MATH 51 Homework #18

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1.

The general formula can be described as $n(t) = 2n(t-1) * (n(-1) + n(0) + n(1) + ... + n(t-2)) + n(t-1)^2$.

We know $n(-1) = 1,\ n(0) = 1,\ n(1) = 3,\ n(2) = 21$

(a) $n(3) = 2n(2)*(n(-1)+n(0)+n(1))+n(2)^2 = 2*21*(1+1+3)+21^2 = 651$

(b) $n(4) = 2n(3) * (n(-1) + n(0) + n(1) + n(2)) + n(3)^2 = 2 * 651(1 + 1 + 3 + 21) + 651^2 = 457653$

2.

(a) $n(k+1) = 2 * n(k) * N(k-1) + n(k)^2$

(b) $n(k+1) = 2(N(k) - N(k-1)) * N(k-1) + (N(k) - N(k-1))^2 = N(k)^2 - N(k-1)^2$

(c) $N(k+1) = n(k+1) + N(k) \equiv N(k+1) = N(k)^2 - N(k-1)^2 + N(k)$

(d) $N(3) = N(2+1) = N(2)^2 - N(1)^2 + N(2) = 26^2 - 5^2 + 26 = 677$

$N(4) = N(3+1) = N(3)^2 - N(2)^2 + N(3) = 677^2 - 26^2 + 677 = 458330$

3.

We can assume:

$v(G_1 * G_2) = v(G_1) + v(G_2) + 1$

$\ell(G_1 * G_2) = \ell(G_1) + \ell(G_2)$

$e(G_1 * G_2) = e(G_1) + e(G_2) + 2$

(a) Base case: $G = \circ$

$n\ell(G) + 1 = 0 + 1 = 1$ is true, as there is only 1 leaf (root).

Inductive step: Assume that $G_1, G_2 \in FBRT$ with both $n\ell(G_1) + 1 = \ell(G_1)$ and $n\ell(G_2) + 1 = \ell(G_2)$ true.

Thus, we can state that the proof works for $n\ell(G_1 * G_2) + 1 = \ell(G_1 * G_2)$.

$$\begin{align}
n\ell(G_1 * G_2) + 1 &= \ell(G_1 * G_2) \tag{1}\\
n\ell(G_1 * G_2) &= \ell(G_1 * G_2) - 1 \tag{2}\\
&= \ell(G_1) + \ell(G_2) - 1 \tag{3}\\
&= (n\ell(G_1) + 1) + (n\ell(G_2) + 1) - 1 \tag{4}\\
\therefore n\ell(G_1 * G_2) &= n\ell(G_1) + n\ell(G_2) + 1 \tag{5}
\end{align}$$

Therefore, from this, we can see that any new element in G can be built from subsequent graphs already existing in G. The relationship between $n\ell(G)$ and $v(G)$ is that $v(G) = n\ell(G) + \ell(G)$.

(b) This does not work for EBRTs because FBRT base case starts at a single vertex. We can just check the induction base case with EBRT:

$n\ell(G) + 1 = \ell(G) \to n\ell(\varnothing) + 1 = \ell(\varnothing) \to 1 \neq 0.$

The base case does not hold for EBRTs.

4.

(a) $e(G_1 * G_2) = e(G_1) + e(G_2) + 2$

(b) Base case: $G = \circ$

$v(\circ) = e(\circ) + 1 \to 1 = 0 + 1 \to 1 = 1$. Base case is true.

Inductive step: Assume that $G_1, G_2 \in FBRT$ with both $v(G_1) = e(G_1)+1$ and $v(G_2) = e(G_2) + 1$ true.

Thus, we can state that the proof works for $v(G_1 * G_2) = e(G_1 * G_2) + 1$.

$$\begin{aligned} v(G) &= e(G) + 1 && (1) \\ v(G_1 * G_2) &= v(G_1) + v(G_2) + 1 && (2) \\ &= [e(G_1) + 1] + [e(G_2) + 1] + 1 && (3) \\ &= e(G_1) + e(G_2) + 3 && (4) \\ v(G_1 * G_2) = e(G_1 * G_2) + 1 &= [e(G_1) + e(G_2) + 2] + 1 && (5) \end{aligned}$$

From our inductive step, we found the formula we stated in part (a) for $e(G_1 * G_2) = e(G_1) + e(G_2) + 2$, which we have assumed to be true. From this, we have also proved that the relation $v(G) = e(G) + 1$ is true for all cases.