Redo 4.(a) $\mathbb{E}_{p_A,p_B}[\text{points for B}] = p_A^T M p_B$ $\quad M := \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$ $\quad p_C = \begin{bmatrix} p_{C_{rock}} \\ p_{C_{paper}} \\ p_{C_{scissors}} \end{bmatrix}$

$\left(= P_{A1}(P_{B3} - P_{B2}) + P_{A2}(P_{B1} - P_{B3}) + P_{A3}(P_{B2} - P_{B1}) \quad \text{(as I did originally)} \right)$

- Clearly, $p_A^T M p_B \geq \underset{p_A \in \Delta^3}{\text{minimize}}\ p_A^T M p_B \Rightarrow \underset{p_B \in \Delta^3}{\text{maximize}}\ p_A^T M p_B \geq \underset{p_B}{\text{maximize}}\ \underset{p_A}{\text{minimize}}\ p_A^T M p_B$

$\Rightarrow \underset{p_A}{\text{minimize}}\ \underset{p_B}{\text{maximize}}\ p_A^T M p_B \geq \underset{p_A}{\text{minimize}}\ \underset{p_B}{\text{maximize}}\ \underset{p_A}{\text{minimize}}\ p_A^T M p_B$ by above

$= \underset{p_B}{\text{maximize}}\ \underset{p_A}{\text{minimize}}\ p_A^T M p_B$ $\quad (*)$

$$\underset{p_A}{\text{minimize}}\ \underset{p_B}{\text{maximize}}\ p_A^T M p_B \leq \underset{p_B}{\text{maximize}}\ p_A^{*} M p_B = 0 \qquad (**)$$

$0 \geq \underset{p_A}{\text{minimize}}\ \underset{p_B}{\text{maximize}}\ p_A^T M p_B$ $\quad$ by $(**)$

$= \underset{p_A}{\text{minimize}}\ \underset{p_B}{\text{maximize}}\ -p_B^T M p_A$ $\quad \because p_A^T M p_B = (p_A^T M p_B)^T = p_B^T M^T p_A = -p_B^T M p_A$

$= -\underset{p_A}{\text{maximize}}\ \underset{p_B}{\text{minimize}}\ p_B^T M p_A$

$= -\underset{p_B}{\text{maximize}}\ \underset{p_A}{\text{minimize}}\ p_A^T M p_B$

$\geq -\underset{p_A}{\text{minimize}}\ \underset{p_B}{\text{maximize}}\ p_A^T M p_B \geq 0$ $\quad$ by $(*)$ and then $(**)$

$\Rightarrow \underset{p_A}{\text{minimize}}\ \underset{p_B}{\text{maximize}}\ p_A^T M p_B = \underset{p_B}{\text{maximize}}\ \underset{p_A}{\text{minimize}}\ p_A^T M p_B = 0 \Leftrightarrow$ existence of saddle pt solution p_A^*, p_B^*

- Show unique by proving $p_A^{*} M p_B \leq p_A^{*T} M p_B^* = 0 \leq p_A^T M p_B^*$ $\quad \forall\, p_A, p_B \in \Delta^3$ only for given p_A^*, p_B^*

Suppose $p_B^* \neq [1/3, 1/3, 1/3]^T \Leftrightarrow M p_B^* = [P_{B_s} - P_{B_p}, P_{B_r} - P_{B_s}, P_{B_p} - P_{B_r}]^T \neq [0,0,0]^T$

$\Rightarrow$ At least one entry of $M p_B^*$ is negative

So pick p_A such that $p_A^T M p_B^* < 0$ to obtain a contradiction with $p_A^T M p_B^* \geq 0\ \forall\, p_A \in \Delta^3$ above.

$\Rightarrow p_B^* = [1/3, 1/3, 1/3]^T$

Analogous argument with $p_A^* M p_B \leq 0$ to show $p_A^* = [1/3, 1/3, 1/3]^T$

So given p_A^*, p_B^* is indeed the unique solution.

(b) (I'd misunderstood the Q the first time - I assumed we had set that B plays p_B^*)
If A chooses $p_A \neq p_A^*$, then B can choose one p_B of $[1,0,0]^T, [0,1,0]^T, [0,0,1]^T$ to make $\mathbb{E}_{p_A,p_B}[\text{points for B}] > 0$! So <u>no</u>, not just any strategy is optimal for A. Only in the specific case where B plays p_B^* is A free to pick any strategy since $\mathbb{E}_{p_A,p_B^*}[\text{points for B}] = 0$.