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# Secant Method

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# 1 SECANT METHOD

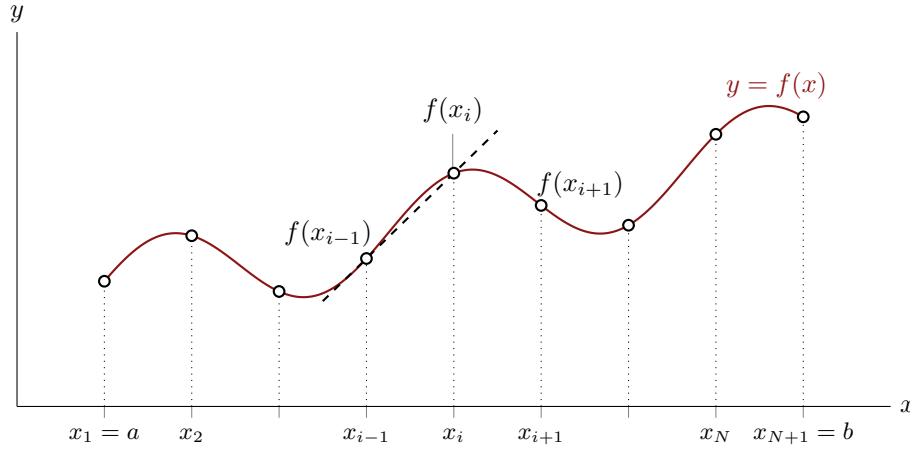
**Newton's method** is a root-finding technique that uses the derivative of a function to find its root<sup>1</sup>. Newton's method is defined iteratively as [2, Eq. (1)]

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad (1)$$

But what if we don't know  $f'(x)$ ? Then we need to approximate it using some numerical method. Specifically, for the secant method, we use the backward approximation of a derivative, given by Eq. (2) below.

$$f'(x_i) \approx \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \quad (2)$$

This approximation can be visualized using the finite difference stencil shown in Fig. 1. Substituting Eq. (2) into Eq.



**Figure 1:** Backward approximation.

(1),

$$\begin{aligned} x_{i+1} &= x_i - \left[ \frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})} \right] f(x_i) = \frac{[f(x_i) - f(x_{i-1})] x_i}{f(x_i) - f(x_{i-1})} - \frac{(x_i - x_{i-1}) f(x_i)}{f(x_i) - f(x_{i-1})} \\ &= \frac{x_i f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})} + \frac{x_{i-1} f(x_i) - x_i f(x_i)}{f(x_i) - f(x_{i-1})} = \frac{x_i f(x_i) - x_i f(x_i) + x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})} \\ &\boxed{x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}} \end{aligned} \quad (3)$$

Equation (3) iteratively defines the **secant method**, which can be essentially thought of as a finite difference approximation of Newton's method for finding the root of a univariate function (based on an initial guess<sup>2</sup>). But how do we actually use Eq. (3)? Given an initial guess  $x_0$ , we can keep coming up with new estimates of the root. But how do we know when to stop? To resolve this issue, we define the **error**<sup>3</sup> as

$$\boxed{\varepsilon = |x_{i+1} - x_i|} \quad (4)$$

<sup>1</sup> For a discussion/MATLAB implementation of Newton's method, see [2].

<sup>2</sup> Often, a function  $f(x)$  will have multiple roots. Therefore, the secant method typically finds the root closest to the initial guess  $x_0$ . However, this is not always the case; the algorithm depends heavily on the derivative of  $f(x)$ , which, depending on its form, may cause it to converge on a root further from  $x_0$ .

<sup>3</sup> Note that  $\varepsilon$  is an *approximate* error. The motivation behind using this definition of  $\varepsilon$  is that as  $i$  gets large (i.e.  $i \rightarrow \infty$ ),  $x_{i+1} - x_i$  approaches  $x_{i+1} - x^*$  (assuming this sequence is convergent), where  $x^*$  is the true root (and therefore  $x_{i+1} - x^*$  represents the *exact* error).

Once  $\varepsilon$  is small enough, we say that the estimate of the root has **converged** to the true root, within some **tolerance** (which we denote as TOL). Therefore, if we predetermine that, at most, we can *tolerate* an error of TOL, then we will keep iterating Eq. (3) until  $\varepsilon < \text{TOL}$ . In some cases, the error may never decrease below TOL, or take too long to decrease to below TOL. Therefore, we also define the **maximum number of iterations** ( $i_{\max}$ ) so that the algorithm does not keep iterating forever, or for too long of a time.

In any implementation, we first have to make an initial guess  $x_0$  for the root. Additionally, we need to set the root estimate at the second iteration (i.e.  $x_2$ ) to a value slightly different than  $x_0$  – otherwise, we will just have  $x_{i+1} = x_i$  for all  $i$  and the algorithm will never “get started” (we can think of this as “kick-starting” the algorithm)<sup>4</sup> [1, 3, 4].

There are two basic algorithms for implementing the secant method. The first implementation, shown in Algorithm 1 below, does *not* store the result of each iteration. On the other hand, the second implementation, shown in Algorithm 2, *does* store the result of each iteration. `secant_method` implements both of these algorithms.

Since Algorithm 2 first needs to preallocate a potentially huge array to store all of the intermediate solutions, Algorithm 1 is significantly faster. Even if  $i_{\max}$  (determines size of the preallocated array) is set to be a small number (for example, 10), Algorithm 1 is still faster. The reason we still consider and implement Algorithm 2 is so that convergence studies may be performed.

### Algorithm 1:

Secant method (fast implementation).

#### Given:

- $f(x)$  - function
- $x_0$  - initial guess for root
- TOL - tolerance
- $i_{\max}$  - maximum number of iterations

#### Procedure:

1. Initialize the error so that the loop will be entered.

$$\varepsilon = (2)(\text{TOL})$$

2. Manually set the root estimates at the first and second iterations based on the initial guess.

$$x_{\text{old}} = x_0$$

$$x_{\text{int}} = 1.01x_0$$

3. Initialize  $x_{\text{new}}$  so its scope will not be limited to within the while loop.

$$x_{\text{new}} = 0$$

4. Find the root using the secant method.

**while** ( $\varepsilon > \text{TOL}$ ) **and** ( $i < i_{\max}$ )

<sup>4</sup> The alternative way to view this is by recalling that the derivative approximation is given by

$$f'(x_i) \approx \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

If  $x_i = x_{i-1}$ , then the approximation to  $f'(x_i)$  will become undefined, resulting in an error.

```

(a) Update root estimate.

$$x_{\text{new}} = \frac{x_{\text{old}}f(x_{\text{int}}) - x_{\text{int}}f(x_{\text{old}})}{f(x_{\text{int}}) - f(x_{\text{old}})}$$

(b) Calculate error.

$$\varepsilon = |x_{\text{new}} - x_{\text{int}}|$$

(c) Store current and previous estimates for next iteration.

$$x_{\text{old}} = x_{\text{int}}$$


$$x_{\text{int}} = x_{\text{new}}$$

(d) Increment loop index.

$$i = i + 1$$

end

```

**Return:**

- root =  $x_{\text{new}}$  - converged root

**Algorithm 2:**

Secant method (storing intermediate root estimates).

**Given:**

- $f(x)$  - function
- $x_0$  - initial guess for root
- TOL - tolerance
- $i_{\text{max}}$  - maximum number of iterations

**Procedure:**

1. Initialize the error so that the loop will be entered.

$$\varepsilon = (2)(\text{TOL})$$

2. Preallocate  $\mathbf{x} \in \mathbb{R}^{i_{\text{max}}}$  to store the estimates of the root at each iteration.
3. Manually set the root estimates at the first and second iterations based on the initial guess.

$$x_1 = x_0$$

$$x_2 = 1.01x_0$$

4. Find the root using the secant method.

**while** ( $\varepsilon > \text{TOL}$ ) **and** ( $i < i_{\text{max}}$ )

(a) Update root estimate.

$$x_{i+1} = \frac{x_{i-1}f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$$

(b) Calculate error.

$$\varepsilon = |x_{i+1} - x_i|$$

(c) Increment loop index.

$$i = i + 1$$

end

**Return:**

- **x** - vector where the first element is the initial guess for the root, the subsequent elements are the intermediate root estimates, and the final element is the converged root

## REFERENCES

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- [2] Tamas Kis. *Newton's Method – MATLAB Implementation*. [https://github.com/tamaskis/newtons\\_method-MATLAB](https://github.com/tamaskis/newtons_method-MATLAB). 2021.
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