Introduction to Contraction Theory

Seminar 2008

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Overview

1. Introduction and Basic Theorems

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Introductions

Some Definitions

We are considering *n*-dimensional deterministic nonlinear systems of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$$

- with $\mathbf{x} \in \mathbb{R}^n$ being the the state vector

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- Note: may also represent closed-loop dynamics of system with state feedback $\mathbf{u}(\mathbf{x}, t)$

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Fluid Mechanics Interpretation

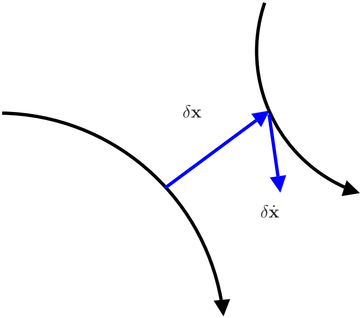
$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ can be seen as an n-dimensional fluid flow, where $\dot{\mathbf{x}}$ is the n-dimensional "velocity" vector at the n-dimensional position $\mathbf{x}$ and time t.

The path to Contraction Theory

With $\delta \mathbf{x}$ being a virtual displacement (= infinitesimal displacement at fixed time) we define a well defined differential relation:

$$\delta \dot{\mathbf{x}} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}, t)\, \delta \mathbf{x}$$

Virtual dynamics of neighboring trajectories



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the associated quadratic tangent form of $\delta \mathbf{x}$ is $\delta \mathbf{x}^T \delta \mathbf{x}$. Looking at rate of change of the quadratic distance between to neighboring trajectories:

$$\frac{d}{dt}(\delta \mathbf{x}^T \delta \mathbf{x}) = 2\delta \mathbf{x}^T \delta \dot{\mathbf{x}} \overset{\text{from above}}{=} 2\delta \mathbf{x}^T \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \delta \mathbf{x}$$

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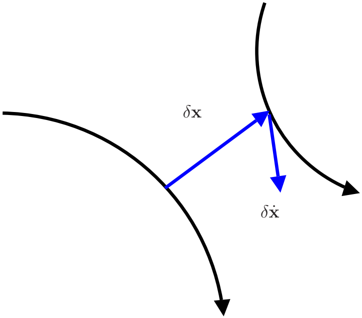

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Now we want to have a negative rate of change = trajectories converge

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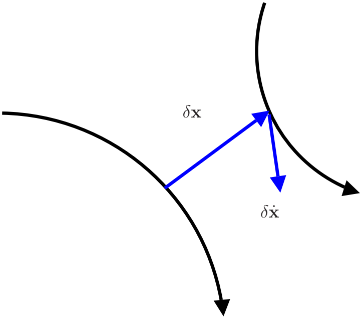

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$$\frac{d}{dt}(\delta\mathbf{x}^T\delta\mathbf{x}) = 2\delta\mathbf{x}^T \underbrace{\frac{\partial\mathbf{f}}{\partial\mathbf{x}}}_{\text{Jacobian}} \delta\mathbf{x}$$

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$$\frac{d}{dt}(\delta \mathbf{x}^T \delta \mathbf{x}) \leq 2\lambda_{max} \delta \mathbf{x}^T \delta \mathbf{x}$$

and hence,

$$\|\delta \mathbf{x}\| \leq \|\delta \mathbf{x}_0\| e^{\int_0^t \lambda_{max}(\mathbf{x},t)dt}$$

if $\lambda_{max}(\mathbf{x}, t)$ is uniformly strictly negative then $\|\delta \mathbf{x}\|$ converges exponentially to zero.

Definition

Given the systems equations $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$, a region of the state space is called a **contraction region** if the Jacobian $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ is uniformly negative definite in that region.

Theorem

Given the systems equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$, any trajectory, which starts in a ball of constant radius centered about a given trajectory and contained at all times in a contraction region, remains in that ball and converges exponentially to this trajectory.
Furthermore, global exponential convergence to the given trajectory is guaranteed if the whole state space is a contraction region.

Examples

Example

Consider the system

$$\dot{x} = -x + e^t$$

and the Jacobian $\frac{\partial f}{\partial x} = -1$ which is globally negative definite.

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Consider the system

$$\dot{x} = -t(x^{3} + x)$$

and the Jacobian $\frac{\partial f}{\partial x} = -t(3x^{2} + 1)$ which is globally negative definite for $t \geq t_0 \geq 0$.

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- where $\mathbf{\Theta}(\mathbf{x}, t)$ is a square matrix and uniformly positive definite.

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$$\delta \mathbf{z}^T \delta \mathbf{z} = \delta \mathbf{x}^T \mathbf{M} \, \delta \mathbf{x}$$

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$$\delta\mathbf{z}^T\delta\mathbf{z} = \delta\mathbf{x}^T\mathbf{M}\,\delta\mathbf{x}$$

- with $\mathbf{M}(\mathbf{x}, t) = \mathbf{\Theta}^T\mathbf{\Theta}$ representing a symmetric and continuously differentiable metric.

Generalization - cont.

Same steps as before:

- Calculating the time derivative of $\delta\mathbf{z}$

$$\frac{d}{dt}\delta\mathbf{z} = \left(\dot{\mathbf{\Theta}} + \mathbf{\Theta}\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)\mathbf{\Theta}^{-1}\delta\mathbf{z} = \mathbf{F}\delta\mathbf{z}$$

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$$\frac{d}{dt}(\delta \mathbf{z}^T \delta \mathbf{z}) = 2 \delta \mathbf{z}^T \mathbf{F} \delta \mathbf{z}$$

Definition

Given the systems equations $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$, a region of the state space is called a **contraction region** with respect to a uniformly positive definite metric $\mathbf{M}(\mathbf{x}, t) = \mathbf{\Theta}^T \mathbf{\Theta}$ if the *generalized Jacobian* $\mathbf{F} = \left(\dot{\mathbf{\Theta}} + \mathbf{\Theta} \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right) \mathbf{\Theta}^{-1}$ is uniformly negative definite in that region.

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Given the systems equations $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$, any trajectory, which starts in a ball of constant radius with respect to the metric $\mathbf{M}(\mathbf{x}, t)$, centered at a given trajectory and containend at all times in a contraction region with respect to $\mathbf{M}(\mathbf{x}, t)$, remains in that ball and converges exponentially to this trajectory. Furthermore, global exponential convergence to the given trajectory is guaranteed if the whole state space is a contraction region with respect the metric $\mathbf{M}(\mathbf{x}, t)$.

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For a linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

the coordinate transformation $\mathbf{z} = \mathbf{\Theta}\mathbf{x}$ (constant!) into a Jordan form.

$$\mathbf{F} = \left(\dot{\mathbf{\Theta}} + \mathbf{\Theta} \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right) \mathbf{\Theta}^{-1} = (\mathbf{0} + \mathbf{\Theta}\mathbf{A}) \mathbf{\Theta}^{-1}$$

and therefore $\mathbf{\Theta}\mathbf{A}\mathbf{\Theta}^{-1}$ has to be uniformly negative definite. This is true if and only if the system is strictly stable.

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FitzHugh-Nagumo model (simplification of the Hodgkin-Huxley model):

$$\begin{aligned} \dot{v} &= c\left(v + w - \frac{1}{3}v^3 + I\right) \\ \dot{w} &= -\frac{1}{c}(v - a + bw) \end{aligned}$$

with c,a and b being some constants, I the input, v the membrane voltage and w the recovery variable. With

$$\mathbf{\Theta} = \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$$

we get following generalized Jacobian:

$$\mathbf{F} = \begin{bmatrix} c(1 - v^2) & 1 \\ -1 & -\frac{b}{c} \end{bmatrix}$$

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- Contractive systems are robust, temporal disturbances vanish exponentially

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Interesting questions raises:

Does the property of contraction hold for bigger systems built out of contractive systems? $\longrightarrow$ the answer is yes!

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<u>**Note**</u>: They can be combined and applied recursively!

One-Way Coupling

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This can be used to extent networks with **chain** or **tree** structures.

Two-Way Coupling

We have two systems coupled like:

$$\dot{\mathbf{x}}_1 - \mathbf{h}(\mathbf{x}_1, t) = \dot{\mathbf{x}}_2 - \mathbf{h}(\mathbf{x}_2, t)$$

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It is possible to design the coupling to have contractive behavior and therefore synchronization.

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So we have a linear combination $\frac{d}{dt}\delta\mathbf{z} = \sum_i \alpha_i(t)\frac{d}{dt}\delta\mathbf{z}_i$ and combined system is contractive again with $\alpha_i > 0$ and same metric.

Parallel Connection Example

Example

Control Primitives with biological control inputs:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + \sum_i \alpha_i(t)\phi_i(\mathbf{x}, t)$$

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$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + \sum_i \alpha_i(t)\phi_i(\mathbf{x}, t)$$

Dynamics $\mathbf{f}$ and primitives ϕ_i all contracting in the same $\mathbf{\Theta}(\mathbf{x})$ and $\alpha_i(t) > 0$ then the whole system is contractive.

Parallel Connection Example

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Note: in general a time-varying combination of stable systems does not have to be stable!

Hierarchical Combination

Consider following virtual dynamics

$$\frac{d}{dt}\begin{pmatrix} \delta \mathbf{z}_1 \\ \delta \mathbf{z}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{F}_{11} & \mathbf{0} \\ \mathbf{F}_{21} & \mathbf{F}_{22} \end{pmatrix} \begin{pmatrix} \delta \mathbf{z}_1 \\ \delta \mathbf{z}_2 \end{pmatrix}$$

and assume the $\mathbf{F}_{21}$ is bounded and $\mathbf{F}_{11}$ and $\mathbf{F}_{22}$ are uniformly negative definite.

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Simple Proof:

The first equation does not depend on the second one and is contractive. $\mathbf{F}_{21}\delta(z)_2$ represents an exponentially decaying disturbance for the second equation. Thus the whole system converges to a single trajectory.

Examples Hierarchies

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Again Motion Primitives:

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the $\alpha_i(t)$ could be outputs of contracting systems of higher up. Again we can guarantee contraction.

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Typical hierarchical processes are chemical chain reactions.

$$\dot{\mathbf{x}} = q(t)(\mathbf{x}_f - \mathbf{x}) + \mathbf{Nr}$$

with $\mathbf{N}$ the reaction rate coefficients, $\mathbf{x} = (c_1 \ldots c_{n-1} T)$ with c_i the chemical concentrations and temperature T, $\mathbf{x}_f$ the corresponding feed vector,$q(t)$ the specific volume flow and r_i the normalized reaction rates.

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Following linear matrix inequalities have to be solved for $\mathbf{M} > \mathbf{0}$

$$\forall i, j \qquad \mathbf{N}_{ij}^T \mathbf{M} + \mathbf{M}\mathbf{N}_{ij} \leq 0$$

Feedback Connection

Two systems

$$\begin{aligned} \dot{\mathbf{x}}_1 &= \mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2, t) \\ \dot{\mathbf{x}}_2 &= \mathbf{f}_2(\mathbf{x}_1, \mathbf{x}_2, t) \end{aligned}$$

in the feedback combination

$$\frac{d}{dt}\begin{pmatrix} \delta\mathbf{z}_1 \\ \delta\mathbf{z}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{F}_1 & \mathbf{G} \\ -\mathbf{G}^T & \mathbf{F}_2 \end{pmatrix}$$

The augmented system is contracting if and only if the separated plants are contracting and under the rather mild assumption:

$$\mathbf{F}_2 < \mathbf{G}^T \mathbf{F}_1^{-1} \mathbf{G}$$

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<u>**Note**:</u> We can usually choose the connection matrix **G**!

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- It is not always easy to find the right metric $\mathbf{M}$.
- It is not trivial to prove negative definiteness of big matrices .
- Lohmiller and Slotine do not show the problems - hard to see what else could be a problem.

For Further Reading

Winfried Lohmiller and Jean-Jacques E. Slotine
On contraction analysis for non-linear systems.
Automatica Vol.34, p683-696, 1998.

J. J. Slotine and W. Lohmiller
Modularity, evolution, and the binding problem: a view from stability theory.
Neural Networks, Vol 14, p137-145, 2001

Wei Wang and Jean-Jacques E Slotine
On partial contraction analysis for coupled nonlinear oscillators.
Biol Cyber, Vol 92, p38-53, 2005

Jean-Jacques E Slotine
Talk about Contraction Theory hold at FIAS Summer School, Theoretical Neuroscience & Complex Systems, Frankfurt, D, August 2007.
can be found in our pdf archive

Winfried Lohmiller and Jean-Jacques E. Slotine
Nonlinear Proces Control using Contraction Theory.
AIChE Journal, Vol 46,Nr:3, p588-596, 2000