Newton's Method

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Contents

1 NEWTON'S METHOD

Newton's method is a technique used to find the root (based on an initial guess[1] x_0) of a *differentiable*, univariate function $f(x)$. The equation of the tangent line to the curve $y = f(x)$ at $x = x_0$ is

$$y = f'(x_0)(x - x_0) + f(x_0)$$

where $f'(x_0)$ is the derivative of $f(x)$ evaluated at x_0. The x-intercept of this tangent line, $x = x_1$, can be solved by setting $y = 0$.

$$0 = f'(x_0)(x_1 - x_0) + f(x_0)$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

x_1 is an updated estimate of the root of $f(x)$. To keep refining our estimate, we can keep iterating through this procedure using Eq. (1).

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}} \tag{1}$$

So how do we actually use Eq. (1)? Given an initial guess x_0, we can keep coming up with new estimates of the root. But how do we know when to stop? To resolve this issue, we define the **error**[2] as

$$\boxed{\varepsilon = |x_{i+1} - x_i|} \tag{2}$$

Once ε is small enough, we say that the estimate of the root has **converged** to the true root, within some **tolerance** (which we denote as TOL). Therefore, if we predetermine that, at most, we can *tolerate* an error of TOL, then we will keep iterating Eq. (1) until $\varepsilon < \text{TOL}$. In some cases, the error may never decrease below TOL, or take too long to decrease to below TOL. Therefore, we also define the **maximum number of iterations** (i_{max}) so that the algorithm does not keep iterating forever, or for too long of a time [1, 2].

There are two basic algorithms for implementing Newton's method. The first implementation, shown in Algorithm 1 below, does *not* store the result of each iteration. On the other hand, the second implementation, shown in Algorithm 2, *does* store the result of each iteration. `newtons_method` implements both of these algorithms.

Since Algorithm 2 first needs to preallocate a potentially huge array to store all of the intermediate solutions, Algorithm 1 is significantly faster. Even if i_{max} (determines size of the preallocated array) is set to be a small number (for example, 10), Algorithm 1 is still faster. The reason we still consider and implement Algorithm 2 is so that convergence studies may be performed.

Algorithm 1:
Newton's method (fast implementation).

Given:
- $f(x)$ - function
- $f'(x)$ - derivative of $f(x)$
- x_0 - initial guess for root
- TOL - tolerance
- i_{max} - maximum number of iterations

[1] Often, a function $f(x)$ will have multiple roots. Therefore, Newton's method typically finds the root closest to the initial guess x_0. *However*, this is not always the case; the algorithm depends heavily on the derivative of $f(x)$, which, depending on its form, may cause it to converge on a root further from x_0.

[2] Note that ε is an *approximate* error. The motivation behind using this definition of ε is that as i gets large (i.e. $i \to \infty$), $x_{i+1} - x_i$ approaches $x_{i+1} - x^*$ (*assuming* this sequence is convergent), where x^* is the true root (and therefore $x_{i+1} - x^*$ represents the *exact* error).

Procedure:

1. Initialize the error so that the loop will be entered.

   $$\varepsilon = (2)(\text{TOL})$$

2. Manually set the root estimate at the first iteration based on the initial guess.

   $$x_{\text{old}} = x_0$$

3. Initialize x_{new} so its scope will not be limited to within the while loop.

   $$x_{\text{new}} = 0$$

4. Initialize the loop index.

   $$i = 1$$

5. Find the root using Newton's method.

   while $(\varepsilon > \text{TOL})$ **and** $(i < i_{\max})$

   (a) Update root estimate.

   $$x_{\text{new}} = x_{\text{old}} - \frac{f(x_{\text{old}})}{f'(x_{\text{old}})}$$

   (b) Calculate error.

   $$\varepsilon = |x_{\text{new}} - x_{\text{int}}|$$

   (c) Store the current root estimate for the next iteration.

   $$x_{\text{old}} = x_{\text{new}}$$

   (d) Increment loop index.

   $$i = i + 1$$

   end

Return:

- root $= x_{\text{new}}$ - converged root

Algorithm 2:

Newton's method (storing intermediate root estimates).

Given:

- $f(x)$ - function
- $f'(x)$ - derivative of $f(x)$
- x_0 - initial guess for root
- TOL - tolerance
- $i_{\max}$ - maximum number of iterations

Procedure:

1. Initialize the error so that the loop will be entered.

$$\varepsilon = (2)(\text{TOL})$$

2. Preallocate $\mathbf{x} \in \mathbb{R}^{i_{\max}}$ to store the estimates of the root at each iteration.
3. Manually set the root estimate at the first iteration based on the initial guess (note that x_1 is the first element of $\mathbf{x}$, while x_0 is the input initial guess).

$$x_1 = x_0$$

4. Initialize the loop index.

$$i = 1$$

5. Find the root using Newton's method.

while $(\varepsilon > \text{TOL})$ **and** $(i < i_{\max})$

   (a) Update root estimate.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

   (b) Calculate error.

$$\varepsilon = |x_{i+1} - x_i|$$

   (c) Increment loop index.

$$i = i + 1$$

end

Return:

- $\mathbf{x}$ - vector where the first element is the initial guess for the root, the subsequent elements are the intermediate root estimates, and the final element is the converged root

REFERENCES

[1] Richard L. Burden and J. Douglas Faires. “Newton’s Method and Its Extensions”. In: *Numerical Analysis*. 9th ed. Boston, MA: Brooks/Cole, Cengage Learning, 2011. Chap. 2.3, pp. 67–78.

[2] *Newton’s method*. Wikipedia. `https://en.wikipedia.org/wiki/Newton%27s_method` (accessed: June 10, 2020).