Chapter 1: The Foundations: Logic and Proofs

1.3 Propositional Equivalences

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Introduction

DEFINITION 1

A compound proposition that is always true, no matter what the truth values of the propositions that occurs in it, is called a *tautology*. A compound proposition that is always false is called a *contradiction*. A compound proposition that is neither a tautology or a contradiction is called a *contingency*.

Examples of a Tautology and a Contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

1.3 Propositional Equivalences

Logical Equivalences

DEFINITION 2

The compound propositions p and q are called *logically equivalent* if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

- Compound propositions that have the same truth values in all possible cases are called logically equivalent.
- Example: Show that $\neg p \vee q$ and $p \rightarrow q$ are logically equivalent.

Truth Tables for $\neg p \vee q$ and $p \rightarrow q$.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Logical Equivalences

- Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.						
p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Logical Equivalences

- Show that p ∨ (q ∧ r) and (p ∨ q) ∧ (p ∨ r) are logically equivalent.

TABLE 5 **A Demonstration That $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ Are Logically Equivalent.**

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Logical Equivalences

Equivalence	***Name***
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws

$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Logical Equivalences

TABLE 7 Logical Equivalences Involving Conditional Statements.

$$p \rightarrow q \equiv \neg p \vee q$$

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

$$p \vee q \equiv \neg p \rightarrow q$$

$$p \wedge q \equiv \neg(p \rightarrow \neg q)$$

$$\neg(p \rightarrow q) \equiv p \wedge \neg q$$

$$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$$

$$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$$

$$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$$

$$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$$

Constructing New Logical Equivalences

- Example: Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent.

  Solution:

  $$\begin{aligned} \neg(p \rightarrow q) &\equiv \neg(\neg p \vee q) && \textit{by } \mathrm{p} \rightarrow \mathrm{q} \equiv \neg \mathrm{p} \vee \mathrm{q} \\ &\equiv \neg(\neg p) \wedge \neg q && \textit{by the second De Morgan law} \\ &\equiv p \wedge \neg q && \textit{by the double negation law} \end{aligned}$$

- *Note: The above example can also be done using truth tables*.

Constructing New Logical Equivalences

- Example: Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

  Solution:

$\neg(p \vee (\neg p \wedge q))$ $\equiv \neg p \wedge \neg(\neg p \wedge q)$ by the second De Morgan law

$\equiv \neg p \wedge [\neg(\neg p) \vee \neg q]$ by the first De Morgan law

$\equiv \neg p \wedge (p \vee \neg q)$ by the double negation law

$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$ by the second distributive law

$\equiv F \vee (\neg p \wedge \neg q)$ because $\neg p \wedge p \equiv F$

$\equiv (\neg p \wedge \neg q) \vee F$ by the commutative law for disjunction

$\equiv \neg p \wedge \neg q$ by the identity law for F

Constructing New Logical Equivalences

- *Example: Show that* $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

  Solution: To show that this statement is a tautology, we will use logical equivalences to demonstrate that it is logically equivalent to T.

  $(p \wedge q) \rightarrow (p \vee q) \equiv \neg(p \wedge q) \vee (p \vee q)$ *by* $p \rightarrow q \equiv \neg p \vee q$

  $\equiv (\neg p \vee \neg q) \vee (p \vee q)$ *by the first De Morgan law*

  $\equiv (\neg p \vee p) \vee (\neg q \vee q)$ *by the associative and communicative law for disjunction*

  $\equiv T \vee T$

  $\equiv T$

- *Note: The above example can also be done using truth tables*.

Satisfiability

- *A compound proposition is satisfiable if there is at least one value that is true in the truth table.*
- *A compound proposition is unsatisfiable when there is not even a single value in the truth table that is true.*

- *Example:* Determine whether the compound propositions $(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p)$ is satisfiable.
- Solution: Due to the $\lor$ (disjunction) within each individual proposition, having same truth values ensures that at least one of the propositional variable within each individual compound proposition will be true. Therefore, the compound proposition propositions $(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p)$ will be true and hence, satisfiable as there is at least one assignment of truth values for p, q, and r that makes it true.

Satisfiability

- *Example:* Determine whether the compound propositions $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ is satisfiable.

- Solution: Note that $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ is true when at least one of p, q, and r is true and at least one is false (see Exercise 43 of Section 1.1). Hence, $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ is satisfiable, as there is at least one assignment of truth values for p, q, and r that makes it true.