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# Newton's Method

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# 1 NEWTON'S METHOD

**Newton's method** is a technique used to find the root (based on an initial guess<sup>1</sup>  $x_0$ ) of a *differentiable*, univariate function  $f(x)$ . The equation of the tangent line to the curve  $y = f(x)$  at  $x = x_0$  is

$$y = f'(x_0)(x - x_0) + f(x_0)$$

where  $f'(x_0)$  is the derivative of  $f(x)$  evaluated at  $x_0$ . The  $x$ -intercept of this tangent line,  $x = x_1$ , can be solved by setting  $y = 0$ .

$$0 = f'(x_0)(x_1 - x_0) + f(x_0)$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$x_1$  is an updated estimate of the root of  $f(x)$ . To keep refining our estimate, we can keep iterating through this procedure using Eq. (1).

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad (1)$$

So how do we actually use Eq. (1)? Given an initial guess  $x_0$ , we can keep coming up with new estimates of the root. But how do we know when to stop? To resolve this issue, we define the **error**<sup>2</sup> as

$$\varepsilon = |x_{i+1} - x_i| \quad (2)$$

Once  $\varepsilon$  is small enough, we say that the estimate of the root has **converged** to the true root, within some **tolerance** (which we denote as TOL). Therefore, if we predetermine that, at most, we can *tolerate* an error of TOL, then we will keep iterating Eq. (1) until  $\varepsilon < \text{TOL}$ . In some cases, the error may never decrease below TOL, or take too long to decrease to below TOL. Therefore, we also define the **maximum number of iterations** ( $i_{\max}$ ) so that the algorithm does not keep iterating forever, or for too long of a time [1, 2].

There are two basic algorithms for implementing Newton's method. The first implementation, shown in Algorithm 1 below, does *not* store the result of each iteration. On the other hand, the second implementation, shown in Algorithm 2, *does* store the result of each iteration. `newtons_method` implements both of these algorithms.

Since Algorithm 2 first needs to preallocate a potentially huge array to store all of the intermediate solutions, Algorithm 1 is significantly faster. Even if  $i_{\max}$  (determines size of the preallocated array) is set to be a small number (for example, 10), Algorithm 1 is still faster. The reason we still consider and implement Algorithm 2 is so that convergence studies may be performed.

## Algorithm 1:

Newton's method [fast implementation].

### Given:

- $f(x)$  - function
- $f'(x)$  - derivative of  $f(x)$
- $x_0$  - initial guess for root
- TOL - tolerance
- $i_{\max}$  - maximum number of iterations

<sup>1</sup> Often, a function  $f(x)$  will have multiple roots. Therefore, Newton's method typically finds the root closest to the initial guess  $x_0$ . However, this is not always the case; the algorithm depends heavily on the derivative of  $f(x)$ , which, depending on its form, may cause it to converge on a root further from  $x_0$ .

<sup>2</sup> Note that  $\varepsilon$  is an *approximate* error. The motivation behind using this definition of  $\varepsilon$  is that as  $i$  gets large (i.e.  $i \rightarrow \infty$ ),  $x_{i+1} - x_i$  approaches  $x_{i+1} - x^*$  (assuming this sequence is convergent), where  $x^*$  is the true root (and therefore  $x_{i+1} - x^*$  represents the *exact* error).

**Procedure:**

1. Initialize the error so that the loop will be entered.

$$\varepsilon = (2)(\text{TOL})$$

2. Manually set the root estimate at the first iteration based on the initial guess.

$$x_{\text{old}} = x_0$$

3. Initialize  $x_{\text{new}}$  so its scope will not be limited to within the while loop.

$$x_{\text{new}} = 0$$

4. Initialize the loop index.

$$i = 1$$

5. Find the root using Newton's method.

**while** ( $\varepsilon > \text{TOL}$ ) **and** ( $i < i_{\text{max}}$ )

(a) Update root estimate.

$$x_{\text{new}} = x_{\text{old}} - \frac{f(x_{\text{old}})}{f'(x_{\text{old}})}$$

(b) Calculate error.

$$\varepsilon = |x_{\text{new}} - x_{\text{int}}|$$

(c) Store the current root estimate for the next iteration.

$$x_{\text{old}} = x_{\text{new}}$$

(d) Increment loop index.

$$i = i + 1$$

**end**

**Return:**

- root =  $x_{\text{new}}$  - converged root

**Algorithm 2:**

Newton's method [storing intermediate root estimates].

**Given:**

- $f(x)$  - function
- $f'(x)$  - derivative of  $f(x)$
- $x_0$  - initial guess for root
- TOL - tolerance
- $i_{\text{max}}$  - maximum number of iterations

**Procedure:**

1. Initialize the error so that the loop will be entered.

$$\varepsilon = (2)(\text{TOL})$$

2. Preallocate  $\mathbf{x} \in \mathbb{R}^{i_{\max}}$  to store the estimates of the root at each iteration.
3. Manually set the root estimate at the first iteration based on the initial guess (note that  $x_1$  is the first element of  $\mathbf{x}$ , while  $x_0$  is the input initial guess).

$$x_1 = x_0$$

4. Initialize the loop index.

$$i = 1$$

5. Find the root using Newton's method.

**while** ( $\varepsilon > \text{TOL}$ ) **and** ( $i < i_{\max}$ )

(a) Update root estimate.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

(b) Calculate error.

$$\varepsilon = |x_{i+1} - x_i|$$

(c) Increment loop index.

$$i = i + 1$$

**end**

**Return:**

- $\mathbf{x}$  - vector where the first element is the initial guess for the root, the subsequent elements are the intermediate root estimates, and the final element is the converged root

## REFERENCES

- [1] Richard L. Burden and J. Douglas Faires. “Newton’s Method and Its Extensions”. In: *Numerical Analysis*. 9th ed. Boston, MA: Brooks/Cole, Cengage Learning, 2011. Chap. 2.3, pp. 67–78.
- [2] *Newton’s method*. Wikipedia. [https://en.wikipedia.org/wiki/Newton%27s\\_method](https://en.wikipedia.org/wiki/Newton%27s_method) (accessed: June 10, 2020).