MATH 51 Homework #17

Tamir Enkhjargal

May 2019

1.

(a) Let $P(n)$ be "if G is connected and has n-1 edges, then G has no cycles"

Base case ($P(1)$): G has 1 vertices and has 0 edges. G also has no cycles.

Inductive step: Assume for some k that $P(k)$ is true. Therefore we need to prove that $P(k) \to P(k+1)$.

Let G be a simple graph with k vertices. Therefore, G is connected and has $k-1$ edges. There are also no cycles in G. If we add one more vertex, q, into G, then we will have $k+1$ vertices, and there will be a new edge formed from any vertex from k to q.

This holds that now G has $k+1$ vertices, is connected, and now has n edges. We can use the argument that since q only has one edge into G, then q has degree of 1. Because q has degree 1, then it holds that there are no cycles to and from q, if there are no cycles in G already.

Therefore, from $P(k)$, and adding in one more vertex, $P(k+1)$ still holds.

(b) Let $P(n)$ be that "if G has no cycles and has n-1 edges, then G has an unique simple path between any two vertices."

Base case ($P(1)$): G has one vertex, so there are no cycles, there are 0 edges, and the simple path is start/end at the vertex.

Inductive step: Assume for some k that $P(k)$ is true. Therefore we need to prove that $P(k) \to P(k+1)$.

Let G be a simple graph with k vertices. Therefore G has no cycles, and has $k-1$ edges. There are unique simple paths between any two vertices in G. If we add one more vertex, q, into G, then we will have $k+1$ vertices, and there will be a new edge formed from any vertex from G to q.

This holds that G already had no cycles and there were unique simple paths between any two vertices. Since we added in q, and there is only one edge from q to G, therefore q has a degree of 1. Because q has a degree of 1, there are no cycles to and from q in G, and q now only has simple paths to any point in G.

Therefore, from $P(k)$, and adding in one more vertex, $P(k+1)$ still holds.

2.

3.

4.

5.

Let $P(n)$ be "if G is a connected graph with n vertices, then G has at least $n-1$ edges".

Base cases: $P(1)$: 1 vertex, 0 edges, $P(2)$: 2 vertices, 1 edge.

We can start at $k = 2$ for the inductive step.

Inductive step: If $k = 2$, then adding in a new vertex, q, means that we will have k ***new edges*** to q. With that, since $k \geqslant k - 1$ it holds for $P(k = 2 + 1)$.

In general, when we add a new vertex, q, into G with k vertices, there will always be k ***new edges*** connecting to q, and as long as $k > 2$, this is true. Therefore when $P(k)$ is true, and we want $P(k + 1)$ this is also true.