Secant Method

MATLAB Implementation

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Contents

secant_method

Calculates the root of a univariate function using the secant method.

Syntax

```
root = secant_method(f,x0)
root = secant_method(f,x0,TOL)
root = secant_method(f,x0,[],imax)
root = secant_method(f,x0,TOL,imax)
root = secant_method(__,'all')
```

Description

`root = secant_method(f,x0)` returns the root of a function $f(x)$ specified by the function handle `f`, where `x0` is an initial guess of the root. The default tolerance and maximum number of iterations are `TOL = 1e-12` and `imax = 1e6`, respectively.

`root = secant_method(f,x0,TOL)` returns the root of a function $f(x)$ specified by the function handle `f`, where `x0` is an initial guess of the root and `TOL` is the tolerance. The default maximum number of iterations is `imax = 1e6`.

`root = secant_method(f,x0,[],imax)` returns the root of a function $f(x)$ specified by the function handle `f`, where `x0` is an initial guess of the root and `imax` is the maximum number of iterations. The default tolerance is `TOL = 1e-12`.

`root = secant_method(f,x0,TOL,imax)` returns the root of a function $f(x)$ specified by the function handle `f`, where `x0` is an initial guess of the root, `TOL` is the tolerance, and `imax` is the maximum number of iterations.

`root = secant_method(__,'all')` returns a vector, where the first element of this vector is the initial guess, all intermediate elements are the intermediate estimates of the root, and the last element is the converged root. This identifier `'all'` may be appended to any of the syntaxes used above.

Examples

Example 1

Find the root(s) of $f(x) = x^2 - 1$.

■ *SOLUTION*

Defining $f(x)$,

```
f = @(x) x^2-1;
```

We know $f(x) = x^2 - 1$ has roots at $x = \pm 1$, but let's pretend we don't know this, and solve this problem using a more general approach. Since $f(x)$ is a quadratic function, we know that it will have either 0 roots (in

the case where $f(x)$ does not cross the x-axis) or 2 roots. Let's assume the latter case (otherwise it would be pointless to try and find roots of $f(x)$). Therefore, we use the secant method twice, with two different guesses. Let's pick -10 and 10 as our initial guesses.

```
% finds first root of f(x)=x^2-1 using the secant method
root1 = secant_method(f,-10)

% finds second root of f(x)=x^2-1 using the secant method
root2 = secant_method(f,10)
```

This yields the result

```
root1 =

    -1


root2 =

     1
```

Example 2

Find a root of $g(x) = h(m(x))$, where $h(x) = 5x^2 - 4$ and $m(x) = \cosh\sqrt{x}$. Additionally, plot the intermediate root estimates vs. iteration number.

■ *SOLUTION*

First, let's define $g(x)$. Instead of defining it as an anonymous function, we define it as a regular MATLAB function (note that we must either put this function in a separate `.m` file or place it at the end of the script).

```
function g = gx(x)
    m = cosh(sqrt(x));
    h = 5*m^2-4;
    g = h;
end
```

However, we cannot use `gx` directly with `secant_method`. Instead, we first have to assign a function handle to `gx` (this allows us to pass the function to another function as an input parameter).

```
g = @(x) gx(x);
```

Due to the complexity of $g(x)$, we have no idea where its root(s) is/are. Let's make the initial guess $x_0 = 5$. Solving for the root with the secant method,

```
root = secant_method(g,5)
```

This yields the result

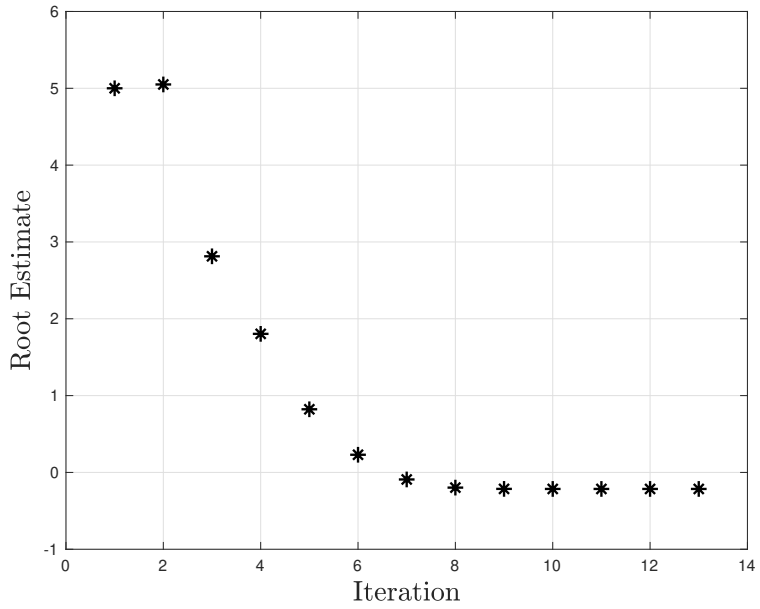
```
root =

   -0.2150
```

To plot the root estimates vs. iteration number,

```
% plots the intermediate root estimates
plot(secant_method(g,5,[],[],'all'),'k*','markersize',9,'linewidth',1.5);
grid on;
xlabel('Iteration','interpreter','latex','fontsize',18);
ylabel('Root Estimate','interpreter','latex','fontsize',18);
```

This yields the following plot:



From this example, we can see that the secant method is much more versatile than Newton's method. The secant method also allows us to find roots of functions that aren't strictly mathematical; that is, we could define a computational function that encapsulates an extremely complicated physical process, and as long as that process has a single input and a single output, we could find its root.

Links

MATLAB® Central's File Exchange:
https://www.mathworks.com/matlabcentral/fileexchange/85745-secant-method-secant_method

GitHub®:
https://github.com/tamaskis/secant_method-MATLAB

Secant Method

Newton's method is a root-finding technique that uses the derivative of a function to find its root[1]. Newton's method is defined iteratively as [2, Eq. (1)]

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \tag{1}$$

But what if we don't know $f'(x)$? Then we need to approximate it using some numerical method. Specifically, for the secant method, we use the backward approximation of a derivative, given by Eq. (2) below.

$$f'(x_i) \approx \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \tag{2}$$

This approximation can be visualized using the finite difference stencil shown in Fig. 1 below.

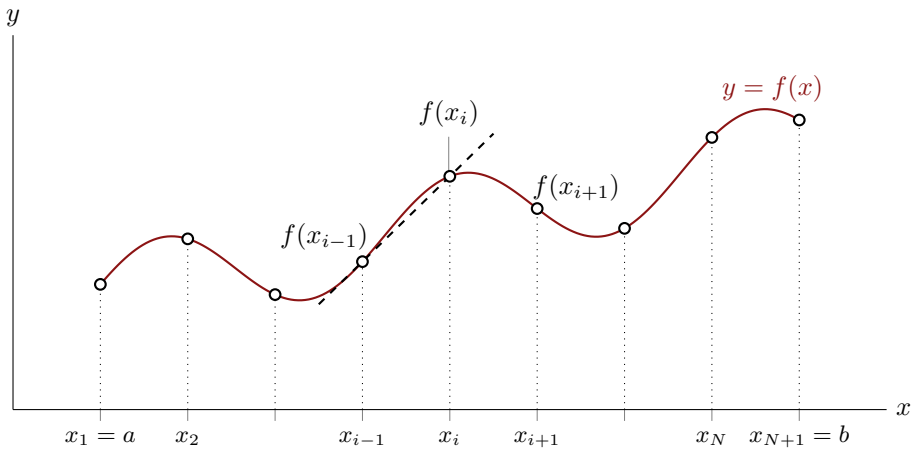


Figure 1: Backward approximation.

Substituting Eq. (2) into Eq. (1),

$$\begin{aligned} x_{i+1} = x_i - \left[\frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})}\right] f(x_i) &= \frac{[f(x_i) - f(x_{i-1})]\, x_i}{f(x_i) - f(x_{i-1})} - \frac{(x_i - x_{i-1}) f(x_i)}{f(x_i) - f(x_{i-1})} \\ = \frac{x_i f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})} + \frac{x_{i-1} f(x_i) - x_i f(x_i)}{f(x_i) - f(x_{i-1})} &= \frac{x_i f(x_i) - x_i f(x_i) + x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})} \end{aligned}$$

$$\boxed{x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}} \tag{3}$$

Equation (3) iteratively defines the **secant method**, which can be essentially thought of as a finite difference approximation of Newton's method for finding the root of a univariate function (based on an initial guess[2]). But how do we actually use Eq. (3)? Given an initial guess x_0, we can keep coming up with new estimates of the root. But how

[1] For a discussion/MATLAB implementation of Newton's method, see [2].

[2] Often, a function $f(x)$ will have multiple roots. Therefore, the secant method typically finds the root closest to the initial guess x_0. *However*, this is not always the case; the algorithm depends heavily on the derivative of $f(x)$, which, depending on its form, may cause it to converge on a root further from x_0.

do we know when to stop? To resolve this issue, we define the **error**[3] as

$$\boxed{\varepsilon = |x_{i+1} - x_i|} \tag{4}$$

Once ε is small enough, we say that the estimate of the root has **converged** to the true root, within some **tolerance** (which we denote as TOL). Therefore, if we predetermine that, at most, we can *tolerate* an error of TOL, then we will keep iterating Eq. (3) until $\varepsilon < \mathrm{TOL}$. In some cases, the error may never decrease below TOL, or take too long to decrease to below TOL. Therefore, we also define the **maximum number of iterations** ($i_{\max}$) so that the algorithm does not keep iterating forever, or for too long of a time.

In any implementation, we first have to make an initial guess x_0 for the root. Additionally, we need to set the root estimate at the second iteration (i.e. x_2) to a value slightly different than x_0 – otherwise, we will just have $x_{i+1} = x_i$ for all i and the algorithm will never "get started" (we can think of this has "kick-starting" the algorithm)[4] [1, 3, 4].

There are two basic algorithms for implementing the secant method. The first implementation, shown in Algorithm 1 below, does *not* store the result of each iteration. On the other hand, the second implementation, shown in Algorithm 2, *does* store the result of each iteration. `secant_method` implements both of these algorithms.

Since Algorithm 2 first needs to preallocate a potentially huge array to store all of the intermediate solutions, Algorithm 1 is significantly faster. Even if $i_{\max}$ (determines size of the preallocated array) is set to be a small number (for example, 10), Algorithm 1 is still faster. The reason we still consider and implement Algorithm 2 is so that convergence studies may be performed.

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[3] Note that ε is an *approximate* error. The motivation behind using this definition of ε is that as i gets large (i.e. $i \to \infty$), $x_{i+1} - x_i$ approaches $x_{i+1} - x^*$ (*assuming* this sequence is convergent), where x^* is the true root (and therefore $x_{i+1} - x^*$ represents the *exact* error).

[4] The alternative way to view this is by recalling that the derivative approximation is given by

$$f'(x_i) \approx \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

If $x_i = x_{i-1}$, then the approximation to $f'(x_i)$ will become undefined, resulting in an error.

Algorithm 1: Secant method.

Given: $f(x)$, x_0, TOL, $i_{\max}$

// initializes the error so the loop will be entered

$\text{err} = (2)(\text{TOL})$

// sets 1st and 2nd root estimates at the first iteration of the secant method based on the initial guess

$x_{\text{old}} = x_0$

$x_{\text{int}} = 1.01x_0$

// secant method

$i = 1$

while $\varepsilon > \text{TOL}$ ***and*** $i < i_{\max}$ **do**

// updates estimate of root

$$x_{\text{new}} = \frac{x_{\text{old}} f(x_{\text{int}}) - x_{\text{int}} f(x_{\text{old}})}{f(x_{\text{int}}) - f(x_{\text{old}})}$$

// calculates error

$\varepsilon = |x_{\text{new}} - x_{\text{int}}|$

// stores updated root estimates for next iteration

$x_{\text{old}} = x_{\text{int}}$

$x_{\text{int}} = x_{\text{new}}$

// increments loop index

$i = i + 1$

end

// returns root

$\text{root} = x_{\text{old}}$

return root

Algorithm 2: Secant method with intermediate root estimates.

Given: $f(x)$, x_0, TOL, $i_{\max}$

// initializes the error so the loop will be entered

$\text{err} = (2)(\text{TOL})$

Preallocate an $i_{\max} \times 1$ vector $\mathbf{x}$, where $\mathbf{x}$ is the vector storing the estimates of the root at each iteration.

// inputs 1st and 2nd guesses for root into $\mathbf{x}$ *vector (note that* x_1 *and* x_2 *are the first and second elements of* $\mathbf{x}$*, while* x_0 *is the input initial guess)*

$x_1 = x_0$

$x_2 = 1.01x_0$

// secant method

$i = 2$

while $\varepsilon > \text{TOL}$ ***and*** $i < i_{\max}$ **do**

// updates estimate of root

$$x_{i+1} = \frac{x_{i-1}f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$$

// calculates error

$\varepsilon = |x_{i+1} - x_i|$

// increments loop index

$i = i + 1$

end

// stores intermediate root estimates (where last element of root *is the converged root)*

$\text{root} = (x_1, ..., x_i)^T$

// returns root

return root

References

[1] James Hateley. *Nonlinear Equations*. MATH 3620 Course Reader (Vanderbilt University). 2019.

[2] Tamas Kis. *Newton's Method – MATLAB Implementation*. https://github.com/tamaskis/newtons_method-MATLAB. 2021.

[3] *Newton's method*. https://en.wikipedia.org/wiki/Newton%27s_method. (accessed: June 10, 2020).

[4] *Secant method*. https://en.wikipedia.org/wiki/Secant_method. (accessed: January 15, 2020).