

Non-Stationary Stochastic Volatility Model for Dynamic Feedback and Skewness

Sujay K Mukhoti*

March 8, 2015

Abstract

In this paper I present a new single factor stochastic volatility model for asset return observed in discrete time and its latent volatility. This model unifies the feedback effect and return skewness using a common factor for return and its volatility. Further, it generalizes the existing stochastic volatility framework with constant feedback to one with time varying feedback and as a consequence time varying skewness follows. However, presence of dynamic feedback effect violates the weak-stationarity assumption usually considered for the latent volatility process. The concept of bounded stationarity has been proposed in this paper to address the issue of non-stationarity. A characterization of the error distributions for returns and volatility is provided on the basis of existence of conditional moments. Finally, an application of the model has been explained using

*e-mail:sujay.mukhoti@gmail.com

S&P100 daily returns under the assumption of Normal error and half Normal common factor distribution.

1 Introduction

Research in financial econometrics has seen a surge in the area of time-varying volatility models for asset returns over last three decades. Stochastic volatility (SV) model ((Taylor 1982)) has been one of the key instruments to address this issue. In addition SV model explains some interesting aspects of asset returns observed empirically and known as “stylized facts”. Some of the important stylized facts are mean reversion of returns, volatility clustering indicating periods of similar volatility occurring together and the negative relation between the return and its volatility divulging their movement in opposite direction. The work of (Taylor 1982) models the time varying volatility of financial returns as a latent auto-regressive process to account for the volatility clustering. Since then multitude of SV models have been developed to explain different stylized facts about asset returns. A comprehensive review of the SV models can be obtained from the works of (Shephard & Andersen 2009) and (Chib, Omori & Asai 2009).

Recent works in this context emphasize on two important aspects of return-volatility relation *viz.* the correlation between current volatility and future returns (or the feedback effect) and the negative correlation between current asset return and its future volatility (or leverage effect). Different types of SV models have been developed to explain the time varying volatility of asset return in presence of leverage effect. (Renault 2009) provides a comprehensive account of feedback and leverage effect in SV models. Another related stylized fact, *viz.* return skewness, has gained importance due to its role in asset and option pricing ((Christoffersen, Heston & Jacobs 2006), (Renault 2009)). However, no work has been done so far to establish the connection between feedback effect, return skewness and leverage effect in discrete time general SV models to the best of the knowledge of the present author. In this paper, I develop a parsimonious generalized single factor SV model to explain

the relation between conditional feedback and return skewness and extend it to an SV model with time varying feedback and skewness.

Time series data on asset returns provide evidences of its correlation with its volatility ((Nelson 1991)) and skewness in asset returns ((Harvey & Siddique 1999)). The negative correlation between current volatility and future return (or feedback) may be attributed to the fact that an anticipated increase in volatility results in immediate price fall ((French, Schwert & Stambaugh 1987)). (Bollerslev, Litvinova & Tauchen 2006) shows that a stronger signal of the feedback effect is reflected through the contemporaneous correlation between asset return and its volatility. They conclude in favor of the contemporaneous correlation as a measure of volatility feedback effect. On the other hand, the fact that a decline in current price would lead to increase in future volatility, could be attributed to changes in financial leverage ((Nelson 1991)) and such a correlation is called the leverage effect. (Renault 2009) points out the possibility of an alternative explanation to the feedback using the return skewness. The intuition behind such possibility could be justified by the following argument. The magnitude of volatility increase due to price fall is much higher than the magnitude by which volatility decreases in case of price increase. Thus the conditional volatility for negative returns is more compared to the same for positive returns. This fact leads to the skewness in the return distribution. (Tsiotas 2012) and (Feunou & Tédongap 2012) provide an account of SV models developed so far with leverage and skewed return distributions.

In this paper I propose a parsimonious representation of such an interlocked explication of feedback effect and skewness. Both of them could be looked upon as the resultant of a common positive stochastic factor acting on both return and volatility shocks which are symmetric. A financial justification of the presence of such a stochastic factor could be in assuming the presence of a market sentiment influencing both return and volatility in different magnitudes. Directions of market sentiment impact could be similar or opposite. I model the influence of market sentiment on return and volatility as product of a common positive latent random variable and corresponding weights. This mechanism generates perturbation to symmetric return shocks by a positive random variable and generates asymmetry in returns

whereas the shared factor generates the feedback effect.

(Bollerslev, Sizova & Tauchen 2012) provides empirical evidence of the dynamic nature of the correlation between return and volatility. On the other hand (Harvey & Siddique 1999) and recently (Boyer, Mitton & Vorkink 2010) provide evidences of time dependent conditional return skewness. Based on the above findings I assume the weights of the factor on return and volatility to be time varying so that the feedback effect and conditional skewness are dynamic. Individual impact of the stochastic factor on return and volatility are measured by the corresponding time dependent coefficients which will be referred to as impact parameters here onwards. The underlying reason of different directions and magnitudes of the time varying conditional skewness and the feedback effect could be then comprehended in terms of the impact parameters.

The main complexity of the proposed model is that it violates weak-stationarity condition of the volatility process. Weak stationarity is crucial to a stochastic process as it restricts the process to increase indefinitely in expectation with time lag. In this paper I introduce the concept of bounded stationarity in terms of 1^{st} and 2^{nd} order moments of a stochastic process to relax the existing weak-stationarity condition yet ensure that the process does not explode. I also provide here a characterization of the auto-regressive volatility process of order one, which is most commonly used to describe volatility process in SV models, in the light of bounded stationarity.

The proposed model is developed under general distributions for return, volatility and the common factor. Many of the existing SV models has been shown to be particular cases of this generalized SV model. An immediate characterization of the plausible distributions for return, volatility and the common factor has been given based on the existence of return moments and the feedback effect. Further, I provide explanation of the skewness and feedback effect in terms of the influence of market sentiment assuming the usual Gaussian framework. The affine combination of Normal return shocks and Half-Normal common factor distribution used in the proposed model results in a variant of a general class of distributions containing standard Normal known as skew-normal distribution ((Azzalini & Dalla Valle 1996)).

Parameter estimation in SV model is a challenging task due to non-identically and non-independently distributed (non-iid) returns with complicated likelihood and high dimension of the parameter space. (Jacquier, Polson & Rossi 1994) developed a Monte Carlo Markov Chain (MCMC) method to estimate the SV model parameters. As pointed out by (Eraker, Johanners & Polson 2003), MCMC method provides an edge over other methods of estimation in different ways. First, MCMC provides estimates of latent volatilities which are as many in numbers as the number of observed returns. Second, it is computationally efficient and easily implementable ((Meyer & Yu 2000)). Another major advantage of the MCMC method is that it allows to incorporate restrictions on the model parameters through selection of relevant prior distributions. Hence, I use MCMC method in this paper to estimate the model parameters and latent volatilities.

Two major concerns in MCMC simulation are to measure the convergence and model adequacy. (Gelman & Rubin 1992) suggested potential scale reduction factor (psrf) as a measure of convergence which is calculated from more than one parallel MCMC simulations. In this paper, Gelman-Rubin psrf has been used to measure convergence of MCMC simulations. On the other hand, recent progress in MCMC estimation and its implementation has become compelling for researchers to fit models with large number of parameters to explore real life complexities more closely. As a result, measurement of model adequacy and complexity has become increasingly important. (Spiegelhalter, Best, Carlin & Van Der Linde 2002) proposed the deviance information criteria (DIC) as a measure of model adequacy. DIC is calculated based on separate measures of model fit and model complexity. The discrepancy between data and model could be measured by the posterior mean of log-likelihood in an MCMC simulation, which is a measure of model fit. Model complexity could be measured using the log-likelihood at the posterior means of the model parameters. DIC combines the two measures to arrive at a measure of model adequacy ((Spiegelhalter et al. 2002)). Observing that the number of unknown quantities involved in the model (including latent volatility) it is crucial to measure the complexity in the model. In this paper I report both DIC and measure of model complexity to gauge the usefulness of the proposed model.

Rest of the paper is organized as follows. In section 2 the general framework for SV model with common factor and time dependent impact parameters is described. The concept of bounded stationarity is introduced in this section to tackle non-stationarity. Section 3 presents an example of generalized SVDF model with half-normal and Gaussian distributions. The expressions for the dynamic feedback, leverage and skewness are presented here. Also necessary and sufficient conditions for negative feedback has been discussed. In section 4 SV models with constant and time-varying feedback effect have been tested with S&P100 daily returns. Estimation results from the proposed models have been compared with the same from some comparable models. Section 5 concludes the paper with a discussion on the proposed model and its applications.

2 Dynamic Feedback SV Model with Common Market Factor

Let P_t be the daily price of an asset and $\log \frac{P_t}{P_{t-1}}$ be the log return. The time series of mean-corrected daily log returns is denoted by y_t and the underlying latent volatilities by θ_t . Let us start with the SV model proposed by (Jacquier et al. 1994) which is given as follows:

$$y_t = e^{\frac{\theta_t}{2}} \epsilon_t, \quad (2.1)$$

$$\theta_t = \alpha + \phi(\theta_{t-1} - \alpha) + \eta_t, t = 1, \dots, T \quad (2.2)$$

ϵ_t and η_t being independent sequences of independently and identically distributed (iid) random shocks (or innovations) with 0 means and variances 1 and σ^2 respectively. ϕ is the volatility clustering parameter which reflects the stylized fact that volatility pattern (high or low) cluster together. Subsequently SV models with contemporaneous correlation (ρ) between ϵ_t and η_t has been discussed by, *e.g.* (Jacquier, Polson & Rossi 2004) among others. SV model with the feedback effect ρ relates the changes in volatility to the sign and magnitude of price changes which helps in pricing the options more accurately.

In this paper, I consider a new SV model with independent symmetric random shocks

ϵ_t and η_t and a general positive common factor for market sentiment, say γ_t , which impacts the return and its latent volatility at each time point. However, such impact on return and its volatility may be different in magnitude and direction and may vary over time ((Boyer et al. 2010)). Let $\lambda_{y,t} \in \mathcal{R}$ and $\lambda_{\theta,t} \in \mathcal{R}$ be the dynamic impacts of the market factor on the return and its latent volatility respectively. Thus the new single factor SV model with time varying feedback (SVDF) is given as

$$y_t = \mu_{y,t} + e^{\frac{\theta_t}{2}} (\lambda_{y,t} \gamma_t + \epsilon_t) \quad (2.3)$$

$$\theta_t = \alpha + \phi(\theta_{t-1} - \alpha) + \mu_{\theta,t} + (\lambda_{\theta,t} \gamma_t + \eta_t) \quad (2.4)$$

where y_t , θ_t are same as in equations (

The affine combination of positive factor with symmetric innovation results in a skewed family of distributions. The impact parameters determine the amount and direction of conditional skewness in the corresponding process and hence will be interchangeably called as skewness parameters and impact parameters here onwards. The presence of common factor in both return and volatility induces the correlation or the feedback effect. The time-dependent impact parameters cause the feedback to be dynamic. It may be remarked here that considering $\lambda_{y,t} = \lambda_y$ and $\lambda_{\theta,t} = \lambda_\theta$, constant feedback model (SVCF) can be obtained. Clearly the volatility asymmetry can now be interpreted in terms of the market sentiment impacts which has been discussed in detail in subsection 2.3.

The SVDF model postulated in equations (

The assumption of dynamic nature of impact parameters ($\lambda_{y,t}$ and $\lambda_{\theta,t}$) in SVDF model immediately results in a serious issue of violating the weak stationarity of the auto-regressive volatility process as stated in (

2.1 Bounded Stationarity For Non-Stationary Process

The bounded stationarity of a discrete time stochastic process is defined as follows.

Bounded Stationarity: Let X_t be a discrete time stochastic process such that its 1st and 2nd moments exist. The process is defined to be *bounded stationary* if $E[X_t] < M$ and

$Cov(X_t, X_{t-k}) < V$; M and V being finite real numbers and k is any integer.

Taking $k = 0$ in the above definition we get the condition $V(y_t) < V$ on the variance for bounded stationarity.

Remark: Notice that, if the 1st and 2nd order moments of a bounded stationary time series are constant, then the series is weak stationary. Further suppose the 1st and 2nd moments of a locally weak stationary series, viz. $y_{\tau_1}, y_{\tau_2}, \dots, y_{\tau_T}$, $\tau \in I$ (an index set), be given by μ_τ and σ_τ^2 . If μ_τ and σ_τ^2 are finite for all $\tau \in I$, then setting $M = \sup_{\tau \in I} \mu_\tau$ and $V = \sup_{\tau \in I} \sigma_\tau^2$ we observe that a locally weak stationary series is bounded stationary.

Based on the above definition, the conditions of bounded stationarity for the volatility process in (

The auto-covariance function of lag k is given by $\Upsilon_t(k) = Cov(\theta_{t+k}, \theta_t)$

$$\Upsilon_t(k) = Cov(\theta_{t+k}, \theta_t) = \phi^k \Upsilon_t(0) \leq \phi^k \frac{\sigma^2 + \delta^2 \lambda^2}{1 - \phi^2} \quad \forall k. \quad (2.5)$$

Proof. Let $V(\eta_t) = \sigma^2$ and observe that $E[a_t] = 0$ and $V(a_t) = \sigma^2 + \lambda_t^2 \delta^2$, $\forall t = 1, 2, \dots$. The proof of the results are given as below.

1. Notice that ,

$$\begin{aligned} E[\theta_t] &= \alpha(1 - \phi) + \phi E[\theta_{t-1}] \\ &= \alpha(1 - \phi)(1 + \phi + \phi^2 + \phi^3 + \dots) \end{aligned}$$

so that $E[\theta_t]$ exists finitely if $|\phi| < 1$.

2.

$$\begin{aligned} \Upsilon_t(0) &= \phi^2 V(\theta_{t-1}) + [\sigma^2 + \delta^2 \lambda_t^2] \\ &= \sigma^2(1 + \phi^2 + \phi^4 + \dots) + \delta^2[\lambda_t^2 + \phi^2 \lambda_{t-1}^2 + \phi^4 \lambda_{t-2}^2 \dots] \end{aligned}$$

and if $|\phi| < 1$ then

$$= \frac{\sigma^2}{1 - \phi^2} + \delta^2 \sum_{k=0}^{\infty} \phi^{2k} \lambda_{t-k}^2.$$

Further if for a finite $\lambda \in \mathcal{R}$, the condition $|\lambda_t| \leq \lambda$ hold for all t , then the bound is immediate from the expression of $\Upsilon_t(0)$.

3. The autocovariance function $\Upsilon_t(k)$ is given by

$$\begin{aligned}\Upsilon_t(k) &= E[(\theta_{t+k} - \alpha)(\theta_t - \alpha)] \\ &= \phi E[(\theta_t - \alpha)E[(\theta_{t+k-1} - \alpha)]]\end{aligned}$$

since a_{t+k} is independent of $\theta_j \forall j < t+k$. Thus, by repeated substitutions we get

$$\Upsilon_t(k) = \phi^k \Upsilon_t(0).$$

The bound on the auto-covariance function follows from (

- Remark:**
1. The condition for bounded stationarity in this case is given by $|\lambda_t| \leq \lambda, \lambda \in \mathcal{R}$.
 2. The auto-correlation function is time invariant and depends only on the lag which is similar to the weak stationary time series.
 3. The upper bound of the auto-covariance function dampens to zero as the lag increases. Thus, similar to weakly stationary series, the impact of the past realizations decreases with the time horizon. However, unlike the weak stationary series, the auto-covariance of a bounded stationary AR process may not reduce to a time invariant constant with increasing lag.
 4. The k-period ahead forecast for such a series is given by $\hat{\theta}_{t+k} = \alpha + \phi^k(\theta_t - \alpha)$ so that $\lim_{k \rightarrow \infty} \hat{\theta}_{t+k} = \alpha$. The forecast error is given by

$$\hat{e}_t(k) = \sum_{j=0}^{k-1} \phi^j a_{t+k-j}$$

5. The forecast error variance is given by $V(\hat{e}_t(k)) = \sum_{j=0}^{k-1} \phi^{2j} V(a_{t+k-j})$. The bounded stationarity condition on λ leads to the following upper bound on forecast error variance.

$$V(\hat{e}_t(k)) \leq \frac{1 - \phi^{2k}}{(1 - \phi^2)^2} (\sigma^2 + \delta^2 \lambda)$$

Notice that the bound tends to $\frac{\sigma^2 + \delta^2 \lambda}{(1 - \phi^2)^2}$, as $k \rightarrow \infty$, *i.e* the bound increases with lag. Further, if $\lambda \rightarrow 0$, the bound on forecast error reduces to $\frac{\sigma^2}{(1 - \phi^2)^2}$. The error variance also increases with the volatility persistence parameter ϕ .

The above discussion ensures that although the auto-regressive volatility process in the proposed SV model is not weakly stationary but the first two moments of the process are bounded and hence the process and its forecast does not explode with increasing lag. In the following section we discuss on the feedback effect for the proposed SVDF model with general innovation distribution under the assumption of bounded stationarity.

2.2 Time Varying Feedback in SVDF Model

The following lemma provides means and variances of the return and volatility under the model postulated in (

Proof. To prove (

To obtain the return variance, denote $e^{\frac{\theta_t}{2}} Z_{y,t}$ in (

Observing that $E[\theta_t | \mathcal{F}_{t-1}] = \alpha + \phi(\theta_{t-1} - \alpha)$ the result (

The above corollary may be helpful in determining the minimum risk premium for options based on returns y_t . Next I provide an expression for the dynamic feedback effect for SVDF model

Theorem 2.1 *Under the model and the assumptions postulated in lemma*

As a consequence of the above theorem the distributions of return and volatility shocks can be characterized as follows.

Corollary 2.2 *The class of distributions that can be considered to model the market factor γ_t and the volatility shock η_t , $\forall t = 1, 2, \dots, T$, are the ones admitting MGF so that the feedback effect and hence the conditional or unconditional return moments exist. However, ϵ_t need not be restricted by such property.*

Remark: *Notice that the feedback will not vanish even if there is no impact of market sentiment on volatility. If the market sentiment has influence only on returns, then the future marginal distribution of return or volatility does not change whereas the future marginal distribution of volatility changes in presence of market sentiment impact on volatility. SV models with only non-zero return impact may be classified as transient feedback SV models where as the same with non-zero impact on both return and volatility could be classified as SV model with persistent feedback.*

In the remaining part of this paper I assume that the MGF of γ_t and η_t exists $\forall t = 1, 2, \dots, T$. Next I describe the leverage effect under the proposed model.

2.3 Time Varying Leverage in SVDF Model

The presence of conditional leverage effect in the proposed model is reflected through the impact of current return on future volatility ((Renault 2009)). The following theorem shows that the conditional expectation of future volatility depends linearly on the current asset return and the direction of the dependence is determined by the return impact parameter as well as the volatility clustering parameter (ϕ).

Theorem 2.3 *In the model described by (*

2.4 Time Varying Skewness in SVDF Model

This subsection attempts to explain the conditional return skewness in terms of the impact parameters. The following theorem provides an expression for the conditional skewness.

Theorem 2.4 *In the model described in (*

In the above expression we notice that the conditional skewness is not dependent on the expected volatility or the persistence. Only the impact parameters and the variance of the volatility distribution contributes to the conditional return skewness. Thus the model disentangles the effect of past volatility from the return skewness.

It is difficult to gain further insight on the dynamic leverage effect without assuming particular distributions for γ_t , ϵ_t and η_t . In the following section we make specific assumptions about the distributions of the market factor and return and volatility innovations.

3 SVDF Model with Gaussian Error Distributions

The SVDF model proposed above aims to capture the skewness in returns and the dynamic nature of the feedback effect together. In this section I first inspect the SVDF model for skewed returns. In particular, I provide the expression for the feedback effect and conditional skewness and their interpretation in terms of the impact parameters using a Gaussian framework.

3.1 Gaussian SVDF Model

I assume that $\gamma_t \sim HN(0,1)$, $HN(0,1)$ being the standard half-normal distribution and $\epsilon_t \sim N(0,1)$, $\eta_t \sim N(0,\sigma^2)$, $\forall t = 1, 2, \dots$ in addition to the assumptions made in the SVDF model. The expression of feedback effect (ρ_t) can be derived in a similar manner as in theorem

In the following theorem the expression for dynamic leverage (ρ_t) is derived under the model postulated in (

As evident from figure

It is clear from the above figures that the interval within which $\kappa_t < 0$ reduces with increasing σ . Thus necessary and sufficient condition for feedback to be negative is translates to either $\lambda_{y,t} < 0$ and $\lambda_{\theta,t}$ lies out side the interval $(\lambda_{\theta,t}^1, \lambda_{\theta,t}^2)$ or the other way around where the limits $\lambda_{\theta,t}^i$, $i = 1, 2$ depend on the variance of the volatility process.

Remark: Figures (

It may be noticed from the above figures that as $\sigma \rightarrow \infty$, the impact surface closes to the constant plane at zero. Observing that very high volatility variance induces positive probability for the event that realization of conditional volatility is far away from its conditional mean. Such a case may happen in times of bubbles and crashes. One possible explanation for such minuscule feedback effect could be that during such time, market factor impacts are outperformed by the random shocks and hence feedback appears insignificant. In the particular case of no impact of market factor on volatility ($\lambda_{\theta,t} \rightarrow 0$), simple algebraic calculation will reveal that $\rho_t \rightarrow \frac{\lambda_{y,t}\sigma}{\sqrt{2\pi e^{\frac{\sigma^2}{4}}(\lambda_{y,t}^2+1)-2\lambda_{y,t}^2}}$, which tends to 0 with increasing σ .

Remark: Notice that for a standard half-normal random variable X ,

$$M_X'''(u) = \frac{d^3}{du^3}M_X(u) = u(u^2 + 3)M_X(u) + \sqrt{\frac{2}{\pi}}(u + 1) \quad (3.6)$$

Hence, from theorem

4 Estimating Stochastic Volatility Feedback and Return Skewness

In this section, I describe likelihood based MCMC estimation of the parameters of the proposed SV model. Let Θ indicate the set of model parameters, *viz.* $\{\lambda_{\mathbf{y}}, \lambda_{\boldsymbol{\theta}}, \alpha, \phi, \sigma^2\}$ where $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_T)'$ and $\mathbf{y} = (y_1, y_2, \dots, y_T)'$. To ensure bounded stationarity of the volatility process in SVDF model, the volatility impact parameters needs to be bounded and the auto-regression parameter ϕ is restricted within (-1,1) a priori. The same could be ensured using suitable prior distributions of λ_{θ_t} with a finite support and ϕ with support (-1,1). The sample information about Θ and the latent volatility $\boldsymbol{\theta}$ is described through the posterior distribution which combines the likelihood, $f(\mathbf{y} | \boldsymbol{\theta}, \Theta)$ and the prior $f(\Theta)$ as follows :

$$f(\Theta | \mathbf{y}, \boldsymbol{\theta}) \propto f(\mathbf{y} | \boldsymbol{\theta}, \Theta)f(\boldsymbol{\theta} | \Theta)f(\Theta)$$

4.1 Convergence of MCMC

In such iterative simulations one critical issue is to correctly assess the convergence of the method. As proposed by (Gelman & Rubin 1992) potential scale reduction factor (psrf) based on multiple independent chains is used in this paper to measure the convergence of iterative simulation. To obtain psrf first n observations are simulated from $m(> 2)$ independent sequences after a sufficient burn-in for each parameter. The variance of the target posterior distribution (V) is then estimated by

$$\hat{V} = \frac{n-1}{n}W + \frac{B}{n}$$

where W is the average within sequence variance of the m independent chains and B is the variance of the means between chains. Notice that expectation of W asymptotically approaches V . Gelman and Rubin's psrf is defined as $\sqrt{\hat{R}} = \sqrt{\frac{n-1}{n} + \frac{B}{nW} \frac{m+1}{m}} \hat{\nu}(W, B, m, n)$ where $\hat{\nu}(W, B, m, n)$ is an adjustment factor tending to unity as $n \rightarrow \infty$ (see Eq. 4 in (Gelman & Rubin 1992)). Since $\sqrt{\hat{R}}$ declines to 1 asymptotically, psrf closer to 1 would indicate convergence of the MCMC simulation of the corresponding parameter.

4.2 Model Adequacy and Complexity

To measure how good a model fits to the data, the proposed one could be compared with a saturated model. In a saturated model a perfect fit to data is obtained by using as many parameters as the number of observations. Rooting from this concept, a frequentist measure of a model fit is defined as the departure of the model from saturated model which is known as deviance. For the SV model proposed in this paper the deviance would be given by

$$D(\Theta) = -2 \log\{f(\mathbf{y} \mid \Theta)\} + 2 \log\{h(\mathbf{y})\}.$$

Here $f(\mathbf{y} \mid \Theta)$ is the conditional likelihood function of the data given the set of parameters Θ and $h(\mathbf{y})$ is a fully specified standardizing term depending only on the observations. On the other hand, model complexity depends on the number of parameters in the model along with the data and priors. Thus large number of parameters add to the complexity of a model

as well. (Spiegelhalter et al. 2002) proposed a measure of model adequacy, called deviance information criteria (DIC), based on posterior mean deviance along with a penalty for model complexity. DIC for the model proposed in this paper is given by

$$DIC = \overline{D(\Theta)} + p_D,$$

where $\overline{D(\Theta)}$ is the posterior mean of deviance and p_D is the penalty for model complexity. p_D is also interpreted as the effective number of parameters which is measured with deviance at posterior mean ($D(\overline{\Theta})$) as follows:

$$p_D = \overline{D(\Theta)} - D(\overline{\Theta}).$$

DIC and p_D can be easily computed from the MCMC output.

The number of unknowns (parameters and volatilities) in a typical SV model are more than the number of observations. Thus p_D and DIC separately play important roles in selecting most appropriate SV model from a set of candidate models ((Berg, Meyer & Yu 2004), (Abanto-Valle, Bandyopadhyay, Lachos & Enriquez 2010), (Tsiotas 2012)). In this paper, I report both DIC and p_D , the former to measure model adequacy and the latter for model complexity.

4.3 Empirical Results

The proposed SV models have been tested with S&P100 daily returns. S&P100 data has been used earlier by (Blair, Poon & Taylor 2001), (Harvey, Ruiz & Shephard 1994) and (Berg et al. 2004) to examine its heteroskedastic volatility. The data considered here is the same as that in (Berg et al. 2004). The data contains 1516 mean corrected daily log-returns on S&P100 observed during the period January, 1993 to December, 1998. Figure

4.3.1 Estimation in Constant Feedback Models

In this sub-section I test the proposed SVCF model and compare it with two more similar SV models. The SVCF model (M1) is as described in (

4.3.2 Estimation Result for SVCF

Table

The model adequacy measure (DIC) reported in the table shows that the proposed SVCF model provides the best fit among the three models considered here. Also the effective number of parameters (p_D) is lowest for the proposed model among the three. This indicates that the proposed SVCF model fits to the S&P100 data with maximum parsimony.

From the figures given in table

Posterior median of the return impact parameter (λ_1) is less in the proposed SVCF model compared to M2 and M3. Further, posterior median of return and volatility impact parameters are of opposite signs in the proposed SVCF model where as M2 indicates both the impacts in the same direction.

Next I compare the model based estimates of S&P100 volatility with the implied volatilities of the same obtained from Chicago board of options exchange (CBOE). CBOE provides annualized implied volatility percentage using S&P100 options (VXO) (CBOE 2009). I transform VXO to daily implied volatility by dividing it with $100 \cdot \sqrt{12}$. On the other hand model based volatilities are estimated using the posterior median of log-volatilities. The model based volatility estimates have been plotted along with VXO in figure

4.4 Estimation Result for SVDF

In this section I consider the dynamic feedback models and apply different extensions of the models described in the previous section incorporating dynamic feedback and skewness parameters. The proposed common factor model is easily obtained from

The psrf figures in the table indicate that simulation has converged for M4 and M5 whereas M6 and M7 could not achieve the same level of convergence. Notice that M4 provides best fit to S&P100 daily returns with lowest DIC and p_D among the four variants considered here. However, model adequacy reduces in dynamic feedback models compared to the constant feedback counterparts and at the same time complexity increases in terms of effective number of parameters (p_D). Observing the fact that convergence may not have

been achieved in models M6 and M7, the results should be interpreted with caution.

Similar to the constant feedback model, posterior medians are considered to be the point estimates of the unknown parameters. Table

The four plot in figure

For the models M4 and M5, I compute the feedback effect from (

5 Discussion

This paper presents a parsimonious single factor SV model that leads to four major insights related to return skewness and feedback.

1. The inter connection between feedback effect and return skewness has been established. Precisely, the skewness of returns has been shown as a perturbation of symmetric return error with a positive “market sentiment” factor common to both return and volatility and the feedback is generated as a result of the shared factor.
2. The model accommodates the dynamic nature of the skewness and feedback effect as mentioned in (Boyer et al. 2010). In particular, the concept of bounded stationarity has been introduced as a generalization of weakly stationary process and the non-stationarity arising out of dynamic skewness has been tackled with the bounded stationarity which enables finite forecasts of risk.
3. The proposed model leads to a simple characterization of the admissible distributions for return and volatility shocks. The findings establish the fact that the common factor and the volatility shocks can’t be characterized by a heavy-tail distribution for which the MGF does not exist.
4. The reaction of the feedback effect to the variance of the volatility process elicits from this single factor model. The interesting fact that could be noticed from the feedback surface plot is that if the volatility process itself has very high variance then the feedback effect is infinitesimal. In particular, large variance of the volatility shock

leads to a non-informative distribution. As per the plot, market sensitivity loses its importance on the risk or volatility in such a condition and in turn generates an infinitesimal effect on future price.

Application of the proposed single factor SV models on S&P100 daily returns shows some additional advantages. Results from MCMC estimation method shows that the single factor model is computationally more efficient compared to some comparable models discussed in this paper in terms of time to convergence and degree of convergence as measured by psrf. Further, the single factor models provide better fit and the complexity measure p_D is lower compared to the comparable models. In-sample volatility estimates seem to replicate the implied volatility pattern over time.

The single factor model gives rise to an interesting problem. Construction of a portfolio involves understanding the risk or volatility of multiple assets and an experienced portfolio manager may have prior knowledge available about the correlation between asset returns and their volatilities within and between different industry sectors. Such expertise may help in eliciting priors for individual feedback in a Bayesian multivariate SV framework. However, assigning priors on individual elements of the correlation matrix does not ensure that at an intermediate stage of simulation the resulting correlation matrix would remain positive definite. On the other hand, if sector specific factors are shared between within-industry assets and the priors for the impact parameters are elicited from the expert knowledge the positive definiteness still may be achieved while incorporating the individual priors.

References

- Aas, K., & Haff, I. H. (2006), “The generalized hyperbolic skew Student’s t-distribution,” *Journal of Financial Econometrics*, 4(2), 275–309.
- Abanto-Valle, C. A., Bandyopadhyay, D., Lachos, V. H., & Enriquez, I. (2010), “Robust Bayesian analysis of heavy-tailed stochastic volatility models using scale mixtures of normal distributions,” *Computational Statistics and Data Analysis*, 54(12), 2883–2898.
- Azzalini, A., & Dalla Valle, A. (1996), “The multivariate skew-normal distribution,” *Biometrika*, 83(4), 715–726.
- Berg, A., Meyer, R., & Yu, J. (2004), “Deviance Information Criterion for Comparing Stochastic Volatility Models,” *Journal of Business and Economic Statistics*, 22(1), 107–120.
- Blair, B., Poon, S., & Taylor, S. (2001), “Modelling S&P100 Volatility: The Information Content of Stock Returns,” *Journal of Banking and Finance*, 25, 1665–1679.
- Bollerslev, T., Litvinova, J., & Tauchen, G. (2006), “Leverage and feedback effects in high frequency data,” *Journal of Financial Econometrics*, 4, 353–384.
- Bollerslev, T., Sizova, N., & Tauchen, G. (2012), “Volatility in Equilibrium: Asymmetries and Dynamic Dependencies,” *Review of Finance*, 16, 31–80.
- Boyer, B., Mitton, T., & Vorkink, K. (2010), “Expected Idiosyncratic skewness,” *The Review of Financial Studies*, 23(1), 169–202.
- CBOE (2009), “The CBOE Volatility Index- VIX,” <http://www.cboe.com/micro/vix/vixwhite.pdf>, .
- Chib, S., Nardari, F., & Shephard, N. (2002), “Markov chain Monte Carlo methods for stochastic volatility models,” *Journal of Econometrics*, 108(2), 281–316.

- Chib, S., Omori, Y., & Asai, M. (2009), "Multivariate Stochastic Volatility," in *Handbook of Financial Time Series*, eds. T. Mikosch, J.-P. Krei, R. A. Davis, & T. G. Andersen, Berlin Heidelberg: Springer, pp. 365–400.
- Christoffersen, P., Heston, S., & Jacobs, K. (2006), .
- Eraker, B., Johanners, M., & Polson, N. G. (2003), "The impact of jumps in volatility and returns," *Journal of Finance*, 53(3), 1269–1300.
- Feunou, B., & Tédongap, R. (2012), "A stochastic volatility model with conditional skewness," *Journal of Business & economic statistics*, 30(4), 576–591.
- French, R. K., Schwert, G. W., & Stambaugh, F. R. (1987), "Expected Stock Returns and Volatility," *Journal of Financial Economics*, 19, 3–29.
- Gelman, A., & Rubin, D. (1992), "Inference from Alternative Simulation Using Multiple Sequences," *Statistical Science*, 7, 457–472.
- Harvey, A. C., Ruiz, E., & Shephard, N. (1994), "Multivariate stochastic variance models," *Review of Economic Studies*, 61, 247–264.
- Harvey, C., & Siddique, A. (1999), "Autoregressive conditional skewness," *Journal of Finance and Quantitative Analysis*, 34, 465–487.
- Jacquier, E., Polson, N. G., & Rossi, P. E. (1994), "Bayesian analysis of stochastic volatility models," *Journal of Business and Economic Statistics*, 12, 371–389.
- Jacquier, E., Polson, N. G., & Rossi, P. E. (2004), "Bayesian analysis of stochastic volatility models with fat-tails and correlated errors," *Journal of Econometrics*, 122(1), 185–212.
- Kim, S., Shephard, N., & Chib, S. (1998), "Stochastic Volatility: Likelihood Inference and Comparison with ARCH Models," *Review of Economic Studies*, 65(3), 361–393.
- Meyer, R., & Yu, J. (2000), "BUGS for a Bayesian analysis of stochastic volatility models," *Econometrics Journal*, 3(2), 198–215.

- Nelson, B. D. (1991), “Conditional heteroskedasticity in asset pricing: a new approach,” *Econometrica*, 59, 347–370.
- Renault, E. (2009), “Moment Based Estimation of Stochastic Volatility Models,” in *Handbook of Financial Time Series*, eds. T. Mikosch, J.-P. Krei, R. A. Davis, & T. G. Andersen, Berlin Heidelberg: Springer, pp. 269–311.
- Shephard, N., & Andersen, G. (2009), “Stochastic Volatility: Origins and Overview,” in *Handbook of Financial Time Series*, eds. T. Mikosch, J.-P. Krei, R. A. Davis, & T. G. Andersen, Berlin Heidelberg: Springer, pp. 237–254.
- Spiegelhalter, D. J., Best, N. G., Carlin, B. P., & Van Der Linde, A. (2002), “Bayesian measures of model complexity and fit,” *Journal of the Royal Statistical Society. Series B: Statistical Methodology*, 64(4), 583–616.
- Taylor, S. J. (1982), “Financial returns modelled by the product of two stochastic processes a study of daily sugar prices 1961-79,” in *Time Series Analysis: Theory and Practice 1*, ed. O. D. Anderson, Amsterdam: North-Holland, pp. 203–226.
- Tsiotas, G. (2012), “On generalised asymmetric stochastic volatility models,” *Computational Statistics and Data Analysis*, 56(1), 151–172.

A

Table 1: Summary Statistics for S&P100 returns

Series	Mean	Median	Min	Max	Skewness	Kurtosis
S&P100	0.0003191	0.0003041	-0.03264	0.02435	-0.5878	11.90426

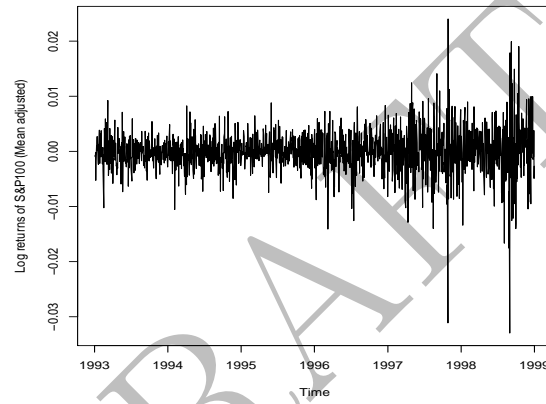


Figure 1: Time plot of S&P100 returns over 1993-1998

Table 2: Parameter Estimates for S&P100 Returns

Parameter	M1	M2	M3
α	-11.716 (0.1058, 1.0003)	-9.6294 (2.848,1.001)	-11.8 (0.1667, 1.0521)
ϕ	0.8277 (0.0306, 1.0008)	0.94193 (0.0087,1.0008)	0.8148 (0.0377, 1.0638)
σ	0.4998 (0.062, 1.0016)	0.53443 (0.05402,1.01)	0.5582 (0.074, 1.1326)
λ_1	-0.4072 (0.1911, 1.0018)	-0.6925 (0.291,1.0149)	-0.5866 (0.3378, 1.0149)
λ_2	0.098 (0.1585, 1.008)	-0.0148 (0.1053, 1.0015)	- (-)
ρ	- (-, -)	-0.0054 (0.00047, 1.0043)	0.0002 (0.0004, 1.043)
p_D	522.2921	1759.247	778.2841
DIC	-12883.86	-8941.212	-12754.21
Iterations	30000 (1,2.8 Hrs)	30000 (40,1.7 days)	90000 (20,1.7 days)

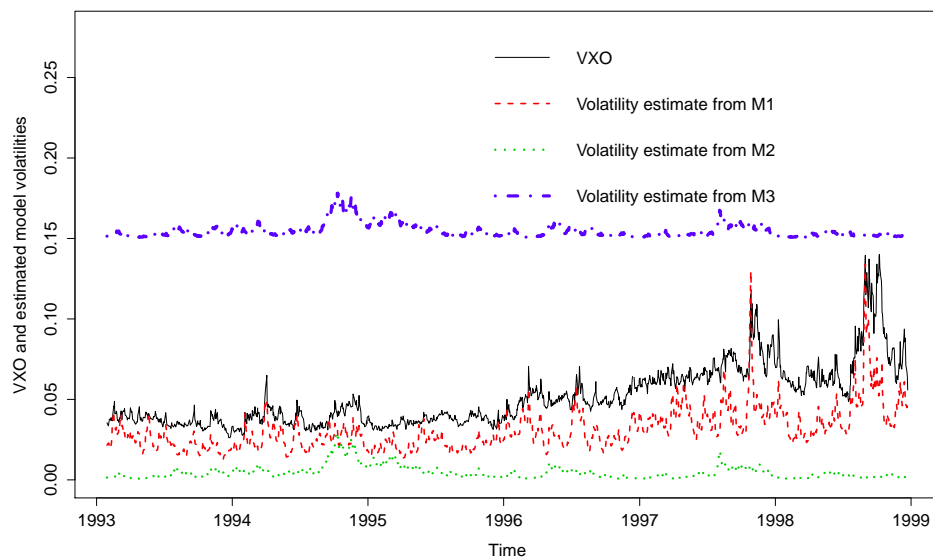


Figure 2: Implied volatility (VXO) and model based estimated volatility of of S&P100

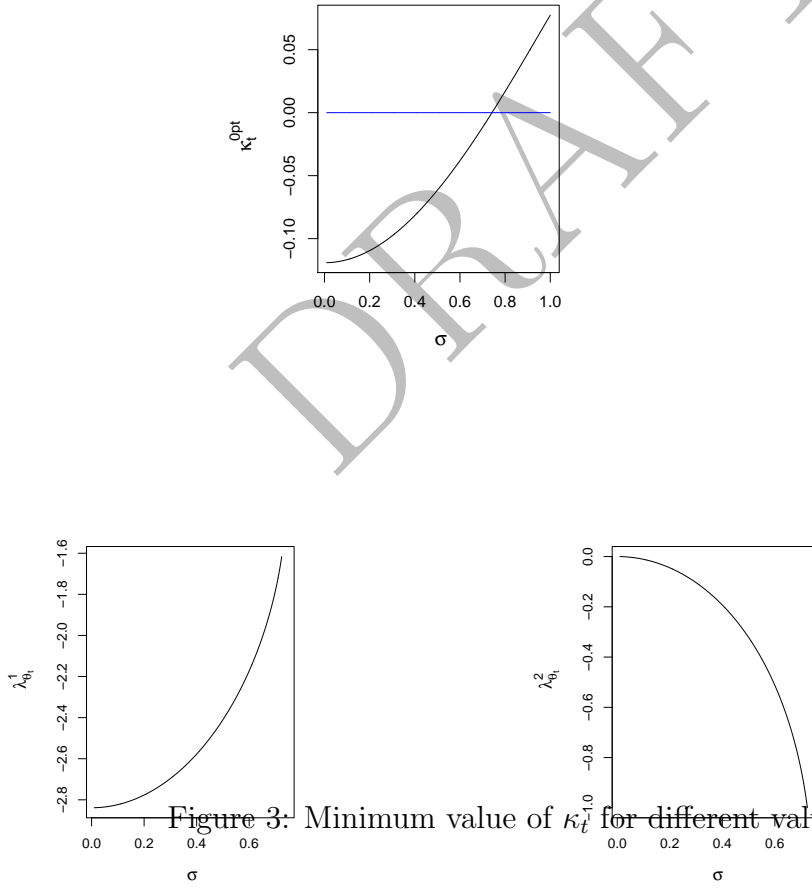


Figure 3: Minimum value of κ_t for different values of σ

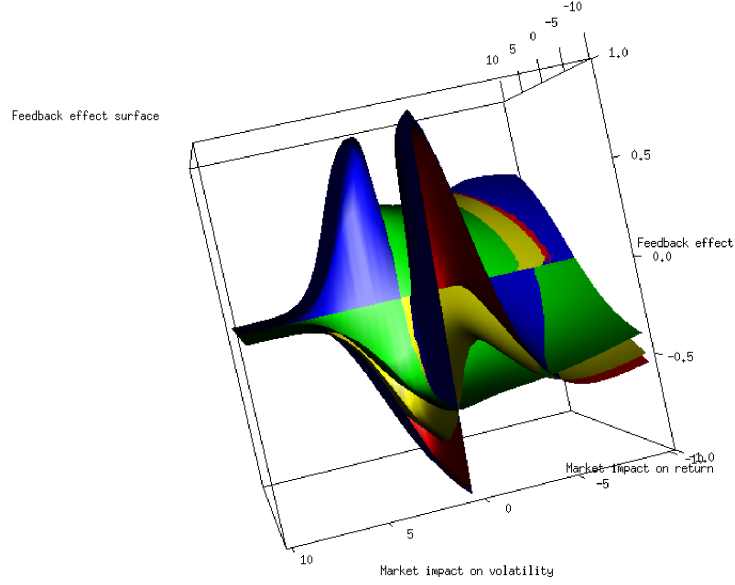


Figure 4: Feedback surface plots for $\sigma=0.001$ (blue), 0.1(yellow), 1(red), 2(green)

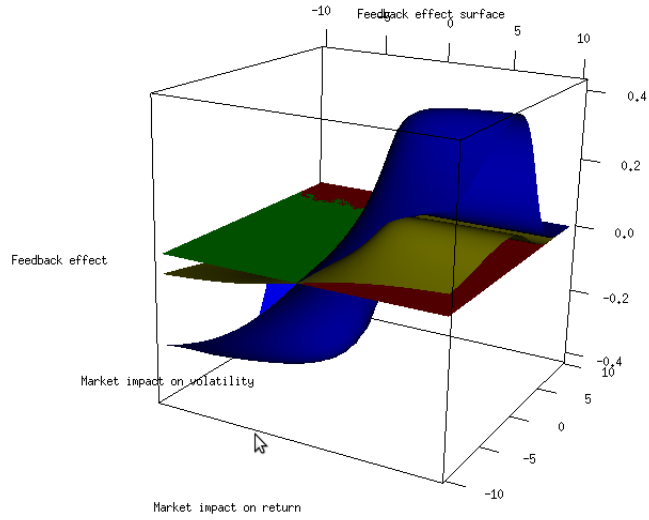


Figure 5: Feedback surface plots for $\sigma=3$ (blue), 5 (yellow), 10 (red), 20 (green)

Table 3: Parameter Estimates for S&P100 Returns with dynamic Feedback

Parameter	M4	M5	M6	M7
α	-12.004 (0.0955, 1.0)	-12.22 (0.0767,1.0006)	-10.99 (0.4294,1.333)	-12.28 (0.5881,1.37)
ϕ	0.8174 (0.0339, 1.0034)	0.5865 (0.04399,1.0008)	0.5723 (0.08335, 1.7118)	0.6704 (0.0351,1.60)
σ	0.5161 (0.0631,1.0026)	0.4981 (0.09288,1.001)	0.9783 (0.13138, 1.0758)	0.6019 (0.11934, 1.15)
λ_1	- (-, -)	- (-, -)	-0.00305 (0.1317, 1.001)	- -
λ_2	-0.152 (0.1584, 1.0008)	- (-, -)	- (-, -)	- (-, -)
p_D	1012.442	3244.661	2348.584	2835.136
DIC	-12832.39	-11075.3	-10379.52	-11783.02
Sample Size	30000 (20,10.6 Hrs)	10000 (5, 4.3 hrs)	100000 (1,6.5 hours)	300000 (1,16 hours)

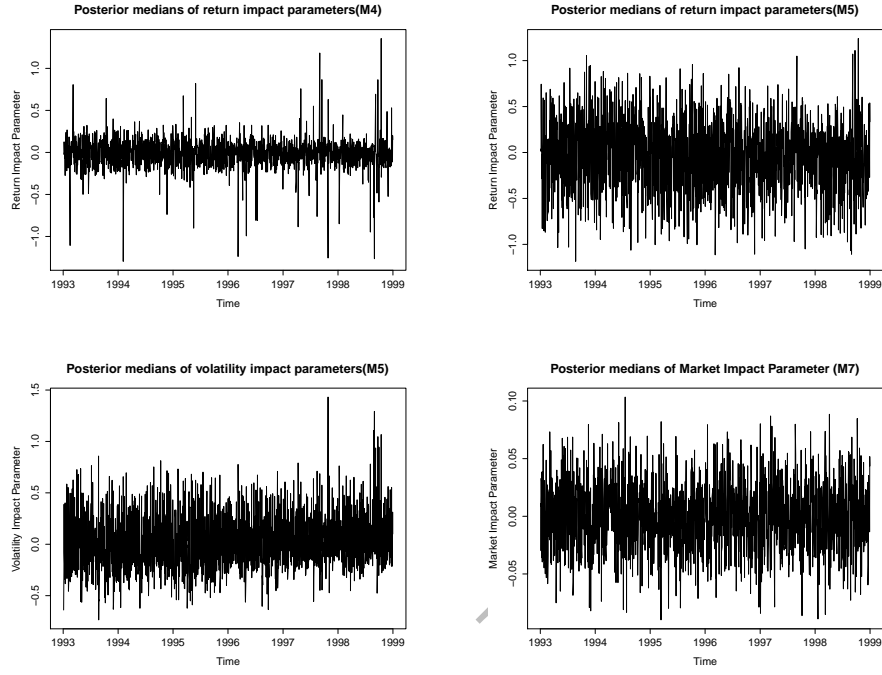


Figure 6: Plot of time-varying impact parameters in M4-M7.

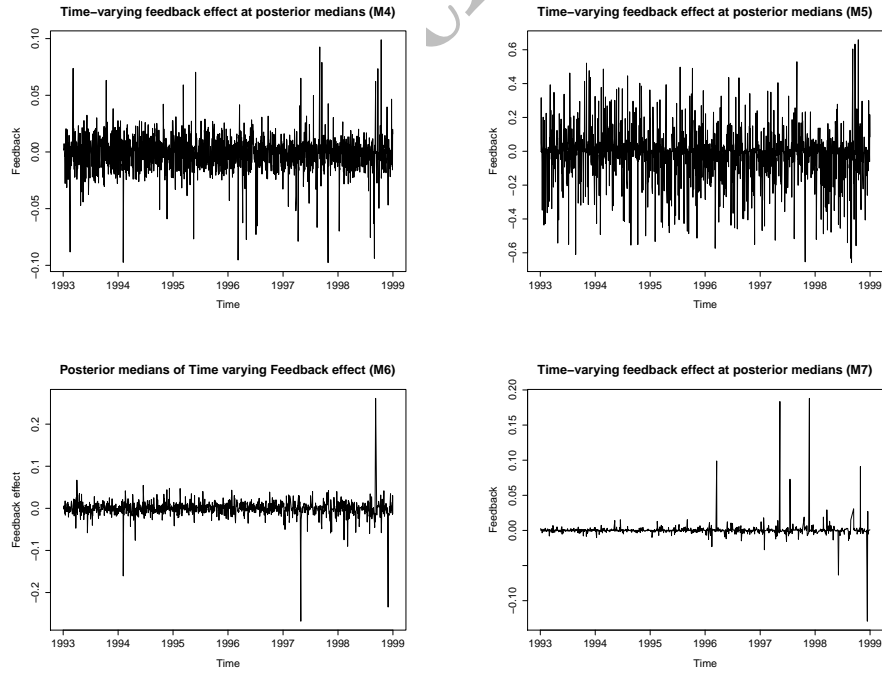


Figure 7: Plot of ρ_t in M4-M7.

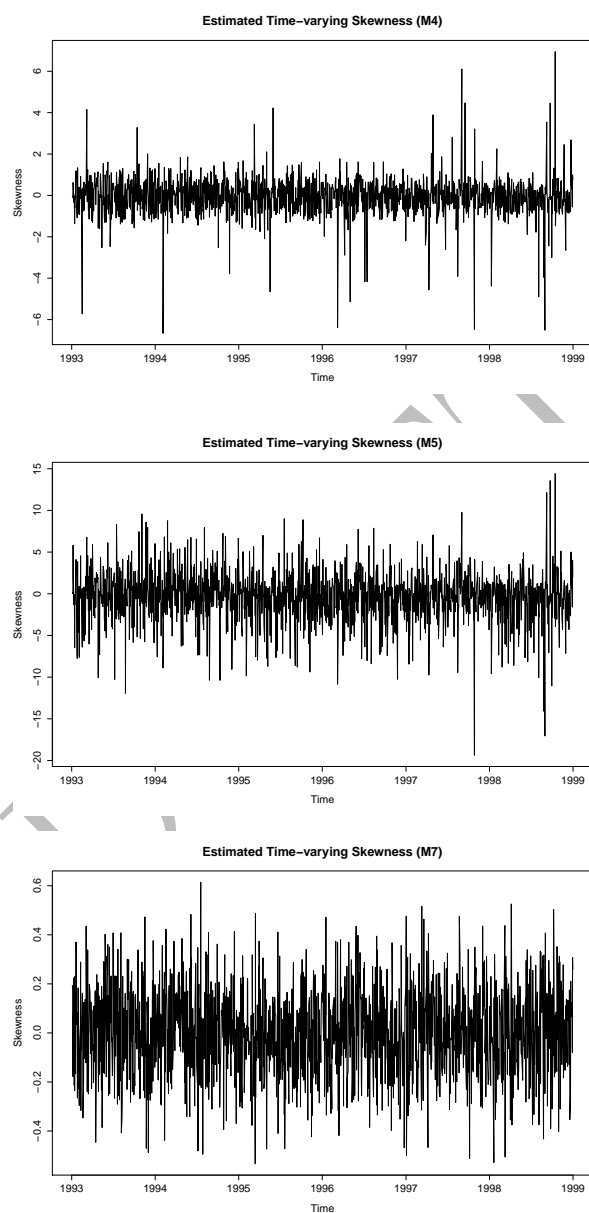


Figure 8: Plot of time-varying skewness in M4, M5 and M7.