Integration

Integration originates from two different concepts. So, the method of integration are two:

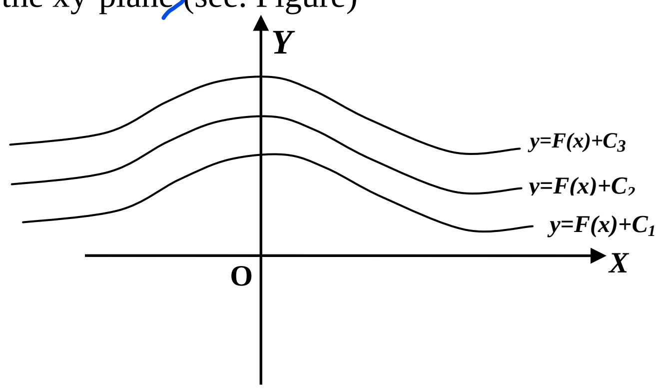
- Integration is the inverse (reverse) process of differentiation, which is the subject matter of indefinite Integration
- Integration is a process of summation which is the subject matter of definite integration.

If $F(x)$ is the derivatives of $f(x)$ for general real values of x i.e. $\frac{d}{dx}\{F(x)\} = f(x)$, then *F(x)* is called the indefinite integral or antiderivative of f(x) and is denoted by

$$F(x) = \int f(x)\, dx$$

Geometric Interpretation of Indefinite Integral

If $F'(x) = f(x)$ then $\int f(x)\, dx = F(x) + C$. Now, if $F(x) + C$ is equated to y *i.e.* $y = F(x) + C$, then it represents the equation of curves for different values of C and these curves are parallel curves. So, indefinite integrals represent a family or set of parallel curves in the xy-plane (see. Figure)



Techniques of Integration

1 ❖ Evaluate $\int\left(\sqrt{\frac{1-x}{1+x}}+\frac{x}{\sqrt{1-x^2}}\right)dx$

Rationalizing and adding

$$\int\left(\sqrt{\frac{1-x}{1+x}}+\frac{x}{\sqrt{1-x^2}}\right)dx=\int\frac{1-x+x}{\sqrt{1-x^2}}\,dx=\int\frac{1}{\sqrt{1-x^2}}\,dx=\sin^{-1}x+C\,.$$

2 ❖ Find $\phi(x)$, if $\phi(0)=\phi'(0)=0$ and $\phi''(x)=\cos^2 x+5$.

Solution: $\because\ \phi''(x)=\cos^2 x+5$

$$\phi'(x)=\int(\cos^2 x+5)\,dx$$

$$=\int\left\{\frac{1}{2}(1+\cos 2x)+5\right\}dx=\frac{1}{2}.\frac{1}{2}\sin 2x+\frac{11x}{2}+A\,,$$

where A is integrating constant.

When $\phi'(0)=0$, then $\phi'(0)=\frac{1}{2}.\frac{1}{2}.0+\frac{11.0}{2}+A\ \Rightarrow A=0$

$$\phi'(x)=\frac{1}{2}.\frac{1}{2}\sin 2x+\frac{11x}{2}$$

On integration, $\phi(x)=-\frac{1}{8}.\cos 2x+\frac{11}{4}x^2+B\,,$

where B is the integrating constant.

Since $\phi(0)=0,\ then\ \phi(0)=-\frac{1}{8}.1+\frac{11}{4}.0+B\Rightarrow B=\frac{1}{8}$

Hence, $\phi(x)=-\frac{1}{8}.\cos 2x+\frac{11}{4}x^2+\frac{1}{8}=\frac{1}{8}(1+22x^2-\cos 2x)\,.$

3 ❖ Evaluate $\int\frac{dx}{\cos(x+a)\cos(x+b)}$

$$=\frac{1}{\sin(a-b)}\int\frac{\sin(a-b)}{\cos(x+a)\cos(x+b)}\,dx=\frac{1}{\sin(a-b)}\int\frac{\sin[(x+a)-(x+b)]}{\cos(x+a)\cos(x+b)}\,dx$$

$$=\frac{1}{\sin(a-b)}\int\frac{\sin(x+a)\cos(x+b)-\cos(x+a)\sin(x+b)}{\cos(x+a)\cos(x+b)}\,dx$$

$$=\frac{1}{\sin(a-b)}\int[\tan(x+a)-\tan(x+b)]\,dx$$

$$=\frac{1}{\sin(a-b)}[\ln\{\sec(x+a)\}-\ln\{\sec(x+b)\}+c=\frac{1}{\sin(a-b)}\ln\left[\frac{\sec(x+a)}{\sec(x+b)}\right]+c$$

Method of Substitutions

✓ ➤ $\int f(g(x))g'(x)\,dx = \int f(u)du$

✓ ➤ $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$

4 ❖ Find $\int \frac{x}{\sqrt{x^2+1}}\,dx$

5 ❖ Find $\int x\cos(3x^2+2)\,dx$, $\int \sin^5 x \cos^2 x\,dx$ X

6 ❖ $\int x^2 \sin x^3 \cos x^3\,dx = \frac{1}{3}\int 3x^2.\sin x^3 \cos x^3\,dx = \frac{1}{3}\int \sin x^3\,d(\sin x^3)$

$$= \frac{1}{3}.\frac{(\sin x^3)^2}{2} + c = \frac{1}{6}\sin^2 x^3 + c$$

7 ❖ $\int \frac{\sqrt{x}}{\sqrt{a^3-x^3}}\,dx = \int \frac{x^{1/2}}{\sqrt{(a^{3/2})^2-(x^{3/2})^2}}\,dx = \frac{2}{3}\int \frac{dz}{\sqrt{k^2-z^2}}$

put, $x^{3/2} = z$ *and* $a^{3/2} = k \Rightarrow x^{1/2}\,dx = \frac{2}{3}\,dz$

$$= \frac{2}{3}\sin^{-1}\frac{z}{k} + c = \frac{2}{3}\sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right) + c = \frac{2}{3}\sin^{-1}\left(\frac{x}{a}\right)^{3/2} + c$$

8 ❖ $I = \int \frac{dx}{x\sqrt{4-9(\ln x)^2}}$

9 ❖ Evaluate $\int \sqrt{1+x^2}\;x^5\,dx$

Let $u = 1+x^2 \Rightarrow du = 2x\,dx \quad \Rightarrow x\,dx = du/2$

Now, $x^5 = x^4.x$

Also, $x^2 = u-1 \Rightarrow x^4 = (u-1)^2$ *and* $xdx = du/2$

$\int \sqrt{1+x^2}\;x^5\,dx = \int \sqrt{1+x^2}\;x^4.x\,dx$

$$= \int \sqrt{u}\,(u-1)^2.\,du/2 = \frac{1}{2}\int \sqrt{u}\,(u^2-2u+1).\,du$$

$$= \frac{1}{2}\int (u^{5/2}-2u^{3/2}+u^{1/2}).\,du = \ldots.$$

$$= \frac{1}{7}(1+x^2)^{7/2} - \frac{2}{5}(1+x^2)^{5/2} + \frac{1}{3}(1+x^2)^{3/2} + C$$

10 ❖ Evaluate the integral $\int \frac{x^3}{(a+bx)^4}\,dx$

[Rule: $\int \frac{x^m}{(a+bx)^n}\,dx,\ m+$ integer , then substitute $(a+bx)=u \quad \Rightarrow x=(u-a)/b$]

Let , $a+bx=u \Rightarrow x=\frac{u-a}{b}$ and $dx=\frac{1}{b}du$

Therefore, $\int \frac{x^3}{(a+bx)^4}\,dx=\int \frac{(u-a)^3}{b^3\,u^4}\,\frac{1}{b}du=\frac{1}{b^4}\int \frac{(u-a)^3}{u^4}\,du$

$$=\frac{1}{b^4}\int \frac{u^3-3u^2a+3ua^2-a^3}{u^4}\,du$$

$$=\frac{1}{b^4}\int\left(\frac{1}{u}-\frac{3a}{u^2}+\frac{3a^2}{u^3}-\frac{a^3}{u^4}\right)du$$

$$=\frac{1}{b^4}\left[\log u+\frac{3a}{u}-\frac{3a^2}{2u^2}+\frac{a^3}{3u^3}\right]+C$$

$$=\frac{1}{b^4}\left[\log(a+bx)+\frac{3a}{a+bx}-\frac{3a^2}{2(a+bx)^2}+\frac{a^3}{3(a+bx)^3}\right]+C$$

11 ❖ Evaluate the integral $\int \frac{1}{x^4\sqrt{x^2-1}}\,dx$

[Rule: $\int \frac{dx}{x^n\sqrt{a+bx^2}}$, n positive even integer , then

substitute $x=\frac{1}{u} \quad \Rightarrow dx=-\frac{1}{u^2}du \quad x=\frac{1}{u} \quad \Rightarrow dx=-\frac{1}{u^2}du$]

Now,

$$\int \frac{1}{x^4\sqrt{x^2-1}}\,dx=-\int \frac{u^3du}{\sqrt{1-u^2}}=\int \frac{-u(1-u^2)+u}{\sqrt{1-u^2}}\,du=-\int u\sqrt{1-u^2}\,du+\int \frac{u}{\sqrt{1-u^2}}\,du$$

$$=-\frac{1}{3}(1-u^2)^{\frac{3}{2}}+\sqrt{1-u^2}+C=-\frac{\sqrt{1-u^2}}{3x^3}(2x^2+1)\ \ ++C$$

Integration by Parts

$$\int uv\,dx = u\int v dx - \int\left\{\frac{d}{dx}(u)\int v dx\right\}dx$$

Find the value of $\int xe^x\,dx$, $\int \ln x\,dx$, $\int e^x \cos x\,dx$, $\int_1^2 x^2 \ln x\,dx$, $\int \sin(\ln x)\,dx$

Integrate $\int \log(x+\sqrt{x^2+a^2}\,)dx$

12 ❖ $\int \frac{xe^x}{(x+1)^2}\,dx = \int \frac{(x+1)e^x - e^x}{(x+1)^2}\,dx = \int \frac{e^x}{(x+1)}\,dx - \int \frac{e^x}{(x+1)^2}\,dx$ …..

Integrating by parts the first integral…..

13 ❖ Find $f(x)$ if $f'(x) = e^x(\sin x - \cos x)$ and $f(0) = 1$.

$\because \quad f'(x) = e^x(\sin x - \cos x)$

$f(x) = \int f'(x)dx + C$, where C is constant of integration.

$= \int e^x(\sin x - \cos x)dx + C$

$= \int e^x \sin x dx - \int e^x \cos x dx + C$ [first term integration by parts…..]

14 ❖ $\int xe^x \cos x\,dx$ (Ref. Das & Mukherjee, p-78)

Let, $I = \int e^x \cos x\,dx$

Partial fractions

15 ❖ $\int \frac{5x-3}{x^2-2x-3}\,dx = \int\left(\frac{2}{x+1} + \frac{3}{x-3}\right)dx$

16 ❖ $\int \frac{2x^2+3}{x(x-1)^2}\,dx$

17 ❖ $\int \frac{x^2+4}{3x^3+4x^2-4x}\,dx$

18 ❖ Evaluate $\int \frac{xdx}{x^4-5x^2+6}$ [Hints: x^2=y, then dx=dy/2]

$$I = \int \frac{xdx}{x^4-5x^2+6} = \frac{1}{2}\int \frac{dy}{y^2-5y+6} = \frac{1}{2}\int \frac{dy}{(y-3)(y-2)}$$

$$= \frac{1}{2}\int\left\{\frac{1}{(y-3)(3-2)} + \frac{1}{(2-3)(y-2)}\right\}dy = \frac{1}{2}\int\left\{\frac{1}{(y-3)} - \frac{1}{(y-2)}\right\}dy$$

$$= \frac{1}{2}\{\ln(y-3) - \ln(y-2)\} + C$$

$$= \frac{1}{2}\ln\frac{y-3}{y-2} + C = \frac{1}{2}\ln\frac{x^2-3}{x^2-2} + C\,.$$

Integrals involving quadratics

19 ❖ $\int \frac{1}{2x^2-3x+2}\,dx$

$$2x^2-3x+2=2\left(x^2-\frac{3}{2}x+1\right)=2\left(x^2-\frac{3}{2}x+\frac{9}{16}-\frac{9}{16}+1\right)$$

$$2x^2-3x+2=2\left(x^2-\frac{3}{2}x+1\right)=2\left(x^2-\frac{3}{2}x+\frac{9}{16}-\frac{9}{16}+1\right)$$

$$=2\left(\left(x-\frac{3}{4}\right)^2+\frac{7}{16}\right)$$

$$\int \frac{1}{2x^2-3x+2}\,dx=\frac{1}{2}\int \frac{1}{\left(x-\frac{3}{4}\right)^2+\frac{7}{16}}\,dx$$

$$=\frac{1}{2}.\frac{4}{\sqrt{7}}\tan^{-1}\left(\frac{4x-3}{\sqrt{7}}\right)+c$$

20 ❖ $\int \frac{7x-9}{x^2-2x+35}\,dx$

21 ❖ $\int \frac{dx}{\sqrt{4x^2-12x+7}}=\int \frac{dx}{\sqrt{4(x^2-3x+7/4)}}=\frac{1}{2}\int \frac{dx}{\sqrt{(x-3/2)^2-(1/\sqrt{2})^2}}$

$$=\frac{1}{2}.\cosh^{-1}\left(\frac{x-3/2}{1/\sqrt{2}}\right)+C$$

22 ❖ $\int \sqrt{4-3x-2x^2}\,dx=\sqrt{2}\int \sqrt{\left(\frac{\sqrt{41}}{4}\right)^2-(x-3/4)^2}\,dx\cdots\cdots\cdots$

23 ❖ $\int (x+1)\sqrt{x^2-2x+5}\,dx=\int \left\{\frac{1}{2}(2x-2)+2\right\}\sqrt{x^2-2x+5}\,dx$

$$=\frac{1}{2}\int (2x-2)\sqrt{x^2-2x+5}\,dx+2\int \sqrt{(x-1)^2+2^2}\,dx$$

$$=\frac{1}{2}\int \sqrt{x^2-2x+5}\,d(x^2-2x+5)+2\int \sqrt{(x-1)^2+2^2}\,dx$$

$$=\frac{1}{2}\frac{(x^2-2x+5)^{3/2}}{3/2}+2\left[\frac{(x-1)\sqrt{x^2-2x+5}}{2}+\frac{2^2}{2}\sinh^{-1}\frac{(x-1)}{2}\right]+C$$

24 ❖ $$\int \frac{2x}{\sqrt{3x-x^2-2}}\,dx = \int \frac{-1(3-2x)+3}{\sqrt{3x-x^2-2}}\,dx$$

$$= -\int \frac{(3-2x)}{\sqrt{3x-x^2-2}}\,dx + 3\int \frac{1}{\sqrt{-2+\frac{9}{4}-(x-3/2)^2}}\,dx$$

$$= -\int \frac{(3-2x)}{\sqrt{3x-x^2-2}}\,dx + 3\int \frac{1}{\sqrt{(1/2)^2-(x-3/2)^2}}\,dx$$

$$= -2\sqrt{3x-x^2-2} + 3\sin^{-1}\frac{x-3/2}{1/2} + C$$

25 ❖ $$I = \int \frac{x^2+4}{x^2+2x+3}\,dx = \int \left(1 - \frac{2x-1}{x^2+2x+3}\right) dx = \int dx - \int \frac{2x-1}{x^2+2x+3}\,dx$$

Second integral

$$= \int \frac{(2x+2)-3}{x^2+2x+3}\,dx = \int \frac{2x+2}{x^2+2x+3}\,dx - 3\int \frac{1}{(x+1)^2+(\sqrt{2})^2}\,dx$$

$$= \log(x^2+2x+3) - \frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$$

$$\therefore \quad I = x - \log(x^2+2x+3) + \frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$$

26 ❖ *Try,* $\cdots$ $\quad I = \int \frac{x^2+x+1}{\sqrt{x^2+2x+3}}\,dx = \cdots$

27 ❖ Integrate $\int \frac{4x-3}{9x^2+4}\,dx$

$$\int \frac{4x-3}{9x^2+4}\,dx = \int \frac{\frac{2}{9}\frac{d}{dx}(9x^2+4)-3}{9x^2+4}\,dx$$

$$= \frac{2}{9}\int \frac{\frac{d}{dx}(9x^2+4)}{9x^2+4}\,dx - 3\int \frac{1}{9x^2+4}\,dx + C$$

$$= \frac{2}{9}\log(9x^2+4) - \frac{1}{3}\int \frac{1}{x^2+\left(\frac{2}{3}\right)^2}\,dx + C$$

$$= \frac{2}{9}\log(9x^2+4) - \frac{1}{3}\tan^{-1}\left(\frac{3x}{2}\right) + C$$

২৮ ❖ Compute $\int \frac{1}{\sqrt{(x-4)(6-x)}}\,dx$

Put, $6-x=u^2 \Rightarrow dx=-2udu$

Therefore, $\int \frac{1}{\sqrt{(x-4)(6-x)}}\,dx = \int \frac{-2udu}{\sqrt{(6-u^2-4)u^2}} + C$

$$= -2\int \frac{du}{\sqrt{2-u^2}} + C$$

Again, let $u=\sqrt{2}\ \sin\theta \Rightarrow du=\sqrt{2}\ \cos\theta\, d\theta$

So, $\int \frac{1}{\sqrt{(x-4)(6-x)}}\,dx = -2\int \frac{\sqrt{2}\cos\theta\, d\theta}{\sqrt{2}\ \sqrt{1-\sin^2\theta}} + C = -2\int d\theta + C = -2\theta + C$

$$= -2\sin^{-1}\frac{u}{\sqrt{2}} + C = -\sin^{-1}\left(\frac{\sqrt{6-x}}{\sqrt{2}}\right) + C.$$

২৯ ❖ Evaluate $\int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$

Let, $I = \int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$

and $x=\alpha\cos^2\theta+\beta\sin^2\theta$

$\therefore\ dx=\alpha.2.\cos\theta(-\sin\theta)\,d\theta+\beta.2.\sin\theta.\cos\theta\, d\theta$

$= (\beta-\alpha).2.\cos\theta\ \sin\theta\ d\theta$

Now, $x-\alpha=\alpha\cos^2\theta+\beta\sin^2\theta-\alpha=(\beta-\alpha)\sin^2\theta$

$$\sqrt{(x-\alpha)} = \sqrt{(\beta-\alpha)}\sin\theta \Rightarrow \theta=\sin^{-1}\sqrt{\frac{x-\alpha}{(\beta-\alpha)}}$$

and $\beta-x=\beta-\alpha\cos^2\theta-\beta\sin^2\theta=(\beta-\alpha)\cos^2\theta$

$\therefore\ \sqrt{\beta-x}=\sqrt{(\beta-\alpha)}\cos\theta$

$$\therefore\ I=\int\frac{(\beta-\alpha).2.\sin\theta.\cos\theta\, d\theta}{\sqrt{(\beta-\alpha)}\sin\theta\ \sqrt{(\beta-\alpha)}\cos\theta} = 2\int d\theta = 2\sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}} + C$$

৩০ ❖ Compute $\int \frac{1}{(x+3)\sqrt{x+2}}\,dx$ [Hints: $x+2=u^2$]

৩১ ❖ Compute $\int \frac{1}{(1-x)\sqrt{1+x}}\,dx$

32 ❖ Evaluate $\int \frac{1}{(x+1)\sqrt{1+2x-x^2}}\,dx$

Hints: If the integral is of the form $\int \frac{1}{(px+q)\sqrt{ax^2+bx+c}}\,dx$, i.e. in the denominator, there is a quadratic expression inside $\sqrt{\ }$ and there is a linear expression outside $\sqrt{\ }$ .Then putting the linear expression $px+q=\frac{1}{z}$, then solve it.

Let, $I=\int \frac{1}{(x+1)\sqrt{1+2x-x^2}}\,dx$ and $x+1=\frac{1}{z} \Rightarrow dx=-\frac{1}{z^2}dz$

$$\therefore \quad I=\int \frac{1}{z^2(1/z)\sqrt{1+2(1/z-1)-(1/z-1)^2}}\,dz$$

$$=\int \frac{dz}{z\sqrt{1+2/z-2-1/z^2+2/z-1}}$$

$$=\int \frac{dz}{z\sqrt{\frac{1}{z^2}+\frac{4}{z}-2}}=\int \frac{dz}{\sqrt{-1+4z-2z^2}}$$

$$=\int \frac{dz}{\sqrt{2(-\frac{1}{2}+2z-z^2)}}=-\frac{1}{\sqrt{2}}\int \frac{dz}{\sqrt{-\frac{1}{2}+1-(z-1)^2}}$$

$$=-\frac{1}{\sqrt{2}}\int \frac{dz}{\sqrt{(1/\sqrt{2})^2-(z-1)^2}}=\ldots\ldots\ldots\ldots try\ldots$$

$$=\ldots\ldots\ldots\ldots try\ldots$$

$$So,\quad I=\frac{1}{\sqrt{2}}\sin^{-1}\frac{x\sqrt{2}}{x+1}+C$$

33 ❖ $$\int \frac{(x+2)dx}{(x^2+3x+3)\sqrt{x+1}}$$

Note: If the integral in a form $\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$, Then putting $px+q=t^2$

$$I=\int \frac{(x+2)dx}{(x^2+3x+3)\sqrt{x+1}} \quad , \quad put,\ x+1=t^2 \Rightarrow dx=2t.dt$$

$$=\int \frac{(t^2-1+2)\ 2tdt}{\{(t^2-1)^2+3(t^2-1)+3\}t}$$

$$=2\int \frac{(t^2+1)\ dt}{t^4+t^2+1}=2\int \frac{t^2(1+1/t^2)\ dt}{t^2(t^2+t+1/t^2)}$$

$$=2\int \frac{(1+1/t^2)\ dt}{(t-1/t)^2+3}, \quad put,\ t-1/t=z \Rightarrow (1+1/t^2)dt=dz$$

$$I=2\int \frac{dz}{z^2+(\sqrt{3})^2}=\frac{2}{\sqrt{3}}\tan^{-1}\frac{z}{\sqrt{3}}+C$$

$$=\frac{2}{\sqrt{3}}\tan^{-1}\frac{t-1/t}{\sqrt{3}}+C=\frac{2}{\sqrt{3}}\tan^{-1}\frac{z^2-1}{\sqrt{3}\ z}+C$$

$$=\frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3x+3}}+C$$

X 34 ❖ $$I=\int \frac{x^2+x+1}{\sqrt{x^2+2x+3}}\ dx=\int \frac{(x^2+x+3)-x-2}{\sqrt{x^2+2x+3}}\ dx$$

$$=\int \sqrt{x^2+2x+3}\ dx-\int \frac{x+2}{\sqrt{x^2+2x+3}}\ dx$$

$$=\int \sqrt{(x+1)^2+(\sqrt{2})^2}\ dx-\frac{1}{2}\int \frac{(2x+2)+1}{\sqrt{x^2+2x+3}}\ dx$$

$$=\int \sqrt{(x+1)^2+(\sqrt{2})^2}\ dx-\frac{1}{2}\int \frac{(2x+2)}{\sqrt{x^2+2x+3}}\ dx-\int \frac{1}{\sqrt{(x+1)^2+(\sqrt{2})^2}}\ dx$$

$$=\cdots \quad \cdots \quad \cdots$$

$$I=\frac{1}{2}(x-1)\sqrt{x^2+2x+3}+C$$

35 ❖ Evaluate $I=\int \sqrt{\frac{x-1}{x+1}}\ dx=\int \frac{x-1}{\sqrt{x^2-1}}\ dx=\int \frac{x}{\sqrt{x^2-1}}\ dx+\int \frac{1}{\sqrt{x^2-1}}\ dx$

36 ❖ Evaluate $\int \frac{1}{(x^2+1)\sqrt{x^2+4}}\ dx$

❖ Note: If the integral in a form $\int \frac{dx}{(ax^2+b)\sqrt{cx^2+d}}$, then at first putting $x=1/u$. Then after simplification, putting the expression inside $\sqrt{\ } = t^2$, then solve…..

$$I = \int \frac{1}{(x^2+1)\sqrt{x^2+4}}\, dx$$

Put, $x = 1/u \Rightarrow dx = -(1/u^2)\, du$

$$\therefore \quad I = -\int \frac{1}{u^2(1/u^2+1)\sqrt{1/u^2+4}}\, du = -\int \frac{u}{(1+u^2)\sqrt{1+4u^2}}\, du$$

$put, \quad 1+4u^2 = t^2 \Rightarrow 2udu = (2t\,dt)/4$

$$\therefore \quad I = -\frac{1}{4}\int \frac{t\,dt}{\{1+(t^2-1)/4\}t} = -\frac{1}{4}\int \frac{4}{4+t^2-1}\, dt$$

$$= -\int \frac{dt}{\{(\sqrt{3})^2+t^2} = -\frac{1}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}} + C$$

$$= -\frac{1}{\sqrt{3}}\tan^{-1}\frac{\sqrt{1+4u^2}}{\sqrt{3}} + C = -\frac{1}{\sqrt{3}}\tan^{-1}\frac{\sqrt{1+4/x^2}}{\sqrt{3}} + C$$

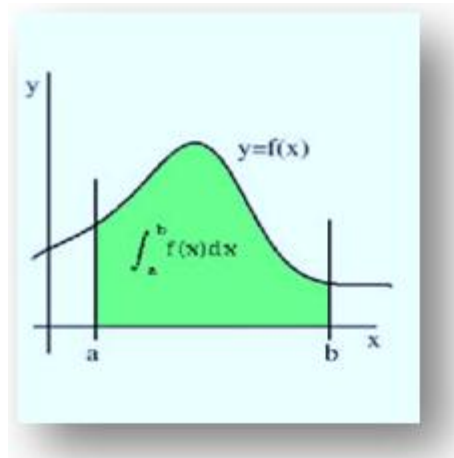
$$\therefore \quad I = -\frac{1}{\sqrt{3}}\tan^{-1}\frac{\sqrt{x^2+4}}{x\sqrt{3}} + C$$

Let's Try:

37 Evaluate $\int \frac{1}{(1+x^2)\sqrt{1-x^2}}\, dx$ $\quad [Firstly, \quad put, x=1/z; \quad Secondly, put, z^2-1=u^2]$

Definite Integral

Definite integral consists of a function $f(x)$ which is continuous in a closed interval $[a, b]$ and the meaning of the definite integral is assumed to be in context of area covered by the function f from a to b.



An alternative way of describing $\int_a^b f(x)\,dx$ is that the definite integral $\int_a^b f(x)\,dx$ is a limiting case of the summation of an infinite series, provided $f(x)$ is continuous on $[a, b]$ i.e.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} h\sum_{r=0}^{n-1} f(a+rh), \quad where\ h = \frac{b-a}{n}$$

$$i.e.\quad \int_a^b f(x)\,dx = \lim_{n\to\infty} h[f(a)+f(a+h)+f(a+2h)+\cdots+f(a+(n-1)h)]$$

FUNDAMENTAL THEOREM OF CALCULUS:

If $f(x)$ is a continuous function on $[a, b]$ and $F(x)$ is any anti derivative of $f(x)$ on $[a, b]$ i.e. $F'(x) = f(x)\ \forall\ x \in (a,b)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

38 ❖ Evaluate $\int_0^2 (3x+1)\,dx$ using the limit of Riemann Sums.

Solution: If $f(x)$ is integrable on $[a,b]$, then $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum f(x_i)\,\Delta_x$,

where $\Delta_x = \dfrac{b-a}{n}$ *and* $x_i = a + i\Delta_x$.

Here, $f(x) = 3x+1$, let $a = 0$ & $b = 2$

$$\Delta_x = \frac{2-0}{n} = \frac{2}{n} \quad and \quad x_i = 0 + i\left(\frac{2}{n}\right) = \frac{2i}{n}$$

$$\therefore \quad f(x_i) = (3x_i + 1) = 3\left(\frac{2i}{n}\right) + 1 \text{ and } f(x_i)\Delta_x = \left(3\left(\frac{2i}{n}\right)+1\right)\left(\frac{2}{n}\right)$$

Now, $\int_0^2 (3x+1)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(3\left(\frac{2i}{n}\right)+1\right)\left(\frac{2}{n}\right)$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{6i}{n}+1\right)\left(\frac{2}{n}\right)$$

$$= \lim_{n\to\infty} \left[\frac{12}{n^2}\sum_{i=1}^{n} i + \frac{2}{n}\sum_{i=1}^{n} 1\right]$$

$$= \lim_{n\to\infty} \left[\frac{12}{n^2}.\frac{n(n+1)}{2} + \frac{2}{n}.n\right] \quad \{\text{Note: } \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \}$$

$$= \lim_{n\to\infty} \left[6.\frac{n^2+n}{n} + 2\right]$$

$$= \lim_{n\to\infty} \left[6.\left(1+\frac{1}{n}\right) + 2\right] = lt_{n\to\infty}\left(8 + \frac{6}{n}\right) = 8$$

39 ❖ Evaluate $\int_1^4 (x^2 - x)\,dx$ using the limit of a Sum.

we know,

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty} \frac{1}{n}[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)]$$

where, $h = \dfrac{b-a}{n}$. Here $a = 1, b = 4 \Rightarrow h = \dfrac{3}{n}$.

$$\int_1^4 (x^2 - x)\,dx = 3\times \lim_{n\to\infty} \frac{1}{n}\left[f(1) + f\left(1+\frac{3}{n}\right) + f\left(1+\frac{6}{n}\right) + \cdots + f\left(1+(n-1)\frac{3}{n}\right)\right]$$

$$= 3\times \lim_{n\to\infty} \frac{1}{n}\left[0+\left(1+\frac{3}{n}\right)^2-\left(1+\frac{3}{n}\right)+\left(1+\frac{6}{n}\right)^2-\left(1+\frac{6}{n}\right)\cdots(n-1)terms\right]$$

$$= 3\times \lim_{n\to\infty} \frac{1}{n}\left[0+\left(1+\frac{9}{n^2}+\frac{6}{n}\right)-\left(1+\frac{3}{n}\right)+\left(1+\frac{36}{n^2}+\frac{12}{n}\right)-\left(1+\frac{6}{n}\right)+\cdots(n-1)terms\right]$$

$$= 3\times \lim_{n\to\infty} \frac{1}{n}\left[0+\left(\frac{9}{n^2}+\frac{3}{n}\right)+\left(\frac{36}{n^2}+\frac{6}{n}\right)+\cdots(n-1)terms\right]$$

$$= 3\times \lim_{n\to\infty} \frac{1}{n}\left[0+\left(\frac{9}{n^2}+\frac{36}{n^2}+\cdots(n-1)terms\right)+\frac{3}{n}+\frac{6}{n}+\cdots(n-1)terms\right]$$

$$= 3\times \lim_{n\to\infty} \frac{1}{n}\left[\frac{9}{n^2}\{1^2+2^2+3^2+\cdots(n-1)terms\}+\frac{3}{n}\{1+2+3+\cdots(n-1)terms\}\right]$$

$$= 3\times \lim_{n\to\infty} \frac{1}{n}\left[\frac{9}{n^2}\left\{\frac{(n-1).n.(2n-1)}{6}\right\}+\frac{3}{n}\left\{\frac{(n-1)n}{2}\right\}\right] \text{[here, n=n-1]}$$

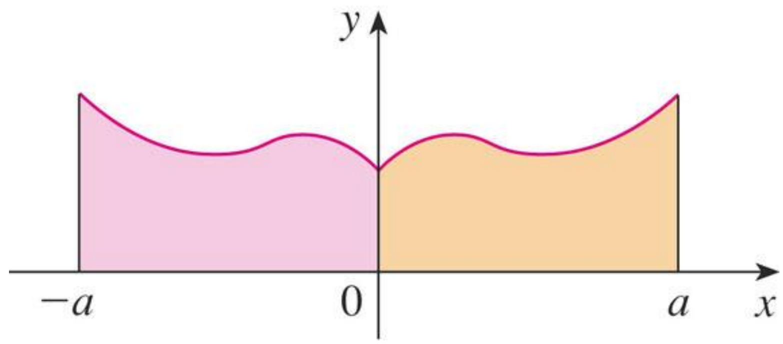
$$= 3\times \lim_{n\to\infty} \left[\frac{9}{6}\left(\frac{n-1}{n}\right)\left(\frac{n}{n}\right)\left(\frac{2n-1}{n}\right)+\frac{3}{2}\left\{\frac{(n-1)}{n}\cdot\frac{n}{n}\right\}\right]$$

$$= 3\times \lim_{\frac{1}{n}\to\infty} \left[\frac{9}{6}\left(1-\frac{1}{n}\right)(1)\left(2-\frac{1}{n}\right)+\frac{3}{2}\left(1-\frac{1}{n}\right)(1)\right]$$

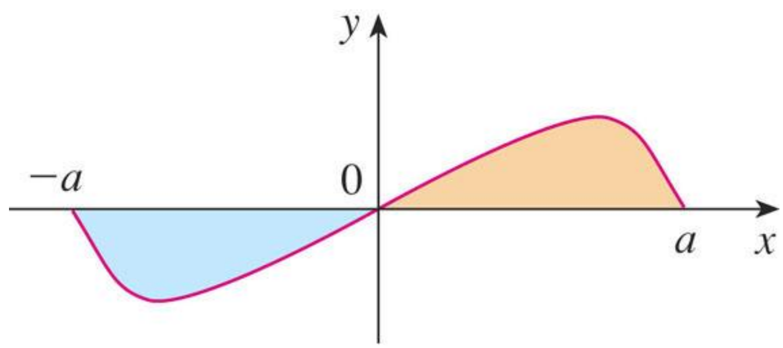
$$= 3\times\left[\frac{9}{6}\times 1\times 2+\frac{3}{2}(1)(1)\right] = 3\times\left(3+\frac{3}{2}\right) = 3\times\frac{9}{2} = \frac{27}{2}$$

❖ Properties of Definite Integrals:

- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^b [f(x)\pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
- $\int_a^b f(x)\, dx = \int_a^b f(t)\, dt$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
- $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$
- $\int_0^{na} f(x)\, dx = n\int_0^a f(x)\, dx, \quad if \quad f(a+x) f(x)$



(a) f even, $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$



(b) f odd, $\int_{-a}^{a} f(x)\,dx = 0$

40 ➢ Prove that $\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} \{f(x) + f(-x)\}\,dx$ and also show that

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_{0}^{a} f(x)\,dx & \text{when f(x) is even function} \\ 0 & \text{when f(x) is odd function} \end{cases}$$

We know, $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$

$$\therefore \int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{-a}^{0} f(x)\,dx \qquad \cdots \quad (1)$$

For the 2nd integral of R.H.S., we put $x = -z \Rightarrow dx = -dz$.
When $x = 0$, *then* $z = 0$ and when $x = a$, *then* $z = a$. So, from(1)

$$\int_{-a}^{a} f(x)\,dx = \int_0^a f(x)\,dx - \int_a^0 f(-z)\,dz = \int_0^a f(x)\,dx + \int_0^a f(-z)\,dz$$

$$= \int_0^a f(x)\,dx + \int_0^a f(-x)\,dx \qquad [\because \int_a^b f(x)\,dx = \int_a^b f(z)\,dz]$$

$$= \int_0^a \{f(x) + f(-x)\}\,dx$$

Second part, If $f(x)$ is even, then $f(x) = f(-x)$. So,

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx.$$

Again, for an odd function $f(-x) = -f(x)$, then $\int_{-a}^{a} f(x)\,dx = 0.$

$$\therefore \int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{when f(x) is even function} \\ 0 & \text{when f(x) is odd function} \end{cases}$$

❖ Evaluate $\int_0^{\pi/2} \frac{d\theta}{1+2\cos\theta}$

$$I = \int_0^{\pi/2} \frac{d\theta}{1+2\cos\theta}$$

$$= \int_0^{\pi/2} \frac{d\theta}{1+2\left(\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right)}$$

$$= \int_0^{\pi/2} \frac{\sec^2\frac{\theta}{2}\,d\theta}{3-\tan^2\frac{\theta}{2}}, \qquad put, \quad \tan\frac{\theta}{2} = z \Rightarrow \sec^2\frac{\theta}{2}\,d\theta = 2dz$$

$$= \int_0^1 \frac{2\,dz}{3-z^2} = 2\int_0^1 \frac{2\,dz}{(\sqrt{3})^2 - z^2}$$

$$= 2\frac{1}{2\sqrt{3}}\left[\log_e \frac{\sqrt{3}+z}{\sqrt{3}-z}\right]_0^1$$

$$= \frac{1}{\sqrt{3}}\ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) - \frac{1}{\sqrt{3}}\ln 1$$

$$= \frac{1}{\sqrt{3}}\ln\left(\frac{(\sqrt{3}+1)^2}{3-1}\right) = \frac{1}{\sqrt{3}}\ln(2+\sqrt{3}).$$

- **Evaluate** $\int_0^1 \frac{dx}{(1+x^2)\sqrt{1-x^2}}$, $\quad put\ x = \sin\theta$
- **Evaluate** $\int_0^1 \sqrt{\frac{a+x}{(a-x)}}\,dx$, $put \quad x = a\cos\theta$
- **Evaluate** $\int_0^1 \frac{dx}{(1+x)\sqrt{1+2x-x^2}}$, $\quad put \quad 1+x = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}dt$
- $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{b^2-a^2}\int_0^\infty \left(\frac{1}{(x^2+a^2)} - \frac{1}{(x^2+b^2)}\right)dx$
- **Evaluate** $\int_3^4 \frac{dx}{(x-3)(4-x)}$

$$put,\ x = 3\cos^2\theta + 4\sin^2\theta \Rightarrow x-3 = \sin^2\theta, \quad 4-x = \cos^2\theta$$

$$\therefore \quad dx = 2\sin\theta\cos\theta$$

When x=3, then $\theta = 0$ and when x=4, then $\theta = \pi/2$

$$\therefore \quad \int_3^4 \frac{dx}{(x-3)(4-x)} = \int_0^{\pi/2} \frac{2\sin\theta\cos\theta\,d\theta}{\sqrt{\sin^2\theta\cos^2\theta}}$$

$$= \int_0^{\pi/2} d\theta = 2[\theta]_0^{\pi/2} = \pi\,.$$

- $\int_3^4 \frac{x^3\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

➢ $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ [Property of Definite Integral]

- Evaluate $\int_0^{\frac{\pi}{2}} \log\sin x\,dx$
- Evaluate $\int_0^\pi \log(1+\cos x)\,dx$
- 229 Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$, $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}}\,dx$
- Evaluate $I = \int_0^{\frac{\pi}{2}} \frac{1}{1+\cot x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\cos x+\sin x}\,dx \quad \cdots(i)$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2}-x\right)}{\cos\left(\frac{\pi}{2}-x\right)+\sin\left(\frac{\pi}{2}-x\right)}\,dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x+\cos x}\,dx \qquad \cdots(ii)$$

Adding (i) and (ii), we have $2I = \int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{\sin x + \cos x}\, dx$

$$i.e.\quad 2I = \int_0^{\frac{\pi}{2}} dx \quad = \frac{\pi}{2} \quad \Rightarrow I = \frac{\pi}{4}.$$

❖ Evaluate $\int_0^1 \frac{dx}{(1+x^2)\sqrt{1-x^2}}\, dx$ [Hints: put $x = \sin\theta$]

❖ Evaluate $\int_0^{\frac{2\pi}{3}} \frac{dx}{5+4\cos x}\, dx$

❖ Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\tan x}}\, dx$

Let, $\mathrm{I} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\tan x}}\, dx \;= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx \quad \cdots(1)$

By the property, $\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$

$$\therefore\quad I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx \quad \cdots(1)$$

>>>>>>>>>>>TRY TO SOLVE>>>>>>>

41 ❖ Show that $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx = \frac{\pi}{4}$

Let, $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx ....(i)$

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}\, dx$$

Or, $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\, dx .....(ii)$

Adding (i) and (ii)

$$\therefore\quad 2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}$$

$$\therefore\quad I = \frac{\pi}{4}.$$

- Evaluate $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}\,dx$

$$I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}\,dx = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}\,dx \quad \cdots(i)$$

………..TRY>>>>>>

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- Show that $\int_0^{1} \frac{\log(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\log 2$.

Solution: Put $x = \tan\theta$; $\therefore dx = \sec^2\theta\, d\theta$

When x=0, θ=0; when x=1, θ=π/4

$$\therefore\quad I = \int_0^{\frac{\pi}{4}} \log(1+\tan\theta)\, d\theta = \int_0^{\frac{\pi}{4}} \log\{1+\tan\left(\frac{\pi}{4}-\theta\right)\}d\theta$$

Now, $1+\tan\left(\frac{\pi}{4}-\theta\right) = 1+\frac{1-\tan\theta}{1+\tan\theta} = \frac{2}{1+\tan\theta}$

$$\therefore\quad I = \int_0^{\frac{\pi}{4}} \log\frac{2}{1+\tan\theta}\, d\theta = \int_0^{\frac{\pi}{4}} \{\log 2 - \log(1+\tan\theta)\}d\theta$$

$$\therefore\quad I = \int_0^{\frac{\pi}{4}} \log 2 - \int_0^{\frac{\pi}{4}} \log(1+\tan\theta)\}d\theta$$

$$\therefore\quad I = \int_0^{\frac{\pi}{4}} \log 2 d\theta - \int_0^{\frac{\pi}{4}} \log(1+\tan\theta)\}d\theta = \frac{1}{4}\pi.\log 2 - I$$

$$\therefore\quad 2I = \frac{1}{4}\pi.\log 2 \quad \Rightarrow \quad I = \frac{\pi}{8}\log 2.$$

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- Evaluate $\int_0^{a} \sqrt{a^2 - x^2}\, dx$

Put $x = a\sin\theta$, $\therefore dx = a\cos\theta\, d\theta$

Also, when x=0, θ=0, and when x=a, θ=π/2.

$$I = \int_0^{\frac{\pi}{2}} a^2\cos^2\theta\, d\theta = a^2 \int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$

Now, $\int \cos^2\theta\, d\theta = \frac{1}{2}\int (1+\cos 2\theta)\, d\theta = \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]$

$$\therefore\quad I = a^2\,\frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} = \frac{1}{4}\pi \quad a^2.$$

43 ✓
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❖ Evaluate $\int_{\alpha}^{\beta}\sqrt{(x-\alpha)(\beta-x)}\ dx$

Put $x=\alpha\cos^2\theta+\beta\sin^2\theta.\quad\therefore\quad dx=2(\beta-\alpha)\sin\theta\cos\theta\ d\theta$

Also, $x-\alpha=\beta\sin^2\theta.-\alpha(1-\cos^2\theta)=(\beta-\alpha)\sin^2\theta,$

$\beta-x=\beta(1-\sin^2\theta.)-\alpha\cos^2\theta=(\beta-\alpha)\cos^2\theta.$

$\therefore$ when $x=\alpha,\quad(\beta-\alpha)\sin^2\theta=0.$

$\therefore\quad\sin\theta=0$, $\sin ce\quad\beta\neq\alpha.\quad\therefore\quad\theta=0$

Similarly, when $x=\beta,\quad(\beta-\alpha)\cos^2\theta=0.$

$$\therefore\quad\cos\theta=0,\quad\therefore\quad\theta=\frac{\pi}{2}$$

$$\therefore\quad I=2(\beta-\alpha)^2\int_0^{\frac{\pi}{2}}\sin^2\theta\cos^2\theta\ d\theta$$

Now,

$$\sin^2\theta\cos^2\theta=\frac{1}{4}.4\sin^2\theta\cos^2\theta=\frac{1}{4}\sin^2 2\theta=\frac{1}{8}(1-\cos 4\theta).$$

Also, $\int(1-\cos 4\theta)d\theta=\theta-\frac{1}{4}\sin 4\theta.$

$$\therefore\quad I=2(\beta-\alpha)^2\,\frac{1}{8}\int_0^{\frac{\pi}{2}}(1-\cos 4\theta)\ d\theta=\frac{1}{4}(\beta-\alpha)^2\left[\theta-\frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{2}}$$

$$\therefore\quad I=\frac{1}{4}(\beta-\alpha)^2\left[\frac{1}{2}\pi-\frac{1}{4}\sin 2\pi\right]=\frac{1}{8}\pi(\beta-\alpha)^2$$

❖ Evaluate $\int_1^2\frac{dx}{(3-5x)^2}$, let u=3-5x

Reduction Formula

168 ❖ Find the reduction formula for $\int (x^2+a^2)^n \, dx$

❖ If $I_n = \int_0^a (a^2-x^2)^n \, dx,\ n>0$, then show that $I_n = \dfrac{2na^2}{2n+1} I_{n-1}$.

$$I_n = \int_0^a (a^2-x^2)^n \, dx$$
$$= \left[(a^2-x^2)^n \, x\right]_0^a - \int_0^a n(a^2-x^2)^{n-1}\,(-2x)x\,dx$$
$$= 0 + 2n\int_0^a (a^2-x^2)^{n-1} \, x^2 \, dx$$
$$= 2n\int_0^a (a^2-x^2)^{n-1} \, \{a^2-(a^2-x^2)\}dx$$
$$= 2na^2\int_0^a (a^2-x^2)^{n-1} \, dx - 2n\int_0^a (a^2-x^2)^n \, dx$$
$$= 2na^2 I_{n-1} - 2nI_n$$
$$\Rightarrow \quad I_n(2n+1) = 2na^2 I_{n-1}$$
$$\therefore \quad I_n = \frac{2na^2}{2n+1} I_{n-1}$$

185 ❖ Find the reduction formula for $\int x^m e^x \, dx$ and use the formula to evaluate $\int_0^1 x^4 e^x \, dx$.

❖ If $I_m = \int_0^\infty e^{-x}\sin^m x \, dx,\ m\ge 2$, then prove that $(1+m^2)I_m = m(m-1)I_{m-2}$ and hence find the value of I_4.

Given that, $I_m = \int_0^\infty e^{-x}\sin^m x \, dx,\ m\ge 2$

$$= [-e^{-x}\sin^m x]_0^\infty + \int_0^\infty m\sin^{m-1} x\cdot\cos x\cdot e^{-x} \, dx$$
$$= 0 + m\left[-\sin^{m-1} x\cos x\cdot e^{-x}\right]_0^\infty$$
$$+ m\int_0^\infty \{(m-1)\sin^{m-2} x\cdot\cos^2 x - \sin^{m-1} x\cdot\sin x\}e^{-x} \, dx$$
$$= 0 + m(m-1)\int_0^\infty e^{-x}\sin^{m-2} x\cdot\cos^2 x dx - m\int_0^\infty e^{-x}\sin^m x \, dx$$
$$= m(m-1)\int_0^\infty e^{-x}\sin^{m-2} x\cdot(1-\sin^2 x)\,dx - mI_m$$
$$= m(m-1)\int_0^\infty e^{-x}\sin^{m-2} x\,dx - m(m-1)\int_0^\infty e^{-x}\sin^m x\,dx - mI_m$$
$$= m(m-1)I_{m-2} - m(m-1)I_m - mI_m$$
$$= m(m-1)I_{m-2} - m^2 I_m.$$
$$\therefore \quad I_m(m^2+1) = m(m-1)I_{m-2}$$

Second Part,

We know, $I_m(m^2+1)=m(m-1)I_{m-2}$

Putting $m=4,\,2$, we have

$$17I_4=4.3.I_2 \qquad \cdots(i) \quad \text{and}$$
$$5I_2=2.I_0 \qquad \cdots(ii)$$

Now, $\quad I_m=\int_0^\infty e^{-x}\sin^m x\,dx$

$$I_0=\int_0^\infty e^{-x}\,dx=0+1=1$$

$$(ii)\Rightarrow\quad I_2=\frac{2}{5}\cdot.I_0=\frac{2}{5}\cdot 1$$

$$\therefore\quad I_4=\frac{12}{17}\cdot.I_2=\frac{12}{17}\cdot\frac{2}{5}=\frac{24}{85}$$

❖ Obtain a reduction formula for $\int_0^\infty e^{-ax}\cos^n x\,dx,\ a>0$ and hence find the value of $\int_0^\infty e^{-4x}\cos^5 x\,dx$.

17) ❖ Obtain the reduction formula of $\int_0^1 x^m(\ln x)^n dx$. Hence show that $\int_0^1 x^m(\ln x)^n dx=\dfrac{(-1)^n\ n!}{(m+1)^{n+1}}$.

Solution: Let, $\quad I_{m,n}=\int_0^1 x^m(\ln x)^n dx$

$$=\left[(\ln x)^n.\frac{x^{m+1}}{m+1}\right]_0^1-n\int_0^1((\ln x)^{n-1}.\frac{1}{x}\cdot\frac{x^{m+1}}{m+1}dx$$

$$=0-\frac{n}{m+1}\int_0^1 x^m(\ln x)^{n-1}.\,dx$$

$$\Rightarrow\int_0^1 x^m(\ln x)^n dx=\frac{-n}{m+1}\int_0^1 x^m(\ln x)^{n-1}dx$$

Which is the reduction formula.

Second part:

We know, $\int_0^1 x^m(\ln x)^n dx=\dfrac{-n}{m+1}\int_0^1 x^m(\ln x)^{n-1}dx$

$$\Rightarrow I_{m,n}=\frac{-n}{m+1}I_{m,n-1} \qquad \cdots(1)$$

Putting n=n-1 in (1) , we have

$$I_{m,n-1}=\frac{-(n-1)}{m+1}I_{m,n-2} \qquad \cdots(2)$$

$$\therefore \quad (1) \Rightarrow \quad I_{m,n} = \frac{(-1)^2 \; n(n-1)}{(m+1)^2} I_{m,n-2} \qquad \cdots(3)$$

Again, putting $n = n-2$ in (2),

$$I_{m,n-2} = \frac{-(n-2)}{m+1} I_{m,n-3} \qquad \cdots(4)$$

$$\therefore \quad (3) \Rightarrow \quad I_{m,n} = \frac{(-1)^3 \; n(n-1)(n-2)}{(m+1)^3} I_{m,n-3}$$

Similarly, $\quad I_{m,n} = \dfrac{(-1)^n \; n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{(m+1)^n} I_{m,0}$

$$= \frac{(-1)^n \; n!}{(m+1)^n} I_{m,0} \qquad \cdots \quad (5)$$

Now, $I_{m,0} = \displaystyle\int_0^1 x^m \, dx = \left[\frac{x^{m+1}}{m+1}\right]_0^1 = \frac{1}{m+1}$

$$(5) \Rightarrow \quad I_{m,n} = \frac{(-1)^n \; n!}{(m+1)^n} \cdot \frac{1}{m+1}$$

$$\therefore \quad \int_0^1 x^m (\ln x)^n dx = \frac{(-1)^n \; n!}{(m+1)^{n+1}}.$$

186 ❖ If $I_n = \displaystyle\int_0^{\pi/2} x^n \sin x \, dx$, then show that $I_n + n(n-1)I_{n-2} = n\left(\dfrac{\pi}{2}\right)^{n-1}$ and hence find the value of $\displaystyle\int_0^{\pi/2} x^5 \sin x \, dx$ i.e. I_5.

Solution: Given that, $I_n = \displaystyle\int_0^{\pi/2} x^n \sin x \, dx = \left[-x^n \cos x\right]_0^{\pi/2} + n\int_0^{\pi/2} x^{n-1} \cos x \, dx$

$$= 0 + n\left[x^{n-1} \sin x\right]_0^{\pi/2} - n(n-1)\int_0^{\pi/2} x^{n-2} \sin x \, dx$$

$$\Rightarrow I_n + n(n-1)I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$$

Second Part:

$$I_n + n(n-1)I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$$

Putting, $n = 5, 3, 1$, we have $I_5 + 5\cdot 4\cdot I_3 = 5\left(\dfrac{\pi}{2}\right)^4 \quad \cdots \quad \cdots(1)$

$$I_3 + 3\cdot 2\cdot I_1 = 3\left(\frac{\pi}{2}\right)^2 \quad \cdots \quad \cdots(2)$$

$$I_1 + 0 = \left(\frac{\pi}{2}\right)^0 \quad \cdots \quad \cdots(3)$$

$$\therefore \quad I_1 = 1 \qquad \therefore \quad (2) \Rightarrow \quad I_3 = -6 + 3\left(\frac{\pi}{2}\right)^2$$

$$\therefore \quad (1) \Rightarrow \quad I_5 = -20\left\{-6 + 3\left(\frac{\pi}{2}\right)^2\right\} + 5\left(\frac{\pi}{2}\right)^4$$

$$= 120 - 60\left(\frac{\pi}{2}\right)^2 + 5\left(\frac{\pi}{2}\right)^4$$

$$i.e. \quad \int_0^{\pi/2} x^5 \sin x \, dx = 5\left(\frac{\pi}{2}\right)^4 - 60\left(\frac{\pi}{2}\right)^2 + 120$$

Walle's Formula:

If n is positive integer, then show that

I(8)

$$I = \int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx = \begin{cases} \dfrac{(n-1)(n-2)(n-3)\cdots 5\cdot 3\cdot 1}{n(n-2)(n-4)\cdots 6\cdot 4\cdot 2} \cdot \dfrac{\pi}{2} & \textit{when n is even} \\[2ex] \dfrac{(n-1)(n-3)\cdots 6\cdot 4\cdot 2}{n(n-2)(n-4)\cdots 5\cdot 3\cdot 1} \cdot 1 & \textit{when n is odd} \end{cases}$$

❖ Using Walle's formula, evaluate the following:

$$\text{(i)} \quad \int_0^{\pi/2} \cos^7 x \, dx \quad \text{(ii)} \quad \int_0^{\pi} \sin^8 x \, dx$$

$$\int_0^{\pi} \sin^8 x \, dx = 2\int_0^{\pi/2} \sin^8 x \, dx = 2.\frac{7\cdot 5\cdot 3\cdot 1}{8\cdot 6\cdot 4\cdot 2} \cdot \frac{\pi}{2} = \frac{35\pi`}{128}$$

IMPROPER INTEGRALS

It is a definite integral evaluated to ∞ or $-\infty$ (or both), of a definite integral that is discontinuous (ex. has a vertical asymptote) over the interval.

There are two types:

(i) The limit goes to ∞ or $-\infty$

The integral represents an area under curve
Some converse $\longrightarrow$ finite area
Some don't converse $\longrightarrow$ infinite area

(ii) The integral is discontinuous

Improper Integrals

$$1. \int_a^{\infty} f(x)dx = \lim_{b \to \infty} \int_a^b f(x)dx$$

$$2. \int_{-\infty}^{b} f(x)dx = \lim_{a \to -\infty} \int_a^b f(x)dx$$

$$3. \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_c^{\infty} f(x)dx$$

If the limit exists then the improper integral converges.
If the limit does not exists then the improper integral diverges.

Examples:

$$\int_0^{\infty} 4e^{-2x}dx = \lim_{b \to \infty} \int_0^b 4e^{-2x}dx$$
$$= \lim_{b \to \infty} \left[-2e^{-2x}\right]_0^b$$
$$= \lim_{b \to \infty} \left[-2e^{-2b} - \left(-2e^{0}\right)\right]$$
$$= 2$$ (converges)

$$\int_1^{\infty} \frac{1}{x}dx = \lim_{b \to \infty} \int_1^b \frac{1}{x}dx$$
$$= \lim_{b \to \infty} \left[\ln x\right]_1^b$$
$$= \lim_{b \to \infty} \left[\ln b - \ln 1\right]$$
(diverges)

❖ Evaluate $\int_1^\infty \frac{x+2}{x(x+1)}dx$

The integrand is continuous in $[1, \infty)$ and the upper limit of the integral s infinite. So, it is an improper integral of the first kind. Therefore,

$$\int_1^\infty \frac{x+2}{x(x+1)}dx = \lim_{b\to\infty}\int_1^b \frac{x+2}{x(x+1)}dx$$

$$= \lim_{b\to\infty}\int_1^b \left(\frac{2}{x}-\frac{1}{x+1}\right)dx = \lim_{b\to\infty}\left[\log\frac{x^2}{x+1}\right]_1^b$$

$$= \lim_{b\to\infty}\left[\log\frac{b^2}{b+1}+\log 2\right] = \lim_{b\to\infty}\left[\log\frac{1}{\frac{1}{b}+\frac{1}{b^2}}+\log 2\right]$$

$$= \log 0 + \log 2 = -\infty + \log 2 = -\infty = \text{infinite}$$

Therefore, the given integral is divergent as it has no finite value.

❖ Evaluate $\int_{-\infty}^\infty \frac{x+}{x^4+1}dx$

The integrand is continuous in $(-\infty, \infty)$ but both upper and the lower limits are infinite. So, it is an improper integral of the first kind. Therefore,

$$I = \int_{-\infty}^\infty \frac{x}{x^4+1}dx = \lim_{a\to-\infty}\int_a^c \frac{x}{x^4+1}dx + \lim_{b\to-\infty}\int_c^b \frac{x}{x^4+1}dx$$

$$= \lim_{a^2\to-\infty}\frac{1}{2}\int_{a^2}^{c^2}\frac{du}{u^2+1} + \lim_{b^2\to-\infty}\frac{1}{2}\int_{c^2}^{b^2}\frac{du}{u^2+1}\ , \quad where\ x^2 = u$$

and as $a \to -\infty, a^2 \to \infty$ and as $b \to \infty, b^2 \to \infty$.

$$\therefore\ I = \frac{1}{2}\lim_{a^2\to-\infty}\left[\tan^{-1}u\right]_{a^2}^{c^2} + \frac{1}{2}\lim_{b^2\to-\infty}\left[\tan^{-1}u\right]_{c^2}^{b^2}$$

$$= \frac{1}{2}\left[\lim_{a^2\to-\infty}(\tan^{-1}c^2 - \tan^{-1}a^2) + \lim_{b^2\to-\infty}(\tan^{-1}b^2 - \tan^{-1}c^2)\right]$$

$$= \frac{1}{2}\left[\tan^{-1}c^2 - \frac{\pi}{2} + \frac{\pi}{2} - \tan^{-1}c^2\right] = 0 = \textit{finite}$$

Therefore, the improper integral is convergent.

❖ Show that $\int_0^\infty \frac{x}{(1+x)^3}dx = \frac{1}{2}\int_0^\infty \frac{x}{(1+x)^2}dx = \frac{1}{2}$

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Using integration by parts for the first integral,

$$\int_0^\infty \frac{x}{(1+x)^3}dx = \lim_{b\to\infty}\left[-\frac{x}{2(1+x)^2}\right]_0^b + \frac{1}{2}\int_0^\infty \frac{1}{(1+x)^2}dx$$

$$= \lim_{b\to\infty}\left[-\frac{b}{2(1+b)^2}\right] + \frac{1}{2}\int_0^\infty \frac{1}{(1+x)^2}dx$$

$$= -\frac{1}{2}\lim_{b\to\infty}\left[\frac{1}{\frac{1}{b}+b+2}\right] + \frac{1}{2}\int_0^\infty \frac{1}{(1+x)^2}dx$$

$$= 0 + \frac{1}{2}\int_0^\infty \frac{1}{(1+x)^2}dx = \frac{1}{2}\int_0^\infty \frac{1}{(1+x)^2}dx$$

$$\therefore \int_0^\infty \frac{x}{(1+x)^3}dx = \frac{1}{2}\int_0^\infty \frac{x}{(1+x)^2}dx$$

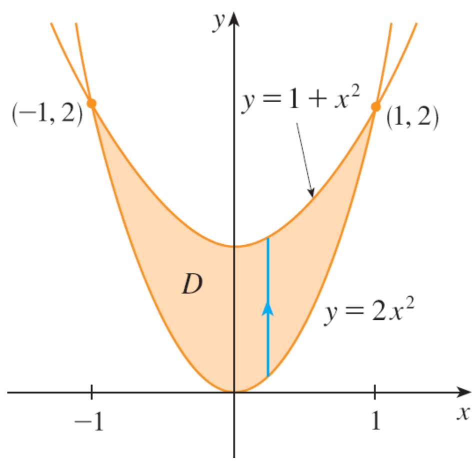
Again, $\int_0^\infty \frac{x}{(1+x)^2}dx = \lim_{b\to\infty}\int_0^b \frac{1}{(1+x)^2}dx = \lim_{b\to\infty}\left[-\frac{1}{1+x}\right]_0^b$

$$= \lim_{b\to\infty}\left[-\frac{1}{1+b}+1\right] = 1.$$

Hence, $\int_0^\infty \frac{x}{(1+x)^3}dx = \frac{1}{2}\int_0^\infty \frac{x}{(1+x)^2}dx = \frac{1}{2}$

Double Integral:

Problem: Evaluate $\iint_D (x+2y)dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.



Solution: The parabolas intersect when $2x^2 = 1 + x^2 \;\Rightarrow\; x^2 = 1$ *i.e.* $x = \pm 1$.

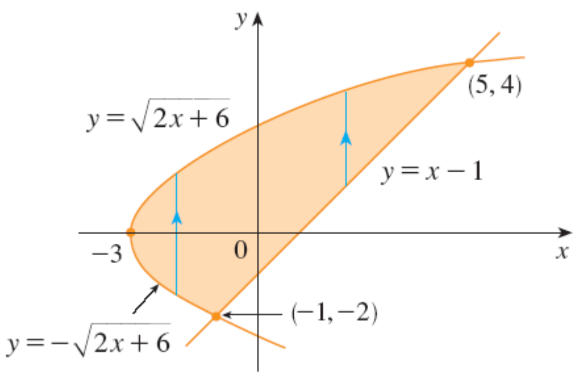
We note that the region D is a type I region but not a type II region.
So, we can write,

$$D = \{(x, y) \mid -1 \le x \le 1,\quad 2x^2 \le y \le 1 + x^2\}$$

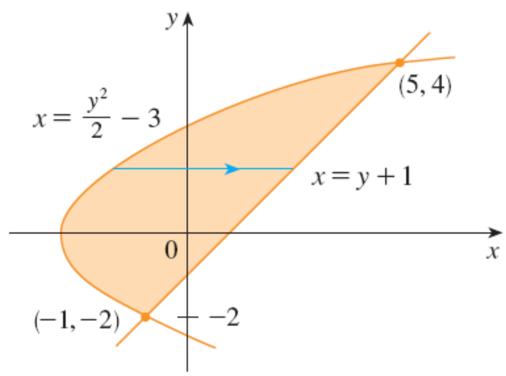
$$\iint_D (x+2y)dA = \int_{-1}^{1}\int_{2x^2}^{1+x^2} (x+2y)\, dy\, dx$$
$$= \int_{-1}^{1} \left[xy + y^2\right]_{y=2x^2}^{y=1+x^2} dx$$
$$= \int_{-1}^{1} \left[x(1+x^2)+(1+x^2)^2 - x(2x^2)-(2x^2)^2\right]dx$$
$$= \int_{-1}^{1} \left[-3x^4 - x^3 + 2x^2 + x + 1\right]dx$$
$$= \frac{32}{15}$$

Problem-2: Evaluate $\iint_D xy\, dA$ where D is the region bounded by the line y=x-1 and the parabola $y^2=2x+6$.

Solution: The region is shown:



(a) *D* as a type I region



(b) *D* as a type II region

Hence, we prefer to express *D* as a type II region:

$D = \{(x, y) \mid -2 \le y \le 4,\ 1/2y^2 - 3 \le x \le y + 1\}$

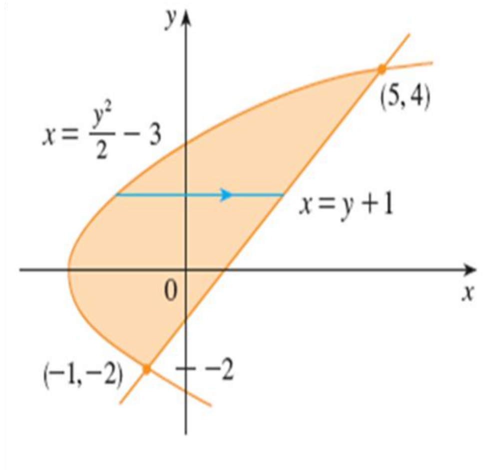
$$\iint_D xy\,dA = \int_{-2}^{4}\int_{\frac{1}{2}y^2-3}^{y+1} xy\,dx\,dy$$

$$= \int_{-2}^{4}\left[\frac{x^2}{2}y\right]_{x=\frac{1}{2}y^2-3}^{x=y+1} dy$$

$$= \tfrac{1}{2}\int_{-2}^{4} y\left[(y+1)^2 - (\tfrac{1}{2}y^2-3)^2\right]dy$$

$$= \tfrac{1}{2}\int_{-2}^{4}\left(-\frac{y^5}{4}+4y^3+2y^2-8y\right)dy$$

$$= \tfrac{1}{2}\left[-\frac{y^6}{24}+y^4+2\frac{y^3}{3}-4y^2\right]_{-2}^{4} = 36$$



D as a type II region

❖ **Find the value of** $\int_1^2\int_{\frac{y}{2}}^{y} e^{2x-y}\,dx\,dy$

Let, $A = \int_1^2\int_{\frac{y}{2}}^{y} e^{2x-y}\,dx\,dy$

$$= \int_1^2\left\{\int_{\frac{y}{2}}^{y} e^{2x-y}\,dx\right\}dy$$

$$= \int_1^2\left[\frac{e^{2x-y}}{2}\right]_{x=y/2}^{y} dy \quad = \frac{1}{2}\int_1^2 (e^y-1)dy = \frac{1}{2}[e^y - y]_1^2$$

$$= \frac{1}{2}[(e^2-2)-(e^1-1)] = \frac{1}{2}[e^2-e-1)]$$

❖ **Evaluate** $\int_0^1 \int_0^x y\sqrt{x^2 - y^2}\; dy\, dx$

Solution: $\int_0^1 \int_0^x y\sqrt{x^2 - y^2}\; dy\, dx = \int_0^1 \left[\int_0^x y(x^2 - y^2)^{\frac{1}{2}}\; dy \right] dx$

$$= -\frac{1}{2} \int_0^1 \left[\int_0^x (x^2 - y^2)^{\frac{1}{2}}\; d(-y^2) \right] dx$$

$$= -\frac{1}{2} \int_0^1 \left[\int_0^x (x^2 - y^2)^{\frac{1}{2}}\; d(x^2 - y^2) \right] dx$$

$$= -\frac{1}{2} \int_0^1 \left[\frac{(x^2 - y^2)^{\frac{3}{2}}}{3/2} \right]_{y=0}^{x} dx$$

$$= -\frac{1}{3} \int_0^1 \left[0 - (x^2 - 0)^{3/2} \right] dx$$

$$= \frac{1}{3} \int_0^1 x^3\; dx = \frac{1}{3} \left[\frac{x^4}{4} \right]_0^1 = \frac{1}{12}.$$

❖ Evaluate the double integral $\iint_R (2x - y^2)\, dA$, where R is the region bounded by $y = -x + 1, y = x + 1$ *and* $y = 3$.

Solution: Given that

$$y = -x + 1 \cdots (1), y = x + 1 \cdots (2), \quad y = 3 \cdots (3)$$

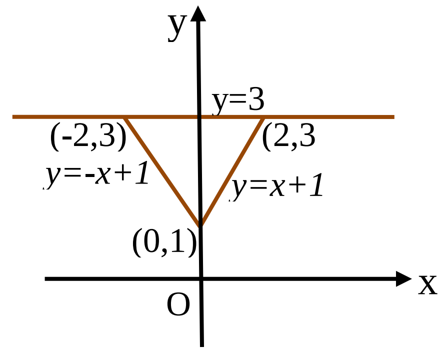
From (1) and (2), we get

$$x + 1 = -x + 1 \Rightarrow x = 0$$

From (1), $y = 0 + 1 = 1$

The lines (1) and (2) intersect at the point (0,1). Here, the region R lies between $y = 1,\ y = 3$ *and* $x = y - 1$

$$i.e. \quad R = \{(x, y) : 1 \le y \le 3, 1 - y \le x \le y - 1\}$$



$$\therefore \quad \iint_R (2x - y^2)\,dA = \int_1^3 \int_{1-y}^{y-1} (2x - y^2)\,dx\,dy$$

$$= \int_1^3 \left[x^2 - y^2 x\right]_{x=1-y}^{y-1} dy$$

$$= \int_1^3 \left[1 - 2y + 2y^2 - y^3 - 1 + 2y - y^3\right] dy$$

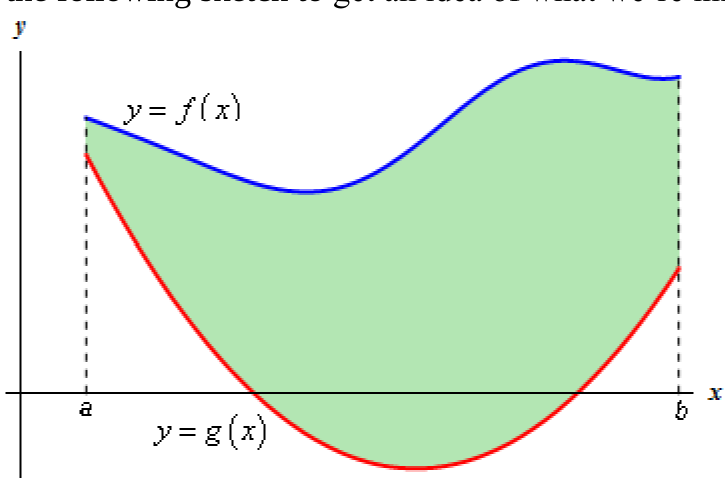
$$= \int_1^3 \left[2y^2 - 2y^3\right] dy = \left[\frac{2y^3}{3} - \frac{y^4}{2}\right]_1^3 = -\frac{68}{3}$$

Applications of integral calculus

Area between Curves

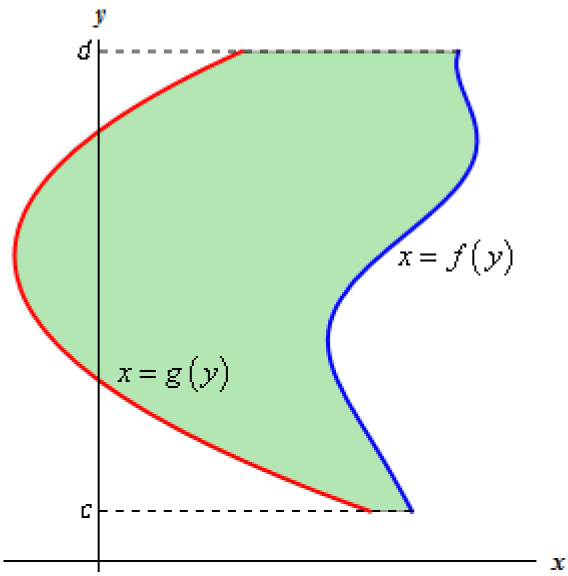
In this section we are going to look at finding the area between two curves. There are actually two cases that we are going to be looking at.

In the first case we want to determine the area between $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$. We are also going to assume that $f(x) \ge g(x)$. Take a look at the following sketch to get an idea of what we're initially going to look at.



The area can be defined in this case $A = \int_a^b f(x) - g(x)\,dx$ (1)

The second case is almost identical to the first case. Here we are going to determine the area between $x = f(y)$ and $x = g(y)$ on the interval $[c,d]$ with $f(y) \geq g(y)$.



In this case the formula is,

$$A = \int_c^d f(y) - g(y)\, dy \qquad (2)$$

Now (1) and (2) are perfectly serviceable formulas, however, it is sometimes easy to forget that these always require the first function to be the larger of the two functions. So, instead of these formulas we will instead use the following "word" formulas to make sure that we remember that the area is always the "larger" function minus the "smaller" function.

In the first case we will use,

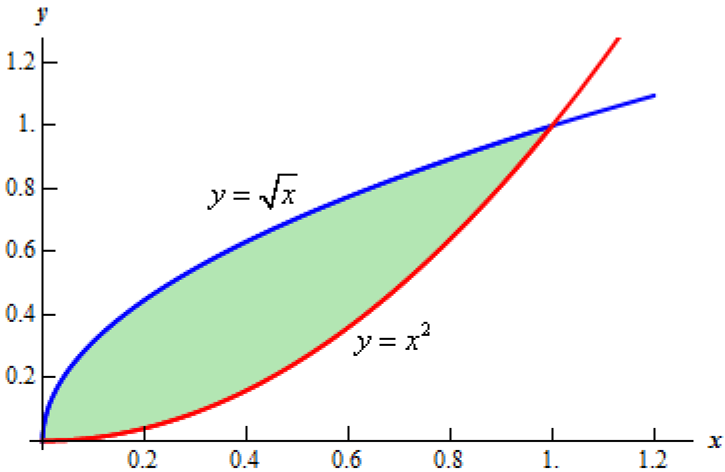
$$A = \int_a^b \begin{pmatrix} \textit{upper} \\ \textit{function} \end{pmatrix} - \begin{pmatrix} \textit{lower} \\ \textit{function} \end{pmatrix} dx, \qquad a \leq x \leq b \qquad (3)$$

In the second case we will use,

$$A = \int_c^d \begin{pmatrix} \textit{right} \\ \textit{function} \end{pmatrix} - \begin{pmatrix} \textit{left} \\ \textit{function} \end{pmatrix} dy, \qquad c \leq x \leq d \qquad (4)$$

Example 1 Determine the area of the region enclosed by $y = x^2$ and $y = \sqrt{x}$.

Solution: First of all, just what do we mean by "area enclosed by". This means that the region we're interested in must have one of the two curves on every boundary of the region. So, here is a graph of the two functions with the enclosed region shaded.



Note that, we don't take any part of the region to the right of the intersection point of these two graphs. In this region there is no boundary on the right side and so is not part of the enclosed area. Remember that one of the given functions must be on the each boundary of the enclosed region.

Also from this graph it's clear that the upper function will be dependent on the range of x's that we use. Because of this you should always sketch of a graph of the region. Without a sketch it's often easy to mistake which of the two functions is the larger. In this case most would probably say that $y = x^2$ is the upper function and they would be right for the vast majority of the x's. However, in this case it is the lower of the two functions.

The limits of integration for this will be the intersection points of the two curves. In this case it's pretty easy to see that they will intersect at $x = 0$ and $x = 1$. So these are the limits of integration.

So, the integral that we'll need to compute to find the area is,

$$A = \int_a^b \begin{pmatrix} upper \\ function \end{pmatrix} - \begin{pmatrix} lower \\ function \end{pmatrix} dx$$

$$= \int_0^1 (\sqrt{x} - x^2) dx = 1/3.$$

Example 2 Determine the area of the region bounded by $y = 2x^2 + 10$ and $y = 4x + 16$.

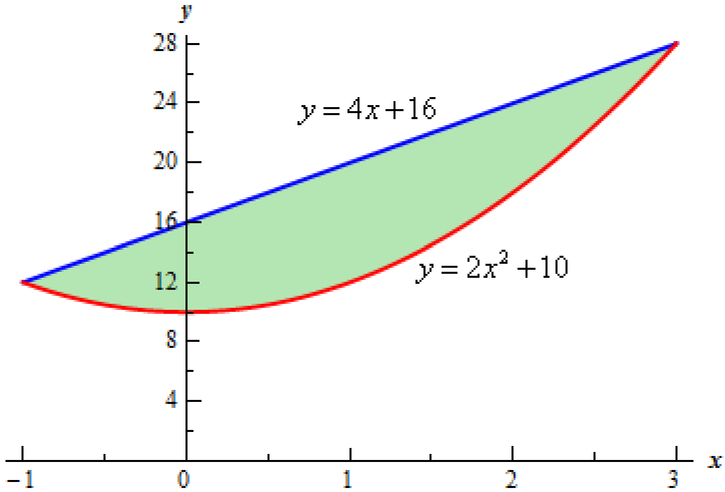
Solution: In this case the intersection points (which we'll need eventually) are not going to be easily identified from the graph so let's go ahead and get them now. Note that for most of these problems you'll not be able to accurately identify the intersection points from the graph and so you'll need to be able to determine them by hand. In this case we can get the intersection points by setting the two equations equal.

$$2x^2 + 10 = 4x + 16$$

$$\Rightarrow\ 2x^2 - 4x - 6 = 0$$
$$\Rightarrow 2(x+1)(x-3) = 0.$$

So it looks like the two curves will intersect at $x = -1$ and $x = 3$. If we need them we can get the *y* values corresponding to each of these by plugging the values back into either of the equations. We'll leave it to you to verify that the coordinates of the two intersection points on the graph are (-1,12) and (3,28).

Note as well that if you aren't good at graphing knowing the intersection points can help in at least getting the graph started. Here is a graph of the region.



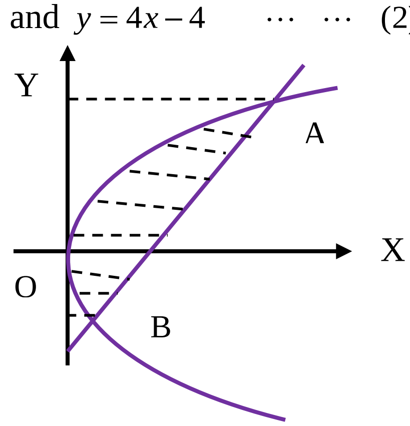
With the graph we can now identify the upper and lower function and so we can now find the enclosed area.

$$A = \int_a^b \begin{pmatrix} upper \\ function \end{pmatrix} - \begin{pmatrix} lower \\ function \end{pmatrix} dx$$
$$= \int_{-1}^{3} 4x + 16 - (2x^2 + 10)\, dx = \int_{-1}^{3} -2x^2 + 4x + 6\, dx = 64/3$$

Example: Determine the area of the region bounded by the parabola $y^2 = 8x$ and the line $y = 4x - 4$.

Solution: Given that $y^2 = 8x$ $\cdots\ \cdots$ (1) and $y = 4x - 4$ $\cdots\ \cdots$ (2)

Here, the vertex of the parabola (1) is $O(0, 0)$ .After solving equations (1) and (2), we find the two intersection points are A(2,4) and B(1/2, -2).



From (1) and (2), we get

$$x = \frac{y^2}{8} = f(y) \quad \text{and}$$
$$x = \frac{1}{4}(y+4) = \phi(y)$$

Required Area , $R = \left| \int_{-2}^{4} \{ f(y) - \phi(y) \} \, dy \right|$

$$= \left| \int_{-2}^{4} \left\{ \frac{y^2}{8} - \frac{1}{4}(y+4) \right\} dy \right|$$

$$= \left| \frac{1}{8}\left[\frac{y^3}{3} \right]_{-2}^{4} - \frac{1}{4}\left[\frac{1}{2}y^2 + 4y \right]_{-2}^{4} \right| = \left| \frac{1}{8}(24) - \frac{1}{4}(30) \right| = \left| 3 - \frac{15}{2} \right|$$

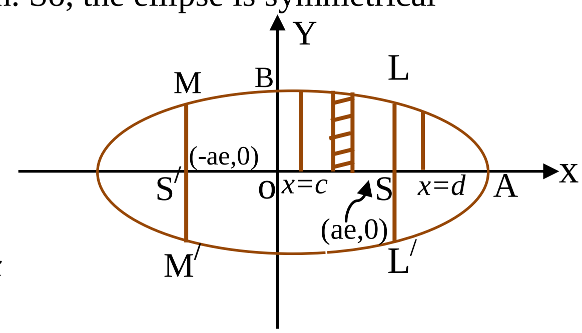
$$= \frac{9}{2} \text{ square unit.}$$

❖ (a) Find the area under the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, x-axis and the ordinates $x = c$ & $x = d$;

(b) find the total area of the ellipse;

(c) find the area between its latus recta.

Solution: The power of both x and y are even. So, the ellipse is symmetrical about both axes. Also,



$$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

$$\therefore \quad y = \pm \frac{b}{a}\sqrt{(a^2 - x^2)}$$

(a) Required area $= \int_c^d y \, dx = \frac{b}{a}\int_c^d \sqrt{a^2 - x^2} \, dx$

y- is positive in XOY quadrant.

$$= \frac{b}{2a}\left[d\sqrt{a^2 - x^2} - c\sqrt{a^2 - x^2} + a^2\left(\sin^{-1}\frac{d}{a} - \sin^{-1}\frac{c}{a} \right) \right] \text{ sq. units.}$$

(b) Here, ordinates are $x = c = 0$ and $x = d = a$. So, from (a), total area of the ellipse =4(area BOAB)

$$= 4 \cdot \frac{b}{2a}\left[a\sqrt{a^2 - a^2} - 0\sqrt{a^2 - 0} + a^2\left(\sin^{-1}\frac{a}{a} - \sin^{-1}\frac{0}{a} \right) \right]$$

$$= \pi ab \text{ sq. units.}$$

(c) Since the coordinates of the foci are $(\pm ae, 0)$. So, from (a), using $x = c = -ac$ and $x = d = ae$, the area

$$MS'OSLBM = \frac{b}{2a}\left[ae\sqrt{a^2 - a^2e^2} + ae\sqrt{a^2 - a^2e^2} + a^2\left(\sin^{-1}\frac{ac}{a} - \sin^{-1}\frac{(-ae)}{a}\right)\right]$$

$$= ab(e\sqrt{1-e^2} + \sin^{-1} e) \text{ sq. units}$$

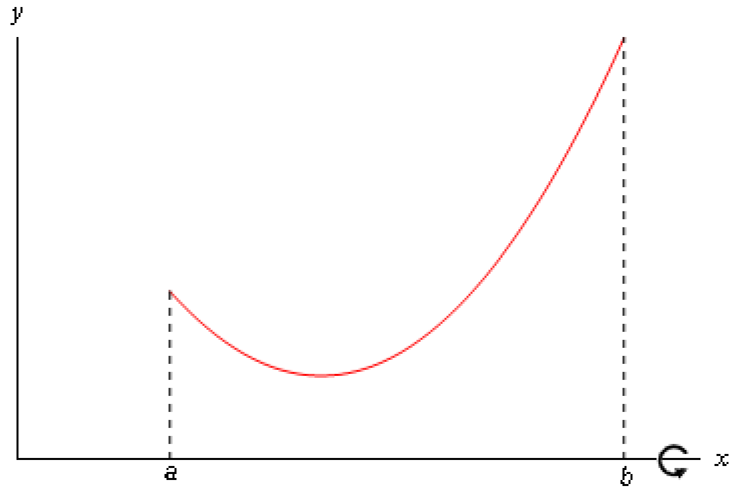
Therefore, required area $= 2(\textit{area of } MS'OSLBM)$

$$= 2ab(e\sqrt{1-e^2} + \sin^{-1} e) \text{ sq. units}$$

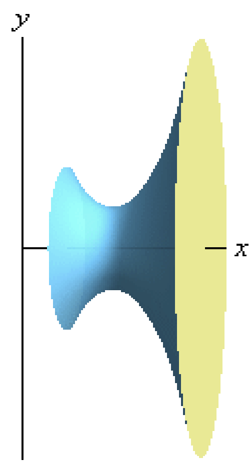
❖ Determine the area bounded by the parabola $y^2 = 4ax$ and any double ordinate of it, say $x = x_1$.(Ref. Das and Mukherjee, p-321)

Volumes of Solids of Revolution / Method of Rings

In this section we will start looking at the volume of a solid of revolution. We should first define just what a solid of revolution is. To get a solid of revolution we start out with a function, $y = f(x)$ on an interval [a,b].



We then rotate this curve about a given axis to get the surface of the solid of revolution. For purposes of this discussion let's rotate the curve about the *x*-axis, although it could be any vertical or horizontal axis. Doing this for the curve above gives the following three dimensional region.



What we want to do over the course of the next two sections is to determine the volume of this object.
In the final the Area and Volume Formulas section of the Extras chapter we derived the following formulas for the volume of this solid.

$$V = \int_a^b A(x)\,dx \qquad\qquad V = \int_c^d A(y)\,dy$$

where, $A(x)$ and $A(y)$ is the cross-sectional area of the solid. There are many ways to get the cross-sectional area and we'll see two (or three depending on how you look at it) over the next two sections. Whether we will use $A(x)$ or $A(y)$ will depend upon the method and the axis of rotation used for each problem.

One of the easier methods for getting the cross-sectional area is to cut the object perpendicular to the axis of rotation. Doing this the cross section will be either a solid disk if the object is solid (as our above example is) or a ring if we've hollowed out a portion of the solid (we will see this eventually).

In the case that we get a solid disk the area is,

$$A = \pi(\text{radius})^2$$

where the radius will depend upon the function and the axis of rotation.

In the case that we get a ring the area is,

$$A = \pi\left(\begin{pmatrix}\text{outer}\\\text{radius}\end{pmatrix}^2 - \begin{pmatrix}\text{inner}\\\text{radius}\end{pmatrix}^2\right)$$

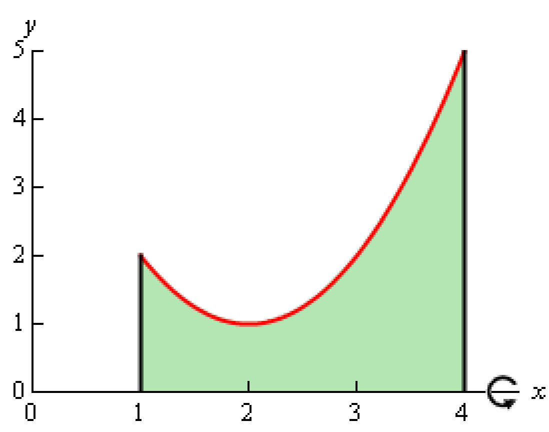
where again both of the radii will depend on the functions given and the axis of rotation. Note as well that in the case of a solid disk we can think of the inner radius as zero and we'll arrive at the correct formula for a solid disk and so this is a much more general formula to use.

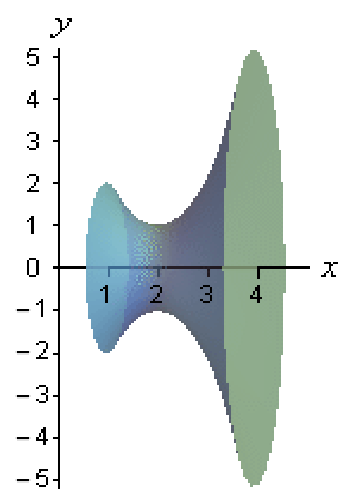
Also, in both cases, whether the area is a function of *x* or a function of *y* will depend upon the axis of rotation as we will see.

This method is often called the **method of disks** or the **method of rings**.

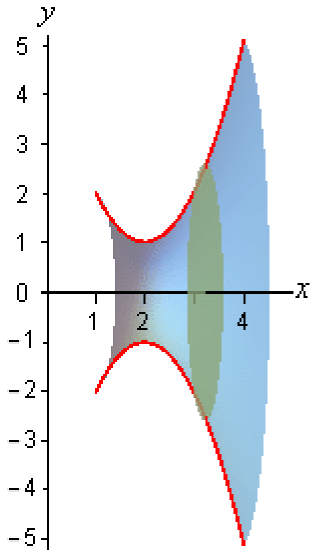
Example 1 Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 4x + 5$, $x = 1$, $x = 4$, and the *x*-axis about the *x*-axis.

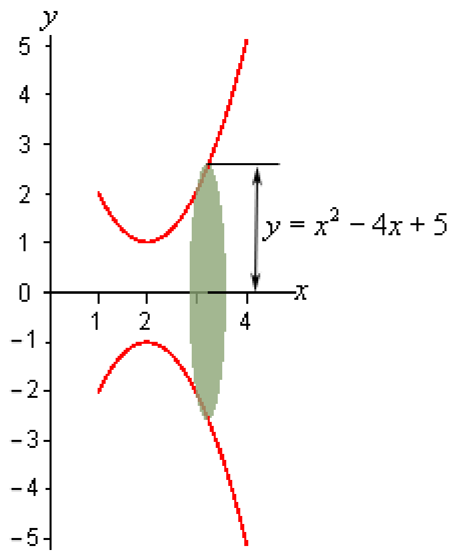
Solution: The first thing to do is get a sketch of the bounding region and the solid obtained by rotating the region about the *x*-axis. Here are both of these sketches.





Okay, to get a cross section we cut the solid at any *x*. Below are a couple of sketches showing a typical cross section. The sketch on the right shows a cut away of the object with a typical cross section without the caps. The sketch on the left shows just the curve we're rotating as well as its mirror image along the bottom of the solid.





In this case the radius is simply the distance from the *x*-axis to the curve and this is nothing more than the function value at that particular *x* as shown above. The cross-sectional area is then,

$$A(x) = \pi\left(x^2 - 4x + 5\right)^2 = \pi\left(x^4 - 8x^3 + 26x^2 - 40x + 25\right)$$

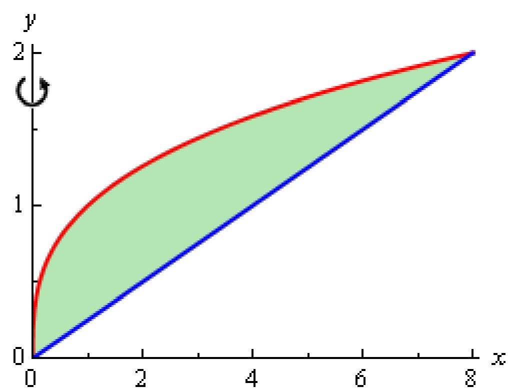
Next we need to determine the limits of integration. Working from left to right the first cross section will occur at $x = 1$ and the last cross section will occur at $x = 4$. These are the limits of integration.

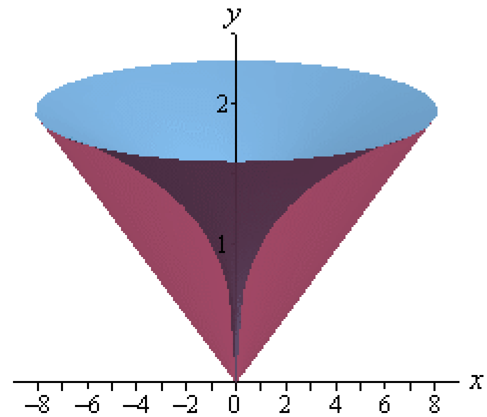
The volume of this solid is then,

$$\begin{aligned} V &= \int_a^b A(x)\,dx \\ &= \pi\int_1^4 x^4 - 8x^3 + 26x^2 - 40x + 25\,dx \\ &= \pi\left(\frac{1}{5}x^5 - 2x^4 + \frac{26}{3}x^3 - 20x^2 + 25x\right)\Big|_1^4 \\ &= \frac{78\pi}{5} \end{aligned}$$

Example 2 Determine the volume of the solid obtained by rotating the portion of the region bounded by $y = \sqrt[3]{x}$ and $y = \frac{x}{4}$ that lies in the first quadrant about the *y*-axis.

Solution: First, let's get a graph of the bounding region and a graph of the object. Remember that we only want the portion of the bounding region that lies in the first quadrant. There is a portion of the bounding region that is in the third quadrant as well, but we don't want that for this problem.





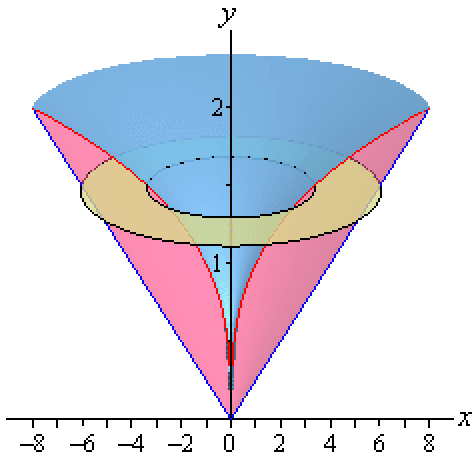
There are a couple of things to note with this problem. First, we are only looking for the volume of the "walls" of this solid, not the complete interior as we did in the last example.

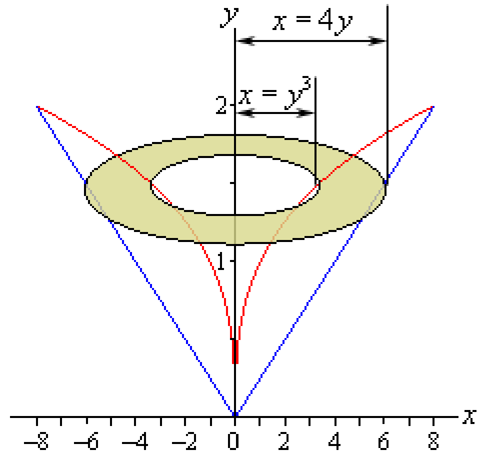
Next, we will get our cross section by cutting the object perpendicular to the axis of rotation. The cross section will be a ring (remember we are only looking at the walls) for this example and it will be horizontal at some *y*. This means that the inner and outer radius for the ring will be *x* values and so we will need to rewrite our functions into the form $x = f(y)$. Here are the functions written in the correct form for this example.

$$y = \sqrt[3]{x} \qquad \Rightarrow \qquad x = y^3$$

$$y = \frac{x}{4} \qquad \Rightarrow \qquad x = 4y$$

Here are a couple of sketches of the boundaries of the walls of this object as well as a typical ring. The sketch on the left includes the back portion of the object to give a little context to the figure on the right.





The inner radius in this case is the distance from the *y*-axis to the inner curve while the outer radius is the distance from the *y*-axis to the outer curve. Both of these are then *x* distances and so are given by the equations of the curves as shown above.

The cross-sectional area is then,

$$A(y) = \pi\left((4y)^2 - \left(y^3\right)^2\right) = \pi\left(16y^2 - y^6\right)$$

Working from the bottom of the solid to the top we can see that the first cross-section will occur at $y=0$ and the last cross-section will occur at $y=2$. These will be the limits of integration. The volume is then,

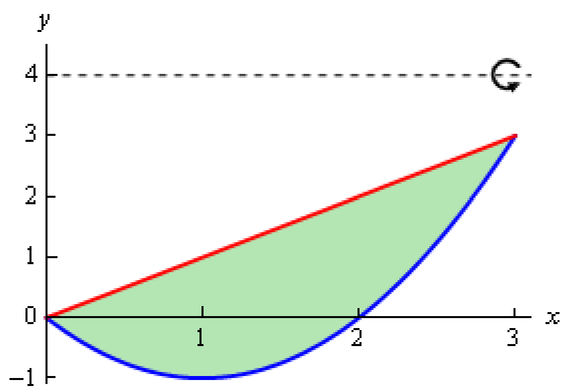
$$\begin{aligned} V &= \int_c^d A(y)\,dy \\ &= \pi\int_0^2 16y^2 - y^6\,dy \\ &= \pi\left(\frac{16}{3}y^3 - \frac{1}{7}y^7\right)\Big|_0^2 \\ &= \frac{512\pi}{21} \end{aligned}$$

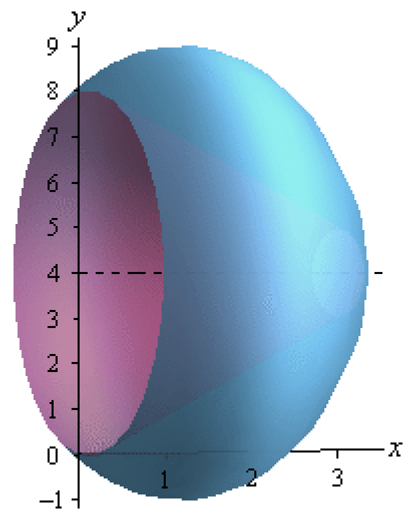
With these two examples out of the way we can now make a generalization about this method. If we rotate about a horizontal axis (the *x*-axis for example) then the cross sectional area will be a function of *x*. Likewise, if we rotate about a vertical axis (the *y*-axis for example) then the cross sectional area will be a function of *y*.

The remaining two examples in this section will make sure that we don't get too used to the idea of always rotating about the *x* or *y*-axis.

Example 3 Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 2x$ and $y = x$ about the line $y = 4$.

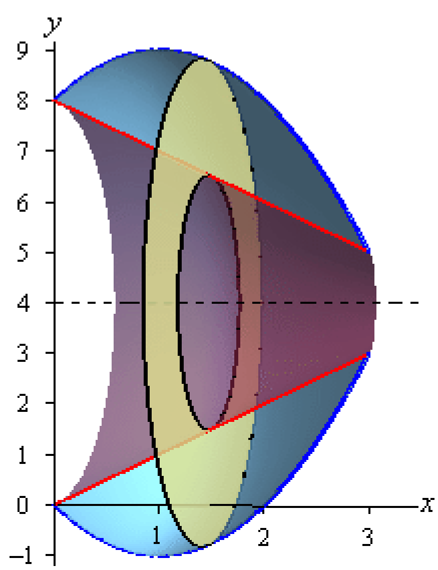
Solution: First let's get the bounding region and the solid graphed.

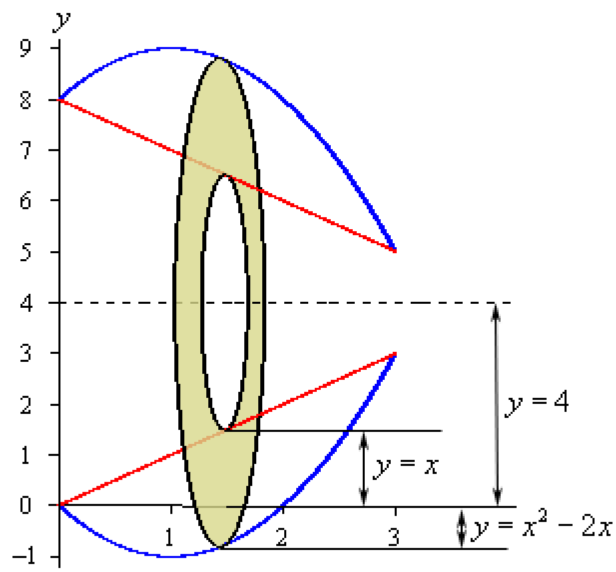




Again, we are going to be looking for the volume of the walls of this object. Also since we are rotating about a horizontal axis we know that the cross-sectional area will be a function of *x*.

Here are a couple of sketches of the boundaries of the walls of this object as well as a typical ring. The sketch on the left includes the back portion of the object to give a little context to the figure on the right.





Now, we're going to have to be careful here in determining the inner and outer radius as they aren't going to be quite as simple they were in the previous two examples.

Let's start with the inner radius as this one is a little clearer. First, the inner radius is NOT *x*. The distance from the *x*-axis to the inner edge of the ring is *x*, but we want the radius and that is the distance from the axis of rotation to the inner edge of the ring. So, we know that the distance from the axis of rotation to the *x*-axis is 4 and the distance from the *x*-axis to the inner ring is *x*. The inner radius must then be the difference between these two. Or,

$$\text{inner radius} = 4 - x$$

The outer radius works the same way. The outer radius is,

$$\text{outer radius} = 4 - \left(x^2 - 2x\right) = -x^2 + 2x + 4$$

Note that given the location of the typical ring in the sketch above the formula for the outer radius may not look quite right but it is in fact correct. As sketched the outer edge of the ring is below the *x*-axis and at this point the value of the function will be negative and so when we do the subtraction in the formula for the outer radius we'll actually be subtracting off a negative number which has the net effect of adding this distance onto 4 and that gives the correct outer radius. Likewise, if the outer edge is above the *x*-axis, the function value will be positive and so we'll be doing an honest subtraction here and again we'll get the correct radius in this case.

The cross-sectional area for this case is,

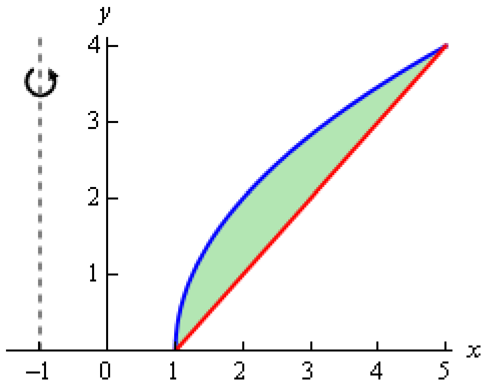
$$A(x) = \pi\left(\left(-x^2 + 2x + 4\right)^2 - \left(4 - x\right)^2\right) = \pi\left(x^4 - 4x^3 - 5x^2 + 24x\right)$$

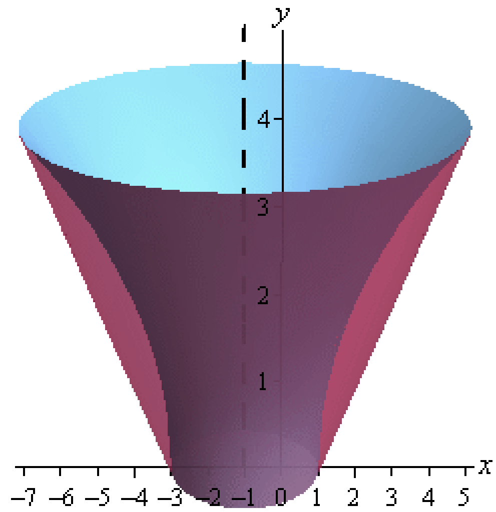
The first ring will occur at $x = 0$ and the last ring will occur at $x = 3$ and so these are our limits of integration. The volume is then,

$$\begin{aligned} V &= \int_a^b A(x)\,dx \\ &= \pi\int_0^3 x^4 - 4x^3 - 5x^2 + 24x\,dx \\ &= \pi\left(\frac{1}{5}x^5 - x^4 - \frac{5}{3}x^3 + 12x^2\right)\Bigg|_0^3 \\ &= \frac{153\pi}{5} \end{aligned}$$

Example 4 Determine the volume of the solid obtained by rotating the region bounded by $y = 2\sqrt{x-1}$ and $y = x - 1$ about the line $x = -1$.

Solution: As with the previous examples, let's first graph the bounded region and the solid.



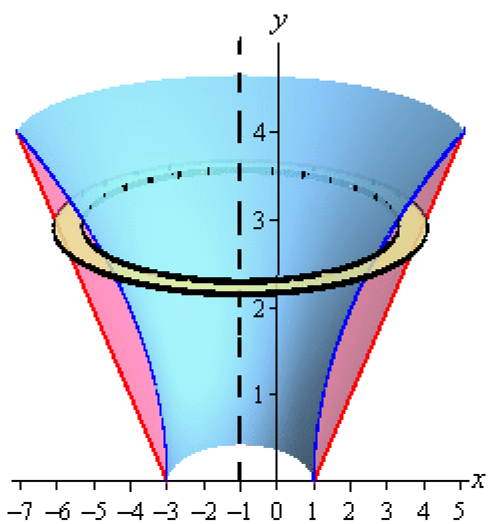


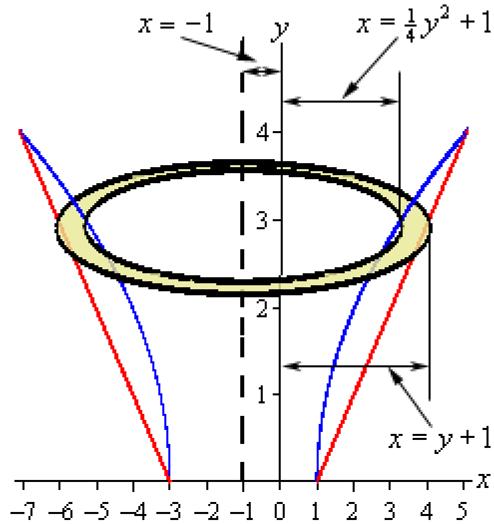
Now, let's notice that since we are rotating about a vertical axis and so the cross-sectional area will be a function of *y*. This also means that we are going to have to rewrite the functions to also get them in terms of *y*.

$$y = 2\sqrt{x-1} \qquad \Rightarrow \qquad x = \frac{y^2}{4} + 1$$

$$y = x - 1 \qquad \Rightarrow \qquad x = y + 1$$

Here are a couple of sketches of the boundaries of the walls of this object as well as a typical ring. The sketch on the left includes the back portion of the object to give a little context to the figure on the right.





The inner and outer radius for this case is both similar and different from the previous example. This example is similar in the sense that the radii are not just the functions. In this example the functions are the distances from the *y*-axis to the edges of the rings. The center of the ring however is a distance of 1 from the *y*-

axis. This means that the distance from the center to the edges is a distance from the axis of rotation to the *y*-axis (a distance of 1) and then from the *y*-axis to the edge of the rings.

So, the radii are then the functions plus 1 and that is what makes this example different from the previous example. Here we had to add the distance to the function value whereas in the previous example we needed to subtract the function from this distance. Note that without sketches the radii on these problems can be difficult to get.

So, in summary, we've got the following for the inner and outer radius for this example.

$$\text{outer radius} = y + 1 + 1 = y + 2$$
$$\text{inner radius} = \frac{y^2}{4} + 1 + 1 = \frac{y^2}{4} + 2$$

The cross-sectional area is then,

$$A(y) = \pi\left((y+2)^2 - \left(\frac{y^2}{4}+2\right)^2\right) = \pi\left(4y - \frac{y^4}{16}\right)$$

The first ring will occur at $y = 0$ and the final ring will occur at $y = 4$ and so these will be our limits of integration.

The volume is,

$$\begin{aligned} V &= \int_c^d A(y)\,dy \\ &= \pi\int_0^4 4y - \frac{y^4}{16}\,dy \\ &= \pi\left(2y^2 - \frac{1}{80}y^5\right)\Big|_0^4 \\ &= \frac{96\pi}{5} \end{aligned}$$

Beta and Gamma Functions

A function given by

$$B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx,\ m>0,\ n>0$$

is called **Beta function** and the integral is called First Eulerian integral.

A function given by

$$\Gamma n=\int_0^\infty e^{-x}x^{n-1}\,dx,\ n>0$$

is called **gamma function** and the integral is called Second Eulerian integral.

Properties:

$B(m,n)=B(n,m)$

$\Gamma(1)=1$

$\Gamma(n+1)=n\Gamma(n)=n!$

$B(m,n)=\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ (Relation between beta and gamma functions)

Problem: (i) State the relation between Beta and Gamma functions and use it to show that $\int_0^1 x^{\frac{3}{2}}(1-x)^{\frac{3}{2}}\,dx=\dfrac{3\pi}{128}$.

Express $\int_0^{\frac{\pi}{2}}\sin^4\theta\cos^6\theta\,d\theta$ as a beta function and hence evaluate it.

Show that $\int_0^1 \dfrac{xdx}{\sqrt{1-x^5}}=\dfrac{1}{5}B\left(\dfrac{2}{5},\dfrac{1}{2}\right)$.

Solution: **(i)** The Relation between beta function and gamma function is

$$B(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Here, $I=\int_0^1 x^{\frac{3}{2}}(1-x)^{\frac{3}{2}}\,dx=.\int_0^1 x^{\frac{5}{2}-1}(1-x)^{\frac{5}{2}-1}\,dx=B(\frac{5}{2},\frac{5}{2})$

$$=\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(5)}=\frac{\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right).\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{4.3.2.1}=\frac{3}{128}\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2=\frac{3\pi}{128}$$

(iv)Let $x^5=\sin^2\theta\Rightarrow x=\sin^{\frac{2}{5}}\theta\Rightarrow dx=\frac{2}{5}.\sin^{-\frac{3}{5}}\theta.\cos\theta\,d\theta$

As $x\to 0$, $\theta\to 0$, and when $x\to 1$, $\theta\to\dfrac{\pi}{2}$

$$I=\int_0^{\frac{\pi}{2}}\frac{\sin^{\frac{2}{5}}\theta.\frac{2}{5}\sin^{-\frac{3}{5}}\theta\cos\theta\,d\theta}{\cos\theta}$$

$$=\frac{2}{5}.\int_0^{\frac{\pi}{2}}\sin^{-\frac{1}{5}}\theta\cos^0\theta\,d\theta=\frac{1}{5}.2\int_0^{\frac{\pi}{2}}\sin^{\left(2\times\frac{2}{5}-1\right)}\theta\,\cos^{\left(2\times\frac{1}{2}-1\right)}\theta\,d\theta=\frac{1}{5}.B\left(\frac{2}{5},\frac{1}{2}\right).$$

Evaluate (i) $\int_0^{\frac{\pi}{2}}\sin^2\theta\cos^4\theta\,d\theta=\frac{5\pi}{256}$ (ii) $\int_0^{\frac{\pi}{6}}\cos^4\theta\sin^2 6\theta\,d\theta=\frac{5\pi}{192}$

(iii) $\int_0^{\frac{\pi}{2}}\sqrt{\tan\theta}\,d\theta=\frac{\pi}{\sqrt{2}}$

Solution: (ii) $\int_0^{\frac{\pi}{6}}\cos^4\theta\sin^2 6\theta\,d\theta=\int_0^{\frac{\pi}{6}}\cos^4\theta\,(2\sin 3\theta\cos 3\theta)^2\,d\theta$

$$=4\int_0^{\frac{\pi}{6}}\sin^2 3\theta\cos^6 3\theta\,d\theta=\frac{4}{3}\int_{3\theta=0}^{3\theta=\frac{\pi}{2}}\sin^2(3\theta)\cos^6(3\theta)\,d(3\theta)$$

$\because\theta=0\Rightarrow 3\theta=0;\quad \theta=\pi/6\Rightarrow 3\theta=\pi/2$

$$=\frac{4}{3}\frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{6+1}{2}\right)}{2\Gamma\left(\frac{2+6+2}{2}\right)}=\frac{2\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{7}{2}\right)}{3\Gamma(5)}=\frac{5\pi}{192}$$

(iii) $$\int_0^{\frac{\pi}{2}}\sqrt{\tan\theta}\,d\theta=\int_0^{\frac{\pi}{2}}\sin^{\frac{1}{2}}\theta\cos^{-\frac{1}{2}}\theta\,d\theta=\frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)}{2\Gamma\left(\frac{\frac{1}{2}-\frac{1}{2}+2}{2}\right)}$$

$$=\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{4}\right)}{2\Gamma(1)}=\frac{\pi}{2}.$$

Show that $\int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta \, d\theta = \dfrac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$

Proof: Try yourself

Put, $I = \int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta \, d\theta$

$$= \int_0^{\frac{\pi}{2}} \sin^{p-1} \theta \cos^{q-1} \theta \sin\theta \cos\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} (\sin^2 \theta)^{(p-1)/2} \theta \, (\cos^2 \theta)^{(q-1)/2} \sin\theta \cos\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} (\sin^2 \theta)^{(p-1)/2} \theta \, (1-\sin^2 \theta)^{(q-1)/2} \sin\theta \cos\theta \, d\theta$$

Put, $\sin^2 \theta = t$, then $2\sin\theta \cos\theta \, d\theta = dt$

When $\theta = 0$, then $t = 0$

When $\theta = \dfrac{\pi}{2}$, then $t = 1$

So, $I = \dfrac{1}{2}\int_0^1 t^{(p-1)/2} \theta \, (1-t)^{(q-1)/2} \, dt = \dfrac{1}{2}\int_0^1 t^{(p+1)/2-1} \theta \, (1-t)^{(q+1)/2-1} \, dt$

$$= \frac{1}{2}.\, \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \frac{\Gamma\left(\frac{p+1}{2}\right), \Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+1}{2}\right)} \quad \text{(Proved)}$$

Evaluate (i) $\int_0^{\pi} x \cos^4 x \, dx$ (ii) $\int_0^{\pi} x \sin^2 x \cos^4 x \, dx$

Solution: (ii) Put, $I = \int_0^{\pi} x \sin^2 x \cos^4 x \, dx$ …..(i)

$$= \int_0^{\pi} (\pi - x) \{\sin(\pi - x)\}^2 \{\cos(\pi - x)\}^4 \, dx$$

$$= \int_0^{\pi} (\pi - x) \sin^2 x \cos^4 x \, dx \quad \ldots\ldots\{\text{ii})$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} (x + \pi - x) \sin^2 x \cos^4 x \, dx = \pi \int_0^{\pi} \sin^2 x \cos^4 x \, dx = 2\pi \int_0^{\frac{\pi}{2}} \sin^2 x \cos^4 x \, dx$$

$$\therefore \quad I = \pi \int_0^{\pi/2} \sin^2 x \cos^4 x \, dx$$

$$= \frac{\pi . \Gamma\left(\frac{3}{2}\right) . \Gamma\left(\frac{5}{2}\right)}{2\Gamma(4)} = \pi . \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right) . \frac{3}{2} . \frac{1}{2} . \Gamma\left(\frac{1}{2}\right)}{2.3.2.1} = \frac{\pi . \frac{1}{2} . \sqrt{\pi} .. \frac{3}{2} . \frac{1}{2} . \sqrt{\pi}}{2.3.2.1} = \frac{\pi^2}{32}$$

So, $\int_0^\pi x \sin^2 x \cos^4 x \, dx = \frac{\pi^2}{32}$.