Chapter 1: The Foundations: Logic and Proofs

1.4 Predicates and Quantifiers

Introduction

- Propositional logic cannot adequately express the meaning of all statements in mathematics and in natural language.
- E.g. “Every computer connected to the university network is functioning properly.”
- “CS2 is under attack by an intruder,” where CS2 is a computer on the university network, to conclude the truth of

“There is a computer on the university network that is under attack by an intruder.

Predicates and Quantifiers

Predicates

- Statements involving variables are neither true nor false.
- E.g. "x > 3", "x = y + 3", "x + y = z"
- "x is greater than 3"
  - "x": subject of the statement
  - "is greater than 3": the ***predicate***
- We can denote the statement "x is greater than 3" by *P(x)*, where *P* denotes the predicate and *x* is the variable.
- Once a value is assigned to the variable *x*, the statement *P(x)* becomes a proposition and has a truth value.

Predicates and Quantifiers

- Example: Let $P(x)$ denote the statement "x > 3." What are the truth values of $P(4)$ and $P(2)$?

  Solution: $P(4)$ – "4 > 3", *true*
  $P(3)$ – "2 > 3", *false*

- Example: Let $Q(x,y)$ denote the statement "x = y + 3." What are the truth values of the propositions $Q(1,2)$ and $Q(3,0)$?

  Solution: $Q(1,2)$ – "1 = 2 + 3" , *false*
  $Q(3,0)$ – "3 = 0 + 3", *true*

Predicates and Quantifiers

- Example: Let *A(c,n)* denote the statement "Computer *c* is connected to network *n*", where *c* is a variable representing a computer and *n* is a variable representing a network. Suppose that the computer MATH1 is connected to network CAMPUS2, but not to network CAMPUS1. What are the values of A(MATH1, CAMPUS1) and A(MATH1, CAMPUS2)?

Solution: *A(MATH1, CAMPUS1) – "MATH1 is connect to CAMPUS1", false*

A(MATH1, CAMPUS2) – "MATH1 is connect to CAMPUS2", true

Predicates and Quantifiers

- A statement involving n variables $x_1, x_2, \ldots, x_n$ can be denoted by $P(x_1, x_2, \ldots, x_n)$.

- A statement of the form $P(x_1, x_2, \ldots, x_n)$ is the value of the propositional function P at the n-tuple $(x_1, x_2, \ldots, x_n)$, and P is also called a ***n*-place predicate** or a ***n*-ary predicate**.

Predicates and Quantifiers

Quantifiers

- **Quantification**: express the extent to which a predicate is true over a range of elements.
- **Universal quantification**: a predicate is true for every element under consideration
- **Existential quantification**: a predicate is true for one or more element under consideration
- A domain must be specified.

Predicates and Quantifiers

DEFINITION 1

The *universal quantification* of *P(x)* is the statement

"*P(x)* for all values of *x* in the domain."

The notation $\forall x P(x)$ denotes the universal quantification of *P(x)*. Here $\forall$ is called the **Universal Quantifier**. We read $\forall x P(x)$ as "for all *xP(x)*" or "for every *xP(x)*."

Example: Let *P(x)* be the statement "$x + 1 > x$." What is the truth value of the quantification $\forall x P(x)$, where the domain consists of all real numbers?

Solution: *Because P(x) is true for all real numbers, the quantification is true.*

Predicates and Quantifiers

- A statement $\forall x P(x)$ is false, if and only if $P(x)$ is not always true where x is in the domain. One way to show that is to find a counterexample to the statement $\forall x P(x)$.
- Example: Let $Q(x)$ be the statement "$x < 2$". What is the truth value of the quantification $\forall x Q(x)$, where the domain consists of all real numbers?

Solution: *$Q(x)$ is not true for every real numbers, e.g. $Q(3)$ is false. $x = 3$ is a counterexample for the statement $\forall x Q(x)$. Thus the quantification is false.*

Predicates and Quantifiers

- Example: What does the statement $\forall x N(x)$ mean if $N(x)$ is "Computer x is connected to the network" and the domain consists of all computers on campus?

Solution: *"Every computer on campus is connected to the network."*

Predicates and Quantifiers

DEFINITION 2

The *existential quantification* of $P(x)$ is the statement

"There exists an element x in the domain such that $P(x)$."

We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the **Existential Quantifier**.

- The existential quantification $\exists x P(x)$ is read as

  "There is an x such that $P(x)$," or

  "There is at least one x such that $P(x)$," or

  "For some x, $P(x)$."

Predicates and Quantifiers

- Example: Let *P(x)* denote the statement "$x > 3$". What is the truth value of the quantification $\exists x P(x)$, where the domain consists of all real numbers?

Solution: *"$x > 3$" is sometimes true – for instance when $x = 4$. The existential quantification is true.*

- $\exists x P(x)$ is false if and only if $P(x)$ is false for every element of the domain.
- Example: Let $Q(x)$ denote the statement "$x = x + 1$". What is the true value of the quantification $\exists x Q(x)$, where the domain consists for all real numbers?

Solution: *$Q(x)$ is false for every real number. The existential quantification is false.*

Predicates and Quantifiers

- If the domain is empty, $\exists xQ(x)$ is false because there can be no element in the domain for which $Q(x)$ is true.
- The existential quantification $\exists xP(x)$ is the same as the disjunction $P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$

Quantifiers		
Statement	***When True?***	***When False?***
$\forall xP(x)$	*xP(x) is true for every x.*	*There is an x for which xP(x) is false.*
$\exists xP(x)$	*There is an x for which P(x) is true.*	*P(x) is false for every x.*

∃

Predicates and Quantifiers

Translating from English into Logical Expressions

- Example: Express the statement "Every student in this class has studied calculus" using predicates and quantifiers.

  Solution:

  If the domain consists of students in the class –

  $$\forall x C(x)$$

  where $C(x)$ is the statement "x has studied calculus.

  If the domain consists of all people –

  $$\forall x(S(x) \rightarrow C(x))$$

  where $S(x)$ represents that person x is in this class.

  If we are interested in the backgrounds of people in subjects besides calculus, we can use the two-variable quantifier $Q(x,y)$ for the statement "student x has studied subject y." Then we would replace $C(x)$ by $Q(x, \text{calculus})$ to obtain $\forall x Q(x, \text{calculus})$ or

  $$\forall x(S(x) \rightarrow Q(x, \text{calculus}))$$

Predicates and Quantifiers

- Example: Consider these statements. The first two are called *premises* and the third is called the *conclusion*. The entire set is called an *argument*.

  "All lions are fierce."

  "Some lions do not drink coffee."

  "Some fierce creatures do not drink coffee."

Solution: Let $P(x)$ be "x is a lion."

$Q(x)$ be "x is fierce."

$R(x)$ be "x drinks coffee."

$\forall x(P(x) \rightarrow Q(x))$

$\exists x(P(x) \land \neg R(x))$

$\exists x(Q(x) \land \neg R(x))$