Alexandr Mikhailovich Liapunov, The general problem of the stability of motion (1892)

J. Mawhin
Université de Louvain, Institut mathématique, B-1348 Louvain-la-Neuve, Belgium
Telephone +32 10 47 31 71 Fax +32 10 47 25 30 e-mail : mawhin@math.ucl.ac.be

This memoir is recognized as the first extensive treatise on the stability theory of solutions of ordinary differential equations. It is the source of the so-called Liapunov first and second methods.

First publication. *The general problem of stability of motion* (in Russian), Doctoral dissertation, University of Kharkov, Kharkov Mathematical Society, 1892. 250 p.
Reprints. Second edition. Moscow-Leningrad : Academy of Science, 1935, 386 pp. With a portrait, additions of the 1907 French version, Russian translation of the Liapunov 1897 paper and an obituary by Steklov.
Third edition. Moscow-Leningrad : GITTL, 1950, 471 pp. With a portrait, the Liapunov 1893a, Liapunov 1893b and Liapunov 1897 papers.
Fourth edition. Academician A.M. Liapunov Collected Works, (Russian), vol 2, Moscow : Academy of Science, 1956, 7-263. This volume of 472 p. contains all published papers of Liapunov on the stability of solutions of ordinary differential equations and an unpublished list of the seven theses joined to the dissertation.
French translation. (Revised and corrected by the author, with an additional note) *Problème général de la stabilité du mouvement,* (trans. E. Davaux), in *Annales de la Faculté des Sciences de Toulouse* (2) 9 (1907), 203-474. [Edition repr. Princeton : Princeton University Press, Annals of Mathematics Studies No. 17, 1949; Paris : J. Gabay, 1988]
English translation. The general problem of the stability of motion (trans. A.T. Fuller), in *International Journal of Control* 55, No. 3 (1992), Liapunov Centenary Issue, London : Taylor and Francis, with an Editorial by A.T. Fuller, a Biography of Liapunov by V.I. Smirnov and a Bibliography of Liapunov's work by J.F. Barrett.
Linked articles. Lagrange 1788, Thomson and Tait, Poincaré, Birkhoff, Volterra.

1. The author

Alexandr Mikhailovich Liapunov was born in 1857, the son of the astronomer Mikhail Vasilievich Liapunov, who worked at Kazan University before becoming the director of a Lyceum in Yaroslavl. Liapunov's brother Sergei was a composer; another brother, Boris, was a specialist in Slavic philology and became a member of the Soviet Academy of Science.

Liapunov received his elementary education at home before graduating from the Gymnasium of Nizhny Novgorod and entering at the Physics and Mathematics Faculty of St. Petersburg University, where P.L. Chebychev greatly influenced him. He graduated in 1880 and obtained his master thesis in 1884 on *The stability of ellipsoidal forms of equilibrium of a rotating liquid*. He teached mechanics as a Privatdocent at Kharkov university and published there in 1892 (in Russian) his classical memoir *The general problem of the stability of motion*, defending it the same year as a doctoral dissertation at Moscow university.

In 1893, Liapunov became a professor at Kharkov and made researches on mathematical physics, in particular on the Dirichlet problem, and the calculus of probability. In 1901, he was elected as a member of the St. Petersburg Academy of Science, at the seat remained vacant for seven years since the death of Chebychev. In 1917, with the hope of improving the health of his wife, who suffered from a serious form of tuberculosis, Liapunov moved to Odessa, where he teached at the university. But his wife died on October 31 1918 and Liapunov shot himself, surviving his wife only three days. For more biographical informations, see e.g. Grigorian 1974, Smirnov 1992.

Liapunov's work on the stability of solutions of ordinary differential equations started with his doctoral dissertation of 1892 (subsequently refered as *Dissertation*) and covered a period of ten years. The nine other contributions, listed in Barrett 1992, give a few additions to the general theory of stability and substantial complements to the study of linear second order equation with periodic coefficients.

2. The aim and the inspiration of the *Dissertation*

The object of Liapunov's *Dissertation* is clearly indicated in its *Preface* :

> In this work are exposed some methods for the resolution of questions concerning the properties of motion and, in particular, of the equilibrium, which are known under the denominations of

> stability and instability [...]. The problem consists in knowing if it is possible to choose the initial values of the solutions x_s small enough so that, for all values of time following the initial instant, those functions remain, in absolute value, smaller than limits given in advance, as small as we want. When we can integrate our differential equations, this problem does not present real difficulties. But it would be important to have methods which would allow to solve it, independently of the possibility of this integration [...].

Then Liapunov analyzes and criticizes the 'linearization method' usually adopted in stability questions, since the pioneering work of J.L. Lagrange, P.S. Laplace and D. Poisson, by authors like W. Thomson and P.G. Tait, E.J. Routh, and N.E. Zhukovski :

> The procedure usually used consists in neglecting, in the considered differential equations, all the terms of order greater than one with respect to the quantities x_s and to consider, instead of the given equations, the linear equations so obtained. [...] But the legitimity of such a simplification is not justified a priori and [...] if the solution of the simplified problem can give an answer to the original one, it is only under certain conditions, which, generally, are not indicated.

Then Liapunov mentions his principal source of inspiration :

> The unique tentative, as far as I know, of rigourous solution of the question belongs to M. Poincaré, who, in a remarkable memoir *'Sur les courbes définies par les équations différentielles',* and in particular in the last two parts, considers stability questions for differential equations of the second order as well as close questions relative to systems of the third order. Although M. Poincaré restricts himself to very special cases, the methods that he uses allow much more general applications and can still provide many new results. This is what will be shown in what follows because, in a large part of my researches, I have been guided by the ideas developed in the quoted Memoir.

Finally, Liapunov explicits the aim of his *Dissertation* :

> The problem that I have posed to myself, in starting the present study, can be formulated as follows : to indicate cases where the first approximation really solves the stability question, and to give procedures which would allow to solve it, at least in some cases, when the first approximation is no more sufficient.

3. Liapunov's concept of stability

The first chapter of Liapunov's monograph, entitled 'Preliminary analysis', contains precise definitions of the used concepts and the development of the general methods applied in the two subsequent chapters. The solution with initial value x_0 at initial time t_0 of the ordinary differential system (written, in contrast to Liapunov, in vector notation)

$$\frac{dx}{dt} = X(x,t). \tag{1}$$

is denoted by $x(t, t_0, x_0)$.

To define and study the concept of *stability* of a solution $\xi(t)$ of (1), Liapunov first observes that the substitution $x \to \xi + x$ reduces the question to the stability of the zero solution of a system of the type (1) satisfying $X(0,t) \equiv 0$. Liapunov calls this zero solution *stable* if for each $\varepsilon > 0$ and each t_0, one can find $\eta > 0$ such that for each x_0 with $\|x_0\| \leq \eta$ and all $t \geq t_0$, one has $\|x(t, t_0, x_0)\| < \varepsilon$. This is essentially the continuous dependence of the solution on initial conditions, *for all values of* t larger than the initial one.

The zero solution is called *unstable* if it is not stable. The related concept of *uniform stability*, in which η is independent of t_0, has been introduced in 1933 by K.P. Persidskii. At the end of Chapter one, Liapunov gives a refinement of the concept of stability, called to-day *asymptotic stability*, in which, in addition to stability, he requires that $x(t, t_0, x_0) \to 0$ when $t \to +\infty$ for each t_0 and each sufficiently small $\|x_0\|$.

After having proved, by the method of majorants, the existence of convergent series for the solutions of (1) of sufficiently small norm, defined over an arbitrary interval of time, Liapunov introduces a terminology still used today, although maybe in a slightly more restricted sense: the set of all procedures of study of the stability depending upon the obtention of solutions of the perturbed motion in the form of infinite series is called the *first method*. The *second method* consists in all types of procedures which are independent of the obtention of solutions of the differential equations of the perturbed motion.

4. The first method of Liapunov

As one can write $X(x,t) = P(t)x + R(x,t)$, where $R(x,t) = O(\|x\|^2)$, the linear system

$$\frac{dx}{dt} = P(t)x. \tag{2}$$

is called the *linearization* or the *variational equation* of (1) around the zero solution, and the first step consists in studying the stability of its trivial solution, in order to deduce possible information on the stability of the trivial solution of (1).

For this, Liapunov introduces the concept of *characteristic number* of a function $x(t)$ such that $x(t)\exp\lambda_1 t \to 0$ and $x(t)\exp\lambda_2 t \to \infty$ as $t \to +\infty$, for some λ_1 and λ_2. Then, a number λ_0 exists such that, for each $\varepsilon > 0$, $x(t)\exp(\lambda_0+\varepsilon)t \to \infty$ and $x(t)\exp(\lambda_0-\varepsilon)t \to 0$ when $t \to +\infty$. λ_0 is called the *characteristic number* of the function $x(t)$. An equivalent definition

$$\lambda_0 = \lambda(x, \exp t) = -\limsup_{t\to+\infty} \frac{\log|x(t)|}{t}, \tag{3}$$

has been given in 1930 by O. Perron, who proved that the set of characteristic numbers of the linear system (2) contains at most n distinct elements. The negative of the Liapunov characteristic numbers and their analogues for discrete dynamical systems play, under the name of *Liapunov exponents,* an important role in the recent researches on chaos.

When P is constant or periodic, the sum of its characteristic numbers is equal to

$$-\limsup_{t\to+\infty} \frac{1}{t}\int_{t_0}^{t} \Re[\mathrm{tr}\ P(\tau)]\, d\tau, \tag{4}$$

and Liapunov calls *regular* a system satisfying this condition. Their study has been continued by O. Perron, N.G. Cetaev and K.P. Persidskii. An important subclass of regular systems introduced by Liapunov are the *reducible systems*, i.e. systems (2) which can be reduced to a system with constant coefficients through a transformation of the type $x = Q(t)y$, where $Q(t)$ is of class C^1, bounded on $[t_0, +\infty[$ together with the determinant of its reciprocal. They have been studied by N.P. Erugin and I.Z. Shtokalo.

Liapunov is then ready to state and prove the basic theorem of his first method : *If the linearized system is regular and if all its characteristic numbers are positive, then the unperturbed motion is stable, and moreover the perturbed motion tends asymptotically to the unperturbed one when t tends to $+\infty$.*

5. The second method of Liapunov

Liapunov then proceeds to his second method, whose aim is to extend the Lagrange-Dirichlet stability theorem to not necessarily conservative systems. In Liapunov's words (Section 16) :

> Everybody knows the theorem of Lagrange on the stability of equilibrium in the case where a potential exists, as well as the elegant proof which has been proposed by Lejeune-Dirichlet. This last proof rests upon considerations which can be used to prove many other analogous theorems.

Recall that P.G. Lejeune-Dirichlet has proved in 1846, by qualitative arguments, that an equilibrium of an autonomous conservative mechanical system is stable if it is a strict minimum of the potential function *V.* Lagrange's earlier proof, based upon linearization, was insufficient.

After introducing and analyzing the concept of positive definite or negative definite function $V(x,t)$ (a *positive definite* $V(x,t)$ is bounded below by a continuous increasing function $\varphi(|x|)$ vanishing at 0), Liapunov proves his fundamental result : *the trivial solution of system (1) is stable if one can find a definite function* $V(x,t)$ *whose derivative along solutions of (1)*

$$\langle V_x'(x,t)|X(x,t)\rangle + V_t'(x,t) \tag{5}$$

has a fixed sign opposite to that of V*, or is identically zero, in some neighborhood of the origin.* The idea of the very simple proof goes back to Dirichlet, and consists, given $\varepsilon > 0$ and t_0, in taking $\eta > 0$ such that $V(t_0, x_0) < \varphi(\varepsilon)$ whenever $\|x_0\| < \eta$. As $V(t, x(t, t_0, x_0))$ is nonincreasing, assuming the existence of a first $t_1 > t_0$ such that $\|x(t_1, t_0, x_0)\| = \varepsilon$ leads to a contradiction.

Liapunov also observes that if, in addition, *V has an infinitesimal upper bound and a defined derivative along solutions of (1), then the zero solution is asymptotically stable.* Liapunov also proves in this setting two sufficient conditions for *instability,* in terms of properties of some Liapunov functions. They will be refined by many authors, starting with N.G. Cetaev in 1934.

Those types of functions V are nowadays called *Liapunov functions,* and a lot of energy has been used to find ways of constructing such functions. Much emphasis was put also on finding suitable types of stability implying the existence of a suitable Liapunov function (*converse theorems*), in the hands of K.P. Persidskii, I.G. Malkin, J.L. Massera, J. Kurzweil, N.N. Krasovskii, E.A. Barbashin. The second method of Liapunov, which is also useful to study various types of *asymptotic behavior of solutions* of differential equations, is often refered as *Liapunov's direct method.*

6. The case of autonomous systems

In Chapter two ('Study of steady motions'), Liapunov applies his second method to the special case where the linear approximation has constant coefficients. He first reproves the simple case where the stability or instability follows from the linear approximation. Incidentally, he rediscovers independently some results of H. Poincaré's Ph.D. thesis (1879).

Liapunov observes in Section 35 that the Lagrange-Dirichlet theorem on the stability of a mechanical systems in the presence of a potential

> gives a sufficient condition for stability, consisting in the fact that the potential must reach a minimum at the equilibrium position. But, in proving that this condition is sufficient, this theorem does not allow to conclude to the necessity of the same condition. This is why the following question can be raised : will the equilibrium position be unstable if the potential is not minimum ? In this general form, this question is not solved up to now. But, under some assumptions of rather general character, one can answer it in a precise way.

After a century of research, despite substantial advances (see Rouche-Habets-Laloy 1977 and Hagedorn-Mawhin 1992 for references), the situation of this problem can still be described exactly in Liapunov's words, except when V is analytical, for which case V.M. Palamodov has proved in 1995 the converse of the Lagrange-Dirichlet theorem.

Liapunov then analyzes in detail the situations where the characteristic equation of the linear approximation has one zero root and the other ones have negative real parts, and the case where it has two purely imaginary roots, the other ones having negative real parts. Those cases are refered nowadays as *critical,* as the linear approximation is no more sufficient to decide of the stability of the trivial solution. Liapunov has considered the case of two zero roots for the characteristic equation in a manuscript which has only been published posthumously (Liapunov 1963), and completed by V.A. Pliss (1964). One finds in Liapunov's treatment the germ of the theory of *center manifolds,* a fundamental tool for many contemporary researches on ordinary differential equations and dynamical systems.

At this occasion, Liapunov also states and proves his famous theorem on the *existence of a family of periodic solutions near the origin in the presence of a first integral.* Consider an autonomous differential system

$$\frac{dx}{dt} = X(x), \tag{6}$$

where X is analytic, $X(0) = 0$, $X'(0)$ has a pair of imaginary eigenvalues $\alpha_1 = i\omega$, $\alpha_2 = -i\omega$ for some $\omega > 0$ and the other eigenvalues such that $\frac{\alpha_k}{\alpha_1}$ is not an integer for $3 \leq k \leq n$ (*nonresonance condition*). Assume moreover that the system (6) admits a first integral G with non-vanishing Hessian on the space E spanned by the eigenfunctions associated to $\pm\alpha_1$.

Then Liapunov proves that *for each sufficiently small* ε, *there exists a unique* T*-periodic solution* $x(t;\varepsilon)$ *near* E *with period* $T(\varepsilon)$ *close to* $\frac{2\pi}{\omega}$, *lying in the set* $G(x) - G(0) = \varepsilon^2$ *and such that* $x(t;\varepsilon) \to 0$ *and* $T(\varepsilon) \to \frac{2\pi}{\omega}$ *as* $\varepsilon \to 0$.

An example of J. Moser shows that the nonresonance condition on the α_k is necessary, but A. Weinstein has proved in 1973, that it is superfluous in the case of a Hamiltonian system with the Hessian of the Hamiltonian definite at zero. *Global* versions of the Liapunov theorem on families of periodic solutions have been obtained in the 1980s for the Hamiltonian case, following P. Rabinowitz (1978), using modern techniques of critical point theory. References on the modern local and global developments of Liapunov's work on periodic solutions can be found in (Starzhinskii 1977, Mawhin-Willem 1989). Those results are important in celestial mechanics.

7. The case of periodic systems

The last chapter of Liapunov's monograph ('Study of periodic motions') concentrates on the case where the system (1) depends periodically on t. Then its linearized system (2) has periodic coefficients, say of period ω.

Liapunov starts by recalling the classical Floquet theory for such systems, stating (except for the matrix notations) that the principal matrix solution of (2) (for which $Y(0) = I$) can always been written in the form

$$Y(t) = Q(t)e^{tM}, \tag{7}$$

for some ω-periodic nonsingular matrix $Q(t)$ and some constant matrix M, whose characteristic roots are called the *characteristic exponents* of (2). Consequently,

$$Y(t + \omega) = CY(t), \tag{8}$$

for all values of t, where $C = e^{\omega M}$, and the characteristic roots of C are called the *characteristic multipliers* of (2). Their explicit determination is of course in general impossible, but Liapunov proves a number of their properties, in particular that *the characteristic multipliers of the adjoint system to (2) are the reciprocal of the characteristic multipliers of (2).* He also finds useful informations on the characteristic multipliers when the coefficients of the system satisfy some symmetry conditions, and when (2) is Hamiltonian.

Liapunov also initiates the study of the *second order linear equation*

$$y'' + p(t)y = 0, \tag{9}$$

with ω-periodic coefficient $p(t)$ and finds explicit conditions upon p providing important information on its characteristic multipliers. He proves that *if* $0 \not\equiv p \leq 0$, *the characteristic multipliers of (9) are real, one larger than one and the other one smaller than one.* On the other hand, *if* $0 \not\equiv p \geq 0$, *and if*

$$\omega \int_0^\omega p(t)\, dt \leq 4, \tag{10}$$

the characteristic multipliers of (9) are imaginary and have modulus one. Those results have been generalized and refined by many authors, including O. Haupt, G. Hamel, G. Borg, I.M. Gelfand, V.B. Lidskii, M.G. Krein, V.A. Yakubovich, V.M. Starzhinskii, H. Hochstadt. Many refinements of *Liapunov inequality* (10) has been obtained. See Yakubovich-Starzhinskii 1975.

Finally Liapunov combines his general results of Chapter 1 with his studies of linear periodic systems to prove that, *when (1) is ω-periodic in t, its* *trivial solution is asymptotically stable when all the characteristic multipliers of its linearization have moduli strictly smaller than one, and is unstable if one of them has modulus strictly larger than one.* Like in the autonomous case, he also discusses in length some difficult *critical* cases, where one characteristic multiplier is equal to one or where two characteristic multipliers are imaginary and of modulus one.

8. The influence of Poincaré's work on Liapunov's *Dissertation*

We have seen that, in the *Preface* of his *Dissertation*, Liapunov generously acknowledges Poincaré's influence. In a footnote to this *Preface*, Liapunov quotes Poincaré's King Oskar Prize *Sur le problème des trois corps et les équations de la Dynamique* (1890), as well as the first volume of the *Méthodes nouvelles de la Mécanique céleste* (1892), just published during the printing of the *Dissertation*. Describing later in the *Preface* his method of development of solutions of ordinary differential in power series, Liapunov mentions in a footnote that

> the series under study have been considered, under special conditions, in my memoir 'Sur les mouvements hélicoïdaux permanents d'un corps solide dans un liquide' (Communications de la Société mathématique de Kharkow, 2e série, t. I, 1888). I have learned after that M. Poincaré had considered those series, under the same hypotheses, in his Thesis 'Sur les propriétés des fonctions définies par les équations aux différences partielles' (1879).

This connection is explicited in Section 24 of Chapter II. In the same Chapter, Liapunov underlines the pioneering contributions of Poincaré in his series of memoirs *Sur les courbes définies par une équations différentielle* (1881-86), to what is called to-day the problem of determining the conditions under which an equilibrium of a planar differential system is a *center*, i.e. is surrounded by a one-parameter family of closed orbits. Other results of this series of memoirs are also mentioned in Section 64 of Chapter III. Furthermore, in a footnote ending Section 45 of Chapter II, devoted to periodic solutions, Liapunov observes :

> The question of periodic solutions of nonlinear differential equations is also considered, although with another viewpoint, in the last memoir of Poincaré: 'Sur le problème des trois corps et les équations de la Dynamique' (Acta Mathematica, t. XIII).

In Chapter III, when he states his theorem that a linear canonical system with periodic coefficients has a reciprocal characteristic equation, Liapunov mentions in a footnote of Section 51 that :

> This theorem is also indicated by M. Poincaré in his memoir 'Sur le problème des trois corps et les équations de la Dynamique' (Acta Mathematica, t. XIII, p. 99-100) [...]. But I knew it before the publication of this memoir and, in February 1900, I have communicated it, in the previous form, at the Mathematical Society of Kharkow, with other propositions related to the characteristic equation (Communications de la Société mathématique de Kharkow, 2e série, t. II; report of the meetings).

Summarizing, we see that Liapunov's work has been influenced by Poincaré's one, and often overlaps with further contributions of the French mathematician. In several occasions, and specially in dealing with the question of stability, Liapunov transforms some remarks, made by Poincaré in special situations, into powerful general methods.

If there is common material in the work of Poincaré and Liapunov, there is a lot of differences in their approach and style. Poincaré's insight is mostly geometrical, and Liapunov's one essentially analytical. With respect to style, it is striking to see how organized is Liapunov's *Dissertation*. This contrasts with Poincaré 1892-99, which is a patchwork of descriptions of tools and results, with an amazing wide scope. The comparison between those two giants of differential equations is somewhat reminiscent of the one between Riemann and Weierstrass in their approaches of complex function theory. Their continuators have taken advantages of both styles.

9. The early reception of the work of Liapunov on stability

The third volume of E. Picard's famous *Traité d'analyse* (Picard 1893-96) is almost entirely devoted to the study of differential equations. Its Chapter VIII describes Poincaré's theory of periodic solutions, with a somewhat more extended discussion of the existence of periodic solutions of an autonomous differential system *around an equilibrium.* Some of his uncorrect conclusions are mentioned to Picard by Liapunov, in a letter of January 20, 1895, reproduced as Appendix III of Mawhin 1994, in which Liapunov provides a nice summary in French of his *Dissertation*, and informs Picard about his own results on periodic solutions. Liapunov notices that Picard's reasoning about the existence of periodic solutions near an equilibrium is not conclusive, except in the presence of a first integral. In 1897, Picard presents to the Academy of Science of Paris a note of P. Painlevé, which exhibits a counterexample to Picard's claim, but again proposes a too optimistic existence condition. Liapunov is not mentioned. Despite of Liapunov and Painlevé's remarks, the sections devoted to the periodic solutions near an equilibrium remain unaltered in the subsequent editions of Picard's *Traité*.

In the second edition (1908) of Volume III of his *Traité* however, Picard adds a section to Chapter VIII entitled *De la stabilité et de l'instabilité des intégrales de certaines équations différentielles; théorème de M. Liapounoff sur l'instabilité de l'équilibre.* He refers only to a note of Liapunov published in the *Journal de Liouville* (Liapunov 1897), summarizing some of the concepts and results of the *Dissertation*, and giving new instability conditions based upon the second method. In his work *Sur certaines propriétés des trajectoires en Dynamique* crowned by the *prix Bordin* of the French Academy of Science in 1896, and published the next year in the same issue of the *Journal de Liouville* as Liapunov's note, J. Hadamard studies the stability and asymptotic behavior of the trajectories of a mechanical system, through auxiliary functions similar to Liapunov's ones. Hadamard mentions that the condition he has found for the instability of the equilibrium of a conservative mechanical system, was obtained by Liapunov in 1892, in a memoir

> unfortunately written in Russian, an extract of which having been published in the journal of Jordan in 1897, whose existence was unknown to me when I communicated the above remarks to the Academy of Science.

See Mawhin 1994 for more details and references.

In Italy, T. Levi-Civita already mentions Liapunov's memoir in a paper of 1897 which criticizes, like Liapunov, the work of the British school based upon unjustified linearization. Inspired by classical mechanics, Levi-Civita requires, in the Liapunov-like definition of stability introduced in his main work of 1901, that the conclusion holds for *all* values of t, and not only in the future. He calls this concept the *unconditional Dirichlet-type stability,* to distinguish it from Liapunov's one. Another important aspect of Levi-Civita's contributions is his detailed study of the stability of "transformations", i.e. of mappings, anticipating the modern theory of dynamical systems. See Dell'Aglio-Israel 1989 for more details and references.

10. The later development of Liapunov stability

The contributions of Liapunov to stability were considered important enough by French mathematicians, to be included in some of their traditional large treatises on analysis published in Paris by Gauthier-Villars, like the second edition (1910-15) of E. Goursat's famous *Cours d'analyse mathématique.* The last treatise keeping this tradition seems to be the *Cours d'analyse de l'Ecole Polytechnique* of J. Favard (1960-63). After Bourbaki's influence, Liapunov stability theory is expelled from general treatises of analysis, but is found, besides the specialized monographs, in most books on ordinary differential equations.

In the former Soviet Union, the interest for Liapunov theory seems to start around 1930, with the work of N.G. Cetaev on instability, of K.P. Persidskii on the first method and of I.G. Malkin on the second method. The first treatise on Liapunov stability is published by Cetaev immediately after the Second World War (Cetaev 1946), and has seen four editions. It has been followed, besides many research papers, by some sixty monographs on stability and its application to mechanics and control theory, among which one must mention the classics Malkin 1952, Letov 1955, Zubov 1957, Krasovskii 1959, Aizerman-Gantmacher 1963, Barbashin 1967.

In United States, G.D. Birkhoff, Poincaré's brilliant follower, makes significant contributions to dynamical systems between 1912 and 1945, but Liapunov's work on stability in only briefly mentioned in one or two memoirs. The introduction of Liapunov theory in United States is due to a topologist and algebraic geometer of Russian origin, S. Lefschetz, who starts, during the Second World War, a new career devoted to differential equations and control theory. He creates a strong interest in stability theory among American mathematicians, as examplified by the publication of about twenty monographs, following the first one of R. Bellman (Bellman 1953), among which one should notice Cesari 1959, LaSalle-Lefschetz 1961, Lefschetz 1965, Bhatia-Szegö 1970. In Europe and Japan, the books Hahn 1959, Yoshizawa 1966, Rouche-Habets-Laloy 1977 have been very influential and are now classical.

The techniques of Liapunov have been successfully applied to other classes of equations, like integral or functional-differential equations, differential or evolution equations in Banach spaces, nonlinear parabolic equations, and to discrete dynamical systems and difference equations. Liapunov's techniques and results have important applications in mechanics, control theory, chaos theory, mathematical biology, population dynamics and economics. More than a century after its publication, Liapunov's *Dissertation* remains an invaluable source of inspiration for mathematiciens specialized in differential equations, dynamical systems and their applications. Its first English translation has been published in 1992 !

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Summary of Liapunov's dissertation

The page numbers come from the French translation (Liapunov 1907).
DE = differential equations.

Pages	Topics and methods
6	PREFACE. Concepts of stability and instability. Earlier work (Thomson-Tait, Routh, Joukovsky, Poincaré). Summary of the memoir.
58	I. PRELIMINARY ANALYSIS.
14	GENERALITIES ON THE CONSIDERED QUESTION (Sections 1-5). Stability, unstability. Solutions of DE by power series.
20	ON SOME SYSTEMS OF LINEAR DE (Sections 6-10). Characteristic numbers. Normal systems. Regular systems.
11	ON A GENERAL CASE OF DE OF PERTURBED MOTION (Sections 11-13). Convergent series solutions of DE. The first method.
13	SOME GENERAL PROPOSITIONS (Sections 14-16). Positive and negative definite functions. The second method.
124	II. STUDY OF STEADY MOTIONS.
17	LINEAR DE WITH CONSTANT COEFFICIENTS (Sections 17-21). Construction of a Liapunov function. Canonical systems.
91	DE OF THE PERTURBED MOTION (Sections 22-41). Sufficient conditions for stability and instability. Inversion of Lagrange-Dirichlet stability theorem. Linearization with one zero root or two imaginary roots.
16	PERIODIC SOLUTIONS OF THE PERTURBED MOTION (Sections 42-45).
72	III. STUDY OF PERIODIC MOTIONS.
7	LINEAR DE WITH PERIODIC COEFFICIENTS (Sections 46-47). Floquet theory.
27	SOME PROPOSITIONS ON THE CHARACTERISTIC EQUATION (Sections 48-53). Second order equation. Canonical systems.
31	STUDY OF THE DE OF THE PERTURBED MOTION (Sections 54-64). Sufficient conditions for stability and instability. Linearization with one characteristic factor equal to one. Linearization with two imaginary characteristic factors of modulus one.
7	A GENERALIZATION (Section 65).