# MATH 51 Homework #15

Tamir Enkhjargal

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# 5.2 - Strong Induction and Well-Ordering

## 4.

For all integers $n \geqslant 18$, $P(n)$ holds, where $P(n)$ is "a postage of $n$ cents can be formed using just 4-cent and 7-cent stamps."

a)
$$P(18) = 1 * 4 + 2 * 7 \tag{1}$$
$$P(19) = 3 * 4 + 1 * 7 \tag{2}$$
$$P(20) = 5 * 4 + 0 * 7 \tag{3}$$
$$P(21) = 0 * 4 + 3 * 7 \tag{4}$$

We see that the basic cases for $P(18)$ to $P(21)$ are true.

b) Assume that $P(j)$ is true where $18 \leqslant j \leqslant k$, and $k \geqslant 21$.

c) Proving the inductive hypothesis will let us prove for $P(k+1)$

d) For $P(k+1)$ starting from $k \geqslant 21$, we know $P(k-3) = 18,\ P(k-2) = 19,\ P(k-1) = 20,\ P(k) = 21$. We already proved that the cases are true for these numbers, and then $P(k+1)$ is just $P(k-3)$ with another 4-cent inclusion.

e) Therefore, every number $n \geqslant 18$ work for $P(n)$, as we add a 4-cent stamp from 18 to 21.

## 14.

Let $P(n)$ be the "sum of the products is $n(n-1)/2$ where $n \geqslant 2$.

Base case:

$$n(n-1)/2 = 2(2-1)/2 = 2/2 = 1 \tag{1}$$

When $n = 2$, we can only split the pile into two piles of 1 rocks each. Therefore $P(2)$ is true. Inductive step: We can assume $P(2),\ P(3), ...P(j)$ are all true, where $2 \leqslant j \leqslant k$. We need to now prove that $P(k+1)$ is true.

If we have a pile of $k+1$ stones, we can split into a pile of $j$ stones, and $k+1-j$ stones. We know that for a pile of $j$ stones, $P(j)$ is true, and since $k+1-j$ is within the range of $2 \leqslant j \leqslant q$, then we can state that $P(k+1-j)$ is also true.

$$\frac{k+1-j}{2} + \frac{j(j-1)}{2} + j(k+1-j) \tag{1}$$
$$\frac{k^2 - jk + k - jk + j^2 - j + j^2 - j + 2jk - 2j^2 + 2j}{2} \tag{2}$$
$$\frac{k^2+k}{2} = \frac{k(k+1)}{2} \tag{3}$$
$$\frac{(k+1)((k+1)-1)}{2} \tag{4}$$

The sum of products for $k+1$ stones is $\frac{(k+1)((k+1)-1)}{2} = \frac{(k+1)(k)}{2}$. From the inductive step, we found that $P(k+1)$ is also true.

## 5.3 - Recursion and Structural Induction

### 8.

a) $a_{n+1} = a_n + 4$ for $n \geqslant 1$ and $a_1 = 2$

b) $a_{n+1} = 2 - a_n$ for $n \geqslant 1$ and $a_1 = 0$

c) $a_{n+1} = a_n + 2n + 2$ for $n \geqslant 1$ and $a_1 = 2$

d) $a_{n+1} = a_n + 2n + 1$ for $n \geqslant 1$ and $a_1 = 1$

### 12.

$f_0 = 0,\ f_1 = 1,\ f_n = f_{n-1} + f_{n-2}$ for $n \geqslant 2$.

Base step: $f_1^2 = f_1 * f_2 \to 1 = 1 * 1$. The base case is true.

Inductive step: We can now assume that $P(k)$ is true. Therefore:

$$f_1^2 + f_2^2 + ... + f_k^2 = f_k * f_{k+1} \tag{1}$$
$$f_1^2 + f_2^2 + ... + f_k^2 + f_{k+1}^2 = f_k * f_{k+1} + f_{k+1}^2 \tag{2}$$
$$f_1^2 + f_2^2 + ... + f_k^2 + f_{k+1}^2 = f_k * f_{k+1} + f_{k+1} * f_{k+1} \tag{3}$$
$$f_1^2 + f_2^2 + ... + f_k^2 + f_{k+1}^2 = f_{k+1}[f_k + f_{k+1}] \tag{4}$$
$$f_1^2 + f_2^2 + ... + f_k^2 + f_{k+1}^2 = f_{k+1} * f_{k+2} \tag{5}$$

Therefore $P(k+1)$ is true.

### 24.

a) $1 \in S$ and if $x \in S$, then $x + 2 \in S$

b) $3 \in S$ and if $x \in S$, then $3x \in S$

c) $1 \in S$ and $p, q \in S$.

If $p = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n$ and $q = b_0 + b_1x + b_2x^2 + b_3x^3 + ... + b_nx^n$, where $a, b \in \mathbb{Z}$ and $p, q \in S$.

Then $p + q \in S$ is a polynomial, and $p - q \in S$ and $p * q \in S$ are also polynomials.