MATH 51 Homework #10

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1. Determine if the following functions are surjective (onto). If a function is surjective, prove it. If a function is not surjective, explicitly demonstrate an element which is not in the range of the function.

   (a) The function $f : \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ where $f(x) = x^2$ is surjective.
   If $\forall y \in \mathbb{R}_{\geqslant 0}, \exists x \in \mathbb{R}$.

$$\begin{align} f(x) &= y \tag{1} \\ x^2 &= y \tag{2} \\ x &= \sqrt{y} \tag{3} \end{align}$$

   Therefore $f(\sqrt{y}) = x$ holds that $f(\sqrt{y}) = y$ at the end, and the domain and codomain properly hold.

   (b) The function $g : \mathbb{Q} \to \mathbb{Q}_{\geqslant 0}$ where $g(x) = x^2$ is *not* surjective.
   If $f(x) = 2$, then:

$$\begin{align} f(x) &= 2 \tag{1} \\ x^2 &= 2 \tag{2} \\ x &= \sqrt{2} \tag{3} \end{align}$$

   The number $\sqrt{2}$ is not a rational number, so the image 2 does not get mapped.

   (c) The function $h : \mathbb{R} - \{1\} \to \mathbb{R}$ where $h(x) = \frac{x+1}{x-1}$ is *not* surjective.
   If $\forall y \in R, \exists x \in \mathbb{R} - \{1\}$.

$$\begin{align} f(x) &= y \tag{1} \\ \frac{x+1}{x-1} &= y & x \neq 1 \tag{2} \\ \frac{y+1}{y-1} &= x & y \neq 1 \tag{3} \end{align}$$

   Here, we see that there is a requirement now that $y \neq 1$, which changes our codomain.

   (d) From part (c) above, we saw that the image also can not be equal to 1, then this should hold.
   $\forall y \in \mathbb{R} - \{1\}, \exists x \in \mathbb{R} - \{1\}$.

$$\begin{align} f(x) &= y \tag{1} \\ \frac{x+1}{x-1} &= y & x \neq 1 \tag{2} \\ \frac{y+1}{y-1} &= x & y \neq 1 \tag{3} \end{align}$$

   Here, $k(\frac{y+1}{y-1}) = y$ holds, because we had previously stated that both elements in x, y can not be 1.

2. Let $f : A \to B$ and $g : B \to C$ be functions.

   (a) $\forall b \in B, \exists a \in A, f(a) = b$ and $\forall c \in C, \exists b \in B, f(b) = c$.

$$
\begin{aligned}
f(a) = b \text{ is onto and} \quad & (1) \\
g(b) = c \text{ is also onto} \quad & (2) \\
g \circ f = g(f(a)) = c \text{ is therefore onto} \quad & (3)
\end{aligned}
$$

   $g \circ f$ is onto where $\forall c \in C, \exists a \in A \to g \circ f = g(f(a)) = c$

   (b) If $A = \{3\}, B = \{2, 3\}, C = \{3\}$, then $f(3) = 3$ and $g(2) = 2$ and $g(3) = 2$. And by definition $g \circ f : A \to C, g(f(2)) = 2$. Here, we see that $g(f(2))$ is injective (one-to-one), but g is not injective because both elements $\{2, 3\}$ point to the same image 2.

3. Let $f : A \to B$ be a function.

   (a) If $f : S \to T$ where $f(x) = x^2 - 3$.

$$
\begin{aligned}
S &= \{-2, -1\} && (1) \\
T &= \{-2, 1\} && (2) \\
S \cap T &= \{-2\} && (3) \\
f(S \cap T) &= \{1\} && (4) \\
f(S) &= \{-2, 1\} && (5) \\
f(T) &= \{-2, 1\} && (6) \\
f(S) \cap f(T) &= \{-2, 1\} && (7) \\
\{1\} &\neq \{-2, 1\} && (8)
\end{aligned}
$$

   (b) If f is injective then $(\forall S \in A) \wedge (\forall T \in A)$, $f(S) = f(T) \to S = T$

$$
\begin{aligned}
S &= T && (1) \\
\therefore S \cap T &= T = S && (2) \\
f(S) &= f(T) && (3) \\
\therefore f(S) \cap f(T) &= f(T) && (4) \\
\therefore f(S \cap T) &= f(T) && (5) \\
f(S \cap T) &= f(S) \cap f(T) && (6)
\end{aligned}
$$