
Tridiagonal Matrix Algorithm (Thomas Algorithm)

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1 TRIDIAGONAL MATRIX ALGORITHM (THOMAS ALGORITHM)

1.1 Tridiagonal Linear Systems

A **tridiagonal linear system** is one of the form

$$\boxed{\mathbf{Ax} = \mathbf{d}} \quad (1)$$

where

$$\underbrace{\begin{bmatrix} b_1 & c_1 & & & \\ a_1 & b_2 & c_2 & & \\ & a_2 & \ddots & \ddots & \\ & & \ddots & \ddots & c_{n-2} \\ & & & a_{n-2} & b_{n-1} & c_{n-1} \\ & & & & a_{n-1} & b_n \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n-1} \\ d_n \end{bmatrix}}_{\mathbf{d}} \quad (2)$$

and where $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{x}, \mathbf{d} \in \mathbb{R}^n$. Owing to the fact that it only has three nonzero diagonals, the matrix \mathbf{A} is referred to as a **tridiagonal matrix**¹.

The **tridiagonal matrix algorithm** (also known as the **Thomas algorithm**) is an algorithm that can efficiently solve the tridiagonal linear system for \mathbf{x} . In this document, we introduce three different ways² to implement the tridiagonal matrix algorithm (Algorithms 1, 2, and 3). The first two implementations use three vectors, \mathbf{a} , \mathbf{b} , and \mathbf{c} , which we define as [1]

$$\mathbf{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_{n-1} \end{bmatrix} \quad (3)$$

1.2 Slower Implementation

The tridiagonal matrix algorithm is shown below [1, 3, 4].

¹ In many references, a tridiagonal matrix is often defined with one of the following two convention:

$$\mathbf{A} = \begin{bmatrix} a_1 & b_1 & & & \\ c_1 & a_2 & b_2 & & \\ & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-2} \\ & & & c_{n-2} & a_{n-1} & b_{n-1} \\ & & & & c_{n-1} & a_n \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & a_3 & \ddots & \ddots & \\ & & \ddots & \ddots & c_{n-2} \\ & & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & & a_n & b_n \end{bmatrix}$$

The first is typically used when defining a tridiagonal linear system [2], while the second is used almost exclusively when defining the tridiagonal matrix algorithm [3, 4]. However, for the second convention above, the a_i 's range from a_2 to a_n , which is inconvenient from a programming standpoint; therefore, I defined them here as ranging from a_1 to a_{n-1} . This convention is also reflected in Algorithms 1 and 2.

² The `tridiagonal` function implements Algorithm 2.

Algorithm 1: tridiagonal_slower

Tridiagonal matrix algorithm (Thomas algorithm) (slower version).

Given:

- $\mathbf{A} \in \mathbb{R}^{n \times n}$ - tridiagonal matrix
- $\mathbf{d} \in \mathbb{R}^n$ - vector

Note:

- \mathbf{A} and \mathbf{d} define the tridiagonal linear system $\mathbf{Ax} = \mathbf{d}$.

Procedure:

1. Determine n , given that $\mathbf{d} \in \mathbb{R}^n$.
2. Preallocate vectors of size $n \times 1$ to store \mathbf{b} and \mathbf{x} .
3. Preallocate vectors of size $(n - 1) \times 1$ to store \mathbf{a} and \mathbf{c} .
4. Extract \mathbf{a} from \mathbf{A} .

```

for  $i = 2$  to  $n$ 
     $a_{i-1} = A_{i,i-1}$ 
end

```

5. Extract \mathbf{b} from \mathbf{A} .

```

for  $i = 1$  to  $n$ 
     $b_i = A_{i,i}$ 
end

```

6. Extract \mathbf{c} from \mathbf{A} .

```

for  $i = 2$  to  $n$ 
     $c_{i-1} = A_{i-1,i}$ 
end

```

7. Forward elimination.

```

for  $2 = 1$  to  $n$ 
     $w = \frac{a_{i-1}}{b_{i-1}}$ 
     $b_i = b_i - wc_{i-1}$ 
     $d_i = d_i - wd_{i-1}$ 
end

```

8. Backward substitution.

```

 $x_n = \frac{d_n}{b_n}$ 
for  $i = n - 1$  to  $1$  by  $-1$ 
     $x_i = \frac{d_i - c_i x_{i+1}}{b_i}$ 
end

```

Return:

- $\mathbf{x} \in \mathbb{R}^n$ - solution of the tridiagonal linear system $\mathbf{Ax} = \mathbf{d}$

1.3 Faster Implementation

We can save some computational effort by reducing the number of for loops in Algorithm 1. For smaller systems, this doesn't make a huge impact, but for larger systems, it can halve the time it takes to solve. We can note that four of the loops go "forward" in i , so we can combine them (with the caveat that we must extract b_1 separately since its loop starts from 1 and not 2). Defining this "faster" algorithm,

Algorithm 2: tridiagonal

Tridiagonal matrix algorithm [Thomas algorithm].

Given:

- $\mathbf{A} \in \mathbb{R}^{n \times n}$ - tridiagonal matrix
- $\mathbf{d} \in \mathbb{R}^n$ - vector

Note:

- \mathbf{A} and \mathbf{d} define the tridiagonal linear system $\mathbf{Ax} = \mathbf{d}$.

Procedure:

1. Determine n , given that $\mathbf{d} \in \mathbb{R}^n$.
2. Preallocate vectors of size $n \times 1$ to store \mathbf{b} and \mathbf{x} .
3. Preallocate vectors of size $(n - 1) \times 1$ to store \mathbf{a} and \mathbf{c} .
4. Extract first element of \mathbf{b} from \mathbf{A} .

$$b_1 = A_{1,1}$$

5. Forward loop.

for $i = 2$ **to** n

(a) Extract relevant elements of \mathbf{a} , \mathbf{b} , and \mathbf{c} from \mathbf{A} .

$$a_{i-1} = A_{i,i-1}$$

$$b_i = A_{i,i}$$

$$c_{i-1} = A_{i-1,i}$$

(b) Forward elimination.

$$w = \frac{a_{i-1}}{b_{i-1}}$$

$$b_i = b_i - wc_{i-1}$$

$$d_i = d_i - wd_{i-1}$$

end

6. Backward loop (backward substitution).

$$x_n = \frac{d_n}{b_n}$$

```

    for  $i = n - 1$  to 1 by  $-1$ 
    |    $x_i = \frac{d_i - c_i x_{i+1}}{b_i}$ 
    end

```

Return:

- $\mathbf{x} \in \mathbb{R}^n$ - solution of the tridiagonal linear system $\mathbf{Ax} = \mathbf{d}$

1.4 Shortest Implementation

Finally, instead of defining/preallocating the vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} , we can access the elements of \mathbf{A} directly. We refer to this as the “shortest” implementation since it results in the least lines of code.

Algorithm 3: tridiagonal_shortest

Tridiagonal matrix algorithm [Thomas algorithm] (shortest implementation).

Given:

- $\mathbf{A} \in \mathbb{R}^{n \times n}$ - tridiagonal matrix
- $\mathbf{d} \in \mathbb{R}^n$ - vector

Note:

- \mathbf{A} and \mathbf{d} define the tridiagonal linear system $\mathbf{Ax} = \mathbf{d}$.

Procedure:

1. Determine n , given that $\mathbf{d} \in \mathbb{R}^n$.
2. Preallocate a vector of size $n \times 1$ to store \mathbf{x} .
3. Forward elimination.

```

    for  $i = 2$  to  $n$ 
    |    $w = \frac{A_{i,i-1}}{A_{i-1,i-1}}$ 
    |    $A_{i,i} = A_{i,i} - w A_{i-1,i}$ 
    |    $d_i = d_i - w d_{i-1}$ 
    end

```

4. Backward substitution.

```

 $x_n = \frac{d_n}{A_{n,n}}$ 
    for  $i = n - 1$  to 1 by  $-1$ 
    |    $x_i = \frac{d_i - A_{i,i+1} x_{i+1}}{A_{i,i}}$ 
    end

```

Return:

- $\mathbf{x} \in \mathbb{R}^n$ - solution of the tridiagonal linear system $\mathbf{Ax} = \mathbf{d}$

For smaller systems, this implementation can actually be the fastest, since you only have to preallocate one vector (\mathbf{x}) instead of four (\mathbf{x} , \mathbf{a} , \mathbf{b} , and \mathbf{c}). However, for large systems, it is costlier to traverse the matrix \mathbf{A} during the substitution process than it is to preallocate, define, and traverse the vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} .

REFERENCES

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