DIFFERENTIAL CALCULUS

Function: If a variable y depends on a variable x in such a way that each value of x determine exactly one value of y, then we say that y is a function of x,

$$y = f(x) = x^3 + 3x^2 - 5.$$

Different Types of Functions:

Constant Function:

Let 'A' and 'B' be any two non–empty sets, then a function '*f*' from 'A' to 'B' is called Constant Function if and only if range of '*f*' is a singleton.

Algebraic Function:

The function defined by algebraic expression are called algebraic function.

e.g. $f(x) = x^2 + 3x + 6$

Polynomial Function:

A function of the form $p(x) = a_m x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$

where 'n' is a positive integer and $a_m, a_{n-1}, ...a_1, a_0$ are real number is called a polynomial function of degree 'n'.

Linear Function:

A polynomial function with degree '*t*' is called a linear function. The most general form of linear function is

$$f(x) = ax + b$$

Quadratic Function:

A polynomial function with degree '2' is called a Quadratic function. The most general form of Quadratic equation is $f(x) = ax^2 + bx + c$

Cubic Function:

A polynomial function with degree '3' is called cubic function. The most general form of cubic function is $f(x) = ax^3 + bx^2 + cx + d$

Identity Function:

Let $f : A \to B$ be a function then '*f*' is called on identity function. If $f(x) = x; \forall x \in A$.

Rational Function:

A function $R(x)$ defined by $R(x) = P(x)/Q(x)$, where both $P(x)$ and $Q(x)$ are polynomial function is called, rational function.

Trigonometric Function:

A function $f(x) = \sin x$, $f(x) = \cos x$ etc, then $f(x)$ is called trigonometric function.

Exponential Function:

A function in which the variable appears as exponent (power) is called an exponential function

e.g. (i) $f(x) = a^x$ (ii) $f(x) = 3^x$.

Logarithmic Function:

A function in which the variable appears as an argument of logarithmic is called logarithmic function. e.g. $f(x) = \log_a(x)$

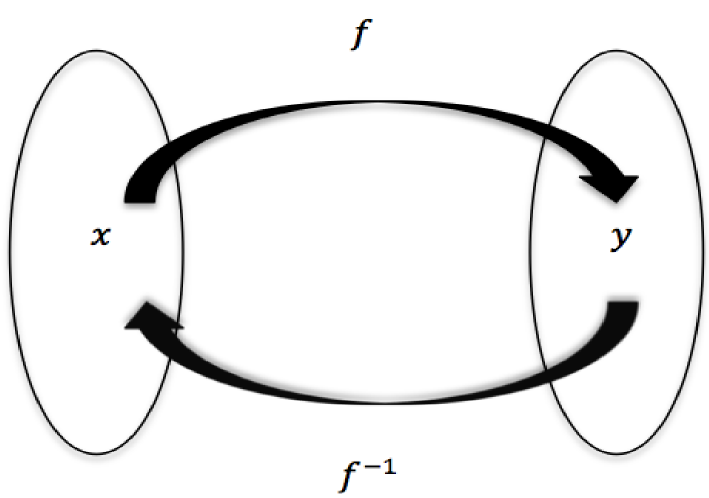
Inverse of function:

Let $f : A \to B$ be a function define by $y = f(x)$, such that $f(x)$ is both one-one and onto. Then there exists an unique function $g : B \to A$, such that

$f(x) = y \Leftrightarrow g(y) = x$, for all $x \in A$ and for all $y \in B$.

In such case a situation g is said to be the inverse of f and we write

$$g = f^{-1} : B \to A.$$



Example: Find the inverse of $y = -2/(x-5)$, and determine whether the inverse is also a function.

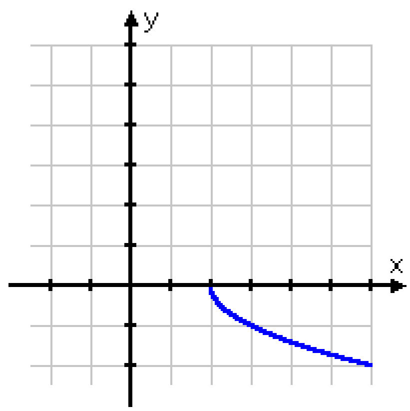
Since the variable is in the denominator, this is a rational function. Here's the algebra:

The original function:	$y = \dfrac{-2}{x-5}$
I multiply the denominator up to the left-hand side of the equation:	$y(x-5) = -2$
I take the *y* through the parentheses:	$xy - 5y = -2$
I get the *x*-stuff by itself on one side of the "equals" sign:	$xy = 5y - 2$
Then I solve for *x*:	$x = \dfrac{5y-2}{y}$
And then switch the *x*'s and *y*'s:	$y = \dfrac{5x-2}{x}$

The inverse function is $y = (5x-2)/x$.

Example-2: Find the inverse of *f*(*x*) = –*sqrt*(*x* – 2), *x* ≥ 2. Determine whether the inverse is also a function, and find the domain and range of the inverse.

The domain restriction comes from the fact that *x* is inside a square root. Usually I wouldn't bother writing down "$x \geq 2$", because I know that *x*-values less than 2 would give me negatives inside the square root. But the restriction is useful in this case because, together with the graph, it will help me determine the domain and range on the inverse:



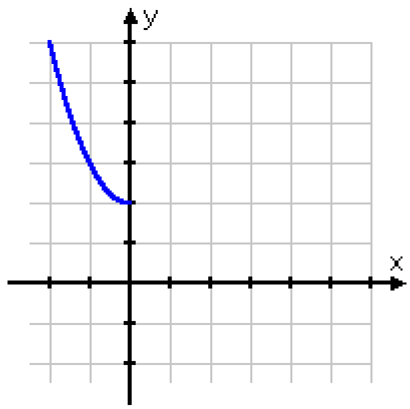
The domain is $x \geq 2$; the range (from the graph) is $y \leq 0$. Then the domain of the inverse will be
$x \leq 0$; the range will be $y \geq 2$. Here's the algebra:

The original function:	$f(x) = -\sqrt{x-2},\quad x \geq 2$
Rename "*f*(*x*)" as "*y*":	$y = -\sqrt{x-2}$

Solve for "$x =$":	$y^2 = x - 2$
	$y^2 + 2 = x$
Switch x and y:	$y = x^2 + 2$
Rename "y" as "f-inverse". Since I already figured out the domain and range, I know which half of the quadratic I have to choose:	$f^{-1}(x) = x^2 + 2, \quad x \le 0$

Then the inverse $y = x^2 + 2$ is a function, with domain $x \le 0$ and range $y \ge 2$.

Here's the graph:



Example 3: Find the inverse of the function $f(x) = \log_e(x+sqrt(x^2+1))$.

Solution: Let $y = \log_e(x + \sqrt{x^2+1})$ then

$$\Rightarrow \; e^y = x + \sqrt{x^2+1}$$

$$\Rightarrow x^2 + 1 = (e^y - x)^2 = e^{2y} - 2xe^y + x^2$$

$$\Rightarrow \; e^{2y} - 2xe^y = 1$$

$$\Rightarrow \; x = \frac{e^{2y} - 1}{2e^y}$$

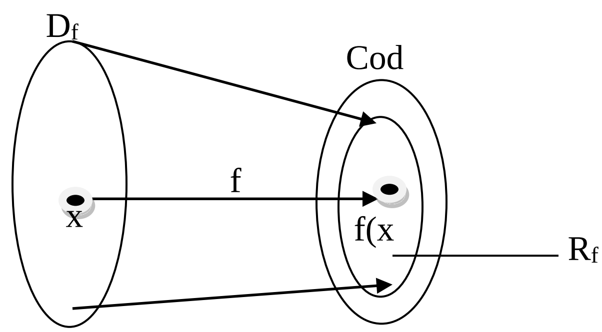
We have, $f^{-1}(x) = \dfrac{e^{2x} - 1}{2e^x} = \dfrac{1}{2}(e^x - e^{-x})$

Domain and Range of a Function

Definition: Domain
The domain of a function is the set of all allowable inputs for a function. For functions whose inputs and outputs are real numbers, the domain is the set of all numbers for which an output number (a real number) can be found.

Definition: Range
The range of a function is the set of all outputs that result from using those inputs that are in the domain. The range is frequently more difficult to determine than the domain.

Domain, Codomain and Range of function



Suppose $f : X \to Y$ is a function. Then X is called domain of f . Y is called codomain or counter domain of f and the subset in Y of all f related element or image is called range of the function which is denoted by D_f, Cod_f and R_f respectively.

Here, $R_f = \{ f(x) \in Cod_f : \forall x \in D_f \}$

i.e. $R_f \subseteq Cod_f$

Some rules to find domain range:
To find domain and range of a constant function of the form $f(x) = c$:
Here $f(x)$ gives real values for all real values of x.

$$D_f = \mathbb{R} \text{ and } R_f = \{c\}$$

To find domain and range of a function of the form

$$f(x) = \frac{1}{ax+b} \; (a \neq 0):$$

Here, $f(x)$ is not defined for $ax + b = 0$ or, $x = -b/a$ and $f(x)$ gives real values i.e. $f(x)$ is defined for all real values of x except $x = -b/a$.

$$D_f = \mathbb{R} - \{-\frac{b}{a}\}$$

Again, $y = f(x) = \dfrac{1}{ax+b} \; (a \neq 0) \Rightarrow ax + b = \dfrac{1}{y}$
Here, x gives real values for all real values of y except $y = 0$.

$$R_f = \mathbb{R} - \{0\}.$$

For example: (i) $f(x) = \dfrac{1}{2x+1}$ (ii) $f(x) = \dfrac{2}{x+3}$

Here, f(x) is not defined for $2x + 1 = 0 \Rightarrow x = \dfrac{1}{2}$ and $f(x)$ gives real values for all real values of x except $x = -1/2$.

$$D_f = \mathbb{R} - \{-\frac{1}{2}\}$$

Again, $y = f(x) = \dfrac{1}{2x+1} \Rightarrow 2x + 1 = \dfrac{1}{y} \Rightarrow x = \dfrac{1}{2}(\dfrac{1}{y} - 1)$
x gives real values for all real values of y except $y = 0$.

$$R_f = \mathbb{R} - \{0\}$$

Class Activity:

Find the domain and range of each function:

(i) $y = \sqrt{x}$

(ii) $y = \sqrt{x-4}$

(iii) $h(x) = 2 + \sqrt{x} - 4$

(iv) $j(x) = \sqrt{x-4} - 5$; Ref: Domain$=[4, \infty)$, Range$=[-5, \infty)$

(v) $y = k(x) = \sqrt{x+3} + \sqrt{4-x}$; [$D_f = [-3, 4]$, $R_f = [2.64575, 3.74166]$]

(vi) $y = |x|$

(vii) $y = -x^2 + 5x - 2$

(viii) $y = \dfrac{x-3}{2x+1}$ [$D_f = \mathbb{R} - \{-1/2\}$, $R_f = \mathbb{R} - \{1/2\}$]

(ix) $f(x) = \dfrac{x-3}{x^2-9}$; [$D_f = \mathbb{R} - \{-3, 3\}$, $R_f = \mathbb{R} - \{0, 1/6\}$]

(x) $y = \dfrac{x^2}{x}$

Example:

$$y = \sqrt{x^2 - 7x + 10}$$

For Domain:

$$y = \sqrt{x^2 - 7x + 10} = \sqrt{(x-2)(x-5)} \ .$$

Here, the value of y will be real if

$$(x-2)(x-5) \geq 0$$
$$\Rightarrow x \leq 2 \ or,\ x \geq 5$$

$$\therefore \quad D_f = \{x : x \leq 2\} \bigcup \{x : x \geq 5\}$$
$$= (-\infty, 2] \bigcup [5, \infty)$$

For Range: $y = \sqrt{x^2 - 7x + 10}$ ....(1)

In (1), the value of y is always positive or zero

$$\Rightarrow y^2 = x^2 - 7x + 10$$
$$. \quad \Rightarrow x^2 - 7x + (10 - y^2) = 0$$

Here, the value of x will be real and it be defined if the discreminent ≥ 0 and y is not less than zero.

$$\Rightarrow 49 - 4(10 - y^2) \geq 0 \qquad [b^2\text{-}4AC>=0]$$

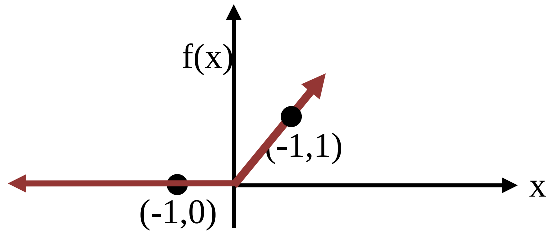
$$\Rightarrow 4y^2 + 9 \geq 0 \Rightarrow y \geq 0$$

Which is true for both inequlities.

$$\therefore R_f = \{y : y \geq 0\} = [0, \infty).$$

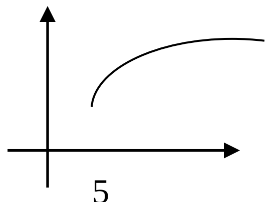
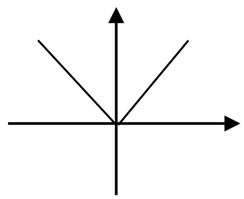
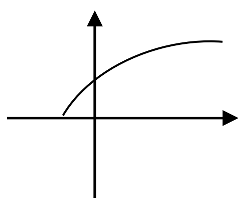
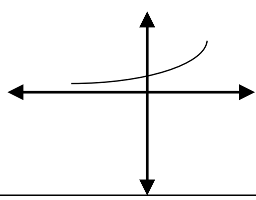
Example: **Draw the graph of the function** $f(x) = \frac{1}{2}(|x| + x)$ **and find its domain and range.**

$$f(x) = \frac{1}{2}(|x| + x) = \begin{cases} \frac{1}{2}(x + x) & when\ x \geq 0 \\ \frac{1}{2}(-x + x) & when\ x < 0 \end{cases} = \begin{cases} x & when\ x \geq 0 \\ 0 & when\ x < 0 \end{cases}$$



$D_f = \mathrm{R}$ and $R_f = [0, \infty)$

Graphical Representation of Functions

$f(x) = \sqrt{x-5} + 1$ 	
$f(x) = \lvert x \rvert$ 	Try yourself: $f(x) = \lvert x-3 \rvert$ $f(x) = \lvert x-3 \rvert + 4$ $y = -\lvert x-3 \rvert - 2$
$f(x) = \sqrt{x+1}$ 	
$f(x) = e^x$, 	$f(x) = \log_a x$

Limit:

A function $f(x)$ is said to tend towards or approach a limit l as x approaches a number 'a' if and only if the absolute value of the difference between $f(x)$ and l is less than any preassigned positive number ε, however small, whenever x approaches 'a' but not equal to 'a'. This is expressed symbolically as

$$Lim_{x \to a} \; f(x) = l$$

Compute $Lim_{x\to\infty} \dfrac{3x^2+3}{5x^2+7x-39}$

Divide top and bottom by x^2, and we get

$$Lim_{x\to\infty} \frac{3x^2+3}{5x^2+7x-39} = Lim_{x\to\infty} \frac{3+3/x^2}{5+7/x-39/x^2}$$
$$= \frac{Lim_{x\to\infty} 3+3/x^2}{Lim_{x\to\infty} 5+7/x-39/x^2} = 3/5.$$

Evaluate $Lim_{x\to 0}\left(\dfrac{\tan x}{x}\right)^{\frac{1}{x}}$ (Try yourself)

Definition of limit $(\delta-\varepsilon)$:
Let $f(x)$ be defined for all x in some open interval containing the number a, with the possible exception that $f(x)$ need not be defined at a. We will write

$$\lim_{x\to a} f(x) = L$$

If given any number $\varepsilon > 0$ we can find a number $\delta > 0$ such that

$$|f(x)-L| < \varepsilon \text{ if } 0 < |x-a| < \delta.$$

Using the definition of $(\delta-\varepsilon)$, prove that $Lim_{x\to 2}(x^3-3x+7) = 9$.

Let $f(x) = x^3-3x+7$ and $|x-2| < \delta, \delta > 0$.

Since $x \to 2$, so $|x-2| < \delta$, where δ is very small positive number.

Then $|x-2| < \delta < 1$.

Now, $|f(x)-9| = |x^3-3x+7-9|$

$$= |x^3-3x+-2|$$
$$= |(x-2)^3+6(x-2)^2+9(x-2)|$$
$$\le |(x-2)|^3+6|(x-2)|^2+9|x-2|$$
$$< |(x-2)|+6|(x-2)|+9|x-2| \quad [\because |x-2|<1]$$

$\therefore\ |f(x)-9| < 16|x-2|$

$\Rightarrow\ |f(x)-9| < 16\delta$

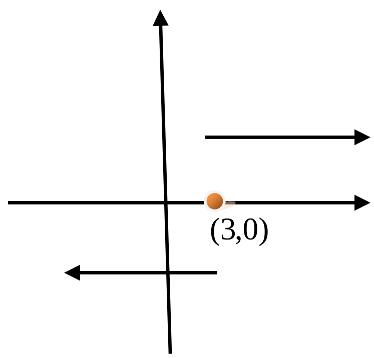
$\Rightarrow\ |f(x)-9| < \varepsilon$, where $\delta = \varepsilon/16$.

So, $\varepsilon > 0$ there exist $\delta > 0$.

Hence, $|x-2|<\delta \Rightarrow |f(x)-9|<\varepsilon$.
Therefore, $Lim_{x\to 2}(x^3-3x+7)=9$.

Question: Let $f(x)=\begin{cases} \dfrac{|x-3|}{x-3}, & x\neq 3 \\ 0, & x=3 \end{cases}$

(a) Graph the function (b) Find $Lim_{x\to 3^+} f(x)$
(c) Find $Lim_{x\to 3^-} f(x)$ (d) Find $Lim_{x\to 3} f(x)$



Solution: (a) For $x>3$, $\dfrac{|x-3|}{x-3}=\dfrac{x-3}{x-3}=1$

For $x<3$, $\dfrac{|x-3|}{x-3}=\dfrac{-(x-3)}{x-3}=-1$

Then the graph, shown in the above figure, consists of lines $y=1,\ x>3;\quad y=-1,\ x<3$ and at the point (3,0).

(b) As $x\to 3$ from the right , $f(x)\to 1$, i.e. $Lim_{x\to 3+} f(x)=1$ as seems clear from the graph. To prove this we must show that given any $\varepsilon>0$, we can find $\delta>0$ such that $|f(x)-1|<\varepsilon$ whenever $0<x-1<\delta$.

(c) As $x\to 3$ from the left, $f(x)\to -1$ i.e. $Lim_{x\to 3^-} f(x)=-1$
(d) Since $Lim_{x\to 3^+} f(x)\neq Lim_{x\to 3^-} f(x)$, $Lim_{x\to 3} f(x)$ does not exist.

Continuity:

A function $f(x)$ is said to be continuous at a point $x=a$ if the following 3 conditions are satisfied:
$f(a)$ is defined that is the value of $f(x)$ at $x=a$ is $f(a)$ is finite.
$Lim_{x\to a} f(x)$ exists.
$Lim_{x\to a} f(x)=f(a)$

A function $f(x)$ is said to be continuous at the point $x=a$ if $f(a)$ is defined i.e. the value of $f(x)$ at $x=a$ is $f(a)$ is finite and

$$Lim_{x\to a+} f(x)=Lim_{x\to a-} f(x)=f(a)$$

Theorem: If a function f is differentiable at some a in its domain, then f is also continuous at a.

Proof: We are given that $Lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ exists, and

we must show that $Lim_{x\to a} f(x) = f(a)$.

This follows from the following computation

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{x\to a}(f(x)-f(a)+f(a)) \\
&= \lim_{x\to a} \frac{f(x)-f(a)}{x-a}.(x-a)+f(a) \\
&= \left\{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}\right\}.\lim_{x\to a}(x-a)+\lim_{x\to a} f(a) \\
&= f'(a).0+f(a) = f(a)
\end{aligned}$$

Every finitely derivable function is continuous.

The following functions are continuous but not differentiable.

Some non-differentiable functions

A graph with a corner.

Consider the function

$$f(x) = |x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

The function is continuous at all x, but it is not differentiable at $x = 0$.
To see this try to compute the derivative at 0,

$$f'(0) = \lim_{x\to 0} \frac{|x|-|0|}{x-0} = \lim_{x\to 0} \frac{|x|}{x} = \lim_{x\to 0} \operatorname{sign}(x)$$

which shows that the limit does not exist.

If we look at the graph of $f(x) = |x|$ then we see what is wrong: the graph has a corner at the origin and it is not clear which line, if any, deserves to be called the tangent to the graph at the origin (Figure 1).

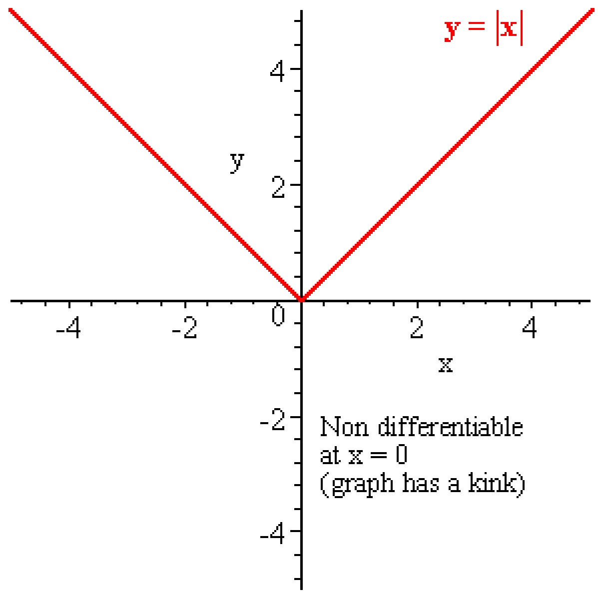


Figure-1

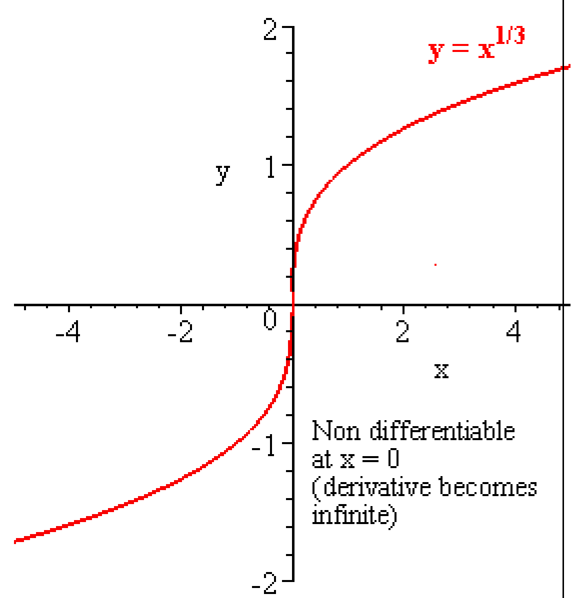
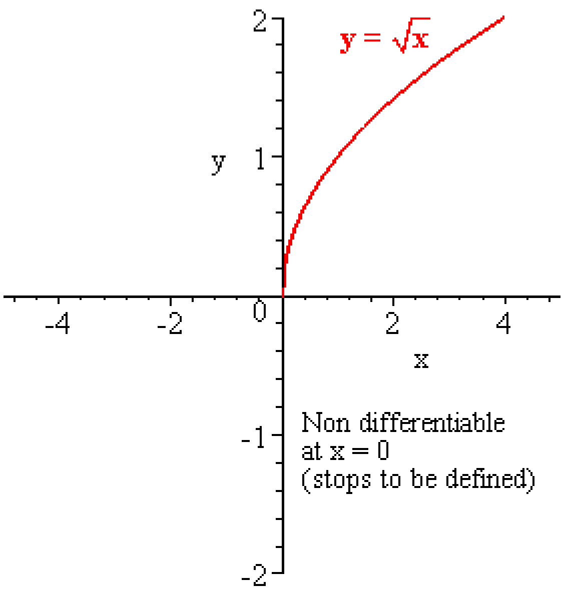


Figure-2

The function can be defined and finite but its derivative can be infinite. An example is $x^{\frac{1}{3}}$ at $x = 0$ (Figure 2).



Problem: Show that the function $f(x)=\sqrt{9-x^2}$ is continuous in the interval $[-3, 3]$.

Solution: Suppose $a \in [-3,3]$

$Lim_{x\to a}\ f(x) = Lim_{x\to a}\ \sqrt{9-x^2} = \sqrt{9-a^2} = f(a)$

Again, $Lim_{x\to -3}\ f(x) = Lim_{x\to -3}\ \sqrt{9-x^2} = 0 = f(-3)$

and $\quad Lim_{x\to 3}\ f(x) = Lim_{x\to 3}\ \sqrt{9-x^2} = 0 = f(3)$

Hence the given function is continuous in the interval $[-3, 3]$

Example: Show that the function $f(x)=|x|+|x-1|+|x-2|$ is continuous at the points $x = 0, 1, 2$.

Solution: Here $$f(x)=\begin{cases} -x-(x-1)-(x-2)=-3x+3 & for\ x<0 \\ x-(x-1)-(x-2)=-x+3 & for\ \ 0\le x<1 \\ x+(x-1)-(x-2)=x+1 & for\ \ 1\le x<2 \\ x+(x-1)+(x-2)=3x-3 & for\ \ x\ge 2 \end{cases}$$

Now, $Lim_{x\to 0^+}\ f(x) = Lim_{x\to 0^+}\ (-x+3) = 3$

$Lim_{x\to 0^-}\ f(x) = Lim_{x\to 0^-}\ (-3x+3) = 3$ and

$f(0) = 3$

Since, $Lim_{x\to 0^+}\ f(x) = Lim_{x\to 0^-}\ f(x) = f(0)$, $f(x)$ is continuous at $x = 0$

Similarly, others…………..

Differentiability:

A function f is said to be differentiable at x_0

if the limit $f'(x_0) = Lim_{h\to 0} \dfrac{f(x_0+h)-f(x_0)}{h}$ exist.

i.e. $Lim_{h\to 0+} \dfrac{f(x_0+h)-f(x_0)}{h} = Lim_{h\to 0-} \dfrac{f(x_0+h)-f(x_0)}{h}$.

Example: If $f(x) = \begin{cases} x & when\ 0 \le x < 1/2 \\ 1-x & when\ 1/2 \le x < 1 \end{cases}$, then show that the function $f(x)$ is continuous at $x = 1/2$ but not differentiable.

Solution: Continuity at $x = 1/2$:

$$Lim_{x\to(1/2)+} f(x) = Lim_{x\to(1/2)+} (1-x) = 1-\frac{1}{2} = \frac{1}{2}$$

$$Lim_{x\to(1/2)-} f(x) = Lim_{x\to\frac{1}{2}-} x = \frac{1}{2}$$

$$f\left(\frac{1}{2}\right) = 1-\frac{1}{2} = \frac{1}{2}$$

Since $Lim_{x\to(1/2)+} f(x) = Lim_{x\to(1/2)-} f(x) = f(x)$. Hence the given function is continuous at $x = 1/2$.

Differentiability at $x = 1/2$:

$$Rf'(1/2) = Lim_{h\to 0+} \frac{f(1/2+h)-f(1/2)}{h}$$

$$= Lim_{h\to 0+} \frac{1-(1/2+h)-(1/2)}{h} = Lim_{h\to 0+} \frac{(-h)}{h} = -1.$$

$$Lf'(1/2) = Lim_{h\to 0-} \frac{f(1/2+h)-f(1/2)}{h} = Lim_{h\to 0-} \frac{(1/2+h)-(1/2)}{h}\ Lim_{h\to 0-} \frac{h}{h} = 1$$

Since $Rf'(1/2) \ne Lf'(1/2)$, so the given function is not differentiable at $x = 1/2$.

Example-2: If $f(x) = |x-2|$, then check the continuity and differentiability of the function $f(x)$ at x=2.

We can define the function in the following way:

$$f(x) = \begin{cases} x-2 & \text{when } x \ge 2 \\ -x+2 & \text{when } x < 2 \end{cases}$$

Continuity at $x = 2$:

when $x > 2$,then $f(x) = x-2$

$$Lim_{x\to 2+}\ f(x) = Lim_{x\to 2+}\ (x-2) = 2-2 = 0$$

when $x<2$,then $f(x) = -x+2$

$$Lim_{x\to 2-}\ f(x) = Lim_{x\to 2-}\ (-x+2) = -2+2 = 0$$

when $x=2$,then $f(x) = x-2$, so $f(2) = 2-2 = 0$

Since $Lim_{x\to 2+}\ f(x) = Lim_{x\to 2-}\ f(x) = f(2)$

So, the function $f(x)$ is continuous at $x=2$.

Differentiability at $x=2$:

when $x>2$,then $f(x) = x-2$. So

$$Rf'(2) = \lim_{h\to 0+} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0+} \frac{(2+h)-2}{h}$$

$$= \lim_{h\to 0+} \frac{h}{h} = 1$$

when $x<2$,then $f(x) = -x+2$. So

$$Lf'(2) = \lim_{h\to 0-} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0-} \frac{(2-h)-2}{h}$$

$$= \lim_{h\to 0-} \frac{-h}{h} = -1$$

Since $Rf'(2) \neq Lf'(2)$. So, the function $f(x)$ is not differentiable at $x=2$.

Problem: If $f(x) = |x-1| + |x+3|$ then discuss the continuity and differentiability of the function $f(x)$ at $x=-3$ and $x=1$.

Given function is

$$f(x) = |x-1| + |x+3|$$

$$= \begin{cases} x-1+x+3 & when \quad x \geq 1 \\ -(x-1)+x+3 & when \quad -3 \leq x < 1 \\ -(x-1)-(x+3) & when \quad x < -3 \end{cases} = \begin{cases} 2x+2 & when \quad x \geq 1 \\ 4 & when \quad -3 \leq x < 1 \\ -2x-2 & when \quad x < -3 \end{cases}$$

To test continuity at x=3:

R.H.L=

Example: Discuss the continuity and differentiability at $x = 0$ and $x = \pi/2$ of the function $f(x) = \begin{cases} 1 & \text{when } x < 0 \\ 1 + \sin x & \text{when } x \le x < \dfrac{\pi}{2} \\ 2 + \left(x - \dfrac{\pi}{2}\right)^2 & \text{when } x \ge \dfrac{\pi}{2} \end{cases}$

Example (Ref. PK-101): A function is defined as $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{when } x \ne 0 \\ 0 & \text{when } x = 0 \end{cases}$

Show that f(x) is continuous and differentiable at x=0.

Continuity Vs Differentiability

Differentiability implies Continuity, Continuity does not imply Differentiability.
If a function f (x) is differentiable at $x = c$ then it must be continuous also at $x = c$
However, if a function is continuous at $x = c$, it need not be differentiable at $x = c$.
And, if a function is not continuous, then it can't be differentiable at $x = c$.

Derivatives:

Definition: Let f be a function which is defined on some interval (c, d) and let a be some number in this interval.
The derivative of the function f at a is the value of the limit

$$f'(a) = Lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \dots\dots\dots.(1)$$

f is said to be differentiable at a if this limit exists.
f is called differentiable on the interval (c, d) if it is differentiable at every point a in (c, d).

- Determine the differential coefficient of (i) $e^{\tan^{-1} x}$ (ii) $x^{\sin x}$ by the first principle rule.

Other notations: putting $x = a + h$ in (1) and let $h \to 0$ instead of $x \to a$, then

$$f'(a) = Lim_{h\to 0}\ \frac{f(a+h)-f(a)}{h}$$

The Differential Rules

Common functions	Function	Derivative
Power rule	x^n	nx^{n-1}
Constant rule	c	$\frac{dc}{dx}=0$
Sum rule:	$(u\pm v)'$	$u'\pm v'$
Product rule:	$(u\,.\,v)'$	$u'.v+u.v'$
Quotient rule:	$\left(\frac{u}{v}\right)'$	$\dfrac{v\frac{d}{dx}(u)-u\frac{d}{dx}(v)}{v^2}$

The chain rule:

If $y=f(u)$, $u=g(v)$ and $v=h(x)$ then $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}$.

For example, if $y=\frac{1}{1+\sqrt{9+x^2}}$, then $y=\frac{1}{1+u}$ where $u=1+\sqrt{v}$ and $v=9+x^2$

So, $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}=-\frac{1}{(1+u)^2}\cdot\frac{1}{2\sqrt{v}}\cdot 2x$

Evaluate the derivatives of the following:

$f(x)=3\left(x^2+x\right)\left(\sin(e^x)\right)$

Solution $\frac{df}{dx}$

$$\begin{aligned}\frac{df}{dx} &= 3\left[\left(x^2+x\right)\left(\sin(e^x)\right)\right]' \\ &= 3\left[\left(x^2+x\right)'\left(\sin(e^x)\right)+\left(x^2+x\right)\left(\sin(e^x)\right)'\right] \\ &= 3\left[\left(2x+1\right)\left(\sin(e^x)\right)+\left(x^2+x\right)\left(\sin(e^x)\cdot e^x\right)\right]\end{aligned}$$

$$= 3\sin(e^x)\left[2x+1+e^x\left(x^2+x\right)\right]$$

$f(x)=\sin\left(e^{(3x+5)^2+\cos(2x)}\right)$

Solution

$$\frac{df}{dx} = \cos\left(e^{(3x+5)^2+\cos(2x)}\right)\cdot\left[e^{(3x+5)^2+\cos(2x)}\right]'$$

$$= \cos\left(e^{(3x+5)^2+\cos(2x)}\right)\cdot\left(e^{(3x+5)^2+\cos(2x)}\right)\cdot\left[(3x+5)^2+\cos(2x)\right]'$$

$$= \cos\left(e^{(3x+5)^2+\cos(2x)}\right)\cdot\left(e^{(3x+5)^2+\cos(2x)}\right)\cdot\left[\left((3x+5)^2\right)'+\left(\cos(2x)\right)'\right]$$

$$= \cos\left(e^{(3x+5)^2+\cos(2x)}\right)\cdot\left(e^{(3x+5)^2+\cos(2x)}\right)\cdot\left[\left(2(3x+5)\cdot 3\right)+\left(-2\sin(2x)\right)\right]$$

$$= \cos\left(e^{(3x+5)^2+\cos(2x)}\right)\cdot\left(e^{(3x+5)^2+\cos(2x)}\right)\cdot\left(6(3x+5)-2\sin(2x)\right)$$

$f(x)=\dfrac{x\sin x}{e^{\cos x}}$

Solution

$$\frac{df}{dx} = \frac{(x\sin x)'e^{\cos x}-(x\sin x)\left(e^{\cos x}\right)'}{\left(e^{\cos x}\right)^2}$$

$$= \frac{\left(x'\cdot\sin x+x\cdot(\sin x)'\right)e^{\cos x}-(x\sin x)\left(e^{\cos x}\cdot(\cos x)'\right)}{\left(e^{\cos x}\right)^2}$$

$$= \frac{\left(1\cdot\sin x+x\cdot(\cos x)\right)e^{\cos x}-(x\sin x)\left(e^{\cos x}\cdot(-\sin x)\right)}{\left(e^{\cos x}\right)^2}$$

$$= \frac{\left(\sin x+x\cos x\right)e^{\cos x}+\left(x\sin^2 x\right)\left(e^{\cos x}\right)}{e^{2\cos x}}$$

$$= \frac{e^{\cos x}\left(\sin x+x\cos x+x\sin^2 x\right)}{e^{2\cos x}}$$

$$= \frac{\left(\sin x + x\cos x + x\sin^2 x\right)}{e^{\cos x}}$$

Derivative of a function of a function:

$y = f(u), u = \phi(x)$**,** then y is called a function of a function.

Differentiate (i) $\sqrt{(3x^2 - 7)}$ (ii) $y = \sin(x^2 + 2x - 5)^7$ w.r. to x.

Solution: $y = \sin(x^2 + 2x - 5)^7$;

Put $u = (x^2 + 2x - 5)$, $v = u^7$ then $y = \sin v$

Therefore, $\frac{du}{dx} = 2x + 2$, $\frac{dv}{du} = 7u^6 = 7(x^2 + 2x - 5)^6$

and $\frac{dy}{dv} = \cos v = \cos u^7 = \cos(x^2 + 2x - 5)^7$

By using the chain rule, we get $\frac{dy}{dx} = \frac{dy}{dv} \times \frac{dv}{du} \times \frac{du}{dx}$

$$= \cos(x^2 + 2x - 5)^7 \times 7(x^2 + 2x - 5) \times (2x + 2)$$

Find the differential coefficients of :

(i) $y = (x^2 + 1)\sin^{-1} x + e^{\sqrt{1+x^2}}$, (ii) $y = \log(x\cos x)$,

(iii) $y = \frac{1}{\sqrt[3]{2x^4 + 3x^2 - 5x + 6}}$,

(iv) $y = \sqrt{\left\{x^2 + \sqrt{x^2 - 1}\right\}}$, (v) $x^2 - y^2 + 3x = 5y$, (vi) $y = (2x)^{(5x+2)}$,

(vii) $y = \sqrt{x^2 + e^{4x}}$, (ix) $\frac{\sqrt{y}}{\sqrt{x}} + \frac{\sqrt{x}}{\sqrt{y}} = a$ (x) $e^{xy} - 4xy = 2$

(viii) $y = (x^2 + 1)^{\sin x}$ (Hints: Taking log on both sides, then differentiate)

(xi) Find $\frac{dy}{dx}$ if $y = e^{\sin x} \sin a^x$.

❖ $y = \tan^{-1}\left(\frac{\cos x}{1 + \sin x}\right) = \tan^{-1}\left\{\frac{\sin\left(\frac{\pi}{2} - x\right)}{1 + \cos\left(\frac{\pi}{2} - x\right)}\right\} = \tan^{-1}\left\{\frac{2\sin\left(\frac{\pi}{4} - \frac{x}{2}\right)\cos\left(\frac{\pi}{4} - \frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}\right\}$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right\} = \frac{\pi}{4} - \frac{x}{2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}$$

❖ $y = \left(\frac{n}{x}\right)^{nx}\left(1 + \ln\frac{x}{n}\right)$

❖ $y = x^{\ln x} + x^{\cos^{-1} x}$

$$\Rightarrow \frac{dy}{dx} = x^{\ln x}\frac{d}{dx}(\ln x.\ln x) + x^{\cos^{-1} x}\frac{d}{dx}(\cos^{-1} x.\ln x) \quad \left[\quad \frac{d}{dx}(u^v) = u^v\frac{d}{dx}(v \cdot \ln u)\right]$$

$$\Rightarrow \frac{dy}{dx} = x^{\ln x}.\frac{2\ln x}{x} + x^{\cos^{-1} x}\left(\frac{\cos^{-1} x}{x} - \frac{\ln x}{\sqrt{1-x^2}}\right).$$

❖ $y = (\sin x)^{\cos x} + (\cos x)^{\sin x}$

❖ $y = \sin^{-1}\left(\frac{ax}{y}\right) \Rightarrow \frac{ax}{y} = \sin y \Rightarrow ax = y\sin y \Rightarrow a = \sin y\frac{dy}{dx} + y\cos y.\frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx}(\sin y + y\cos y) = a \quad \Rightarrow \frac{dy}{dx} = \frac{a}{\sin y + y\cos y}$$

Parametric functions differentiations:

❖ $x = a(t - \sin t),\ y = a(1 - \cos t)$;

❖ $x = e^t \sin t,\ y = e^t \cos t$.

❖ $y = \sin^{-1}\left(\frac{2t}{1+t^2}\right),\ x = 3\tan^{-1}\left(\frac{2t}{1+t^2}\right)$

❖ $y = (\tan x)^{\cot x} + (\cot x)^{\tan x}$;

❖ $y = (x)^{\tan x} + (\sin x)^{\cos x}$

Hints: Let $u = x^{\tan x}$ and $v = (\sin x)^{\cos x}$, so $y = u + v \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

where

$$\log u = \log(x^{\tan x}) = \tan x \log x \Rightarrow \frac{1}{u}\frac{du}{dx} = \tan x.\frac{1}{x} + \log x.\sec^2 x \Rightarrow \frac{du}{dx} = u(\tan x.\frac{1}{x} + \log x.\sec^2 x)$$

,

❖ Differentiate $y = \sin\left(2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right)$ w.r.to x. [Hints: $x = \cos\theta$]

❖ Differentiate $y = \sin^{-1}\left(\frac{x + \sqrt{1-x^2}}{\sqrt{2}}\right)$ w.r.to x. [Hints: $x = \sin\theta$]

$$\Rightarrow y=\sin^{-1}\left(\frac{\sin\theta+\sqrt{1-\sin^2\theta}}{\sqrt{2}}\right)=\sin^{-1}\left(\frac{1}{\sqrt{2}}\cdot\sin\theta+\frac{1}{\sqrt{2}}\cdot\cos\theta\right)=\sin^{-1}\left(\sin\theta\cos\frac{\pi}{4}+\cos\theta\sin\frac{\pi}{4}\right)$$

$$\Rightarrow y=\sin^{-1}\sin\left(\theta+\frac{\pi}{4}\right)=\theta+\frac{\pi}{4}=\sin^{-1}+\frac{\pi}{4}\Rightarrow\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$$

- ❖ Differentiate $\tan^{-1}\left(\frac{x-\sqrt{a^2-x^2}}{x+\sqrt{a^2-x^2}}\right)$ w.r.to x. {Hints: $x=a\sin\theta$ }
- ❖ Differentiate $\frac{\tan x}{x}\log\frac{e^x}{x^x}$ w.r.to x
- ❖ Differentiate $\tan^{-1}\frac{\sqrt{(1+x^2)}-1}{x}$ w.r.to $\tan^{-1}x$.
- ❖ Differentiate $e^{\sin^{-1}x}$ w.r.to $\cos 3x$.
- ❖
- ❖ Differentiate $x^{\sin x}$ w.r.to $(\sin x)^x$.
- ❖ Differentiate e^t w.r.to $\sqrt{t}$.

L'Hospital's Rule: Indeterminate form

Theorem: Suppose that f and g are differentiable functions on an open interval containing $x=a$, except possibly at $x=a$, and that

$$\lim_{x\to a} f(x)=0 \text{ and } \lim_{x\to a} g(x)=0$$

If $\lim_{x\to a}[f'(x)/g'(x)]$ exists, or if this limit is $+\infty$ or $-\infty$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Find the limit using L'Hospital's rule:

(a) $$\lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{\cos x}=\lim_{x\to\frac{\pi}{2}}\frac{\frac{d}{dx}[1-\sin x]}{\frac{d}{dx}[\cos x]}=\lim_{x\to\frac{\pi}{2}}\frac{-\cos x}{-\sin x}=\frac{0}{-1}=0$$

$$\lim_{x\to 0}\frac{e^x-1}{x^3}$$

The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hospital's rule yields

$$\lim_{x\to 0} \frac{e^x - 1}{x^3} = \lim_{x\to 0} \frac{e^x}{3x^2} = +\infty$$

$$\lim_{x\to +} \frac{x^{-4/3}}{\sin(1/x)}$$

$\lim_{x\to 0+} x\ln x = \lim_{x\to 0+} \frac{\ln x}{1/x}$ *or*, $\lim_{x\to 0+} \frac{x}{1/\ln x}$ [The first being an indeterminate form of ∞/∞ and the second an indeterminate form of type $0/0$]

Now, $\lim_{x\to 0+} \frac{\ln x}{1/x} = \lim_{x\to 0+} \frac{1/x}{-1/x^2} = \lim_{x\to 0+} (-x) = 0$

Evaluate $\lim_{x\to 0+} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$

Show that $\lim_{x\to 0} \left(\frac{(1+x)^{1/x} - e}{x}\right) = -\frac{1}{2}e$

$$\lim_{x\to 0} \left(\frac{(1+x)^{1/x} - e}{x}\right) = \lim_{x\to 0} \left(\frac{e^{\frac{in(1+x)}{x}} - e}{x}\right)$$

$$= \lim_{x\to 0} \frac{1}{x}\left(e^{\frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right)} - e\right)$$

$$= \lim_{x\to 0} \frac{1}{x}\left(e^{1 - \frac{x}{2} + \frac{x^2}{3} - \cdots} - e\right)$$

$$= \lim_{x\to 0} \frac{e}{x}\left(e^{-\frac{x}{2} + \frac{x^2}{3} - \cdots} - 1\right)$$

$$= \lim_{x\to 0} \frac{e}{x}\left[1+\left(-\frac{x}{2}+\frac{x^2}{3}-\cdots\right)+\frac{1}{2!}\left(-\frac{x}{2}+\frac{x^2}{3}-\cdots\right)^2-1\right]$$

$$= \lim_{x\to 0} \frac{e}{x}\left[x\left(-\frac{1}{2}+\frac{x}{3}-\cdots\right)+\frac{x^2}{2!}\left(-\frac{1}{2}+\frac{x}{3}-\cdots\right)^2+\cdots\right]$$

$$= \lim_{x\to 0} e\left[\left(-\frac{1}{2}+\frac{x}{3}-\cdots\right)+\frac{x}{2!}\left(-\frac{1}{2}+\frac{x}{3}-\cdots\right)^2+\cdots\right]=-\frac{1}{2}e$$

Show that $\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)=\frac{1}{2}$

Solution:

$$\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right) \quad (\infty-\infty)\ form$$

$$= \lim_{x\to 1}\frac{x\ln x - x+1}{(x-1)\ln x} \qquad \left(\frac{0}{0}\ form\right)$$

$$= \lim_{x\to 1}\frac{x/x+\ln x-1}{\ln x+(x-1)/x} \quad = \lim_{x\to 1}\frac{x\ln x}{x\ln x+x-1} \quad (\frac{0}{0} \quad form)$$

$$= \lim_{x\to 1}\frac{\ln x + x/x}{\ln x + x/x+1} \quad = \frac{0+1}{0+1+1} \quad = \frac{1}{2}$$

Higher Derivatives

The nth derivative of f is denoted $f^{(n)}$. Thus $f^{(0)} = f$, $f^{(1)} = f'$, $f^{(2)} = f''$, $f^{(3)} = f'''$, …………

Leibniz' notation for the nth derivative of $y = f(x)$ is $\frac{d^n y}{dx^n} = f^n(x)$.

Example: The nth derivatives of $y = (ax+b)^m$, where m is any number.

Solution: $y_1 = ma(ax+b)^{m-1}$;

$$y_2 = m(m-1)a^2(ax+b)^{m-2};$$
$y_3 = m(m-1)(m-2)a^3(ax+b)^{m-3}$; and proceeding similarly,

....

$$y_n = m(m-1)(m-2)\cdots\cdots(m-n+1)a^n(ax+b)^{m-n} = \frac{m!}{(m-n)!}a^n(ax+b)^{m-n}.$$

Example: The nth derivatives of $y = \cos(ax+b)$,

Or, If $y = \cos(ax+b)$, show that $y_n = a^n \cos(n.\frac{\pi}{2} + ax + b)$

Example: If $y = \frac{x^2+x-1}{x^3+x^2-6x}$, find y_n.

Solution: Let $\frac{x^2+x-1}{x^3+x^2-6x} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-2}$

Multiplying both sides by $x(x+3)(x-2)$ and then putting $x = 0, -3, 2$, we get

$A = \frac{1}{6}, B = \frac{1}{3}, C = \frac{1}{2}$

$\therefore\ y = \frac{1}{6}.\frac{1}{x} + \frac{1}{3}.\frac{1}{x+3} + \frac{1}{2}.\frac{1}{x-2}$

$\therefore\ y_n = (-1)^n n! \left\{ \frac{1}{6}.\frac{1}{x^{n+1}} + \frac{1}{3}.\frac{1}{(x+3)^{n+1}} + \frac{1}{2}.\frac{1}{(x-2)^{n+1}} \right\}.$

Example: Differentiate n times the expression $y = \frac{x^2-6x+1}{(x-1)(3x-2)(2x+3)}$.

Leibnitz's Theorem (n-th derivatives of the product of two functions)

Statement: If u and v are two functions of x, each possessing derivatives upto nth order, then the nth derivative of their product, i.e.

$(uv)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots\cdots + {}^nC_r u_{n-r} v_r + \cdots\cdots + u v_n$, where the suffixes of u and v denote the order of differentiations of u and v w.r.to u and x.

Proof: Try yourself

Example: If $y = e^{a\sin^{-1}x}$, then show that $(1-x^2)y_2 - xy_1 = a^2 y$.

Example2: If $y = e^{\cos^{-1}x}$, show that an equation connecting y_n, y_{n+1} and y_{n+2} is given by $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+1)y_n = 0$.

Example: If $y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$, then show that

$(1-x^2)y_2 - 3xy_1 - y = 0$.

$(1-x^2)y_{n+2}-(2n+3)xy_{n+1}-(n+1)^2 y_n=0$.

Example: If $y=\cos(m\sin^{-1}x)$, then show that

$(1-x^2)y_2-xy_1+m^2y=0$.

$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}+(m^2-n^2)y_n=0$.

Try : (1) If $y=\dfrac{x}{1+x}$, then show that $y_5(0)=5!$

(2) If $y=e^{\cos^{-1}x}$, show that an equation connecting y_n, y_{n+1} and y_{n+2} is given by $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}+(n^2+1)y_n=0$.

(3) If $y=(\sin^{-1}x)^2$, then show that

(i) $(1-x^2)y_2-xy_1-2=0$.

(ii) $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n=0$.

(4) If $y=e^{k\sin^{-1}x}$, then show that

(i) $(1-x^2)y_2-xy_1-k^2y=0$.

(ii) $(1-x^2)y_{n+2}-(2n+1)x\,y_{n+1}-(n^2+k^2)y_n=0$.

(5) If $y=\sin(m\sin^{-1}x)$, show that

(i) $(1-x^2)y_2=xy_1-m^2y$.

(ii) $(1-x^2)y_{n+2}=(2n+1)x\,y_{n+1}+(n^2-m^2)y_n=0$.

(6) If $y=\log\{x+\sqrt{(1+x^2}\}^2$ prove that

(i) $(1+x^2)y_2+xy_1=0$.

(ii) $(1+x^2)y_{n+2}+(2n+1)x\,y_{n+1}+n^2y_n=0$.

Theorem : Rolle's Theorem

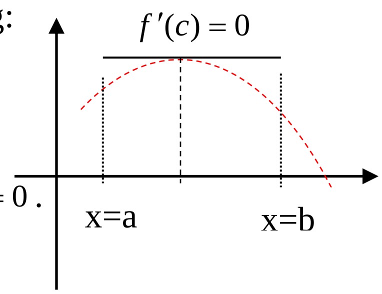
Suppose $f(x)$ is function that satisfies all the following:

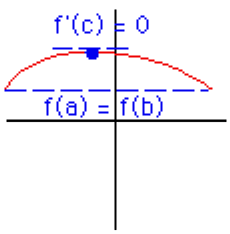
$f(x)$ is continuous on the closed interval $[a,b]$

$f(x)$ is differentiable on the open interval (a,b)

$f(a) = f(b)$

Then there is a number c such that $a < c < b$ and $f'(c) = 0$.





Example: Determine whether Rolle's theorem can be applied to f on the closed interval $[a,b]$. If Rolle's theorem can be applied, find all values of c in the open interval (a,b) such that $f'(c) = 0$.

$$f(x) = x^2 - 5x + 4 \qquad [1, 4]$$

Solution: Rolle's theorem applied if

the function is continuous on the closed interval $[1, 4]$

i.e. $f(1) = 0 = f(4)$

the function differentiable on the open interval

$f'(x) = 2x - 5$

$\Rightarrow f'(c) = 2c - 5$

$\Rightarrow 0 = 2c - 5 \Rightarrow c = 5/2 = 2.5$ which lies in $1 < x < 4$.

Therefore, Rolle's theorem can be applied.

Example : Determine all the number(s) c which satisfy the conclusion of Rolle's Theorem for $f(x) = x^2 - 2x - 8$ on $[-1, 3]$.

Solution: Rolle's Theorem can be used here.

The function is a polynomial which is continuous and differentiable everywhere and so will be continuous on $[-1, 3]$ and differentiable on $(-1, 3)$.

Next, a couple of quick function evaluations shows that $f(-1) = f(3) = -5$.

Therefore, the conditions for Rolle's Theorem are met and so we can actually do the problem.
Now that we know that Rolle's Theorem can be used there really isn't much to do. All we need to do is take the derivative,

$$f'(x) = 2x - 2$$

and then solve $f'(c) = 0$

$$2c - 2 = 0 \qquad \Rightarrow \qquad c = 1$$

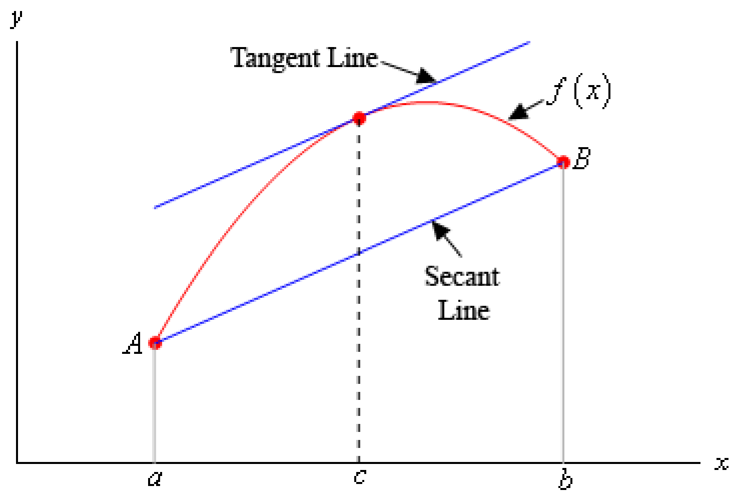
So, we found a single value and it is in the interval so the value we want is,

$$\boxed{c = 1}$$

Example 2: Determine all the number(s) c which satisfy the conclusion of Rolle's Theorem for $g(t) = 2t - t^2 - t^3$ on $[-2, 1]$.
Example-3: Verify Rolle's for $f(x) = 2x^3 + x^2 - 4x - 2$ over $[-\sqrt{2}, \sqrt{2}]$

Theorem : The Mean Value Theorem
If $f(x)$ is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b) , then there exists a number a number c in $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



Likewise, if we draw in the tangent line to $f(x)$ at $x = c$ we know that its slope is $f'(c)$

What the Mean Value Theorem tells us is that these two slopes must be equal or in other words the secant line connecting *A* and *B* and the tangent line at $x = c$ must be parallel. We can see this in the above sketch.

Example 1: Determine all the numbers *c* which satisfy the conclusions of the Mean Value Theorem for the following function.

$$f(x) = x^3 + 2x^2 - x \quad \text{on} \quad [-1,2]$$

Solution: There isn't really a whole lot to this problem other than to notice that since $f(x)$ is a polynomial it is both continuous and differentiable (*i.e.* the derivative exists) on the interval given.

First let's find the derivative.

$$f'(x) = 3x^2 + 4x - 1$$

Now, to find the numbers that satisfy the conclusions of the Mean Value Theorem all we need to do is plug this into the formula given by the Mean Value Theorem.

$$f'(c) = \frac{f(2) - f(-1)}{2 - (-1)}$$

$$3c^2 + 4c - 1 = \frac{14 - 2}{3} = \frac{12}{3} = 4$$

Now, this is just a quadratic equation,

$$3c^2 + 4c - 1 = 4$$

$$3c^2 + 4c - 5 = 0$$

Using the quadratic formula on this we get,

$$c = \frac{-4 \pm \sqrt{16 - 4(3)(-5)}}{6} = \frac{-4 \pm \sqrt{76}}{6}$$

So, solving gives two values of *c*.

$$c = \frac{-4 + \sqrt{76}}{6} = 0.7863 \qquad\qquad c = \frac{-4 - \sqrt{76}}{6} = -2.1196$$

Notice that only one of these is actually in the interval given in the problem. That means that we will exclude the second one (since it isn't in the interval). The number that we're after in this problem is,

$$c = 0.7863.$$

Example: Determine whether Mean value theorem can be applied to f on the closed interval [a,b]. If the Mean value theorem can be applied, find all values of c in the open interval *(a,b)* such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$, where

$$f(x)=x^2 \qquad [-2,1]$$

Solution: Mean value theorem applied if
the function is continuous on the closed interval
the function differentiable on the open interval

$$f'(x)=2x$$

$$\frac{f(b)-f(a)}{b-a}=\frac{1-4}{1-(-2)}=\frac{-3}{3}=-1$$

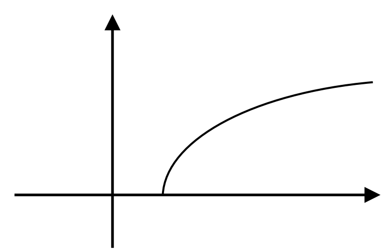
$2x=-1$ [By MVT]

$\Rightarrow\; x=-1/2=c$ which lies in $-2<x<1$.

Therefore, Mean value theorem can be applied.

Example: Determine whether Mean value theorem can be applied to f on the closed interval [a,b]. If the Mean value theorem can be applied, find all values of c in the open interval *(a,b)* such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

$$f(x)=\sqrt{x-2} \qquad [2, 6]$$



Solution: $f(x)=\sqrt{x-2}$

$$\Rightarrow\; f'(x)=\frac{1}{2}(x-2)^{-\frac{1}{2}}$$

$$\Rightarrow\; f'(c)=\frac{1}{2(x-2)^{1/2}}$$

Slope of second line $\dfrac{f(b)-f(a)}{b-a}=\dfrac{2-0}{6-2}=\dfrac{2}{4}=\dfrac{1}{2}$

$$\Rightarrow\; \frac{1}{2(x-2)^{1/2}}=\frac{1}{2} \Longrightarrow \text{MVT}$$

$$\Rightarrow 2=2\sqrt{x-2}\Rightarrow (1)^2=(\sqrt{x-2})^2$$

$$\Rightarrow\; x=3=c$$

Example-3: Using Mean-value theorem, $f(b)-f(a)=(b-a)\,f'(c)$, find the value of c where $f(x)=x(x-1)(x-2)$; $a=0,\quad b=1/2$.

Example-4: Discuss the applicability of the Mean-value theorem
$f(b)-f(a)=(b-a)\,f'(\xi),\, a<\xi<b$

Find ξ, if the theorem is applicable: $f(x)=|x|$, $-1\le x\le 1$.

[Hints: Das & Mukherjee: p-304]

Taylor's series (Extended to infinity)

If $f(x)$, $f'(x)$, $f''(x)$,……. $f^n(x)$ exist infinitely however large n may be in any interval $[x-\delta, x+\delta]$ enclosing the point x and if in addition R_n tends to zero as n tends to infinity, then the Taylor's series can be written as

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \cdots\cdot \text{to } \infty$$

Denoting the first n terms of the expansion by S_n and the remainder by R_n, we have

$$f(x+h) = S_n + R_n$$

Now, let $n \to \infty$; then $R_n \to 0$, we have

$$f(x+h) = \lim_{n\to\infty} S_n = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \cdots\cdot \text{to } \infty$$

Example 1: Find the Taylor Series for $f(x) = \ln(x)$ about $x = 2$.

Solution: Here are the first few derivatives and the evaluations.

$$f^{(0)}(x) = \ln(x) \qquad\qquad f^{(0)}(2) = \ln 2$$

$$f^{(1)}(x) = \frac{1}{x} \qquad\qquad f^{(1)}(2) = \frac{1}{2}$$

$$f^{(2)}(x) = -\frac{1}{x^2} \qquad\qquad f^{(2)}(2) = -\frac{1}{2^2}$$

$$f^{(3)}(x) = \frac{2}{x^3} \qquad\qquad f^{(3)}(2) = \frac{2}{2^3}$$

$$f^{(4)}(x) = -\frac{2(3)}{x^4} \qquad\qquad f^{(4)}(2) = -\frac{2(3)}{2^4}$$

$$f^{(5)}(x) = \frac{2(3)(4)}{x^5} \qquad\qquad f^{(5)}(2) = \frac{2(3)(4)}{2^5}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n} \qquad\qquad f^{(n)}(2) = \frac{(-1)^{n+1}(n-1)!}{2^n} \qquad\qquad n = 1, 2, 3, \ldots$$

Note that while we got a general formula here it doesn't work for $n = 0$. This will happen on occasion so don't worry about it when it does.
In order to plug this into the Taylor Series formula we'll need to strip out the $n = 0$ term first.

$$\begin{aligned}
\ln(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n \\
&= f(2) + \sum_{n=1}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n \\
&= \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{n!\ 2^n}(x-2)^n \\
&= \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\,2^n}(x-2)^n
\end{aligned}$$

Example 2: Find the Taylor Series for $f(x) = x^3 - 10x^2 + 6$ about $x = 3$
Solution: Here are the derivatives for this problem.

$$\begin{array}{ll}
f^{(0)}(x) = x^3 - 10x^2 + 6 & f^{(0)}(3) = -57 \\
f^{(1)}(x) = 3x^2 - 20x & f^{(1)}(3) = -33 \\
f^{(2)}(x) = 6x - 20 & f^{(2)}(3) = -2 \\
f^{(3)}(x) = 6 & f^{(3)}(3) = 6 \\
f^{(n)}(x) = 0 & f^{(4)}(3) = 0 \qquad n \ge 4
\end{array}$$

This Taylor series will terminate after $n = 3$. This will always happen when we are finding the Taylor Series of a polynomial. Here is the Taylor Series for this one.

$$\begin{aligned}
x^3 - 10x^2 + 6 &= \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n \\
&= f(3) + f'(3)(x-3) + \frac{f''(3)}{2!}(x-3)^2 + \frac{f'''(3)}{3!}(x-3)^3 + 0 \\
&= -57 - 33(x-3) - (x-3)^2 + (x-3)^3
\end{aligned}$$

Example 3: Find the Taylor Series for $f(x)=\frac{1}{x^2}$ about $x=-1$.

Solution: Here are the derivatives and evaluations.

$$f^{(0)}(x)=\frac{1}{x^2} \qquad f^{(0)}(-1)=\frac{1}{(-1)^2}=1$$

$$f^{(1)}(x)=-\frac{2}{x^3} \qquad f^{(1)}(-1)=-\frac{2}{(-1)^3}=2$$

$$f^{(2)}(x)=\frac{2(3)}{x^4} \qquad f^{(2)}(-1)=\frac{2(3)}{(-1)^4}=2(3)$$

$$f^{(3)}(x)=-\frac{2(3)(4)}{x^5} \qquad f^{(3)}(-1)=-\frac{2(3)(4)}{(-1)^5}=2(3)(4)$$

$$\vdots \qquad\qquad \vdots$$

$$f^{(n)}(x)=\frac{(-1)^n(n+1)!}{x^{n+2}} \qquad f^{(n)}(-1)=\frac{(-1)^n(n+1)!}{(-1)^{n+2}}=(n+1)!$$

Notice that all the negative signs will cancel out in the evaluation. Also, this formula will work for all *n*, unlike the previous example.

Here is the Taylor Series for this function.

$$\begin{aligned}\frac{1}{x^2}&=\sum_{n=0}^{\infty}\frac{f^{(n)}(-1)}{n!}(x+1)^n\\&=\sum_{n=0}^{\infty}\frac{(n+1)!}{n!}(x+1)^n\\&=\sum_{n=0}^{\infty}(n+1)(x+1)^n\end{aligned}$$

Exercise:

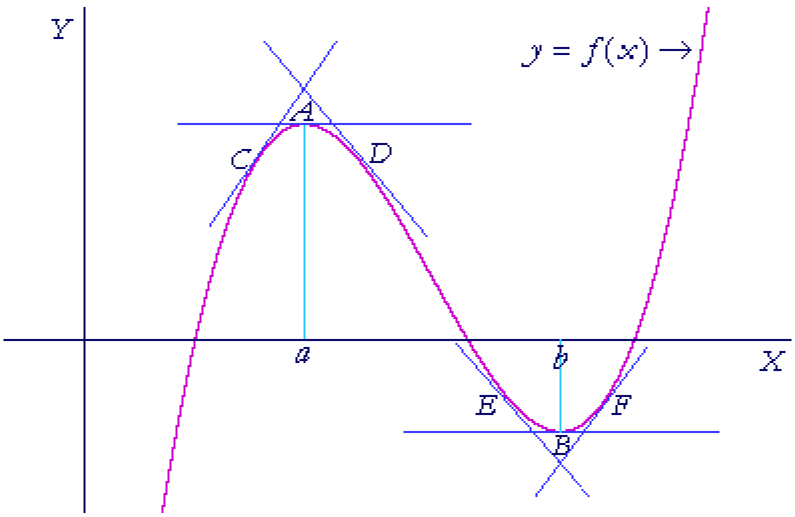
Find the Taylor series for $f(x)=\sin(x)$, $f(x)=\cos(x)$ about $x=0$

Find the Taylor series for $f(x)=e^{-x}$ about $x=0$.

Taylor's Theorem for two variables

$$\phi(x+h, y+k) = \phi(x, y+k) + h\frac{\partial}{\partial x}\phi(x, y+k) + \frac{h^2}{2!}\phi(x, y+k) + \frac{h^3}{3!}\frac{\partial}{\partial x^3}\phi(x, y+k)\cdots\cdots \text{to } \infty$$

1 MAXIMUM AND MINIMUM VALUES



WE SAY THAT A FUNCTION *f*(*x*) has a relative maximum value at $x = a$,
if *f*(*a*) is *greater* than any value immediately preceding or following.

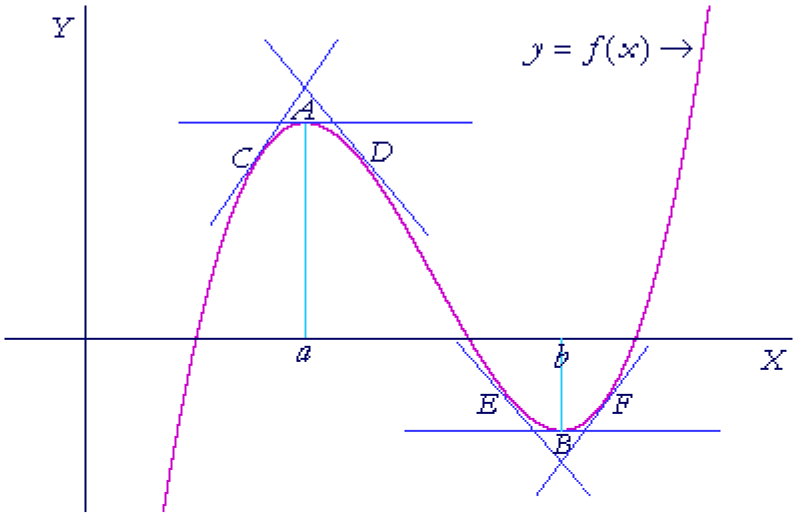
We call it a "relative" maximum because other values of the function may in fact be greater.

We say that a function *f*(*x*) has a relative minimum value at $x = b$,
if *f*(*b*) is *less* than any value immediately preceding or following.

Again, other values of the function may in fact be less. With that understanding, then, we will drop the term relative.

The value of the function, the value of *y*, at either a maximum or a minimum is called an extreme value.

Now, what characterizes the graph at an extreme value?



The tangent to the curve is horizontal. We see this at the points *A* and *B*. The slope of each tangent line -- the *derivative* when evaluated at *a* or *b* -- is 0.

$$f\,'(x) = 0.$$

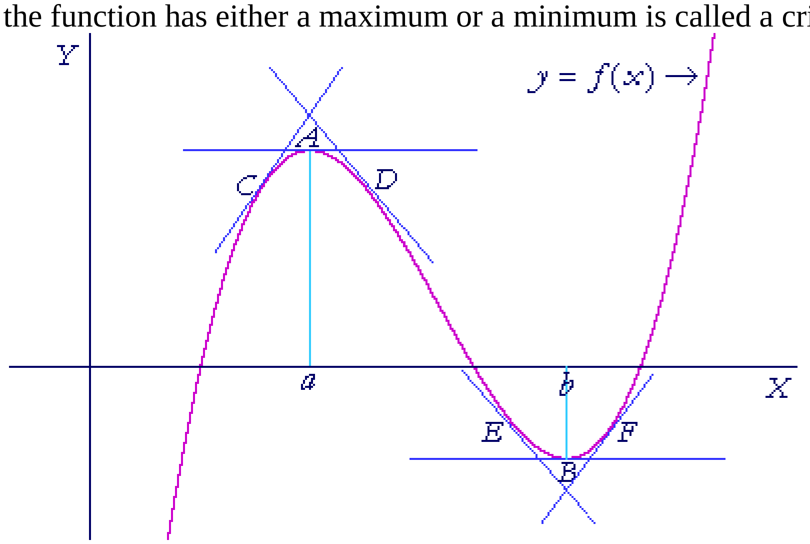
Moreover, at points immediately to the *left* of a maximum -- at a point *C* -- the slope of the tangent is positive: $f\,'(x) > 0$. While at points immediately to the *right* -- at a point *D* -- the slope is negative: $f\,'(x) < 0$.

In other words, at a maximum, $f\,'(x)$ changes sign from + to − .

At a minimum, $f\,'(x)$ changes sign from − to + . We can see that at the points *E* and *F*.

We can also observe that at a maximum, at *A*, the graph is concave downward. While at a minimum, at *B*, it is concave upward.

A value of *x* at which the function has either a maximum or a minimum is called a critical value. In the figure --



-- the critical values are $x = a$ and $x = b$.

The critical values determine turning points, at which the tangent is parallel to the x-axis. The critical values -- if any -- will be the *solutions* to $f\,'(x) = 0$.

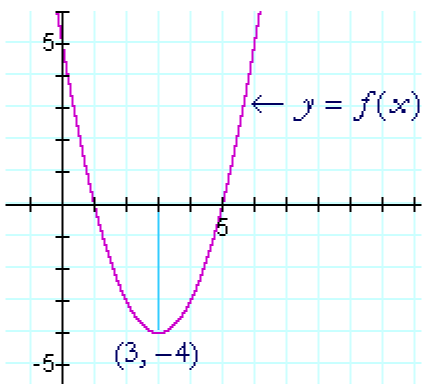
Example 1. Let $f(x) = x^2 - 6x + 5$. Are there any critical values -- any turning points? If so, do they determine a maximum or a minimum? And what are the coördinates on the graph of that maximum or minimum?

Solution. $f\,'(x) = 2x - 6 = 0$ implies $x = 3$.

$x = 3$ is the only critical value. It is the x-coördinate of the turning point. To determine the y-coördinate, *evaluate f at that critical value* -- evaluate $f(3)$:

$$\begin{aligned} f(x) &= x^2 - 6x + 5 \\ f(3) &= 3^2 - 6\cdot 3 + 5 \\ &= -4. \end{aligned}$$

The extreme value is -4. To see whether it is a maximum or a minimum, in this case we can simply look at the graph.



$f(x)$ is a parabola, and we can see that the turning point is a minimum.

By finding the value of x where the derivative is 0, then, we have discovered that the vertex of the parabola is at $(3, -4)$.

But we will not always be able to look at the graph. The algebraic condition for a minimum is that $f\,'(x)$ changes sign from $-$ to $+$. We see this at the points *E, B, F* above. The value of the slope is increasing.

Now to say that the slope is increasing, is to say that, at a critical value, the second derivative -- which is rate of change of the slope -- is *positive*.

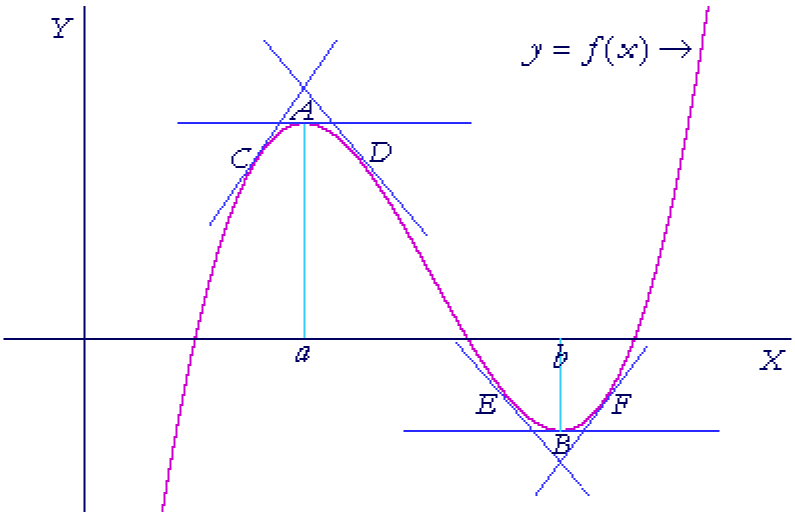
Again, here is $f(x)$:

$$\begin{aligned} f(x) &= x^2 - 6x + 5. \\ f\,'(x) &= 2x - 6. \end{aligned}$$

$$f''(x) = 2.$$

f '' evaluated at the critical value 3 -- *f ''*(3) = 2 -- is positive. This tells us algebraically that the critical value 3 determines a minimum.

Sufficient conditions



We can now state these sufficient conditions for extreme values of a function at a critical value *a*:

The function has a minimum value at $x = a$ if $f'(a) = 0$ and $f''(a)$ = a positive number.

The function has a maximum value at $x = a$ if $f'(a) = 0$ and $f''(a)$ = a negative number.

In the case of the maximum, the slope of the tangent is *decreasing* -- it is going from positive to negative. We can see that at the points *C, A, D*.

Example 2. Let $f(x) = 2x^3 - 9x^2 + 12x - 3$.Are there any extreme values? First, are there any critical values -- solutions to $f'(x) = 0$ -- and do they determine a maximum or a minimum? And what are the coördinates on the graph of that maximum or minimum? Where are the turning points?

Solution. $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2)$

$$= 6(x - 1)(x - 2)$$

$$= 0$$

implies: $x = 1$ or $x = 2$.

Those are the critical values. Does each one determine a maximum or does it determine a minimum? To answer, we must evaluate the second derivative at each value.

$$f\,'(x) = 6x^2 - 18x + 12.$$

$$f\,''(x) = 12x - 18.$$

$$f\,''(1) = 12 - 18 = -6.$$

The second derivative is negative. The function therefore has a maximum at $x = 1$.

To find the y-coördinate -- the extreme value -- at that maximum we evaluate $f(1)$:

$$f(x) = 2x^3 - 9x^2 + 12x - 3$$

$$f(1) = 2 - 9 + 12 - 3$$

$$= 2.$$

The maximum occurs at the point (1, 2).

Next, does $x = 2$ determine a maximum or a minimum?

$$f\,''(x) = 12x - 18.$$

$$f\,''(2) = 24 - 18 = 6.$$

The second derivative is positive. The function therefore has a minimum at $x = 2$.

To find the y-coördinate -- the extreme value -- at that minimum, we evaluate $f(2)$:

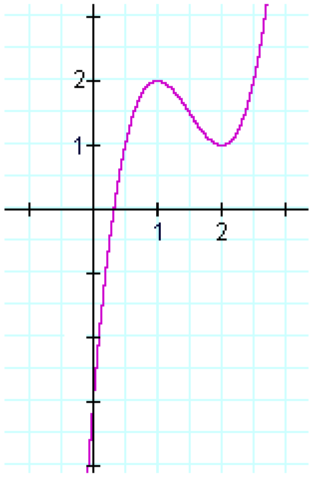
$$f(x) = 2x^3 - 9x^2 + 12x - 3.$$

$$f(2) = 16 - 36 + 24 - 3$$

$$= 1.$$

The minimum occurs at the point (2, 1).

Here in fact is the graph of $f(x)$:



Solutions to $f''(x) = 0$ indicate a point of inflection at those solutions, not a maximum or minimum. An example is $y = x^3$. $y'' = 6x = 0$ implies $x = 0$. But $x = 0$ is a point of inflection in the graph of $y = x^3$, not a maximum or minimum.

Another example is $y = \sin x$. The solutions to $y'' = 0$ are the multiples of π, which are points of inflection.

Example: Find the maximum and minimum of the function defined by

$$f(x) = \frac{40}{3x^4 + 8x^3 - 18x^2 + 60}$$

Solution: We determine the maxima and minima of the function defined by

$$F(x) = 3x^4 + 8x^3 - 18x^2 + 60$$

$$\text{Clearly, } F'(x) = 12x^3 + 24x^2 - 36x = 12x(x^2 + 2x - 3)$$

$$\text{And } F''(x) = 36x^2 + 48x - 36 = 12(3x^2 + 4x - 3)$$

For maxima and minima of F(x), we must have $F'(x) = 0$ i.e.

$$x(x^2 + 2x - 3) = 0 \Rightarrow x(x+3)(x-1) = 0 \Rightarrow x = 0, 1, -3$$

Substituting in $F''(x)$, we obtain

$$F''(-3) = 12(3.9 - 12 - 3) = 144 > 0$$
$$F''(0) = 12(0 + 0 - 3) < 0$$
$$F''(1) = 12(3.1 + 4 - 3) > 0$$

Hence, F(x) has a maximum at x=0 and minimum at both x=-3 and x=1. The given function f(x) has therefore a minimum at x=0 and maximum at both x=-3 and x=1. By substitution, we obtain

$$f(0) = \frac{40}{60} = 2/3; \quad f(-3) = \frac{8}{15}; \quad f(1) = \frac{40}{53}.$$

Example: Find the maximum and minimum values of $f(x) = x^4 - 2x^2$ and confirm that your results are consistent with the graph of $f(x)$.

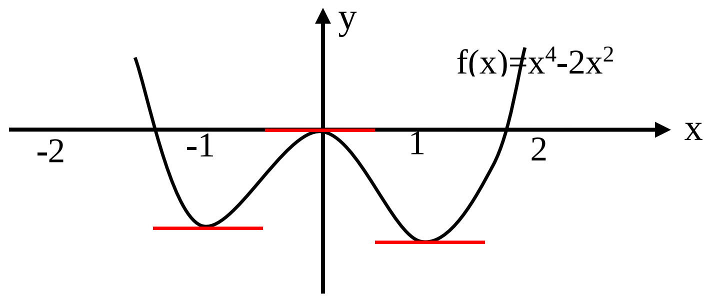
Solution: $f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x-1)(x+1)$

$f''(x) = 12x^2 - 4$

Solving $f'(x)=0$ yield stationary points at $x=0$, $x=1$ and $x=-1$. Evaluating $f''(x)$ at these points yields

$$f''(0)=-4<0\,;\; f''(1)=8>0 \text{ and } f''(-1)=8>0\,.$$

So, there is a maximum at $x=0$ and minimum at $x=-1$ & $x=1$. (Figure)



Problem 1. Examine each function for maxima and minima.

a) $y = x^3 - 3x^2 + 2.$

$y' = 3x^2 - 6x = 3x(x-2) = 0$ implies

$$x = 0 \text{ or } = 2.$$

$$y''(x) = 6x - 6.$$

$$y''(0) = -6.$$

The second derivative is negative. That means there is a maximum at $x = 0$. That maximum value is

$$y(0) = 2.$$

Next,

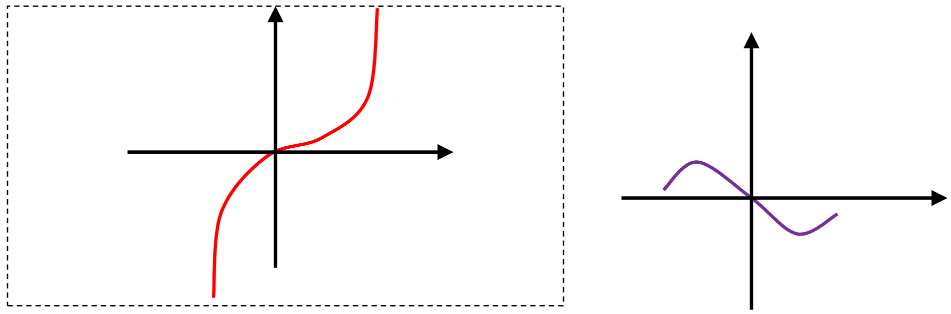
$$y''(2) = 12 - 6 = 6.$$

The second derivative is positive. That means there is a minimum at $x = 2$. That minimum value is

$y(2) = 2^3 - 3\cdot 2^2 + 2 = 8 - 12 + 2 = -2.$

Q. What is a point of inflexion?

The value of x where a graph changes concavity, from concave upword to concave downwor.



In each case graph x=0 is a point of inflection.

b) State Mean-value theorem. Show that $f(x) = \sqrt{25 - x^2}$ satisfies the hypothesis of Mean value theorem on $[-5, 3]$, and find all values of c in $(-5, 3)$ that satisfy the conclusion of the theorem.

At $x = 1$ there is a maximum of $y = 17$.

At $x = -2$ there is a minimum of $y = -10$.

c) $y = 2x^3 + 3x^2 + 12x - 4$.

Since $f'(x) = 0$ has no real solutions, there are no extreme values.

d) $y = 3x^4 - 4x^3 - 12x^2 + 2$.

At $x = 0$ there is a maximum of $y = 2$.

At $x = -1$ there is a minimum of $y = -3$.

At $x = 2$ there is a minimum of $y = -30$.

GEOMETRICAL MEANING OF DERIVATIVE AT POINT

The derivative $[f'(x)$ or $dy/dx]$ of the function $y = f(x)$ at the point $p(x, y)$ (when exist) is equal to the slope (or gradient) of the tangent line to the curve $y = f(x)$ at $p(x, y)$.

Slope of tangent to the curve $y = f(x)$ at the point (x, y) is

$$m = \tan\theta = \left[\frac{dy}{dx}\right]_{(x,y)}$$

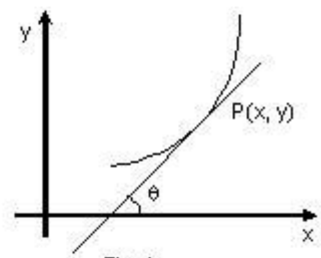


Fig. 1

But what is a tangent line?
It is NOT just a line that meets the graph at one point.
It is the limit of the secant lines joining points $P(x, y)$ and Q on the graph of $f(x)$ as Q approaches P.
The tangent line touches the graph at $(x, f(x))$; the slope of the tangent line matches the direction of the graph at that point.

Equation of Tangent

The equation of tangent to the curve $y = f(x)$ at the point $P(x_1, y_1)$ is given by

$$y - y_1 = \left[\frac{dy}{dx}\right]_{(x,y)} (x - x_1)$$

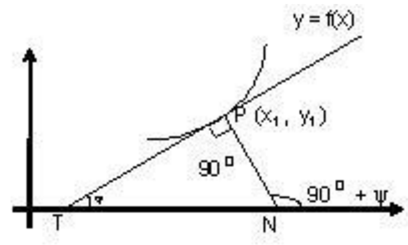


Fig. 2

Notes:

If $dy/dx = 0$ then the tangent to curve $y = f(x)$ at the point (x, y) is parallel to the x-axis.

If $dy/dx \to \infty$, $dx/dy = 0$, then the tangent to the curve $y = f(x)$ at the point (x, y) is parallel to the y-axis.

If $dy/dx = \tan\theta > 0$, then the tangent to the curve $y = f(x)$ at the point (x, y) makes an acute angle with positive x-axis and vice versa.

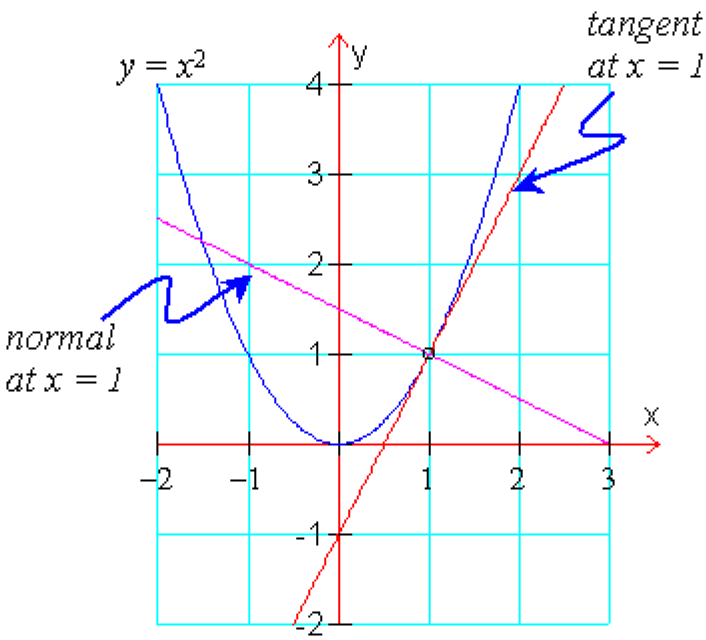
Equation of Normal

The normal to the curve at the point $P(x_1, y_1)$ is a line perpendicular to the tangent at the point $P(x_1, y_1)$ and passing through it. The angle between a tangent and a normal at a point is always 90^0.

The equation of the normal to the curve $y = f(x)$ at a given point $P(x_1, y_1)$ is given by

$$(x - x_1) + (y - y_1)\left[\frac{dy}{dx}\right]_{(x_1,y_1)} = 0 \ .$$

For example, consider the function y = x^2. The tangent and normal lines at the point (1,1) are shown on the diagram below:



The equation of the **tangent line** to $y = f(x)$ at the point (x_1, y_1):

('m' is the gradient at (x_1, y_1))

$$y - y_1 = m(x - x_1)$$

The equation of the **normal line** to $y = f(x)$ at (x_1, y_1) is:

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

The derivative of $y = f(x)$ at (x_1, y_1) gives us the gradient 'm'.

For example: Calculate the tangent and normal lines to the function: $y = x^2$, when x = 1.

Solution: Part 1, tangent line.

At x = 1, $y = 1^2 = 1$. So (x_1, y_1) = (1,1)

The gradient 'm' at x = 1 is found by calculating the derivative of the function at that point.

For $y = x^2$, $dy/dx = 2x$. At x = 1, $dy/dx = 2 \times 1 = 2$. So 'm' = 2.

The tangent line is given by:

$$y - y_1 = m(x - x_1)$$

Substituting, we have: y = 2(x - 1) + 1

Expanding and simplifying: y = 2x - 2 + 1

so: y = 2x - 1

Solution: Part 2, normal line.

At x = 1, the gradient of the tangent, 'm' = 2. (See above.)

The normal line has a gradient of $^{-1}/_{m}$ = $^{-1}/_{2}$, and so has this equation:

$$y = -\frac{1}{m}(x - x_1) + y_1$$

Substituting, we have: y = $-^{1}/_{2}$(x - 1) + 1

Expanding and simplifying: y = -0.5x + 0.5 + 1

so: y = -0.5x

Exercises

1. Write the equation for both the *tangent line* and *normal line* to the curve $y = 3x^2 - x + 1$ at the point where $x = 1$.
2. Write the equation for both the *tangent line* and *normal line* to the curve $y=(x-1)/(x+1)$ at the point where $x=0$.
3. Find the equation of tangent and normal of the curve $y = x^2 + 2x + 3$ at $x = 1$.

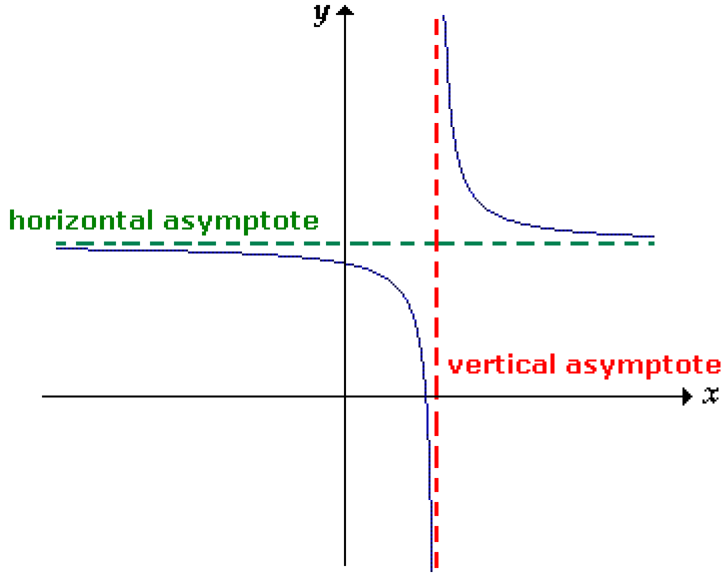
ASYMPTOTES

Definition of Asymptote

Asymptote is a line that a graph gets closer and closer to, but never touches or crosses it.

An asymptote is a **line** that a curve approaches, as it heads towards infinity:

Examples of Asymptote



Types

There are three types: horizontal, vertical and oblique:

Horizontal Asymptotes

It is a Horizontal Asymptote when:

as x goes to infinity (or −infinity) the curve approaches some constant value **b**

Vertical Asymptotes

It is a Vertical Asymptote when:

as x approaches some constant value **c** (from the left or right) then the curve goes towards infinity (or −infinity).

Example: Find the vertical and horizontal asymptotes of the graph of

$f(x) = \dfrac{3x+1}{x^2-4}$.

Solution: The Vertical asymptotes will occur at those values of x for which the denominator is equal to zero:

$x^2 - 4 = 0 \Rightarrow x = 2, -2$

Thus, the graph will have vertical asymptotes at $x = 2$ and $x = -2$.

To find the horizontal asymptote, we note that the degree of the numerator is one and the

degree of the denominator is two. Since the larger degree occurs in the denominator, the
graph will have a horizontal asymptote at $y = 0$ (i.e., the x-axis).

Partial Differentiation

(Functions of two or more variables)

Functions of more than one independent variable i.e. $z = f(x, y)$ is called partial differentiation.
Symbolically, we represent it by

$\left(\frac{\partial z}{\partial x}\right)$ or, simply $\frac{\partial z}{\partial x}$

i.e. $\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$ etc.

Homogeneous Functions:

A function $f(x, y)$ is said to be homogeneous of degree n in the variables x and y, if it can be represented in the form $x^n \phi\left(\frac{y}{x}\right)$, or in the form $y^n \phi\left(\frac{x}{y}\right)$.

For example, Since $ax^2 + 2hxy + by^2 = x^2\left\{a + 2h\frac{y}{x} + b\left(\frac{y}{x}\right)^2\right\} = x^2\phi\left(\frac{y}{x}\right)$.

So, $ax^2 + 2hxy + by^2$ is a homogeneous function of degree 2 in x and y.
Similarly, y/x, $x\tan^{-1}(y/x)$, $x^2 \log(y/x)$ are homogeneous functions of degree 0, 1 and 2 respectively.

Euler's theorem on Homogeneous Functions

Statement: If $f(x, y)$ be homogeneous function of x and y of degree n then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n\, f(x, y).$$

Proof: Since $f(x, y)$ is a homogeneous function of degree n

Let $f(x, y) = x^n\, \phi(y/x) = x^n\, \phi(v)$, where $v = (y/x)$, v/=(1/x)

$$\frac{\partial f}{\partial x} = nx^{n-1}\,\phi(v) + x^n\phi'(v)\frac{\partial v}{\partial x} = nx^{n-1}\,\phi(v) + x^n\phi'(v).\frac{-y}{x^2}$$

$$\frac{\partial f}{\partial y} = x^n\,\phi'(v)\frac{\partial v}{\partial y} = x^n\phi'(v).\frac{1}{x}$$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nx^n\,\phi(v) = n\, f(x, y)$$

Problem: State and prove Euler theorem for a homogeneous function in two variables and hence find $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}u=\frac{\sqrt{xy}}{\sqrt{x}+\sqrt{y}}$.

Example: If $u=\tan^{-1}\frac{x^3+y^3}{x-y}$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u$

Solution: From the given relation, we get

$$\tan u=\frac{x^3+y^3}{x-y}=\frac{x^3\{1+(y/x)^3\}}{x\{1-(y/x)\}}=x^2\phi\left(\frac{y}{x}\right)$$

$\tan u$ is a homogeneous function of degree 2.
Let, $v=\tan u$; therefore by Euler's theorem,

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=2v$$

$$\therefore\quad x\sec^2 u\,\frac{\partial u}{\partial x}+y\sec^2 u\,\frac{\partial u}{\partial y}=2\tan u$$

$$\therefore\quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{2\tan u}{\sec^2 u}=\sin 2u.$$

Example: (i) Show that $f(x,y)=\tan^{-1}\frac{y}{x}+\sin^{-1}\frac{x}{y}$ is a homogeneous function of x and y. Determine the degree of homogeneity. Hence, or otherwise, find the value of $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}$.

(ii) Examine whether the function $u(x,y)=\frac{x^3+y^3}{x^2+y^2}$ is a homogeneous function of x and y.

Solution: (i) $f(x,y)=\tan^{-1}\frac{y}{x}+\sin^{-1}\frac{x}{y}=\tan^{-1}\frac{y}{x}+\operatorname{cosec}^{-1}\frac{y}{x}$

$$=x^0\left\{\tan^{-1}\frac{y}{x}+\operatorname{cosec}^{-1}\frac{y}{x}\right\}=x^0\phi\left(\frac{y}{x}\right).$$

Hence, $f(x,y)$ is a homogeneous function of x and y degree 0.
By Euler's theorem on homogeneous functions,

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=0\times f=0.$$

Example: (i) If $V=2\cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, show that $x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+\cot\frac{V}{2}=0$.

(ii)If $U=xy\,f(y/x)$, prove that $x\frac{\partial U}{\partial x}+y\frac{\partial U}{\partial y}=2U$ by Euler's theorem.

(iii) If $u = \dfrac{x^2+y^2}{\sqrt{x+y}}$, $(x,y) \neq (0,0)$ and $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = ku$, find the value of k.

(iv) $$u = \frac{x^2+y^2}{\sqrt{x+y}} = \frac{x^2\left\{1+\left(\frac{y}{x}\right)^2\right\}}{x^{1/2}\left\{\sqrt{1+\left(\frac{y}{x}\right)}\right\}} = x^{\frac{3}{2}}.\phi\left(\frac{y}{x}\right)$$

So, u is a homogeneous function of x and y of degree (3/2). Hence, by Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3}{2}u \qquad \ldots(1)$$

But given that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = ku \quad \ldots(2)$

From (1) and (2), we have $k = \dfrac{3}{2}$

Example: What is the order of u, if $u = \dfrac{x+y}{x^2+y^2}$. Verify Euler's theorem for u

We have, $u(tx,ty) = \dfrac{tx+ty}{(tx)^2+(ty)^2} = \dfrac{t(x+y)}{t^2(x^2+y^2)} = t^{-1}u(x,y)$

Here, u is a homogeneous function of degree -1.

Now, $$\frac{\partial u}{\partial x} = \frac{1(x^2+y^2)-2x(x+y)}{(x^2+y^2)^2} = \frac{y^2-2xy-x^2}{(x^2+y^2)^2}$$

Similarly, $$\frac{\partial u}{\partial y} = \frac{x^2-2xy-y^2}{(x^2+y^2)^2}$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{x(y^2-2xy-x^2)+(x^2-2xy-y^2)}{(x^2+y^2)^2} = \frac{-xy^2-xy^2-x^3-y^3}{(x^2+y^2)^2}$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{(x+y)(x^2+y^2)}{(x^2+y^2)^2} = (-1)\frac{(x+y)}{(x^2+y^2)} = (-1)u.$$

Which verifies the Euler's theorem.

Example: If $u = \sin^{-1}\dfrac{x^2+y^2}{x+y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$

Solution: $u=\sin^{-1}\dfrac{x^2+y^2}{x+y} \Rightarrow \sin u=\dfrac{x^2+y^2}{x+y}=\dfrac{x^2(1+y^2/x^2)}{x(1+y/x)}=x.\dfrac{(1+y^2/x^2)}{(1+y/x)}$

Which shows that $\sin u$ is a homogeneous function of degree 1 in x and y. Apply Euler's theorem, we get

$$x\frac{\partial}{\partial x}(\sin u)+y\frac{\partial}{\partial y}(\sin u)=\sin u$$

$$i.e.\quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u$$

Example: If $u=\log\sqrt{x^2+y^2+z^2}$, then prove that $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=\dfrac{1}{x^2+y^2+z^2}$.

Example: Using Euler's theorem if $u=\log\dfrac{x^4+y^4}{x-y}$, show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=3$.

Solution: $u=\log\dfrac{x^4+y^4}{x-y} \Rightarrow e^u=\dfrac{x^4+y^4}{x-y}$

This is a homogeneous function of degree 3 in x and y.
By Euler's theorem, we have

$$x\frac{\partial}{\partial x}(e^u)+y\frac{\partial}{\partial y}(e^u)=3e^u \Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3$$

Integration

Integration originates from two different concepts. So, the method of integration are two:

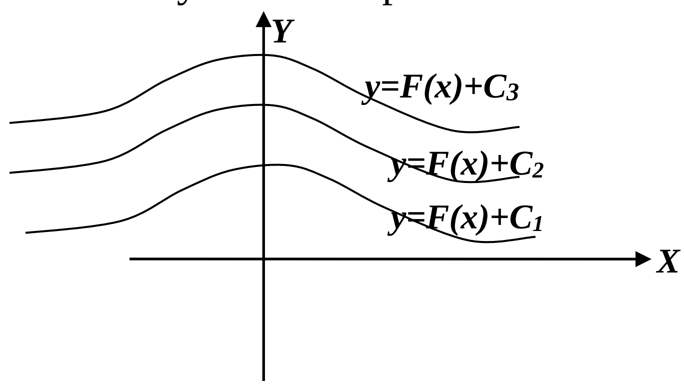
- Integration is the inverse (reverse) process of differentiation, which is the subject matter of indefinite Integration
- Integration is a process of summation which is the subject matter of definite integration.

If *F(x)* is the derivatives of *f(x)* for general real values of *x* i.e. $\frac{d}{dx}\{F(x)\} = f(x)$, then *F(x)* is called the indefinite integral or antiderivative of f(x) and is denoted by

$$F(x) = \int f(x)\,dx$$

Geometric Interpretation of Indefinite Integral

If $F'(x) = f(x)$ then $\int f(x)\,dx = F(x) + C$. Now, if $F(x)+C$ is equated to y *i.e.* $y = F(x)+C$, then it represents the equation of curves for different values of C and these curves are parallel curves. So, indefinite integrals represent a family or set of parallel curves in the xy-plane (see. Figure)



Techniques of Integration

❖ Evaluate $\int\left(\sqrt{\frac{1-x}{1+x}} + \frac{x}{\sqrt{1-x^2}}\right)dx$

Rationalizing and adding

$$\int\left(\sqrt{\frac{1-x}{1+x}} + \frac{x}{\sqrt{1-x^2}}\right)dx = \int \frac{1-x+x}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C\,.$$

❖Find $\phi(x)$, if $\phi(0) = \phi'(0) = 0$ and $\phi''(x) = \cos^2 x + 5$.

Solution: $\because\ \phi''(x) = \cos^2 x + 5$

$$\phi'(x) = \int (\cos^2 x + 5)\,dx$$

$$= \int \left\{\frac{1}{2}(1+\cos 2x) + 5\right\} dx = \frac{1}{2}.\frac{1}{2}\sin 2x + \frac{11x}{2} + A\,,$$

Where A is integrating constant.

When $\phi'(0)=0$, then $\phi'(0)=\frac{1}{2}.\frac{1}{2}.0+\frac{11.0}{2}+A \quad \Rightarrow A=0$

$$\phi'(x)=\frac{1}{2}.\frac{1}{2}\sin 2x+\frac{11x}{2}$$

On integration, $\qquad \phi(x)=-\frac{1}{8}.\cos 2x+\frac{11}{4}x^2+B$,

Where B is the integrating constant.

Since $\phi(0)=0$, then $\phi(0)=-\frac{1}{8}.1+\frac{11}{4}.0+B \Rightarrow B=\frac{1}{8}$

Hence, $\phi(x)=-\frac{1}{8}.\cos 2x+\frac{11}{4}x^2+\frac{1}{8}=\frac{1}{8}(1+22x^2-\cos 2x)$.

❖ Evaluate $\int \frac{dx}{\cos(x+a)\cos(x+b)}$

$$=\frac{1}{\sin(a-b)}\int \frac{\sin(a-b)}{\cos(x+a)\cos(x+b)}\,dx=\frac{1}{\sin(a-b)}\int \frac{\sin[(x+a)-(x+b)]}{\cos(x+a)\cos(x+b)}\,dx$$

$$=\frac{1}{\sin(a-b)}\int \frac{\sin(x+a)\cos(x+b)-\cos(x+a)\sin(x+b)}{\cos(x+a)\cos(x+b)}\,dx$$

$$=\frac{1}{\sin(a-b)}\int [\tan(x+a)-\tan(x+b)]\,dx$$

$$=\frac{1}{\sin(a-b)}[\ln\{\sec(x+a)\}-\ln\{\sec(x+b)\}+c=\frac{1}{\sin(a-b)}\ln\left[\frac{\sec(x+a)}{\sec(x+b)}\right]+c$$

Method of Substitutions

Substitution Rule $\int f(g(x))g'(x)\,dx=\int f(u)du$

$$\int \frac{f'(x)}{f(x)}\,dx=\ln\left|f(x)\right|+c$$

❖ Find $\int \frac{x}{\sqrt{x^2+1}}\,dx$

❖Find $\int x\cos(3x^2+2)\,dx$, $\int \sin^5 x\cos^2 x\,dx$

❖ $\int x^2\sin x^3\cos x^3\,dx=\frac{1}{3}\int 3x^2.\sin x^3\cos x^3\,dx=\frac{1}{3}\int \sin x^3\,d(\sin x^3)$

$$=\frac{1}{3}.\frac{(\sin x^3)^2}{2}+c=\frac{1}{6}\sin^2 x^3+c$$

❖ $\int \frac{\sqrt{x}}{\sqrt{a^3-x^3}}\,dx \;=\int \frac{x^{1/2}}{\sqrt{\left(a^{3/2}\right)^2-\left(x^{3/2}\right)^2}}\,dx=\frac{2}{3}\int \frac{dz}{\sqrt{k^2-z^2}}$

put, $x^{3/2} = z \quad and \quad a^{3/2} = k \quad \Rightarrow x^{1/2}\ dx = \frac{2}{3}\ dz$

$$= \frac{2}{3}\sin^{-1}\frac{z}{k} + c = \frac{2}{3}\sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right) + c = \frac{2}{3}\sin^{-1}\left(\frac{x}{a}\right)^{3/2} + c$$

❖ Evaluate the integral $\int \frac{x^3}{(a+bx)^4}\ dx$

[Rule: $\int \frac{x^m}{(a+bx)^n}\ dx,\ m+$ integer, then substitute $(a+bx) = u \quad \Rightarrow x = (u-a)/b$]

Let , $a + bx = u \Rightarrow x = \frac{u-a}{b}$ and $dx = \frac{1}{b}du$

Therefore, $\int \frac{x^3}{(a+bx)^4}\ dx = \int \frac{(u-a)^3}{b^3\ u^4}\ \frac{1}{u}du = \frac{1}{b^4}\int \frac{(u-a)^3}{u^4}\ du$

$$= \frac{1}{b^4}\int \frac{u^3 - 3u^2a + 3ua^2 - a^3}{u^4}\ du$$

$$= \frac{1}{b^4}\int \left(\frac{1}{u} - \frac{3a}{u^2} + \frac{3a^2}{u^3} - \frac{a^3}{u^4}\right) du$$

$$= \frac{1}{b^4}\left[\log u + \frac{3a}{u} - \frac{3a^2}{2u^2} + \frac{a^3}{3u^3}\right] + C$$

$$= \frac{1}{b^4}\left[\log(a+bx) + \frac{3a}{a+bx} - \frac{3a^2}{2(a+bx)^2} + \frac{a^3}{3(a+bx)^3}\right] + C$$

❖ Evaluate the integral $\int \frac{1}{x^4\sqrt{x^2-1}}\ dx$

[Rule: $\int \frac{dx}{x^n\sqrt{a+bx^2}}$, n positive even integer, then

substitute $x = \frac{1}{u} \quad \Rightarrow dx = -\frac{1}{u^2}du \quad x = \frac{1}{u} \quad \Rightarrow dx = -\frac{1}{u^2}du$]

Now,

$$\int \frac{1}{x^4\sqrt{x^2-1}}\ dx = -\int \frac{u^3 du}{\sqrt{1-u^2}} = \int \frac{-u(1-u^2)+u}{\sqrt{1-u^2}}du = -\int u\sqrt{1-u^2}du + \int \frac{u}{\sqrt{1-u^2}}du$$

$$= -\frac{1}{3}(1-u^2)^{\frac{3}{2}} + \sqrt{1-u^2} + C = -\frac{\sqrt{1-u^2}}{3x^3}(2x^2+1) \quad + + C$$

Integration by Parts

$$\int uv\, dx = u\int vdx - \int\left\{\frac{d}{dx}(u)\int vdx\right\}dx$$

Find the value of $\int xe^x\, dx$, $\int \ln x\, dx$, $\int e^x \cos x\, dx$, $\int_1^2 x^2 \ln x\, dx$, $\int \sin(\ln x)\, dx$

Integrate $\int \log(x+\sqrt{x^2+a^2}\, dx$

❖ $\int \frac{xe^x}{(x+1)^2}\, dx = \int \frac{(x+1)e^x - e^x}{(x+1)^2}\, dx = \int \frac{e^x}{(x+1)}\, dx - \int \frac{e^x}{(x+1)^2}\, dx$ .....

Integrating by parts the first integral.....

❖ Find $f(x)$ if $f'(x) = e^x(\sin x - \cos x)$ and $f(0) = 1$.

$\because \quad f'(x) = e^x(\sin x - \cos x)$

$f(x) = \int f'(x)dx + C$, where C is constant of integration.

$= \int e^x(\sin x - \cos x)dx + C$

$= \int e^x \sin xdx - \int e^x \cos xdx + C$ [first term integration by parts.....]

❖ $\int xe^x \cos x\, dx$ (Ref. Das & Mukherjee, p-78)

Let, $I = \int e^x \cos x\, dx$

Partial fractions

❖ $\int \frac{5x-3}{x^2-2x-3}\, dx = \int\left(\frac{2}{x+1} + \frac{3}{x-3}\right)dx$

❖ $\int \frac{2x^2+3}{x(x-1)^2}\, dx$

❖ $\int \frac{x^2+4}{3x^3+4x^2-4x}dx$

Integrals involving quadratics

❖ $\int \frac{1}{2x^2-3x+2}\, dx$

$$2x^2-3x+2 = 2\left(x^2-\frac{3}{2}x+1\right) = 2\left(x^2-\frac{3}{2}x+\frac{9}{16}-\frac{9}{16}+1\right)$$

$$= 2\left(\left(x-\frac{3}{4}\right)^2+\frac{7}{16}\right)$$

$$\int \frac{1}{2x^2-3x+2}\,dx = \frac{1}{2}\int \frac{1}{\left(x-\frac{3}{4}\right)^2+\frac{7}{16}}\,dx$$

$$= \frac{1}{2}.\,\frac{4}{\sqrt{7}}\tan^{-1}\left(\frac{4x-3}{\sqrt{7}}\right)+c$$

❖ $\displaystyle\int \frac{7x-9}{x^2-2x+35}\,dx$

❖ $\displaystyle I = \int \frac{x^2+4}{x^2+2x+3}\,dx = \int\left(1-\frac{2x-1}{x^2+2x+3}\right)dx = \int dx - \int \frac{2x-1}{x^2+2x+3}\,dx$

Second integral

$$= \int \frac{(2x+2)-3}{x^2+2x+3}\,dx = \int \frac{2x+2}{x^2+2x+3}\,dx - 3\int \frac{1}{(x+1)^2+(\sqrt{2})^2}\,dx$$

$$= \log(x^2+2x+3) - \frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$$

$$\therefore\quad I = x - \log(x^2+2x+3) + \frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$$

❖ Integrate $\displaystyle\int \frac{4x-3}{9x^2+4}\,dx$

$$\int \frac{4x-3}{9x^2+4}\,dx = \int \frac{\frac{2}{9}\frac{d}{dx}(9x^2+4)-3}{9x^2+4}\,dx$$

$$= \frac{2}{9}\int \frac{\frac{d}{dx}(9x^2+4)}{9x^2+4}\,dx - 3\int \frac{1}{9x^2+4}\,dx + C$$

$$= \frac{2}{9}\log(9x^2+4) - \frac{1}{3}\int \frac{1}{x^2+\left(\frac{2}{3}\right)^2}\,dx + C$$

$$= \frac{2}{9}\log(9x^2+4) - \frac{1}{3}\tan^{-1}\left(\frac{3x}{2}\right)+C$$

❖ Compute $\int \frac{1}{\sqrt{(x-4)(6-x)}}\,dx$

Put, $6-x=u^2 \Rightarrow dx=-2udu$

Therefore, $\int \frac{1}{\sqrt{(x-4)(6-x)}}\,dx = \int \frac{-2udu}{\sqrt{(6-u^2-4)u^2}} + C$

$$= -2\int \frac{du}{\sqrt{2-u^2}} + C$$

Again, let $u=\sqrt{2}\ \sin\theta \Rightarrow du=\sqrt{2}\ \cos\theta\ d\theta$

So, $\int \frac{1}{\sqrt{(x-4)(6-x)}}\,dx = -2\int \frac{\sqrt{2}\cos\theta\ d\theta}{\sqrt{2}\ \sqrt{1-\sin^2\theta}} + C = -2\int d\theta + C = -2\theta + C$

$$= -2\sin^{-1}\frac{u}{\sqrt{2}} + C = -\sin^{-1}\left(\frac{\sqrt{6-x}}{\sqrt{2}}\right) + C$$

❖ Compute $\int \frac{1}{(x+3)\sqrt{x+2}}\,dx$ [Hints: $x+2=u^2$]

Definite Integral

$\int_0^{\frac{\pi}{2}} \log\sin x\ dx$

$\int_0^{\pi} \log(1+\cos x)\ dx$

$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$,

$\int_0^1 \frac{dx}{(1+x^2)\sqrt{1-x^2}}\,dx$ [Hints: put $x=\sin\theta$]

$\int_0^{\frac{2\pi}{3}} \frac{dx}{5+4\cos x}\,dx$

$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\tan x}}\,dx$

Let, $\mathrm{I} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\tan x}}\, dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx \quad \cdots (1)$

By the property, $\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$

$\therefore \quad I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx \quad \cdots (1)$

;;

❖ Show that $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx = \frac{\pi}{4}$

Let, $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx$..........$(i)$

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}\, dx$$

Or, $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\, dx$.....$(ii)$

Adding (i) and (ii)

$\therefore \quad 2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}$

$\therefore \quad I = \frac{\pi}{4}$.

❖ Show that $\int_0^1 \frac{\log(1+x)}{1+x^2}\, dx = \frac{\pi}{8}\log 2$.

Solution: Put $x = \tan\theta$; $\therefore dx = \sec^2\theta\, d\theta$

When x=0, θ=0; when x=1, θ=π/4

$\therefore \quad I = \int_0^{\frac{\pi}{4}} \log(1+\tan\theta)\, d\theta = \int_0^{\frac{\pi}{4}} \log\{1+\tan\left(\frac{\pi}{4}-\theta\right)\}d\theta$

Now, $1+\tan\left(\frac{\pi}{4}-\theta\right) = 1+\frac{1-\tan\theta}{1+\tan\theta} = \frac{2}{1+\tan\theta}$

$\therefore \quad I = \int_0^{\frac{\pi}{4}} \log\frac{2}{1+\tan\theta}\, d\theta = \int_0^{\frac{\pi}{4}} \{\log 2 - \log(1+\tan\theta)\}d\theta$

$\therefore \quad I = \int_0^{\frac{\pi}{4}} \log 2 - \int_0^{\frac{\pi}{4}} \log(1+\tan\theta)\}d\theta$

$$\therefore \quad I = \int_0^{\frac{\pi}{4}} \log 2 d\theta - \int_0^{\frac{\pi}{4}} \log(1+\tan\theta)\} d\theta = \frac{1}{4}\pi.\log 2 - I$$

$$\therefore \quad 2I = \frac{1}{4}\pi.\log 2 \quad \Rightarrow \quad I = \frac{\pi}{8}\log 2.$$

❖ Evaluate $\int_0^a \sqrt{a^2 - x^2}\ dx$

Put $x = a\sin\theta, \quad \therefore dx = a\cos\theta\ d\theta$

Also, when x=0, θ=0, and when x=a, θ=π/2.

$$I = \int_0^{\frac{\pi}{2}} a^2 \cos^2\theta\ d\theta = a^2 \int_0^{\frac{\pi}{2}} \cos^2\theta\ d\theta$$

Now, $\int \cos^2\theta\ d\theta = \frac{1}{2}\int (1+\cos 2\theta)\ d\theta = \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]$

$$\therefore \quad I = a^2 \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} = \frac{1}{4}\pi \quad a^2.$$

❖ Evaluate $\int_\alpha^\beta \sqrt{(x-\alpha)(\beta-x)}\ dx$

Put $x = \alpha\cos^2\theta + \beta\sin^2\theta. \quad \therefore \quad dx = 2(\beta-\alpha)\sin\theta\cos\theta\ d\theta$

Also, $x - \alpha = \beta\sin^2\theta. - \alpha(1-\cos^2\theta) = (\beta-\alpha)\sin^2\theta,$

$\beta - x = \beta(1-\sin^2\theta.) - \alpha\cos^2\theta = (\beta-\alpha)\cos^2\theta.$

$\therefore$ when $x = \alpha, \quad (\beta-\alpha)\sin^2\theta = 0.$

$\therefore \quad \therefore \quad \sin\theta = 0 \sin ce \quad \beta \neq \alpha. \quad \therefore \quad \theta = 0$

Similarly, when $x = \beta, \quad (\beta-\alpha)\cos^2\theta = 0.$

$$\therefore \quad \cos\theta = 0, \quad \therefore \quad \theta = \frac{\pi}{2}$$

$$\therefore \quad I = 2(\beta-\alpha)^2 \int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\ d\theta$$

Now,

$$\sin^2\theta\cos^2\theta = \frac{1}{4}.4\sin^2\theta\cos^2\theta = \frac{1}{4}\sin^2 2\theta = \frac{1}{8}(1-\cos 4\theta).$$

Also, $\int (1-\cos 4\theta) d\theta = \theta - \frac{1}{4}\sin 4\theta.$

$$\therefore \quad I = 2(\beta-\alpha)^2 \frac{1}{8}\int_0^{\frac{\pi}{2}} (1-\cos 4\theta)\ d\theta = \frac{1}{4}(\beta-\alpha)^2\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{2}}$$

$$\therefore \quad I = \frac{1}{4}(\beta-\alpha)^2\left[\frac{1}{2}\pi - \frac{1}{4}\sin 2\pi\right] = \frac{1}{8}\pi(\beta-\alpha)^2$$

Find the reduction formula for $\int (x^2 + a^2)^n \, dx$

Find the reduction formula for $\int x^m e^x \, dx$ and use the formula to evaluate $\int_0^1 x^4 e^x \, dx$.

Obtain a reduction formula for $\int_0^\infty e^{-ax} \cos^n x \, dx,\ a > 0$ and hence find the value of $\int_0^\infty e^{-4x} \cos^5 x \, dx$.

Beta and Gamma Functions

A function given by

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} \, dx,\ m > 0,\ n > 0$$

is called **Beta function** and the integral is called First Eulerian integral.

A function given by

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} \, dx,\ n > 0$$

is called **gamma function** and the integral is called Second Eulerian integral.

Properties:

$B(m,n) = B(n,m)$

$\Gamma(1) = 1$

$\Gamma(n+1) = n\Gamma(n) = n!$

$B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ (Relation between beta and gamma functions)

Problem: (i) State the relation between Beta and Gamma functions and use it to show that $\int_0^1 x^{\frac{3}{2}}(1-x)^{\frac{3}{2}}\, dx = \frac{3\pi}{128}$.

Express $\int_0^{\frac{\pi}{2}} \sin^4\theta \cos^6\theta\, d\theta$ as a beta function and hence evaluate it.

Show that $\int_0^1 \frac{xdx}{\sqrt{1-x^5}} = \frac{1}{5} B\left(\frac{2}{5}, \frac{1}{2}\right)$.

Solution: (i) The Relation between beta function and gamma function is

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Here, $I = \int_0^1 x^{\frac{3}{2}}(1-x)^{\frac{3}{2}}\, dx = .\int_0^1 x^{\frac{5}{2}-1}(1-x)^{\frac{5}{2}-1}\, dx = B(\frac{5}{2}, \frac{5}{2})$

$$= \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(5)} = \frac{\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right).\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{4.3.2.1} = \frac{3}{128}\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = \frac{3\pi}{128}$$

(iv)Let $x^5 = \sin^2\theta \Rightarrow x = \sin^{\frac{2}{5}}\theta \Rightarrow dx = \frac{2}{5}.\sin^{-\frac{3}{5}}\theta.\cos\theta\, d\theta$

As $x \to 0$, $\theta \to 0$, and when $x \to 1$, $\theta \to \frac{\pi}{2}$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{2}{5}}\theta.\frac{2}{5}\sin^{-\frac{3}{5}}\theta\cos\theta\, d\theta}{\cos\theta}$$

$$= \frac{2}{5}.\int_0^{\frac{\pi}{2}} \sin^{-\frac{1}{5}}\theta\cos^0\theta\, d\theta = \frac{1}{5}.2\int_0^{\frac{\pi}{2}} \sin^{\left(2\times\frac{2}{5}-1\right)}\theta\, \cos^{\left(2\times\frac{1}{2}-1\right)}\theta\, d\theta = \frac{1}{5}.B\left(\frac{2}{5}, \frac{1}{2}\right).$$

Evaluate (i) $\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^4\theta\, d\theta = \frac{5\pi}{256}$ (ii) $\int_0^{\frac{\pi}{6}} \cos^4\theta\sin^2 6\theta\, d\theta = \frac{5\pi}{192}$

(iii) $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\, d\theta = \frac{\pi}{\sqrt{2}}$

Solution: (ii) $\int_0^{\frac{\pi}{6}} \cos^4\theta\sin^2 6\theta\, d\theta = \int_0^{\frac{\pi}{6}} \cos^4\theta\,(2\sin 3\theta\cos 3\theta)^2\, d\theta$

$$= 4\int_0^{\frac{\pi}{6}} \sin^2 3\theta\cos^6 3\theta\, d\theta = \frac{4}{3}\int_{3\theta=0}^{3\theta=\frac{\pi}{2}} \sin^2(3\theta)\cos^6(3\theta)\, d(3\theta)$$

$\because \theta = 0 \Rightarrow 3\theta = 0; \quad \theta = \pi/6 \Rightarrow 3\theta = \pi/2$

$$= \frac{4}{3}\,\frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{6+1}{2}\right)}{2\Gamma\left(\frac{2+6+2}{2}\right)} = \frac{2\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{7}{2}\right)}{3\Gamma(5)} = \frac{5\pi}{192}$$

(iii) $$\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\; d\theta = \int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}}\theta\cos^{-\frac{1}{2}}\theta\; d\theta = \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)}{2\Gamma\left(\frac{\frac{1}{2}-\frac{1}{2}+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{4}\right)}{2\Gamma(1)} = \frac{\pi}{2}.$$

\# Show that $$\int_0^{\frac{\pi}{2}} \sin^p\theta\cos^q\theta\; d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

Proof: Try yourself

Put, $I = \int_0^{\frac{\pi}{2}} \sin^p\theta\cos^q\theta\; d\theta$

$$= \int_0^{\frac{\pi}{2}} \sin^{p-1}\theta\cos^{q-1}\theta\sin\theta\cos\theta\; d\theta$$

$$= \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{(p-1)/2}\theta\,(\cos^2\theta)^{(q-1)/2}\sin\theta\cos\theta\; d\theta$$

$$= \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{(p-1)/2}\theta\,(1-\sin^2\theta)^{(q-1)/2}\sin\theta\cos\theta\; d\theta$$

Put, $\sin^2\theta = t$, then $2\sin\theta\cos\theta\; d\theta = dt$

When $\theta = 0$, then $t = 0$

When $\theta = \frac{\pi}{2}$, then $t = 1$

So, $$I = \frac{1}{2}\int_0^1 t^{(p-1)/2}\theta\,(1-t)^{(q-1)/2}\; dt = \frac{1}{2}\int_0^1 t^{(p+1)/2-1}\theta\,(1-t)^{(q+1)/2-1}\; dt$$

$$= \frac{1}{2}.\ \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \frac{\Gamma\left(\frac{p+1}{2}\right), \Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+1}{2}\right)} \quad \text{(Proved)}$$

Evaluate (i) $\int_0^{\pi} x\cos^4 x\, dx$ (ii) $\int_0^{\pi} x\sin^2 x \cos^4 x\, dx$

Solution: (ii) Put, $I = \int_0^{\pi} x\sin^2 x \cos^4 x\, dx$ …..(i)

$= \int_0^{\pi} (\pi - x)\, \{\sin(\pi - x)\}^2\, \{\cos(\pi - x)\}^4\, dx$

$= \int_0^{\pi} (\pi - x)\, \sin^2 x \cos^4 x\, dx$ ……{ii)

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} (x + \pi - x) \sin^2 x \cos^4 x\, dx = \pi \int_0^{\pi} \sin^2 x \cos^4 x\, dx = 2\pi \int_0^{\frac{\pi}{2}} \sin^2 x \cos^4 x\, dx$$

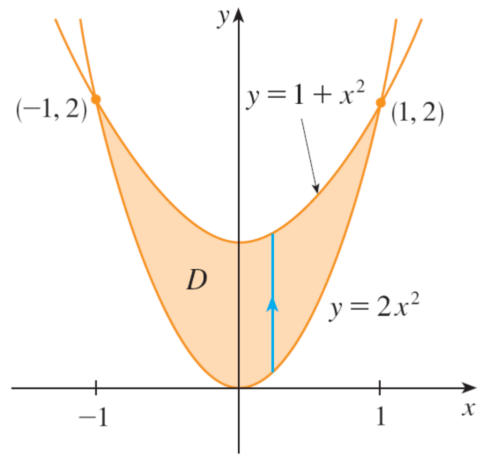
$$\therefore \quad I = \pi \int_0^{\pi/2} \sin^2 x \cos^4 x\, dx$$

$$= \frac{\pi.\ \Gamma\left(\frac{3}{2}\right).\Gamma\left(\frac{5}{2}\right)}{2\Gamma(4)} = \pi.\ \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right).\frac{3}{2}.\frac{1}{2}.\Gamma\left(\frac{1}{2}\right)}{2.3.2.1} = \frac{\pi.\frac{1}{2}.\sqrt{\pi}..\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}}{2.3.2.1} = \frac{\pi^2}{32}$$

So, $\int_0^{\pi} x\sin^2 x \cos^4 x\, dx = \frac{\pi^2}{32}$

Double Integral:

Problem: Evaluate $\iint_D (x + 2y)dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.



Solution: The parabolas intersect when $2x^2 = 1 + x^2 \quad \Rightarrow \quad x^2 = 1 \quad i.e. \quad x = \pm 1$.

We note that the region D is a type I region but not a type II region. So, we can write,

$$D = \{(x, y) \mid -1 \le x \le 1, \quad 2x^2 \le y \le 1 + x^2\}$$

$$\iint_D (x + 2y)\,dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\,dy\,dx$$

$$= \int_{-1}^{1} \left[xy + y^2\right]_{y=2x^2}^{y=1+x^2} dx$$

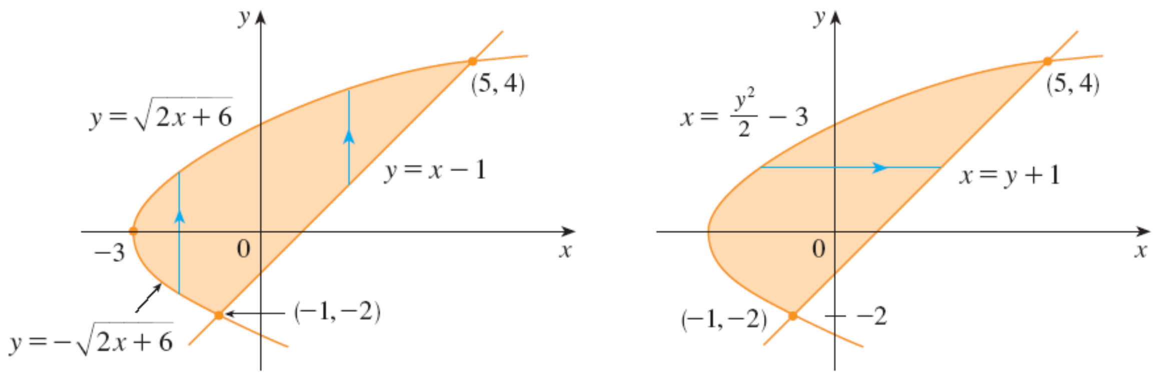
$$= \int_{-1}^{1} \left[x(1 + x^2) + (1 + x^2)^2 - x(2x^2) - (2x^2)^2\right] dx$$

$$= \int_{-1}^{1} \left[-3x^4 - x^3 + 2x^2 + x + 1\right] dx$$

$$= \frac{32}{15}$$

Problem-2: Evaluate $\iint_D xy\,dA$ where D is the region bounded by the line y=x-1 and the parabola $y^2=2x+6$.

Solution: The region is shown:



(a) *D* as a type I region (b) *D* as a type II region

Hence, we prefer to express *D* as a type II region:

$$D = \{(x, y) \mid -2 \le y \le 4,\ 1/2y^2 - 3 \le x \le y + 1\}$$

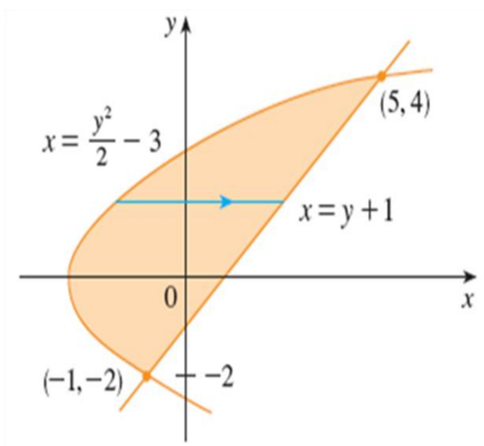
$$\iint_D xy\,dA = \int_{-2}^{4}\int_{\frac{1}{2}y^2-3}^{y+1} xy\,dx\,dy$$

$$= \int_{-2}^{4}\left[\frac{x^2}{2}y\right]_{x=\frac{1}{2}y^2-3}^{x=y+1} dy$$

$$= \tfrac{1}{2}\int_{-2}^{4} y\left[(y+1)^2 - (\tfrac{1}{2}y^2-3)^2\right]dy$$

$$= \tfrac{1}{2}\int_{-2}^{4}\left(-\frac{y^5}{4}+4y^3+2y^2-8y\right)dy$$

$$= \tfrac{1}{2}\left[-\frac{y^6}{24}+y^4+2\frac{y^3}{3}-4y^2\right]_{-2}^{4} = 36$$



D as a type II region

- **Find the value of** $\int_1^2\int_{\frac{y}{2}}^{y} e^{2x-y}\,dx\,dy$

Let, $A=\int_1^2\int_{\frac{y}{2}}^{y} e^{2x-y}\ dx\, dy$

$$=\int_1^2\left\{\int_{\frac{y}{2}}^{y} e^{2x-y}dx\right\}\ dy$$

$$=\int_1^2\left[\frac{e^{2x-y}}{2}\right]_{x=y/2}^{y} dy \quad =\frac{1}{2}\int_1^2 (e^y-1)dy=\frac{1}{2}[e^y-y]_1^2$$

$$=\frac{1}{2}[(e^2-2)-(e^1-1)]=\frac{1}{2}[e^2-e-1)]$$

❖ **Evaluate** $\int_0^1\int_0^x y\sqrt{x^2-y^2}\ dy\, dx$

$$\int_0^1\int_0^x y\sqrt{x^2-y^2}\ dy\, dx=\int_0^1\left[\int_0^x y(x^2-y^2)^{\frac{1}{2}}\ dy\right]dx$$

$$=-\frac{1}{2}\int_0^1\left[\int_0^x (x^2-y^2)^{\frac{1}{2}}\ d(-y^2)\right]dx$$

$$=-\frac{1}{2}\int_0^1\left[\int_0^x (x^2-y^2)^{\frac{1}{2}}\ d(x^2-y^2)\right]dx$$

$$=-\frac{1}{2}\int_0^1\left[\frac{(x^2-y^2)^{\frac{3}{2}}}{3/2}\right]_{y=0}^{x}\quad dx$$

$$=-\frac{1}{3}\int_0^1\left[0-(x^2-0)^{3/2}\right]dx$$

$$=\frac{1}{3}\int_0^1 x^3\ dx=\frac{1}{3}\left[\frac{x^4}{4}\right]_0^1=\frac{1}{12}.$$

❖ Evaluate the double integral $\iint_R (2x-y^2)\ dA$, where R is the region bounded by $\mathrm{y}=-\mathrm{x}+1$, $\mathrm{y}=\mathrm{x}+1$ and $\mathrm{y}=3$.

Solution: Given that

$$y = -x + 1 \cdots (1),\ y = x + 1 \cdots (2), \quad y = 3 \cdots (3)$$

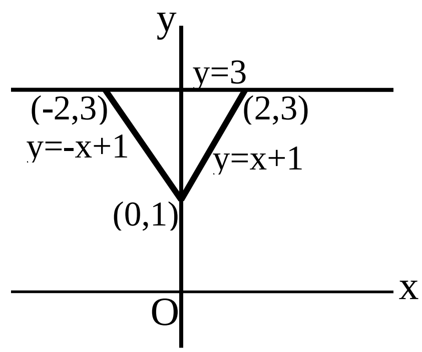
From (1) and (2), we get

$$x + 1 = -x + 1 \Rightarrow x = 0$$

From (1), $y = 0 + 1 = 1$

The lines (1) and (2) intersect at the point (0,1). Here, the region R lies between $y = 1,\ y = 3\ and\ x = y - 1$

$$i.e. \quad R = \{(x, y) : 1 \le y \le 3, 1 - y \le x \le y - 1\}$$



$$\therefore \quad \iint_R (2x - y^2)\,dA = \int_1^3 \int_{1-y}^{y-1} (2x - y^2)\,dx\,dy$$

$$= \int_1^3 \left[x^2 - y^2 x\right]_{x=1-y}^{y-1} dy$$

$$= \int_1^3 \left[1 - 2y + 2y^2 - y^3 - 1 + 2y - y^3\right] dy$$

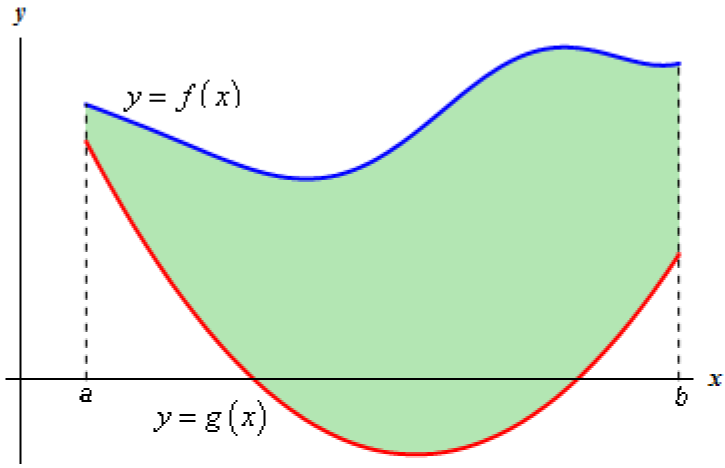
$$= \int_1^3 \left[2y^2 - 2y^3\right] dy = \left[\frac{2y^3}{3} - \frac{y^4}{2}\right]_1^3 = -\frac{68}{3}$$

Applications of integral calculus

Area Between Curves

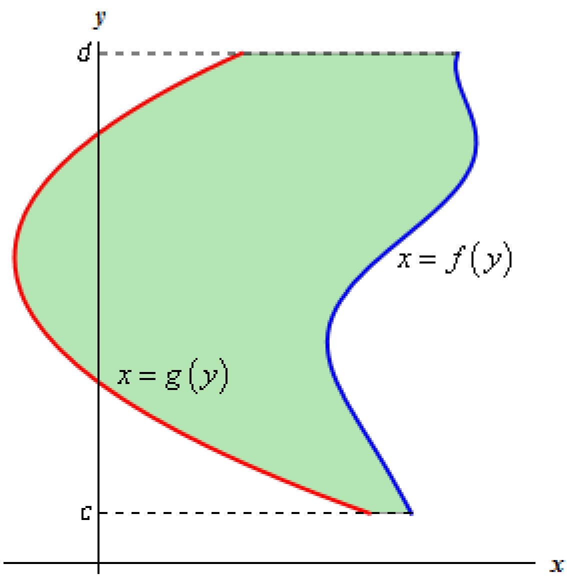
In this section we are going to look at finding the area between two curves. There are actually two cases that we are going to be looking at.

In the first case we want to determine the area between $y = f(x)$ and $y = g(x)$ on the interval $[a,b]$. We are also going to assume that $f(x) \ge g(x)$. Take a look at the following sketch to get an idea of what we're initially going to look at.



The area can be defined in this case $A = \int_a^b f(x) - g(x)\, dx$ (1)

The second case is almost identical to the first case. Here we are going to determine the area between $x = f(y)$ and $x = g(y)$ on the interval $[c,d]$ with $f(y) \geq g(y)$.



In this case the formula is,

$$A = \int_c^d f(y) - g(y)\, dy \qquad (2)$$

Now (1) and (2) are perfectly serviceable formulas, however, it is sometimes easy to forget that these always require the first function to be the larger of the two functions. So, instead of these formulas we will instead use the following "word" formulas to make sure that we remember that the area is always the "larger" function minus the "smaller" function.

In the first case we will use,

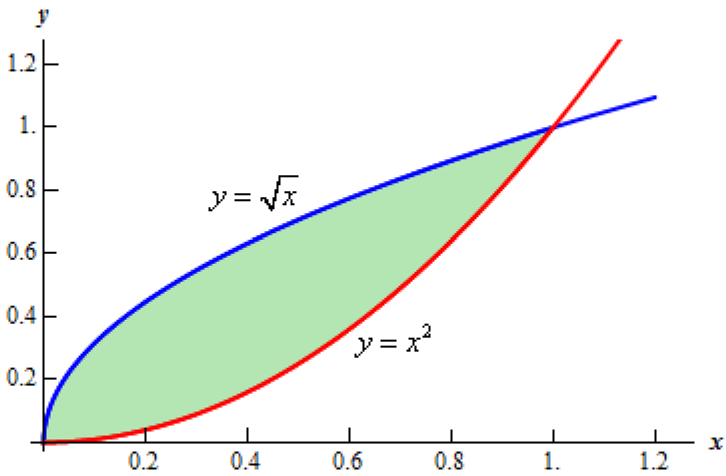
$$A = \int_a^b \begin{pmatrix} upper \\ \\ function \end{pmatrix} - \begin{pmatrix} lower \\ \\ function \end{pmatrix} dx, \qquad a \le x \le b \qquad (3)$$

In the second case we will use,

$$A = \int_c^d \begin{pmatrix} right \\ \\ function \end{pmatrix} - \begin{pmatrix} left \\ \\ function \end{pmatrix} dy, \qquad c \le x \le d \qquad (4)$$

Example 1 Determine the area of the region enclosed by $y = x^2$ and $y = \sqrt{x}$.

Solution: First of all, just what do we mean by "area enclosed by". This means that the region we're interested in must have one of the two curves on every boundary of the region. So, here is a graph of the two functions with the enclosed region shaded.



Note that, we don't take any part of the region to the right of the intersection point of these two graphs. In this region there is no boundary on the right side and so is not part of the enclosed area. Remember that one of the given functions must be on the each boundary of the enclosed region.

Also from this graph it's clear that the upper function will be dependent on the range of x's that we use. Because of this you should always sketch of a graph of the region. Without a sketch it's often easy to mistake which of the two functions is the larger. In this case most would probably say that $y = x^2$ is the upper function and they would be right for the vast majority of the x's. However, in this case it is the lower of the two functions.

The limits of integration for this will be the intersection points of the two curves. In this case it's pretty easy to see that they will intersect at $x = 0$ and $x = 1$. So these are the limits of integration.

So, the integral that we'll need to compute to find the area is,

$$A = \int_a^b \begin{pmatrix} upper \\ \\ function \end{pmatrix} - \begin{pmatrix} lower \\ \\ function \end{pmatrix} dx$$
$$= \int_0^1 (\sqrt{x} - x^2)dx = 1/3.$$

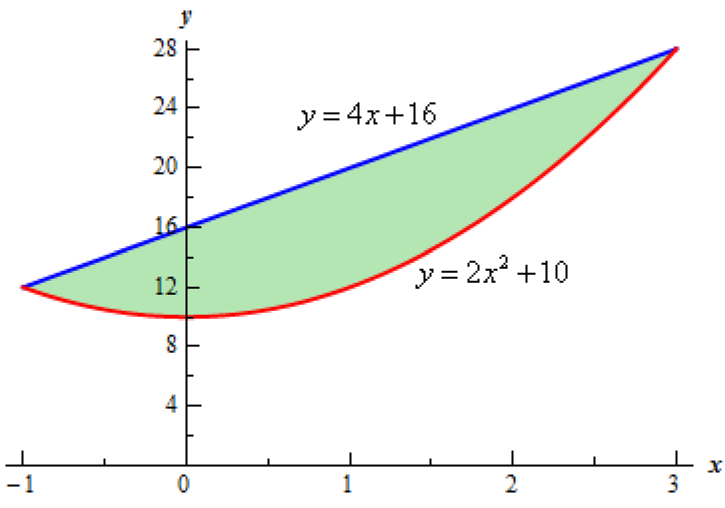
Example 2 Determine the area of the region bounded by $y = 2x^2 + 10$ and $y = 4x + 16$.

Solution: In this case the intersection points (which we'll need eventually) are not going to be easily identified from the graph so let's go ahead and get them now. Note that for most of these problems you'll not be able to accurately identify the intersection points from the graph and so you'll need to be able to determine them by hand. In this case we can get the intersection points by setting the two equations equal.

$$2x^2 + 10 = 4x + 16$$
$$\Rightarrow \; 2x^2 - 4x - 6 = 0$$
$$\Rightarrow 2(x+1)(x-3) = 0.$$

So it looks like the two curves will intersect at $x = -1$ and $x = 3$. If we need them we can get the *y* values corresponding to each of these by plugging the values back into either of the equations. We'll leave it to you to verify that the coordinates of the two intersection points on the graph are (-1,12) and (3,28).

Note as well that if you aren't good at graphing knowing the intersection points can help in at least getting the graph started. Here is a graph of the region.



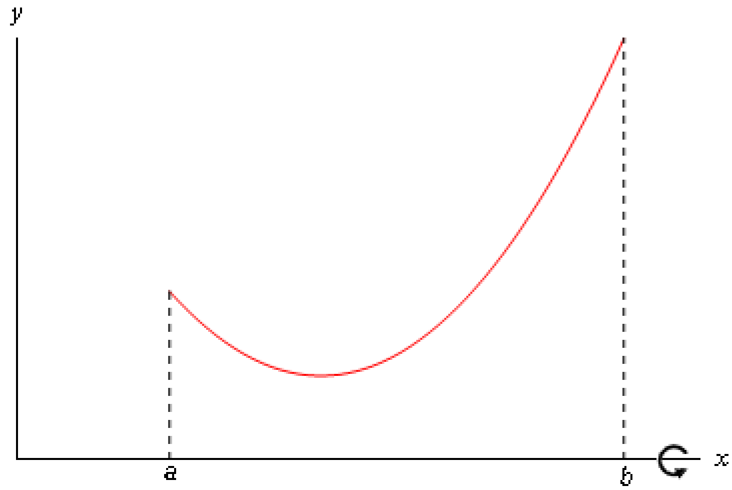
With the graph we can now identify the upper and lower function and so we can now find the enclosed area.

$$A = \int_a^b \begin{pmatrix} upper \\ \\ function \end{pmatrix} - \begin{pmatrix} lower \\ \\ function \end{pmatrix} dx$$

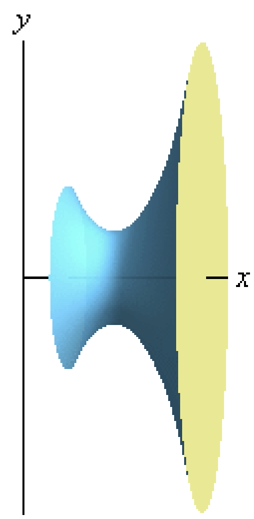
$$= \int_{-1}^{3} 4x + 16 - (2x^2 + 10)\ dx = \int_{-1}^{3} -2x^2 + 4x + 6\ dx = 64/3$$

Volumes of Solids of Revolution / Method of Rings

In this section we will start looking at the volume of a solid of revolution. We should first define just what a solid of revolution is. To get a solid of revolution we start out with a function, $y = f(x)$ on an interval [a,b].



We then rotate this curve about a given axis to get the surface of the solid of revolution. For purposes of this discussion let's rotate the curve about the *x*-axis, although it could be any vertical or horizontal axis. Doing this for the curve above gives the following three dimensional region.



What we want to do over the course of the next two sections is to determine the volume of this object.
In the final the [Area and Volume Formulas]() section of the Extras chapter we derived the following formulas for the volume of this solid.

$$V = \int_a^b A(x)\,dx \qquad\qquad V = \int_c^d A(y)\,dy$$

where, $A(x)$ and $A(y)$ is the cross-sectional area of the solid. There are many ways to get the cross-sectional area and we'll see two (or three depending on how you look at it) over the next two sections. Whether we will use $A(x)$ or $A(y)$ will depend upon the method and the axis of rotation used for each problem.

One of the easier methods for getting the cross-sectional area is to cut the object perpendicular to the axis of rotation. Doing this the cross section will be either a solid disk if the object is solid (as our above example is) or a ring if we've hollowed out a portion of the solid (we will see this eventually).

In the case that we get a solid disk the area is,

$$A = \pi(\text{radius})^2$$

where the radius will depend upon the function and the axis of rotation.

In the case that we get a ring the area is,

$$A = \pi\left(\begin{pmatrix}\text{outer}\\ \text{radius}\end{pmatrix}^2 - \begin{pmatrix}\text{inner}\\ \text{radius}\end{pmatrix}^2\right)$$

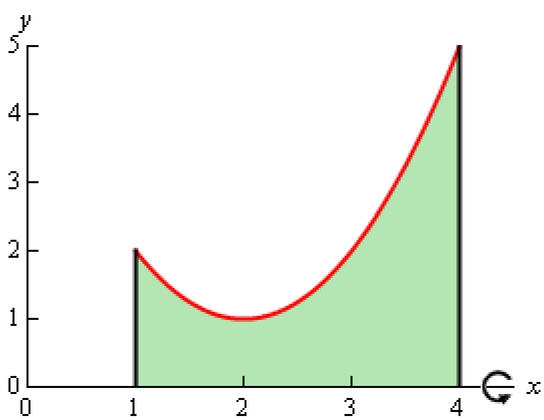
where again both of the radii will depend on the functions given and the axis of rotation. Note as well that in the case of a solid disk we can think of the inner radius as zero and we'll arrive at the correct formula for a solid disk and so this is a much more general formula to use.

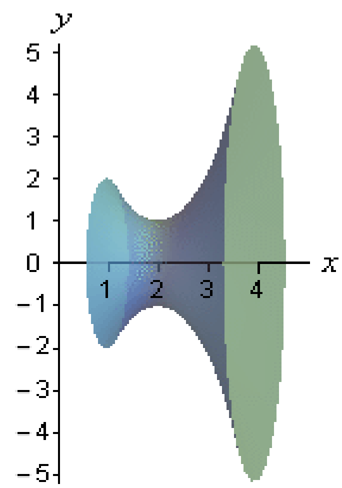
Also, in both cases, whether the area is a function of *x* or a function of *y* will depend upon the axis of rotation as we will see.

This method is often called the **method of disks** or the **method of rings**.

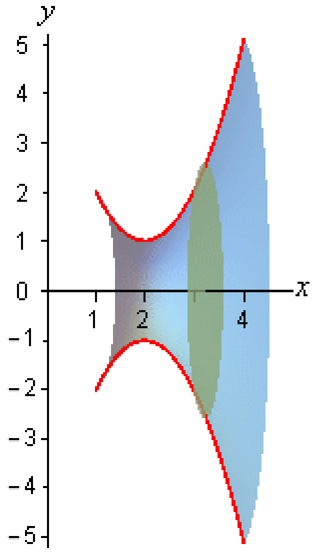
Example 1 Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 4x + 5$, $x = 1$, $x = 4$, and the *x*-axis about the *x*-axis.

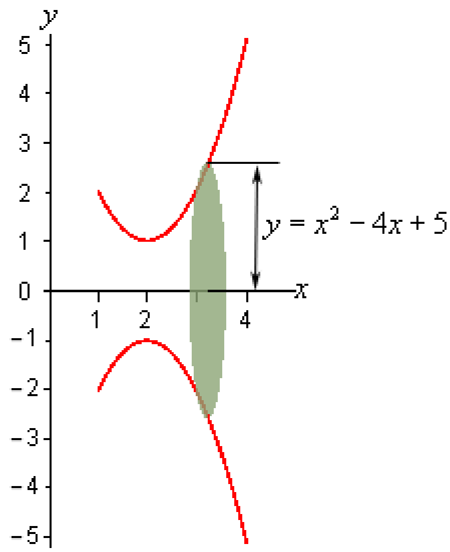
Solution: The first thing to do is get a sketch of the bounding region and the solid obtained by rotating the region about the *x*-axis. Here are both of these sketches.





Okay, to get a cross section we cut the solid at any *x*. Below are a couple of sketches showing a typical cross section. The sketch on the right shows a cut away of the object with a typical cross section without the caps. The sketch on the left shows just the curve we're rotating as well as its mirror image along the bottom of the solid.





In this case the radius is simply the distance from the *x*-axis to the curve and this is nothing more than the function value at that particular *x* as shown above. The cross-sectional area is then,

$$A(x) = \pi\left(x^2 - 4x + 5\right)^2 = \pi\left(x^4 - 8x^3 + 26x^2 - 40x + 25\right)$$

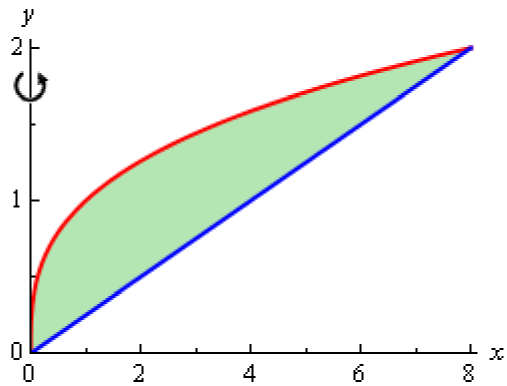
Next we need to determine the limits of integration. Working from left to right the first cross section will occur at $x = 1$ and the last cross section will occur at $x = 4$. These are the limits of integration.

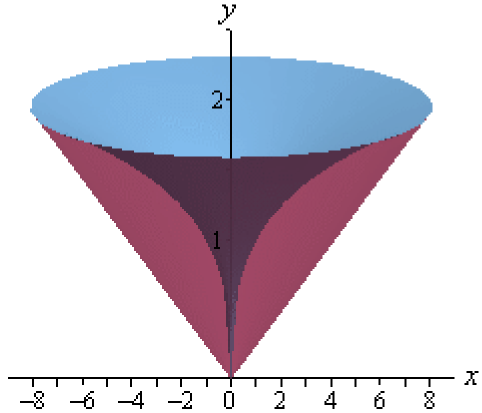
The volume of this solid is then,

$$\begin{aligned} V &= \int_a^b A(x)\,dx \\ &= \pi\int_1^4 x^4 - 8x^3 + 26x^2 - 40x + 25\,dx \\ &= \pi\left(\frac{1}{5}x^5 - 2x^4 + \frac{26}{3}x^3 - 20x^2 + 25x\right)\Bigg|_1^4 \\ &= \frac{78\pi}{5} \end{aligned}$$

Example 2 Determine the volume of the solid obtained by rotating the portion of the region bounded by $y = \sqrt[3]{x}$ and $y = \dfrac{x}{4}$ that lies in the first quadrant about the *y*-axis.

Solution: First, let's get a graph of the bounding region and a graph of the object. Remember that we only want the portion of the bounding region that lies in the first quadrant. There is a portion of the bounding region that is in the third quadrant as well, but we don't want that for this problem.





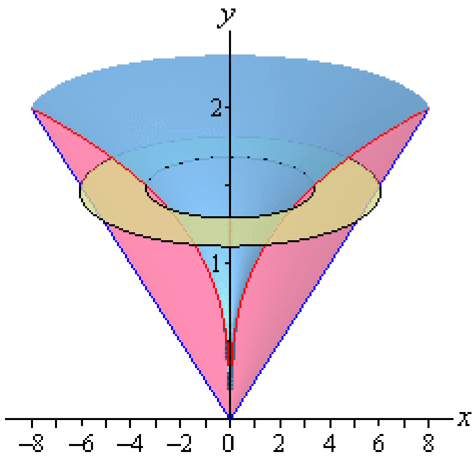
There are a couple of things to note with this problem. First, we are only looking for the volume of the "walls" of this solid, not the complete interior as we did in the last example.

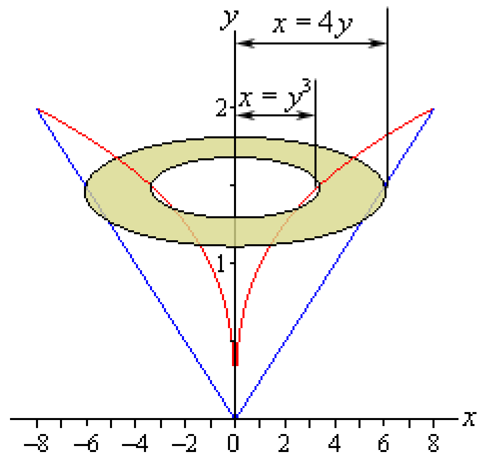
Next, we will get our cross section by cutting the object perpendicular to the axis of rotation. The cross section will be a ring (remember we are only looking at the walls) for this example and it will be horizontal at some y. This means that the inner and outer radius for the ring will be x values and so we will need to rewrite our functions into the form $x = f(y)$. Here are the functions written in the correct form for this example.

$$y = \sqrt[3]{x} \qquad \Rightarrow \qquad x = y^3$$

$$y = \frac{x}{4} \qquad \Rightarrow \qquad x = 4y$$

Here are a couple of sketches of the boundaries of the walls of this object as well as a typical ring. The sketch on the left includes the back portion of the object to give a little context to the figure on the right.





The inner radius in this case is the distance from the *y*-axis to the inner curve while the outer radius is the distance from the *y*-axis to the outer curve. Both of these are then *x* distances and so are given by the equations of the curves as shown above.

The cross-sectional area is then,

$$A(y) = \pi\left((4y)^2 - \left(y^3\right)^2\right) = \pi\left(16y^2 - y^6\right)$$

Working from the bottom of the solid to the top we can see that the first cross-section will occur at $y = 0$ and the last cross-section will occur at $y = 2$. These will be the limits of integration. The volume is then,

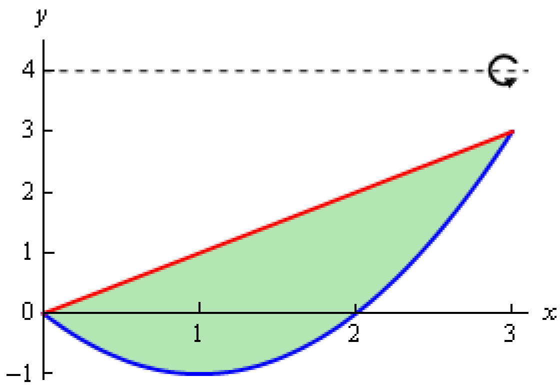
$$\begin{aligned} V &= \int_c^d A(y)\,dy \\ &= \pi\int_0^2 16y^2 - y^6\,dy \\ &= \pi\left(\frac{16}{3}y^3 - \frac{1}{7}y^7\right)\Bigg|_0^2 \\ &= \frac{512\pi}{21} \end{aligned}$$

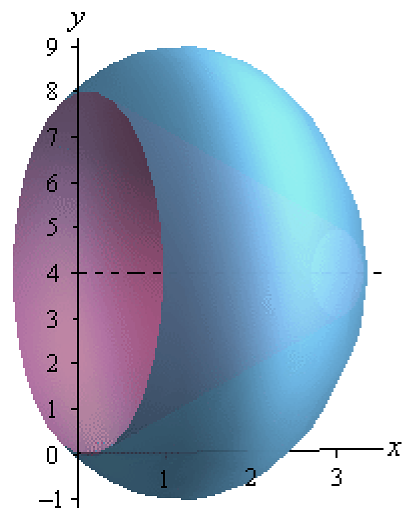
With these two examples out of the way we can now make a generalization about this method. If we rotate about a horizontal axis (the *x*-axis for example) then the cross sectional area will be a function of *x*. Likewise, if we rotate about a vertical axis (the *y*-axis for example) then the cross sectional area will be a function of *y*.

The remaining two examples in this section will make sure that we don't get too used to the idea of always rotating about the *x* or *y*-axis.

Example 3 Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 2x$ and $y = x$ about the line $y = 4$.

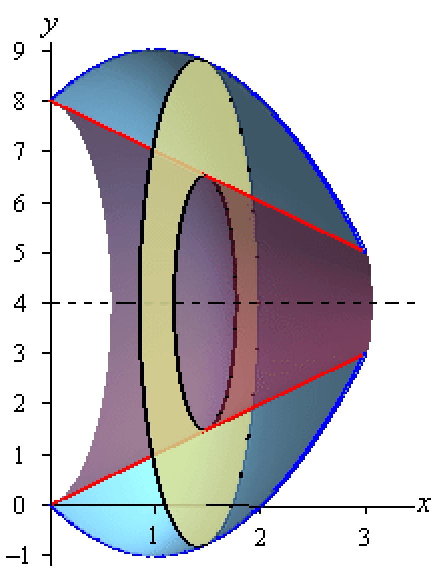
Solution: First let's get the bounding region and the solid graphed.

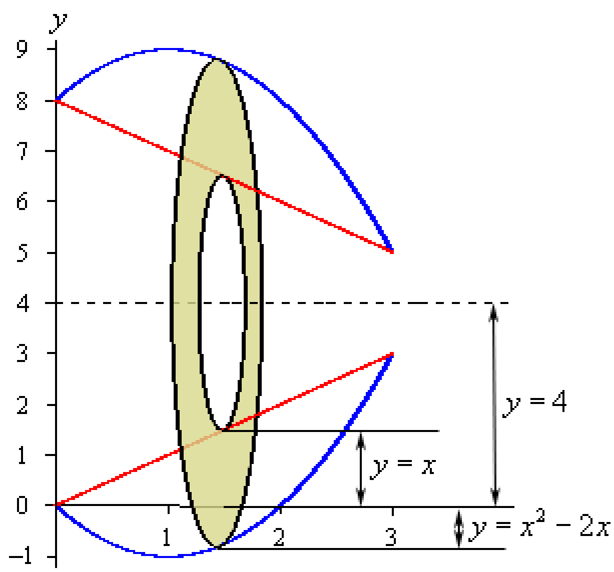




Again, we are going to be looking for the volume of the walls of this object. Also since we are rotating about a horizontal axis we know that the cross-sectional area will be a function of *x*.

Here are a couple of sketches of the boundaries of the walls of this object as well as a typical ring. The sketch on the left includes the back portion of the object to give a little context to the figure on the right.





Now, we're going to have to be careful here in determining the inner and outer radius as they aren't going to be quite as simple they were in the previous two examples.

Let's start with the inner radius as this one is a little clearer. First, the inner radius is NOT *x*. The distance from the *x*-axis to the inner edge of the ring is *x*, but we want the radius and that is the distance from the axis of rotation to the inner edge of the ring. So, we know that the distance from the axis of rotation to the *x*-axis is 4 and the distance from the *x*-axis to the inner ring is *x*. The inner radius must then be the difference between these two. Or,

$$\text{inner radius} = 4 - x$$

The outer radius works the same way. The outer radius is,

$$\text{outer radius} = 4 - \left(x^2 - 2x\right) = -x^2 + 2x + 4$$

Note that given the location of the typical ring in the sketch above the formula for the outer radius may not look quite right but it is in fact correct. As sketched the outer edge of the ring is below the *x*-axis and at this point the value of the function will be negative and so when we do the subtraction in the formula for the outer radius we'll actually be subtracting off a negative number which has the net effect of adding this distance onto 4 and that gives the correct outer radius. Likewise, if the outer edge is above the *x*-axis, the function value will be positive and so we'll be doing an honest subtraction here and again we'll get the correct radius in this case.

The cross-sectional area for this case is,

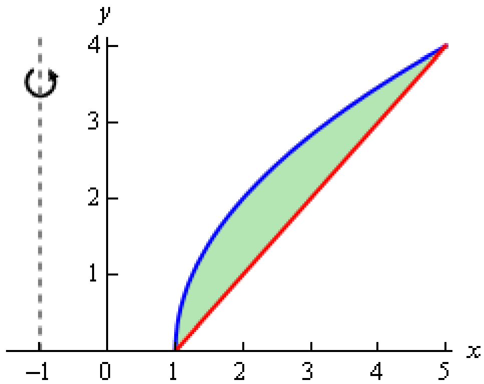
$$A(x) = \pi\left(\left(-x^2 + 2x + 4\right)^2 - \left(4 - x\right)^2\right) = \pi\left(x^4 - 4x^3 - 5x^2 + 24x\right)$$

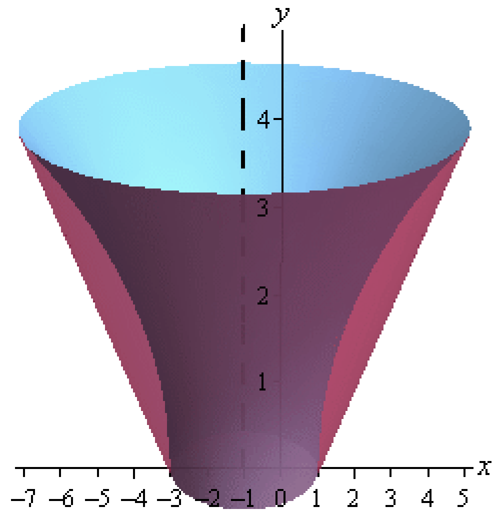
The first ring will occur at $x = 0$ and the last ring will occur at $x = 3$ and so these are our limits of integration. The volume is then,

$$\begin{aligned} V &= \int_a^b A(x)\,dx \\ &= \pi\int_0^3 x^4 - 4x^3 - 5x^2 + 24x\,dx \\ &= \pi\left(\frac{1}{5}x^5 - x^4 - \frac{5}{3}x^3 + 12x^2\right)\Bigg|_0^3 \\ &= \frac{153\pi}{5} \end{aligned}$$

Example 4 Determine the volume of the solid obtained by rotating the region bounded by $y = 2\sqrt{x-1}$ and $y = x - 1$ about the line $x = -1$.

Solution: As with the previous examples, let's first graph the bounded region and the solid.



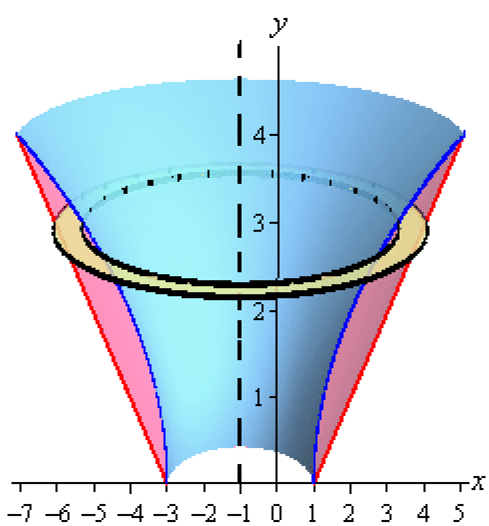


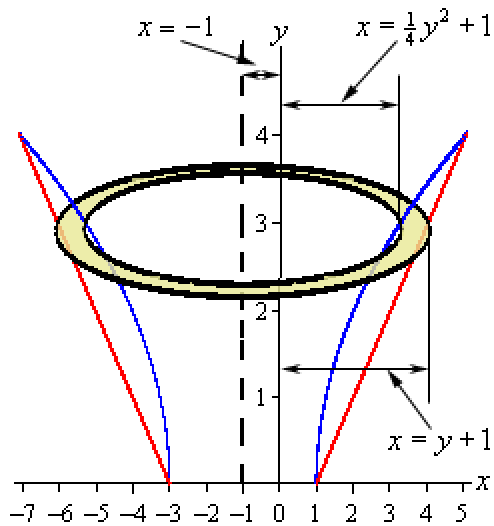
Now, let's notice that since we are rotating about a vertical axis and so the cross-sectional area will be a function of *y*. This also means that we are going to have to rewrite the functions to also get them in terms of *y*.

$$y = 2\sqrt{x-1} \qquad \Rightarrow \qquad x = \frac{y^2}{4} + 1$$

$$y = x - 1 \qquad \Rightarrow \qquad x = y + 1$$

Here are a couple of sketches of the boundaries of the walls of this object as well as a typical ring. The sketch on the left includes the back portion of the object to give a little context to the figure on the right.





The inner and outer radius for this case is both similar and different from the previous example. This example is similar in the sense that the radii are not just the functions. In this example the functions are the distances from the *y*-axis to the edges of the rings. The center of the ring however is a distance of 1 from the *y*-

axis. This means that the distance from the center to the edges is a distance from the axis of rotation to the *y*-axis (a distance of 1) and then from the *y*-axis to the edge of the rings.

So, the radii are then the functions plus 1 and that is what makes this example different from the previous example. Here we had to add the distance to the function value whereas in the previous example we needed to subtract the function from this distance. Note that without sketches the radii on these problems can be difficult to get.

So, in summary, we've got the following for the inner and outer radius for this example.

$$\text{outer radius} = y + 1 + 1 = y + 2$$
$$\text{inner radius} = \frac{y^2}{4} + 1 + 1 = \frac{y^2}{4} + 2$$

The cross-sectional area is then,

$$A(y) = \pi\left((y+2)^2 - \left(\frac{y^2}{4}+2\right)^2\right) = \pi\left(4y - \frac{y^4}{16}\right)$$

The first ring will occur at $y = 0$ and the final ring will occur at $y = 4$ and so these will be our limits of integration.

The volume is,

$$\begin{aligned} V &= \int_c^d A(y)\,dy \\ &= \pi\int_0^4 4y - \frac{y^4}{16}\,dy \\ &= \pi\left(2y^2 - \frac{1}{80}y^5\right)\Bigg|_0^4 \\ &= \frac{96\pi}{5} \end{aligned}$$