
Tridiagonal Matrix Algorithm (Thomas Algorithm)

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1 TRIDIAGONAL MATRIX ALGORITHM (THOMAS ALGORITHM)

A tridiagonal linear system is one of the form

$$\begin{bmatrix} b_1 & c_1 & & & \\ a_1 & b_2 & c_2 & & \\ & a_2 & \ddots & \ddots & \\ & & \ddots & \ddots & c_{n-2} \\ & & & a_{n-2} & b_{n-1} & c_{n-1} \\ & & & & a_{n-1} & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n-1} \\ d_n \end{bmatrix}$$

We can define the \mathbf{x} and \mathbf{d} vectors as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n-1} \\ d_n \end{bmatrix}$$

and the $n \times n$ **tridiagonal matrix**¹, \mathbf{A} , as

$$\mathbf{A} = \begin{bmatrix} b_1 & c_1 & & & \\ a_1 & b_2 & c_2 & & \\ & a_2 & \ddots & \ddots & \\ & & \ddots & \ddots & c_{n-2} \\ & & & a_{n-2} & b_{n-1} & c_{n-1} \\ & & & & a_{n-1} & b_n \end{bmatrix} \quad (1)$$

Now we can write the tridiagonal linear system as

$$\mathbf{Ax} = \mathbf{d} \quad (2)$$

where $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{x}, \mathbf{d} \in \mathbb{R}^n$.

The **tridiagonal matrix algorithm** (also known as the **Thomas algorithm**) is an algorithm that can efficiently solve the tridiagonal linear system (given by Eq. (2)) for \mathbf{x} . This algorithm uses three vectors, \mathbf{a} , \mathbf{b} , and \mathbf{c} , which we define as [1]

$$\mathbf{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_{n-1} \end{bmatrix}$$

¹ In many references, a tridiagonal matrix is defined with the convention

$$\mathbf{A} = \begin{bmatrix} a_1 & b_1 & & & \\ c_1 & a_2 & b_2 & & \\ & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-2} \\ & & & c_{n-2} & a_{n-1} & b_{n-1} \\ & & & & c_{n-1} & a_n \end{bmatrix}$$

When dealing with the tridiagonal matrix algorithm, a convention similar to the one in Eq. (1) is used almost exclusively. However, the convention that most sources have has the a_i 's ranging from a_2 to a_n , which is inconvenient from a programming standpoint; therefore, I defined them here as ranging from a_1 to a_{n-1} . This convention is also reflected in Algorithm 1.

The tridiagonal matrix algorithm is shown below [1–3].

Algorithm 1: tridiagonal

Tridiagonal matrix algorithm (Thomas algorithm).

Given:

- $\mathbf{A} \in \mathbb{R}^{n \times n}$ - tridiagonal matrix
- $\mathbf{d} \in \mathbb{R}^n$ - vector

Note:

- \mathbf{A} and \mathbf{d} define the tridiagonal linear system $\mathbf{Ax} = \mathbf{d}$.

Procedure:

1. Determine n (where $\mathbf{A} \in \mathbb{R}^{n \times n}$).
2. Preallocate vectors of size $n \times 1$ to store \mathbf{b} and \mathbf{x} .
3. Preallocate vectors of size $(n - 1) \times 1$ to store \mathbf{a} and \mathbf{c} .
4. Extract \mathbf{a} from \mathbf{A} .

```

for  $i = 2$  to  $n$ 
     $a_{i-1} = A_{i,i-1}$ 
end

```

5. Extract \mathbf{b} from \mathbf{A} .

```

for  $i = 1$  to  $n$ 
     $b_i = A_{i,i}$ 
end

```

6. Extract \mathbf{c} from \mathbf{A} .

```

for  $i = 2$  to  $n$ 
     $c_{i-1} = A_{i-1,i}$ 
end

```

7. Forward elimination.

```

for  $i = 1$  to  $n$ 
     $w = a_{i-1}/b_{i-1}$ 
     $b_i = b_i - wc_{i-1}$ 
     $d_i = d_i - wd_{i-1}$ 
end

```

8. Backward substitution.

```

for  $i = n - 1$  to  $1$  by  $-1$ 
     $x_i = (d_i - c_i x_{i+1})/b_i$ 
end

```

Return:

- $\mathbf{x} \in \mathbb{R}^n$ - solution of the tridiagonal linear system $\mathbf{Ax} = \mathbf{d}$

REFERENCES

- [1] James Hateley. *Linear Systems of Equations and Direct Solvers*. MATH 3620 Course Reader (Vanderbilt University). 2019.
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- [3] *Tridiagonal matrix algorithm – TDMA (Thomas algorithm)*. CFD Online. [https://www.cfd-online.com/Wiki/Tridiagonal_matrix_algorithm_-_TDMA_\(Thomas_algorithm\)](https://www.cfd-online.com/Wiki/Tridiagonal_matrix_algorithm_-_TDMA_(Thomas_algorithm)) (accessed: January 9, 2021).