Derivation of the Differential Continuity Equation

Eric Mott

2015-10-11

1 Continuity Equation

Starting with the first principle of conservation of mass applied to a control volume:

$$\text{rate of mass accumulation} = \text{mass transfer in} - \text{mass transfer out} \tag{1}$$

This can be expressed mathematically as:

$$\frac{\partial m}{\partial t} = \sum \dot{m}_{in} - \sum \dot{m}_{out} \tag{2}$$

where, $\frac{\partial m}{\partial t}$, is the time rate change of mass, m, inside the control volume, Recall that the mass flow rate, $\dot{m}$, for a fluid is simply the following:

$$\dot{m} = \rho A u \tag{3}$$

where, ρ, is its density, and u, is its velocity flowing through a defined surface of a control volume with cross-sectional area, A.

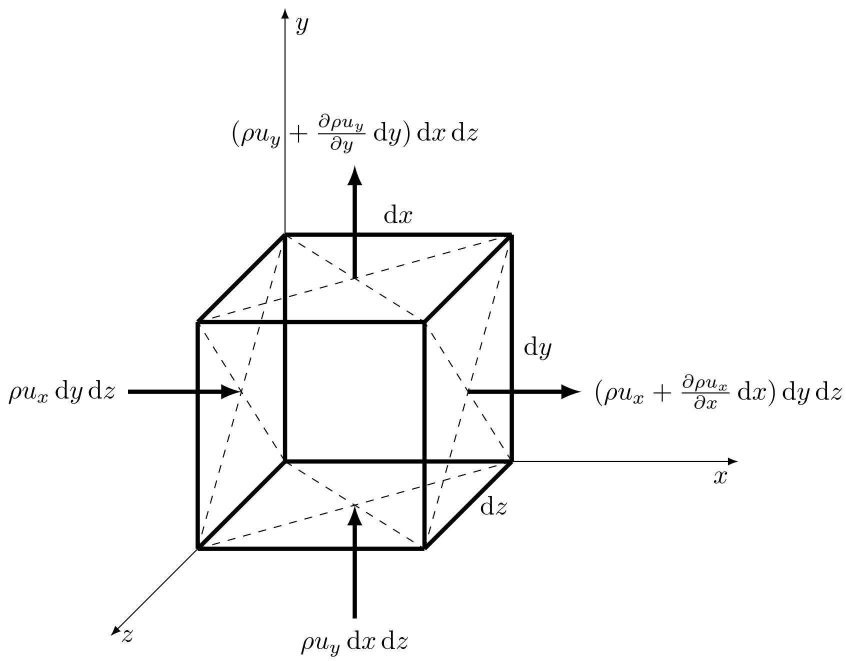
$$\frac{\partial m}{\partial t} = (\rho A u)_{in} - (\rho A u)_{out} \tag{4}$$

For the infinitesimal control volume defined by lengths $\mathrm{d}x$, $\mathrm{d}y$, and $\mathrm{d}z$, mass can be expressed in terms of the density, ρ, as:

$$m = \rho \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \tag{5}$$

$$\frac{\partial \rho}{\partial t} \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \dot{m}_{in} - \dot{m}_{out} \tag{6}$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho \boldsymbol{\nabla} \cdot \boldsymbol{u} \tag{7}$$



y
$(\rho u_y + \frac{\partial \rho u_y}{\partial y}\,\mathrm{d}y)\,\mathrm{d}x\,\mathrm{d}z$
$\mathrm{d}x$
$\mathrm{d}y$
$\rho u_x\,\mathrm{d}y\,\mathrm{d}z$
$(\rho u_x + \frac{\partial \rho u_x}{\partial x}\,\mathrm{d}x)\,\mathrm{d}y\,\mathrm{d}z$
x
$\mathrm{d}z$
z
$\rho u_y\,\mathrm{d}x\,\mathrm{d}z$