converted to a continuous-time transfer function  $G(s)$  in MATLAB using the `tf2ss` function. However, we can simply define the linear system using

$$\text{sys} = \text{ss}(\mathbf{A}, \mathbf{B}, \mathbf{C}, D)$$

and then use the `evalfr` function to evaluate the system at a specific point  $s$  in the frequency domain.

<sup>1</sup> Note that the phase is often defined as

$$\angle z = \arctan\left(\frac{y}{x}\right)$$

This will not always return the correct phase since the more general definition is

$$\tan(\angle z) = \frac{y}{x}$$