Derivatives

Derivatives

Ordinary Derivatives

$$\frac{dv}{dt}$$

v is a function of one independent variable

Partial Derivatives

$$\frac{\partial u}{\partial y}$$

u is a function of more than one independent variable

Differential Equations

Differential Equations

Ordinary Differential Equations

$$\frac{d^2v}{dt^2} + 6tv = 1$$

involve one or more

Ordinary derivatives of

unknown functions

Partial Differential Equations

$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

involve one or more

partial derivatives of

unknown functions

Differential Equations

An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation **(DE)**.

For example, $$\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0$$

$$\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 3x = \sin t$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Example of Differential Equations

ODE

$$\frac{dy}{dx}+2xy=e^{-x^2}$$

$$\frac{d^2y}{dx^2}+5y=0$$

$$2\left(\frac{d^2y}{dx^2}\right)^2=x^2\frac{dy}{dx}$$

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=\left(\frac{d^2y}{dx^2}\right)^2$$

PDE

$$\frac{\partial^2 z}{\partial x^2}-2\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y^2}=0$$

$$f_{xx}+f_{yy}+f_x+f_y=f+xy$$

Recommended Books

- Differential Equations
  …….by Shepley . Ross.

- Ordinary and Partial Differential equations
  ……..by M.D. Raisinghania.

- Differential Equations with Applications
  …….by Dr. Md. Mustafa Kamal Chowdhury.

Ordinary Differential Equations

A differential equation involving derivatives w.r.to single independent variable is called an **ODE.**

A DE involving partial derivatives w.r.to two or more independent variable is called **PDE.**

An ODE is a differential equation for a function of a single variable, e.g., $y(x)$, while a PDE is a differential equation for a function of several variables, e.g., $v(x, y, z, t)$.

Ordinary Differential Equations

Ordinary Differential Equations (ODEs) involve one or more ordinary derivatives of unknown functions with respect to one independent variable

$$Examples:$$

$$\frac{dv(t)}{dt} - v(t) = e^{t}$$

$$\frac{d^2 x(t)}{dt^2} - 5\frac{dx(t)}{dt} + 2x(t) = \cos(t)$$

x(t): unknown function

t: independent variable

Order and Degree of DE

Order: The order of a differential equation is the order of the highest derivative involve.

Degree: The degree of a differential equation is the power (or Degree) of the highest derivative after the equation has been made rational.

Order and Degree of DE

$$\sin\left(\frac{dy}{dx}\right) + x = 0$$

Order:1, Degree:1

$$\frac{dy}{dx} + \sin\left(\frac{dy}{dx}\right) + x + 2y = 0$$

The equ. Can not be expressed as linear function in (dy/dx). So the order and degree of the equation is not defined.

$$\frac{d^2y}{dx^2} = \frac{m}{H}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Order:2, Degree: 2

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = k\frac{d^2y}{dx^2}$$

Order:2, Degree: 2

Order of a Differential Equation

The **order** of an ordinary differential equation is the order of the highest order derivative.

$$Examples:$$

$$\frac{dx(t)}{dt} - x(t) = e^t$$

First order ODE

$$\frac{d^2x(t)}{dt^2} - 5\frac{dx(t)}{dt} + 2x(t) = \cos(t)$$

Second order ODE

$$\left(\frac{d^2x(t)}{dt^2}\right)^3 - \frac{dx(t)}{dt} + 2x^4(t) = 1$$

Second order ODE

Formation of ODE

An ODE is formed by differentiating the equation and eliminating arbitrary constant(s).

Form a DE by eliminating the constants A and B from

01

$$y = e^{mx}(A\cos nx + B\sin nx)$$

Solution: $$\frac{dy}{dx} = e^{mx}(-An\sin nx + Bn\cos nx) + my$$

$$\frac{d^2y}{dx^2} = e^{mx}(-An^2\cos nx - Bn^2\sin nx) + m\left(\frac{dy}{dx} - my\right) + m\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = n^2 y + m\left(\frac{dy}{dx} - my\right) + m\frac{dy}{dx} \Rightarrow \frac{d^2y}{dx^2} - 2m\frac{dy}{dx} + (m^2 + n^2)y = 0$$

Which the required DE

Formation of ODE

Q2

- Form an DE corresponding to $x = 2t + c,\ y = ct + 3$, where t is a parameter.

Solution: The parametric relation has only one arbitrary constant c. Now eliminating t, we get

$$y = \frac{c}{2}(x - c) + 3 \quad \ldots\ldots(1)$$

Differentiating w.r.to x, $\frac{dy}{dx} = \frac{c}{2} \quad \therefore c = 2\frac{dy}{dx}$

Putting the value of c in (1), we obtained the required DE

$$y = \frac{dy}{dx}\left(x - 2\frac{dy}{dx}\right) + 3 \Rightarrow 2\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} 3 - y$$

Solution of a Differential Equation

A **solution** to a differential equation is a function that satisfies the equation.

$$Example:$$

$$\frac{dx(t)}{dt} + x(t) = 0$$

$$Solution \quad x(t) = e^{-t}$$

Proof :

$$\frac{dx(t)}{dt} = -e^{-t}$$

$$\frac{dx(t)}{dt} + x(t) = -e^{-t} + e^{-t} = 0$$

Linear ODE

An ODE is linear if

The unknown function and its derivatives appear to power one

No product of the unknown function and/or its derivatives

Examples:

$$\frac{dx(t)}{dt} - x(t) = e^t$$

Linear ODE

$$\frac{d^2x(t)}{dt^2} - 5\frac{dx(t)}{dt} + 2t^2x(t) = \cos(t)$$

Linear ODE

$$\left(\frac{d^2x(t)}{dt^2}\right)^3 - \frac{dx(t)}{dt} + \sqrt{x(t)} = 1$$

Non-linear ODE

Nonlinear ODE

An ODE is non-linear if

The unknown function and its derivatives appear to power more than one

And product of the unknown function and/or its derivatives

Examples of nonlinear ODE:

$$\frac{dx(t)}{dt} - \cos(x(t)) = 1$$

$$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 + 6y = 3$$

$$\frac{d^2x(t)}{dt^2} - 5\frac{dx(t)}{dt}x(t) = 2$$

because in second term is not of degree one

$$\frac{d^2x(t)}{dt^2} - \left|\frac{dx(t)}{dt}\right| + x(t) = 1$$

First Order and First Degree DE

$$Mdx + Ndy = 0 \quad or,\ M + N\frac{dy}{dx} = 0$$

Differential Equations

ODE

PDE

Equations solvable by Separation of variables

Homogeneous equation

linear Differential Equations

Exact Differential Equations

Variable Separable

If the DE $Mdx + Ndy = 0$ can be put in the form

$$f(y)dy = \phi(x)dx$$

i.e. *dx* all terms containing *x* are on one side linear DE

and *dy* all terms containing *y* are on the other side

Variable Separable

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Solve: $y\sqrt{1+x^2}\,dy - x\sqrt{1+y^2}\,dx = 0$

Solution: **Separating the variables of the equations we have**

$$\frac{y\,dy}{\sqrt{1+y^2}} = \frac{x\,dx}{\sqrt{1+x^2}}$$

Integration yields $\int uv\,dx = u\int v\,dx - \int (u'\int v\,dx)\,dx$

$$\sqrt{1+y^2} = \sqrt{1+x^2} + c$$

Which is the required solution.

Variable Separable(Try yourself)

4 **(1)** $$(x+y)^2\left(x\frac{dy}{dx}+y\right)=xy\left(1+\frac{dy}{dx}\right)$$

5 **(2)** $$\cos(x+y)dy=dx$$

6 **(3)** $$\frac{dy}{dx}=\frac{1-x-y}{x+y}$$

Homogeneous DE

A DE $M(x,y)dx + N(x,y)dy = 0$ is said to be homogeneous if it can be put in the form

$$\frac{dy}{dx} = \frac{\phi(x,y)}{\psi(x,y)}$$

where $\phi(x,y)$ and $\psi(x,y)$ are homogeneous functions of the same degree (say, *n*)

Homogeneous DE (Try Yourself)

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Solve: (1) $(x^2 + y^2)dx - 2xydy = 0$

(2) $(2\sqrt{xy} - y)dx - xdy = 0$

(3) $2y^3dx + (x^2 - 3y^2)xdy = 0$

Equation reducible to homogeneous form

$$\frac{dy}{dx} = \frac{ax + by + c}{a_1x + b_1y + c_1}$$

can be reduced to homogeneous form by substitution

$$x = X + h,\ y = Y + k \Rightarrow \frac{dy}{dx} = \frac{dY}{dX}$$ ***Then Try…..***

Equation reducible to homogeneous form

4A6

Solve: **(1)** $(3x-7y-3)\dfrac{dy}{dx}=3y-7x+7$

(2) $$\frac{dy}{dx}=\frac{x-y-2}{x+y+6}$$

Solution (2): Putting $x=X+h,\ y=Y+k$

$$\frac{dY}{dX}=\frac{X+h-Y-k-2}{X+h+Y+k+6}=\frac{X-Y+(h-k-2)}{X+Y+(h+k+6)}$$

Now we will choose h and k such that

$$h-k-2=0,\ h+k+6=0\Rightarrow h=-2,\ k=-4$$

Linear Differential Equation

A DE is said to be linear if the depenendent variable y and its derivatives occur only in the first degree and are not multiplied together. The **LDE** is of the form

$$\frac{dy}{dx} + Py = Q$$

where ***P*, *Q*** are functions of x or constants.

$$I.F. = ?$$

Linear Differential Equation

:Solution Procedure:

Multiplying both sides of $\dfrac{dy}{dx} + Py = Q$ by $e^{\int P dx}$

Integrating Factor (I.F)

We get,

$$e^{\int P dx}\left[\frac{dy}{dx} + Py\right] = Qe^{\int P dx}$$

$$or, \quad \frac{d}{dx}\left[y\, e^{\int P dx}\right] = Qe^{\int P dx}$$

Integrating both sides,

$$y\, e^{\int P dx} = \int\left(Qe^{\int P dx}\right)dx + c$$

Linear Differential Equation (Example)

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Solve (i) $x\dfrac{dy}{dx}+2y=x^5$ **(ii)** $\dfrac{dy}{dx}+\left(\dfrac{2x+1}{x}\right)y=e^{-2x}$

(i) The given equation can be written as

$$\frac{dy}{dx}+2\frac{y}{x}=x^4 \quad \text{Here, } P=\frac{2}{x},\ Q=x^4$$

$$\text{So,} \quad \text{I.F.}=e^{\int\frac{2}{x}dx}=e^{2\log x}=x^2$$

Therefore, the solution of the given DE is

$$y\,x^2=\int x^4\cdot x^2 dx=\frac{x^7}{7}+c$$

Reducible to Linear Form (Bernoulli's Equation)

An equation of the form

$$\frac{dy}{dx} + Py = Qy^n$$

where *P* and *Q* are constants of *x* alone and n is constant except 0 and 1, is called a **Bernoulli's equation**.

For example, $$\frac{dy}{dx} + y = xy^3$$

Bernoulli's DE → Reduces to → **Linear DE**

Bernoulli's Equation (Example)

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Solve $(1-x^2)\dfrac{dy}{dx}+xy=xy^2$

The given equation can be written as

$$y^{-2}\frac{dy}{dx}+\frac{x}{1-x^2}y^{-1}=\frac{x}{1-x^2}$$

Put, $y^{-1}=v \;\Rightarrow\; -y^{-2}\dfrac{dy}{dx}=\dfrac{dv}{dx}$ and the given eq. reduces

$$\frac{dv}{dx}-\frac{x}{1-x^2}v=\frac{x}{1-x^2}$$

Which is a Linear *v*

Bernoulli's Equation (Example)

$$\frac{dv}{dx} - \frac{x}{1-x^2} v = \frac{x}{1-x^2}$$

I.F. is

$$= e^{\int -\frac{x}{1-x^2} dx} = e^{\frac{1}{2}\ln(1-x^2)} = \sqrt{1-x^2}$$

Multiplying both sides by I.F., we have

$$\frac{d}{dx}(v.\sqrt{1-x^2}) = \frac{x}{1-x^2} \times \sqrt{1-x^2}$$

Integrating, ……………..

Exact Differential Equation

$M(x,y)dx + N(x,y)dy = 0$ is said to be **exact if**

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

: Working Rule :

If $M(x,y)dx + N(x,y)dy = 0$ is an exact DE, then the method of solution is as follows:

Step 1: Integrate M w.r.to x as if y is constant

Step 2: Find out the term which is free from x and integrate those terms w.r.to y considering x as constant.

Step3: Step1+Step2=Constant which is the G.S.

Exact Differential Equation (Try…)

Solve:

1. $(e^{2y} - y\cos xy)dx + (2xe^{2y} - x\cos xy + 2y)dy = 0$

2. $(y^3 - y^2 \sin x - x)dx + (3xy^2 + 2y\cos x)dy = 0$

3. $(5x^4 + 3x^2y^2 - 2xy^3)dx + (2x^3y - 3x^2y^2 - 5y^4)dy = 0$

4. $\left[(\cos x)\log_e(2y-8) + \frac{1}{8}\right]dx + \frac{\sin x}{y-4}dy = 0$

Hints . $\frac{\partial M}{\partial y} = \frac{2\cos x}{2y-8} = \frac{\cos x}{y-4}, \frac{\partial N}{\partial x} = \frac{\cos x}{y-4},$ Which is exact

Equations Reducible to the Exact Differential Equation

Initially this is not exact, but it can be reduced to be exact by using some special I.F

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Rule-I. **If** $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}$ **is a function of x alone, say f(x), then**

$$I.F = e^{\int f(x)dx}$$

For example, Solve $(2x\log x - xy)dy + 2ydx = 0$

Try Yourself…..

i). $(xy^2 + x)dx + (yx^2 + y)dy = 0$

ii). $x\dfrac{dy}{dx} + \cot y = 0$, given $y = \dfrac{\pi}{4}$ where $x = \sqrt{2}$

iii). $\dfrac{dy}{dx} = \dfrac{y}{x} + x\sin\dfrac{y}{x}$

iv). If $\dfrac{dy}{dx} + 2y\tan x = \sin x$, and $y = 0$ for $\mathrm{x} = \dfrac{\pi}{3}$

v). $x\dfrac{dy}{dx} + y\log y = xye^x$

Try Yourself.....

vi). $x\log x\dfrac{dy}{dx}+y=2\log x$

vii). $x\dfrac{dy}{dx}+y=(xy)^{\frac{3}{2}}\ ;y(1)=4.$

$$Solution:\quad y^{-\frac{3}{2}}\frac{dy}{dx}+\frac{1}{xy^{\frac{1}{2}}}=x^{\frac{1}{2}}\quad\cdots(1)$$

$$putting,\quad y^{-\frac{1}{2}}=v\Rightarrow y^{-\frac{3}{2}}\frac{dy}{dx}=-2\frac{dv}{dx}$$

$$So\ eq.(1)\ reduces,\quad -2\frac{dv}{dx}+\frac{v}{x}=x^{\frac{1}{2}}\Rightarrow\frac{dv}{dx}-\frac{v}{2x}=-\frac{x^{\frac{1}{2}}}{2}$$

$$I.F.=e^{-\int\frac{dx}{2x}}=\frac{1}{\sqrt{x}}$$

Try Yourself…..

Therefore, the solution is

$$\frac{v}{\sqrt{x}} = \int \frac{\sqrt{x}}{2\sqrt{x}} dx \Rightarrow \frac{1}{\sqrt{xy}} = -\frac{x}{2} + C$$

$$Given, \quad x = 1, y = 4$$

$$\frac{1}{\sqrt{4}} = -\frac{1}{2} + C \Rightarrow C = 1$$

Therefore, the required solution is

$$\frac{1}{\sqrt{xy}} + \frac{x}{2} = 1$$

Boundary-Value and Initial value Problems

Initial-Value Problems

- The auxiliary conditions are at one point of the independent variable

$$\ddot{x} + 2\dot{x} + x = e^{-2t}$$

$$x(0) = 1, \quad \dot{x}(0) = 2.5$$

same

Boundary-Value Problems

- The auxiliary conditions are not at one point of the independent variable
- More difficult to solve than initial value problems

$$\ddot{x} + 2\dot{x} + x = e^{-2t}$$

$$x(0) = 1, \quad x(2) = 1.5$$

different

Initial Condition Problem

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Problem: Solve $x^2\frac{dy}{dx}-2xy=3y^4$ subject to initial condition y(1)=2.

Solution: The given equation can be written as

$$\frac{1}{y^4}\frac{dy}{dx}-\frac{2}{xy^3}=\frac{3}{x^2}$$

Putting

$$-\frac{1}{y^3}=v\Rightarrow\frac{3}{y^4}\frac{dy}{dx}=\frac{dv}{dx}\Rightarrow\frac{1}{y^4}\frac{dy}{dx}=\frac{1}{3}\frac{dv}{dx}$$

The given equation reduces to

$$\frac{1}{3}\frac{dv}{dx}+\frac{2v}{x}=\frac{3}{x^2}\Rightarrow\frac{dv}{dx}+\frac{6v}{x}=\frac{9}{x^2}$$

Initial Condition Problem

which is linear equation in v

$$I.F. = e^{\int \frac{6}{x} dx} = x^6$$

The required solution of the given equation is

$$v\,x^6 = \int \frac{9}{x^2}\, x^6\, dx = \frac{9}{5} x^5 + c \Rightarrow v = \frac{9}{5x} + \frac{c}{x^6}$$

$$\therefore \quad -\frac{1}{y^3} = \frac{9}{5x} + \frac{c}{x^6}$$

given x=1, y=2 $$-\frac{1}{8} = \frac{9}{5} + c \Rightarrow c = -\frac{77}{40}$$

$$\therefore \quad \frac{1}{y^3} = -\frac{9}{5x} + \frac{77}{40x^6}$$

Second Order Linear Homogeneous Equations with Constant Coefficients

- A **second order ordinary differential equation** has the general form

$$y'' = f(t, y, y')$$

where f is some given function.

- This equation is said to be **linear** if f is linear in y and y':

$$y'' = g(t) - p(t)y' - q(t)y$$

Otherwise the equation is said to be **nonlinear**.

- A second order linear equation often appears as

$$P(t)y'' + Q(t)y' + R(t)y = G(t)$$

- If $G(t) = 0$ for all t, then the equation is called **homogeneous**. Otherwise the equation is **nonhomogeneous**.

Higher order linear non-homogeneous DEs with constant coefficient

- Suppose the roots be

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n y = F(x)$$

where the coefficients a_1, $a_{2,}$ $a_{3,}$ ….. a_n are constants but the non-homogeneous term *F* is a non-constant function of *x*.

The general solution for this case

$$y = y_c + y_p$$

$$= \text{Complement ary Function(C .F)} + \text{Particular Integral(P .I)}$$

For Homogeneous case: Auxiliary Equation

- Let the differential equation be

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n y = 0 \quad \ldots.(1)$$

Let $y = e^{mx}$ be the **trial solution** of equation (1). Then putting $y = e^{mx}$ in equ. (1), we get

$$(a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_n) e^{mx} = 0$$

$$Since\ e^{mx} \neq 0, \quad a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_n = 0$$

This equation is called **auxiliary equation** of equ (1)

Auxiliary Equation

- While solving the auxiliary equation, the following three cases may arises:

- **All the roots are real and distinct**
- **All the roots are real but some are repeating**
- **All the roots are imaginary**

Case-I: Auxiliary equation having real and different roots

- If $m_1, m_2, \cdots, m_n$ be the n different roots of A.E., then

$$y = e^{m_1 x}, \quad y = e^{m_2 x}, \cdots, y = e^{m_n x}$$

Are all independent solution of equ(1). Therefore, the general solution of (1) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$

Example: 45

490 $Solve \quad \frac{d^2 y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$

Case-II: Auxiliary equation having repeated real roots

- We know that the general solution of a higher order differential equation having different roots

$$m_1, m_2, \cdots, m_n \quad \text{is}$$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$

If two roots say m_1 and m_2 are equal then the solution becomes

$$y = (c_1 + x c_2) e^{m_1 x} + c_3 e^{m_2 x} \cdots + c_n e^{m_n x}$$

Example:

A6

$$Solve \quad \frac{d^3 y}{dx^3} - 7\frac{d^2 y}{dx^2} + 16\frac{dy}{dx} - 12y = 0$$

Case-III: Auxiliary equation having imaginary roots

- Suppose the roots be

$$\alpha \pm i\beta$$

Let's Try..., $\dfrac{d^4 y}{dx^4} - 16y = 0$

The general solution is

$$y = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}$$

$$= e^{\alpha x}[c_1 e^{i\beta x} + c_2{}^{-i\beta x}]$$

$$= e^{\alpha x}[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)]$$

$$= e^{\alpha x}[(c_1 + c_2)\cos\beta x + (c_1 - c_2)i\sin\beta x]$$

$$= e^{\alpha x}[A\cos\beta x + B\sin\beta x]$$

For Non-Homogeneous case:

- Suppose the roots be

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n y = F(x)$$

where the coefficients a_1, $a_{2,}$ $a_{3,}$ ….. a_n are constants but the non-homogeneous term *F* is a non-constant function of *x*.

The general solution for this case

$$y = y_c + y_p$$

$$= \text{Complement ary Function(C .F)} + \text{Particular Integral(P .I)}$$

For non-homogeneous DEs :

Methods of determining a P.I.

Type-I: If *F(x)=X*, a polynomial in *x* then

$$y_p = \frac{1}{f(D)} X = [f(D)]^{-1} X$$

Type-II: If *F(x)* = e^{ax}, *a* is constant

$$y_p = \frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)}, \quad f(a) \neq 0$$

If *f(a)=0*, case is failure, then ??????

Higher order linear non-homogeneous DEs with constant coefficient

Type-III: If *F(x)= sinax or, cosax* then

$$\frac{1}{f(D^2)}\sin ax = \frac{1}{f(-a^2)}\sin ax$$

and

$$\frac{1}{f(D^2)}\cos ax = \frac{1}{f(-a^2)}\cos ax$$

except when f(-a²)=0.

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$$Solve\ (D^2+2D+4)y = e^x \cos 2x$$

Example..........

16 **Example:** $$\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=\cos 2x$$

Solution: $(D^2+D+1)y=\cos 2x, \quad D\equiv\dfrac{d}{dx}$

Auxiliary equation is $(m^2+m+1)=0, \quad if \;\; y=e^{mx}\neq 0$

$$m=\frac{-1\pm\sqrt{-3}}{2}, \quad C.F.=e^{\frac{x}{2}}[A\cos\frac{\sqrt{3}}{2}x+B\sin\frac{\sqrt{3}}{2}x]$$

$$P.I.=\frac{1}{D^2+D+1}\cos 2x=\frac{1}{(-2^2)+D+1}\cos 2x=\frac{1}{D-3}\cos 2x$$

$$=\frac{D+3}{D^2-9}\cos 2x=\frac{D+3}{(-2^2)-9}\cos 2x=\frac{1}{-13}(D+3)\cos 2x\cdots Try$$

Examples..........

17 **Solve:** $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = \sin(3x+2)$

18 ❖ $(D^2 + 2D + 1)y = 2x + x^2$

19 ❖ $(D^2 + 2D + 4)y = e^x \cos 2x$

20 ❖ $(D^2 - 6D + 9)y = x^2 e^{3x} \cos 2x$

Example 1: Infinitely Many Solutions

- Consider the second order linear differential equation

$$y'' - y = 0$$

- Two solutions of this equation are

$$y_1(t) = e^t, \quad y_2(t) = e^{-t}$$

- Other solutions include

$$y_3(t) = 3e^t, \quad y_4(t) = 5e^{-t}, \quad y_5(t) = 3e^t + 5e^{-t}$$

- Based on these observations, we see that there are infinitely many solutions of the form

$$y(t) = c_1 e^t + c_2 e^{-t}$$

- It will be shown in Section 3.2 that all solutions of the differential equation above can be expressed in this form.

Example 1: Initial Conditions (2 of 3)

- Now consider the following initial value problem for our equation:

$$y'' - y = 0, \quad y(0) = 3, \; y'(0) = 1$$

We have found a general solution of the form

$$y(t) = c_1 e^t + c_2 e^{-t}$$

Using the initial equations,

$$\left.\begin{aligned} y(0) &= c_1 + c_2 = 3 \\ y'(0) &= c_1 - c_2 = 1 \end{aligned}\right\} \Rightarrow c_1 = 2, \; c_2 = 1$$

Thus, the general solution is

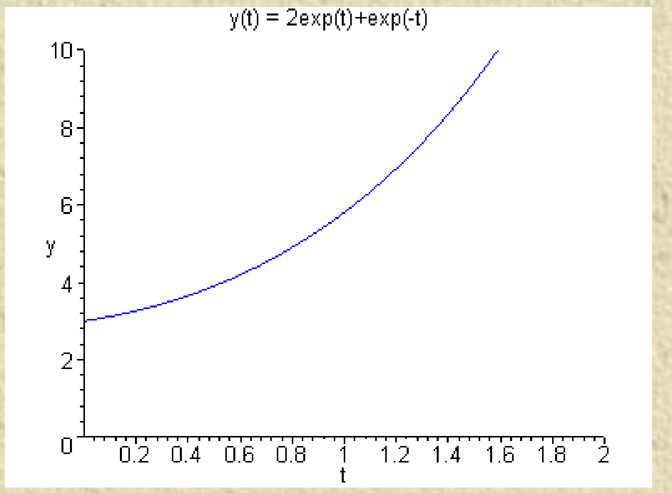
$$y(t) = 2e^t + e^{-t}$$

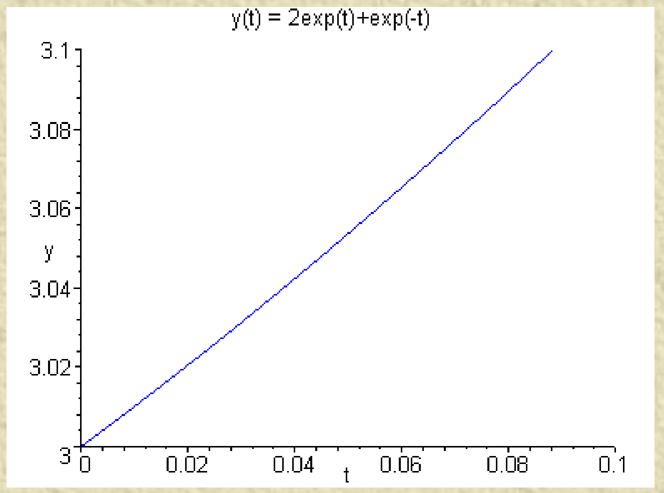
Example 1: Solution Graphs (3 of 3)

- Our initial value problem and solution are

$$y'' - y = 0, \quad y(0) = 3,\ y'(0) = 1 \ \Rightarrow \ y(t) = 2e^{t} + e^{-t}$$

- Graphs of this solution are given below. The graph on the right suggests that both initial conditions are satisfied.





Characteristic Equation

- To solve the 2nd order equation with constant coefficients,

$$ay'' + by' + cy = 0,$$

we begin by assuming a solution of the form $y = e^{rt}$.

- Substituting this into the differential equation, we obtain

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$$

- Simplifying,

$$e^{rt}(ar^2 + br + c) = 0$$

and hence

$$ar^2 + br + c = 0$$

- This last equation is called the **characteristic equation** of the differential equation.
- We then solve for r by factoring or using quadratic formula.

General Solution

- Using the quadratic formula on the characteristic equation

$$ar^2 + br + c = 0,$$

  we obtain two solutions, r_1 and r_2.
- There are three possible results:
  - The roots r_1, r_2 are real and $r_1 \neq r_2$.
  - The roots r_1, r_2 are real and $r_1 = r_2$.
  - The roots r_1, r_2 are complex.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- In this section, we will assume r_1, r_2 are real and $r_1 \neq r_2$.
- In this case, the general solution has the form

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Initial Conditions

- For the initial value problem

$$ay'' + by' + cy = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y'_0,$$

  we use the general solution

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

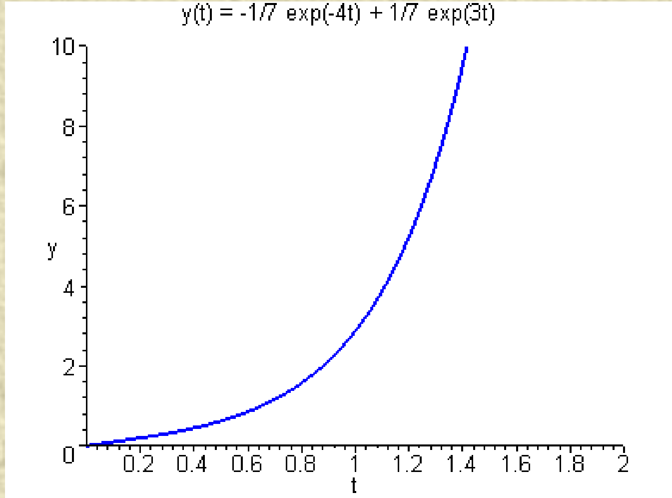
  together with the initial conditions to find c_1 and c_2. That is,

$$\left.\begin{aligned} c_1 e^{r_1 t_0} + c_2 e^{r_2 t_0} &= y_0 \\ c_1 r_1 e^{r_1 t_0} + c_2 r_2 e^{r_2 t_0} &= y'_0 \end{aligned}\right\} \Rightarrow c_1 = \frac{y'_0 - y_0 r_2}{r_1 - r_2} e^{-r_1 t_0}, \; c_2 = \frac{y_0 r_1 - y'_0}{r_1 - r_2} e^{-r_2 t_0}$$

- Since we are assuming $r_1 \neq r_2$, it follows that a solution of the form $y = e^{rt}$ to the above initial value problem will always exist, for any set of initial conditions.

Example 2

- Consider the initial value problem
  $$y'' + y' - 12y = 0, \;\; y(0) = 0, \;\; y'(0) = 1$$
- Assuming exponential soln leads to characteristic equation:
  $$y(t) = e^{rt} \quad \Rightarrow \quad r^2 + r - 12 = 0 \;\Leftrightarrow\; (r+4)(r-3) = 0$$
- Factoring yields two solutions, $r_1 = -4$ and $r_2 = 3$
- The general solution has the form
  $$y(t) = c_1 e^{-4t} + c_2 e^{3t}$$
- Using the initial conditions:
  $$\left.\begin{aligned} c_1 + c_2 &= 0 \\ -4c_1 + 3c_2 &= 1 \end{aligned}\right\} \Rightarrow c_1 = \frac{-1}{7},\; c_2 = \frac{1}{7}$$
- Thus $$y(t) = \frac{-1}{7} e^{-4t} + \frac{1}{7} e^{3t}$$



Example 3:

- Consider the initial value problem

$$2y'' + 3y' = 0,\ \ y(0) = 1,\ \ y'(0) = 3$$

- Then

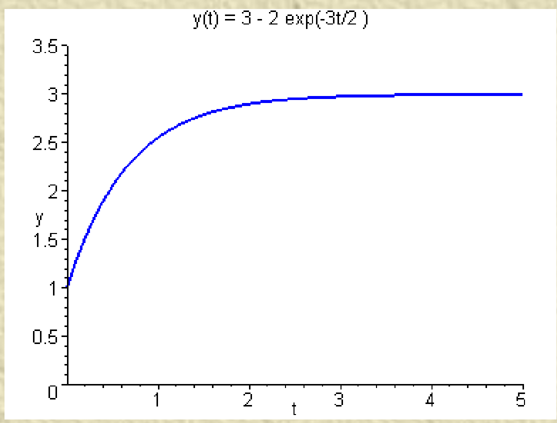
$$y(t) = e^{rt} \quad \Rightarrow \quad 2r^2 + 3r = 0 \ \Leftrightarrow\ r(2r+3) = 0$$

- Factoring yields two solutions, $r_1 = 0$ and $r_2 = -3/2$
- The general solution has the form

$$y(t) = c_1 e^{0t} + c_2 e^{-3t/2} = c_1 + c_2 e^{-3t/2}$$

- Using the initial conditions:

$$\left.\begin{aligned} c_1 + c_2 &= 1 \\ -\frac{3c_2}{2} &= 3 \end{aligned}\right\} \Rightarrow c_1 = 3,\ c_2 = -2$$



- Thus

$$y(t) = 3 - 2e^{-3t/2}$$

Example 4: Initial Value Problem (1 of 2)

- Consider the initial value problem

$$y'' + 5y' + 6y = 0, \quad y(0) = 2, \; y'(0) = 3$$

- Then

$$y(t) = e^{rt} \quad \Rightarrow \quad r^2 + 5r + 6 = 0 \;\Leftrightarrow\; (r+2)(r+3) = 0$$

- Factoring yields two solutions, r_1 = -2 and r_2 = -3
- The general solution has the form

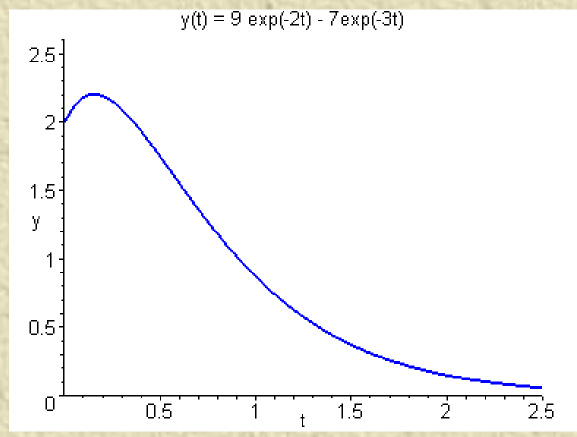
$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}$$

- Using initial conditions:

$$\left.\begin{aligned} c_1 + c_2 &= 2 \\ -2c_1 - 3c_2 &= 3 \end{aligned}\right\} \Rightarrow c_1 = 9, \; c_2 = -7$$

- Thus

$$y(t) = 9e^{-2t} - 7e^{-3t}$$



Example 5: Initial Value Problem

- Consider the initial value problem

$$y'' - 2y' - 3y = 2e^{x} - 10\sin x, \quad y(0) = 2, \; y'(0) = 4$$

Solution: Then

$$y(x) = e^{mx} \quad \Rightarrow \quad m^{2} - 2m - 3 = 0 \;\Leftrightarrow\; (m-3)(m+1) = 0$$

Factoring yields two solutions, m_1 =3 and m_2 = -1

$$\text{C.F.} = c_1 e^{3x} + c_2 e^{-x}$$

$$\text{P.I.} = \frac{1}{2}e^{x} + 2\sin x - \cos x \Rightarrow try \cdots$$

The general solution has the form

$$y(x) = c_1 e^{3x} + c_2 e^{-x} - \frac{1}{2}e^{x} + 2\sin x - \cos x$$

From this, we have

$$y'(x) = 3c_1 e^{3x} - c_2 e^{-x} - \frac{1}{2}e^{x} + 2\cos x + \sin x$$

Example 5: Initial Value Problem

Applying the initial conditions, we have

$$2 = c_1 e^0 + c_2 e^0 - \frac{1}{2} e^0 + 2\sin 0 - \cos 0$$

$$4 = 3c_1 e^0 - c_2 e^0 - \frac{1}{2} e^0 + 2\cos 0 + \sin 0$$

These equations simplify at once to the following:

$$c_1 + c_2 = 7/2, \quad 3c_1 - c_2 = 5/2 \Leftrightarrow c_1 = \frac{3}{2}, c_2 = 2$$

Thus, the general solution of the IVP is

$$y(x) = \frac{3}{2} e^{3x} + 2e^{-x} - \frac{1}{2} e^{x} + 2\sin x - \cos x.$$

Examples:

Example 4.8 Solve: $(D^2+5D+6)y=\sin 2x.$

Solution A. E is $m^2+5m+6=0$

or $$m=-2,-3$$

Thus C. F. is $$y_c=c_1e^{-2x}+c_2e^{-3x}$$

Also $$y_p=\frac{1}{D^2+5D+6}\sin 2x=\frac{1}{-4+5D+6}\sin 2x$$

$$=\frac{1}{5D+2}\sin 2x$$

$$=\frac{5D-2}{25D^2-4}.\sin 2x=\frac{(5D-2)}{-104}.\sin 2x$$

$$=-\frac{1}{104}[10\cos 2x-2\sin 2x]$$

$$=\frac{1}{52}(\sin 2x-5\cos 2x)$$

Hence the complete solution is,

$$y=c_1e^{-2x}+c_2e^{-3x}+\frac{1}{52}(\sin 2x-5\cos 2x)$$

Example 4.9 Solve $(D^2-5D+6)y=\sin(3x+2)$

Solution A.E is $m^2-5m+6=0$ has the roots 2, 3.

Thus $$y_c=c_1\,e^{2x}+c_2\,e^{3x}$$

Also $$y_p=\frac{1}{D^2-5D+6}\sin(3x+2)$$

Examples:

$$= \frac{1}{-3^2 - 5D + 6}\sin(3x+2)$$

$$= \frac{1}{-(5D+3)}\sin(3x+2)$$

$$= \frac{(5D-3)}{9-25D^2}\sin(3x+2)$$

$$= \frac{5D-3}{9+225}\sin(3x+2)$$

$$= \frac{(5D-3)}{234}\sin(3x+2)$$

$$= \frac{1}{234}\left[15\cos(3x+2) - 3\sin(3x+2)\right]$$

$$= \frac{3}{234}\left[5\cos(3x+2) - \sin(3x+2)\right]$$

$$= \frac{1}{78}\left[5\cos(3x+2) - \sin(3x+2)\right]$$

$$\therefore \quad y = c_1 e^{2x} + c_2' e^{3x} + \frac{1}{78}\left[5\cos(3x+2) - \sin(3x+2)\right]$$

Examples:

Example 4.13 Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = x^2 e^{3x}\cos 2x.$

Solution Here $\quad y_c = (c_1 + c_2 x)e^{3x}$

$$y_p = \frac{1}{(D-3)^2}.x^2 e^{3x}\cos 2x = e^{3x}.\frac{1}{(D+3-3)^2}x^2\cos 2$$

$$= e^{3x}\frac{1}{D^2}x^2\cos 2x$$

$$= \text{R.P. of } e^{3x}.\frac{1}{D^2}x^2 e^{2ix}$$

$$= \text{R.P. of } e^{3x}.e^{2ix}\frac{1}{(D+2i)^2}x^2$$

$$= \text{R. P.of } e^{3x}e^{2ix}\frac{1}{D^2+4iD-4}x^2$$

$$= \text{R. P. of } -e^{3x}e^{2ix}\frac{1}{4}\left(1-iD-\frac{D^2}{4}\right)^{-1}x^2$$

Examples:

$$= \text{R. P. of } -e^{3x}e^{2ix}\frac{1}{4}\left(1+iD+\frac{D^2}{4}-D^2\right)x^2$$

$$= \text{R. P. of } -e^{3x}e^{2ix}\frac{1}{4}\left(x^2+2ix-\frac{3}{2}\right)$$

$$= \text{R. P. of } -\frac{e^{3x}}{4}(\cos 2x + i\sin 2x)\left(x^2+2ix-\frac{3}{2}\right)$$

$$= -\frac{e^{3x}}{4}\left[\left(x^2-\frac{3}{2}\right)\cos 2x - 2x\sin 2x\right]$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2x)e^{3x} - \frac{e^{3x}}{4}\left[\left(x^2-\frac{3}{2}\right)\cos 2x - 2x\sin 2x\right]$$

Example

Solve the following DE with the initial conditions:

$$y'' - y' + 2y = e^{-4t}, \quad y(0) = 1, \; y'(0) = 5$$