

Tridiagonal Matrix Algorithm

MATLAB Implementation

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tridiagonal

Solves the tridiagonal linear system $A\mathbf{x} = \mathbf{d}$ for \mathbf{x} using the tridiagonal matrix algorithm (i.e. the Thomas algorithm).

Syntax

`x = tridiagonal(A,d)`

Description

`x = tridiagonal(A,d)` solves the tridiagonal linear system $A\mathbf{x} = \mathbf{d}$ for the vector $\mathbf{x} \in \mathbb{R}^n$, where $A \in \mathbb{R}^{n \times n}$ is a tridiagonal matrix and $\mathbf{d} \in \mathbb{R}^n$ is a vector.

Examples

Example 1

Solve the tridiagonal linear system $A\mathbf{x} = \mathbf{d}$ for \mathbf{x} , where

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 3 & 4 & 5 & 0 & 0 \\ 0 & 6 & 7 & 8 & 0 \\ 0 & 0 & 9 & 1 & 2 \\ 0 & 0 & 0 & 3 & 4 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

■ SOLUTION

Entering A and \mathbf{d} into MATLAB,

```
% defines tridiagonal matrix A
A = [1,2,0,0,0;
     3,4,5,0,0;
     0,6,7,8,0;
     0,0,9,1,2;
     0,0,0,3,4];

% defines vector d
d = [1;
     2;
     3;
     4;
     5];
```

To solve the tridiagonal linear system for \mathbf{x} ,

```
x = tridiagonal(A,d)
```

This yields the result

```
x =
```

```
-0.7229  
0.8614  
0.1446  
-0.3976  
1.5482
```

Links

MATLAB® Central's File Exchange:

<https://www.mathworks.com/matlabcentral/fileexchange/85438-tridiagonal-matrix-algorithm-thomas-alg-tridiagonal>

GitHub®:

<https://github.com/tamaskis/tridiagonal-MATLAB>

Tridiagonal Matrix Algorithm (Thomas Algorithm)

A tridiagonal linear system is one of the form

$$\begin{bmatrix} b_1 & c_1 & & & \\ a_1 & b_2 & c_2 & & \\ & a_2 & \ddots & \ddots & \\ & & \ddots & \ddots & c_{n-2} \\ & & & a_{n-2} & b_{n-1} & c_{n-1} \\ & & & & a_{n-1} & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n-1} \\ d_n \end{bmatrix}$$

We can define the \mathbf{x} and \mathbf{d} vectors as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n-1} \\ d_n \end{bmatrix}$$

and the $n \times n$ **triadiagonal matrix**¹, A , as

$$A = \begin{bmatrix} b_1 & c_1 & & & \\ a_1 & b_2 & c_2 & & \\ & a_2 & \ddots & \ddots & \\ & & \ddots & \ddots & c_{n-2} \\ & & & a_{n-2} & b_{n-1} & c_{n-1} \\ & & & & a_{n-1} & b_n \end{bmatrix} \quad (1)$$

Now we can write the tridiagonal linear system as

$$A\mathbf{x} = \mathbf{d} \quad (2)$$

where $A \in \mathbb{R}^{n \times n}$ and $\mathbf{x}, \mathbf{d} \in \mathbb{R}^n$.

The **tridiagonal matrix algorithm** (also known as the **Thomas algorithm**) is an algorithm that can efficiently solve the tridiagonal linear system (given by Eq. (2)) for \mathbf{x} . This algorithm uses three vectors, \mathbf{a} , \mathbf{b} , and \mathbf{c} , which we

¹ In many references, a tridiagonal matrix is defined with the convention

$$A = \begin{bmatrix} a_1 & b_1 & & & \\ c_1 & a_2 & b_2 & & \\ & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-2} \\ & & & c_{n-2} & a_{n-1} & b_{n-1} \\ & & & & c_{n-1} & a_n \end{bmatrix}$$

However, when dealing with the tridiagonal matrix algorithm, a convention similar to the one in Eq. (1) is used almost exclusively. However, the convention that most sources have has the a_i 's ranging from a_2 to a_n , which is extremely inconvenient from an algorithmic standpoint; therefore, I defined them here as ranging from a_1 to a_{n-1} , and this is also reflected in Algorithm 1.

define as [1]

$$\mathbf{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_{n-1} \end{bmatrix}$$

The tridiagonal matrix algorithm is shown below [1–3].

Algorithm 1: Tridiagonal matrix algorithm (Thomas algorithm).

```

1  Given:  $A$ ,  $\mathbf{x}$ ,  $\mathbf{d}$ 

    // determines  $n$  (where  $A \in \mathbb{R}^{n \times n}$ )
2   $n = \text{size}(A, 1)$ 

3  Preallocate vectors of size  $n \times 1$  to store  $\mathbf{b}$  and  $\mathbf{x}$ .
4  Preallocate vectors of size  $(n - 1) \times 1$  to store  $\mathbf{a}$  and  $\mathbf{c}$ .

    // extracts  $\mathbf{a}$  from  $A$ 
5  for  $i = 2$  to  $n$  do
6     $a_{i-1} = A_{i,i-1}$ 
7  end

    // extracts  $\mathbf{b}$  from  $A$ 
8  for  $i = 1$  to  $n$  do
9     $b_i = A_{i,i}$ 
10 end

    // extracts  $\mathbf{c}$  from  $A$ 
11 for  $i = 2$  to  $n$  do
12    $c_{i-1} = A_{i-1,i}$ 
13 end

    // forward elimination
14 for  $i = 1$  to  $n$  do
15    $w = a_{i-1}/b_{i-1}$ 
16    $b_i = b_i - wc_{i-1}$ 
17    $d_i = d_i - wd_{i-1}$ 
18 end

    // backward substitution
19  $x_n = d_n/b_n$ 
20 for  $i = n - 1$  to  $1$  by  $-1$  do
21    $x_i = (d_i - c_i x_{i+1})/b_i$ 
22 end

23 return  $\mathbf{x}$ 

```

References

- [1] James Hateley. *Linear Systems of Equations and Direct Solvers*. MATH 3620 Course Reader (Vanderbilt University). 2019.
- [2] *Tridiagonal matrix algorithm*. https://en.wikipedia.org/wiki/Tridiagonal_matrix_algorithm. (accessed: January 9, 2021).
- [3] *Tridiagonal matrix algorithm – TDMA (Thomas algorithm)*. [https://www.cfd-online.com/Wiki/Tridiagonal_matrix_algorithm_-_TDMA_\(Thomas_algorithm\)](https://www.cfd-online.com/Wiki/Tridiagonal_matrix_algorithm_-_TDMA_(Thomas_algorithm)). (accessed: January 9, 2021).