# Question 1

## Part (a)

**Theorem 1.** $k-$ COLORING $\leq_p$ HOMOMORPHISM

*Proof.* Let $K_n$ be a graph whose nodes are $\{1, 2, ..., n\}$, and whose edge set is $\{\{i, j\} : i, j \in \{1, 2, ..., n\}, i \neq j\}$. Fix $k \in \mathbb{N}$. Also, let $G_0$ and $G_1$ be two non-isomorphic graphs. Now consider the following polynomial-time computable function $f : \Sigma^* \to \Sigma^*$:

$$f(w) := \begin{cases} \langle G, K_k \rangle & \text{, if } w = \langle G \rangle \\ \langle G_0, G_1 \rangle & \text{, otherwise} \end{cases}$$

$\square$

I will now show that $w \in k-$ COLORING iff. $f(w) \in$ HOMOMORPHISM:

- If $w \in k-$ COLORING, then $w = \langle G \rangle$, for some graph $G = (V, E)$, and $G$ admits a $k$-coloring $c : V \to 1, 2, ..., k$. Now, this implies that for any edge $\{v_1, v_2\} \in E$, we have that $c(v_1) \neq c(v_2)$. But, this implies that $\{c(v_1), c(v_2)\}$ is an edge of $K_k$! Thus, $c$ is also a homomorphism from $G$ to $K_k$. The existence of such a homomorphism implies that $f(w) = \langle G, K_k \rangle \in$ HOMOMORPHISM.
- If $w \notin k-$ COLORING, we then have two cases:
  - If $w = \langle G \rangle$ for some graph $G = (V, E)$, then $f(w) = \langle G, K_k \rangle$. Now, I will show that $f(w)$ is not in HOMOMORPHISM by contradiction: suppose that $f(w) \in$ HOMOMORPHISM. This implies that some homomorphism $h : V \to 1, 2, ..., k$ exists from $G$ to $K_k$. By the definition of homomorphisms, this means that for all edges $\{u, v\} \in E$, we have that $\{h(u), h(v)\}$ is an edge of $K_k$. However, such edges consist of sets of two different nodes; thus, $h(u) \neq h(v)$. This implies that $h$ is also a valid $k$-coloring of $G$. But this contradicts the fact that $\langle G \rangle = w \notin k-$ COLORING. So our supposition is false, and $f(w) \notin$ HOMOMORPHISM.
  - Otherwise, we have that $f(w) = \langle G_0, G_1 \rangle$. Since, by definition, no homomorphism exists from $G_0$ to $G_1$, we have that $f(w) \notin$ HOMOMORPHISM.

  So overall, in this case, $w \notin$ HOMOMORPHISM.

Altogether, this means that $k-$ COLORING $\leq_p$ HOMOMORPHISM.

## Part (b)

To do this, I show:

**Theorem 2.** CLIQUE $\leq_p$ HOMOMORPHISM.

*Proof.* Use $G_0$, $G_1$ and $K_k$ from the previous proof. Now, define a polynomially computable function $f : \Sigma^* \to \Sigma^*$ by:

$$f(w) := \begin{cases} \langle K_k, G \rangle & \text{, if } w = \langle G, k \rangle \\ \langle G_0, G_1 \rangle & \text{, otherwise} \end{cases}$$

I will now show that $w \in$ CLIQUE iff. $f(w) \in$ HOMOMORPHISM:

- If $w \in$ CLIQUE, then $w = \langle G \rangle$ from some graph $G = (V, E)$ that contains a clique of size $k$; suppose that the clique is formed from nodes $v_1, ..., v_k$. Now, in this case, consider the following function $f : 1, ..., k \to V$, defined by $f(x) = v_x$. I now want to show that this defines a homomorphism from $K_k$ to $G$: for any edge $\{i, j\}$ of $K_k$, with $i, j \in \{1, ..., k\}, i \neq j$, since $\{v_1, ..., v_k\}$ is a clique of $G$, we have that $\{v_i, v_j\} \in E$. So $f$ is indeed a homomorphism from $K_k$ to $G$; and thus, $f(w) = \langle K_k, G \rangle \in$ HOMOMORPHISM.
- If $w \notin$ CLIQUE, then there are two cases:
  - $w = \langle G \rangle$ for some graph $G$ with no clique of size $k$. In this case, I show that no homomorphism from $K_k$ to $G$ exists, by contradiction. Suppose $h : 1, ..., k \to V$, a homomorphism between $K_k$ and $G$, exists. Now note that, by the definition of $K_k$, for any $i, j \in 1, ..., k, i \neq j$, we have that $\{i, j\}$ is an edge of $K_k$. This implies, by the definition of homomorphism, that $\{f(i), f(j)\} \in E$. Since $E$ can only contain valid edges (i.e. sets of nodes of cardinality 2), this implies that all the values $f(1), ..., f(k)$ are distinct and have edges between them. But this means that $G$ has a clique of size $k$, a contradiction! Thus our supposition is false, and no homomorphism from $K_k$ to $G$ exists. So $f(w) = \langle K_k, G \rangle \notin$ HOMOMORPHISM.
  - $w \neq \langle G \rangle$ for any graph $G$. In this case, $f(w) = \langle G_0, G_1 \rangle$; and this in outside of HOMOMORPHISM by the definition of $G_0, G_1$.

  So overall, in this case, $f(w) \notin$ HOMOMORPHISM.

Since $f$ is polynomially computable, this implies that CLIQUE $\leq_p$ HOMOMORPHISM. Since CLIQUE is NP-hard, and HOMOMORPHISM is at least as hard as CLIQUE, this implies that HOMOMORPHISM is also NP-hard. $\square$

## Part (c)

This would imply that NP is closed under complement, viz:

**Theorem 3.** *If $A \in NP$, then $\overline{A} \in NP$.*

**Theorem 4.** *First, I assume* HOMOMORPHISM *is $NP$. Since I proved earlier that it is NP-hard, this implies that it is NP-complete. I also assume that* $\overline{\text{3-COLORING}}$ *is co-NP complete (the proof for this is identical to the proof that $3 - coloring$ is NP complete). Now, for any NP problem $A$ I show that $\overline{A} \leq_p A$, as follows: $\overline{A} \leq_p$* $\overline{\text{3-COLORING}}$ $\leq_p$ HOMOMORPHISM $\leq_p$ *$A$. So, as $A$ is NP, so is $\overline{A}$.*

This implies also that NP = co-NP.