The t-distribution Approaches the Normal Distribution as d→∞

It is generally well known that the normal distribution has the pdf $N(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$, with $-\infty < x < \infty$. It is less well known that the *t*-distribution has the pdf $t(x)=\dfrac{\left(\frac{d-1}{2}\right)!}{\sqrt{d\pi}\left(\frac{d-2}{2}\right)!\left(1+\frac{x^2}{d}\right)^{\left(\frac{d+1}{2}\right)}}$, with $-\infty < x < \infty$ and *d* representing the degrees of freedom. With a little calculus and a TI-83, we can provide a convincing argument that *N*(*x*) is the limiting distribution for *t*(*x*) as the degrees of freedom increase without bound.

To show that $\lim\limits_{d\to\infty}\dfrac{\left(\frac{d-1}{2}\right)!}{\sqrt{d\pi}\left(\frac{d-2}{2}\right)!\left(1+\frac{x^2}{d}\right)^{\left(\frac{d+1}{2}\right)}}=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$, we need to take the problem in two parts. Recall that the limit of a product is the product of the limit, providing the two limits exist. So, we will consider two limit problems. First, consider the functional form $\lim\limits_{d\to\infty}\dfrac{1}{\left(1+\frac{x^2}{d}\right)^{\left(\frac{d+1}{2}\right)}}$ and then address the value of the constant $\lim\limits_{d\to\infty}\dfrac{\left(\frac{d-1}{2}\right)!}{\sqrt{d\pi}\left(\frac{d-2}{2}\right)!}$.

$$\lim_{d\to\infty}\frac{1}{\left(1+\frac{x^2}{d}\right)^{\left(\frac{d+1}{2}\right)}} \text{ and } \lim_{d\to\infty}\frac{\left(\frac{d-1}{2}\right)!}{\sqrt{d\pi}\left(\frac{d-2}{2}\right)!}.$$

The Functional Form $\lim\limits_{d\to\infty}\dfrac{1}{\left(1+\frac{x^2}{d}\right)^{\left(\frac{d+1}{2}\right)}}=e^{-\frac{1}{2}x^2}$

This first limit is easier than the second, so we begin by considering the function in the denominator, $\left(1+\frac{x^2}{d}\right)^{\left(\frac{d+1}{2}\right)}$. The $\lim\limits_{d\to\infty}\left(1+\frac{x^2}{d}\right)^{\left(\frac{d+1}{2}\right)}$ is of the indeterminate form 1^∞. So, we proceed in the standard manner. Consider $y=\left(1+\frac{x^2}{d}\right)^{\left(\frac{d+1}{2}\right)}$, so $\ln y=\left(\frac{d+1}{2}\right)\ln\left(1+\frac{x^2}{d}\right)$. This expression is in the indeterminate form $\infty\cdot 0$, so we rewrite to find $\ln y=\frac{\ln\left(1+\frac{x^2}{d}\right)}{\left(\frac{2}{d+1}\right)}$, which has indeterminate form $\frac{0}{0}$. Now, apply L'Hopital's rule.

We consider $\lim\limits_{d\to\infty}\frac{\left(\frac{\left(\frac{-x^2}{d^2}\right)}{\left(1+\frac{x^2}{d}\right)}\right)}{\left(\frac{-2}{(d+1)^2}\right)}=\lim\limits_{d\to\infty}\left(\frac{-x^2}{d^2}\right)\left(\frac{1}{1+\frac{x^2}{d}}\right)\left(\frac{(d+1)^2}{-2}\right)=\frac{x^2}{2}$.

Then $\lim\limits_{d\to\infty}\ln y=\frac{x^2}{2}$ by L'Hopital's rule and $\lim\limits_{d\to\infty}y=e^{\frac{x^2}{2}}$.

Finally, $\lim\limits_{d\to\infty}\frac{1}{\left(1+\frac{x^2}{d}\right)^{\left(\frac{d+1}{2}\right)}}=\frac{1}{e^{\frac{x^2}{2}}}=e^{-\frac{x^2}{2}}$ which we recognize as the functional structure of the Normal distribution.

A Numerical Approach to $\lim\limits_{d\to\infty}\frac{\left(\frac{d-1}{2}\right)!}{\sqrt{d\pi}\left(\frac{d-2}{2}\right)!}=\frac{1}{\sqrt{2\pi}}$

If we can show that $\lim\limits_{d\to\infty}\frac{\left(\frac{d-1}{2}\right)!}{\sqrt{d\pi}\left(\frac{d-2}{2}\right)!}=\frac{1}{\sqrt{2\pi}}$, then we are done. However, this limit is much, much more challenging than the previous one. We can use our TI-83 or 84 (not 89) to approximate this if we choose. Begin by defining the function $y1=\frac{\left(\frac{x-1}{2}\right)!}{\sqrt{x\pi}\left(\frac{x-2}{2}\right)!}$. Now, compute values using the table at integers. You will see that the table looks like the following:

x	1	2	3	4	5	6	7	8
$y1(x)$	0.31831	0.35355	0.36755	0.37500	0.37961	0.38273	0.38499	0.38670
x	9	10	20	30	40	50	60	70
$y1(x)$	0.38803	0.38911	0.39399	0.39563	0.39646	0.39695	0.39728	0.39752
x	80	90	100	110	120	130	140	150
$y1(x)$	0.39770	0.39784	0.39795	0.39804	0.39811	0.39818	0.39823	ERROR

The calculator breaks down at d = 141, however, we can easily believe that the value of the function is converging to something a little less than 0.4. The expected value is $\frac{1}{\sqrt{2\pi}} \approx 0.39894$. So, it is certainly plausible that the product of these two functions converges to $N(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$. For many students, a plausibility argument is enough. Others want a more formal, analytic development.

Also, we don't really need to worry about proving that $\frac{1}{\sqrt{2\pi}}$ this is the correct constant, since all that matters is the form of the limiting density function. We know that the constant must make the integral equal to 1, so since the density function for $N(0,1)$ is proportional to $e^{-t^2/2}$. That information alone requires the constant of proportionality to be $\frac{1}{\sqrt{2\pi}}$.

Analytic approach to $\lim_{d\to\infty} \frac{\left(\frac{d-1}{2}\right)!}{\sqrt{d\pi}\left(\frac{d-2}{2}\right)!} = \frac{1}{\sqrt{2\pi}}$

Warning: The analytic approach is not for the faint of heart. We need Stirling's approximation for the factorials, $n! \approx \sqrt{2\pi n}\, e^{-n} n^n$ (see *Calculus and Factorials* at http://courses.ncssm.edu/math/Talks/index.htm for a derivation of Stirling's formula).

We begin by pulling out the constant $\frac{1}{\sqrt{\pi}}$ to simplify the work. So,

$$\lim_{d\to\infty} \frac{\left(\frac{d-1}{2}\right)!}{\sqrt{d\pi}\left(\frac{d-2}{2}\right)!} = \left(\frac{1}{\sqrt{\pi}}\right) \lim_{d\to\infty} \frac{\left(\frac{d-1}{2}\right)!}{\sqrt{d}\left(\frac{d-2}{2}\right)!}$$

and we need to show that $\lim_{d\to\infty} \frac{\left(\frac{d-1}{2}\right)!}{\sqrt{d}\left(\frac{d-2}{2}\right)!} = \frac{1}{\sqrt{2}}$.

If we rewrite using Stirling's formula and separate the limit of the product into the product of three limits, then

$$\lim_{d\to\infty}\frac{\sqrt{2\pi\left(\frac{d-1}{2}\right)}\,e^{\left(\frac{d-1}{2}\right)}\left(\frac{d-1}{2}\right)^{\left(\frac{d-1}{2}\right)}}{\sqrt{d}\,\sqrt{2\pi\left(\frac{d-2}{2}\right)}\,e^{\left(\frac{d-2}{2}\right)}\left(\frac{d-2}{2}\right)^{\left(\frac{d-2}{2}\right)}}$$

$$=\left(\lim_{d\to\infty}\frac{\sqrt{2\pi\left(\frac{d-1}{2}\right)}}{\sqrt{2\pi\left(\frac{d-2}{2}\right)}}\right)\left(\lim_{d\to\infty}\frac{e^{\left(\frac{d-1}{2}\right)}}{e^{\left(\frac{d-2}{2}\right)}}\right)\left(\lim_{d\to\infty}\frac{\left(\frac{d-1}{2}\right)^{\left(\frac{d-1}{2}\right)}}{\sqrt{d}\left(\frac{d-2}{2}\right)^{\left(\frac{d-2}{2}\right)}}\right).$$

We can take each of the three limits one at a time. The first is quite straightforward,

$$\lim_{d\to\infty}\frac{\sqrt{2\pi\left(\frac{d-1}{2}\right)}}{\sqrt{2\pi\left(\frac{d-2}{2}\right)}}=1.$$

The second requires a bit of rewriting,

$$\lim_{d\to\infty}\frac{e^{-\left(\frac{d-1}{2}\right)}}{e^{-\left(\frac{d-2}{2}\right)}}=\lim_{d\to\infty}\frac{e^{-\left(\frac{d}{2}\right)}e^{\frac{1}{2}}}{e^{-\left(\frac{d}{2}\right)}e^{1}}=e^{-\frac{1}{2}}.$$

The last term requires the most work. We rewrite

$$\lim_{d\to\infty}\frac{\left(\frac{d-1}{2}\right)^{\left(\frac{d-1}{2}\right)}}{\sqrt{d}\left(\frac{d-2}{2}\right)^{\left(\frac{d-2}{2}\right)}}=\lim_{d\to\infty}\frac{\left(\frac{d-1}{2}\right)^{\left(\frac{d-2}{2}\right)}\sqrt{\left(\frac{d-1}{2}\right)}}{\left(\frac{d-2}{2}\right)^{\left(\frac{d-2}{2}\right)}\sqrt{d}}=\lim_{d\to\infty}\left(\frac{d-1}{d-2}\right)^{\left(\frac{d-2}{2}\right)}\lim_{d\to\infty}\sqrt{\left(\frac{d-1}{2d}\right)}.$$

The first term is indeterminate of the form 1^{∞}, so it can be evaluated using L'Hopital's Rule.

Let $y=\left(\frac{d-1}{d-2}\right)^{\left(\frac{d-2}{2}\right)}$, so $\ln y=\frac{\ln\left(\frac{d-1}{d-2}\right)}{\left(\frac{2}{d-2}\right)}$.

Consider, $\lim_{d\to\infty} \dfrac{\left(\dfrac{d-2}{d-1}\right)\left(\dfrac{(d-1)-(d-2)}{(d-2)^2}\right)}{\left(\frac{-2}{(d-2)^2}\right)} = \lim_{d\to\infty} \dfrac{1}{2}\left(\dfrac{d-2}{d-1}\right) = \dfrac{1}{2}$. So, $\lim_{d\to\infty} y = e^{\frac{1}{2}}$.

Finally, we have $\lim_{d\to\infty} \sqrt{\left(\dfrac{d-1}{2d}\right)} = \dfrac{1}{\sqrt{2}}$. Putting it all together, we have

$$\lim_{d\to\infty} \frac{\sqrt{2\pi\left(\frac{d-1}{2}\right)}\, e^{\left(\frac{d-1}{2}\right)} \left(\frac{d-1}{2}\right)^{\left(\frac{d-1}{2}\right)}}{\sqrt{d}\ \sqrt{2\pi\left(\frac{d-2}{2}\right)}\ e^{\left(\frac{d-2}{2}\right)} \left(\frac{d-2}{2}\right)^{\left(\frac{d-2}{2}\right)}}$$

$$= \left(\lim_{d\to\infty} \frac{\sqrt{2\pi\left(\frac{d-1}{2}\right)}}{\sqrt{2\pi\left(\frac{d-2}{2}\right)}}\right)\left(\lim_{d\to\infty} \frac{e^{\left(\frac{d-1}{2}\right)}}{e^{\left(\frac{d-2}{2}\right)}}\right)\left(\lim_{d\to\infty} \frac{\left(\frac{d-1}{2}\right)^{\left(\frac{d-1}{2}\right)}}{\sqrt{d}\left(\frac{d-2}{2}\right)^{\left(\frac{d-2}{2}\right)}}\right)$$

$$= (1)\left(e^{-\frac{1}{2}}\right)\left(e^{\frac{1}{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}.$$

This was the desired result. So,

$$\lim_{d\to\infty} \frac{\left(\frac{d-1}{2}\right)!}{\sqrt{d\pi}\ \left(\frac{d-2}{2}\right)!\left(1+\frac{x^2}{d}\right)^{\left(\frac{d+1}{2}\right)}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2},$$

and the *t*-distribution approaches the standard normal distribution as the degrees of freedom increase without bound.