MATH 51 Homework #14

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May 2019

Section 5.1 - Mathematical Induction

6.

For all positive integers n, $P(n)$ is true, where $P(n)$ is $1*1!+2*2!+...+n*n! = (n+1)!-1$.
Base case: $P(1)$

$$
\begin{aligned}
1*1! &= (1+1)!-1 && (1)\\
1 &= 2-1 && (2)\\
1 &= 1 && (3)
\end{aligned}
$$

We have proved the base case. We can now assume $P(k)$ is true.
Inductive step:

$$
\begin{aligned}
1*1!+2*2!+...+k*k! &= (k+1)!-1 && (1)\\
1*1!+...+k*k!+(k+1)*(k+1)! &= (k+1)!-1+(k+1)*(k+1)! && (2)\\
&= (k+1)![1+k+1]-1 && (3)\\
&= (k+1)!(k+2)-1 && (4)\\
&= (k+2)!-1 && (5)
\end{aligned}
$$

We can see from the inductive step $P(k)$ that we reached the conclusion that $P(k+1)$ also holds true.

14.

For all positive integers n, $P(n)$ true, where $P(n) = \sum_{k=1}^{n} k2^k = (n-1)2^{n+1}+2$.
Base case: $P(1)$

$$
\begin{aligned}
1*2^1 &= (1-1)2^{1+1}+2 && (1)\\
2 &= 0*2^2+2 && (2)\\
2 &= 2 && (3)
\end{aligned}
$$

We have proved the base case. We can now assume $P(k)$ is true.
Inductive step:

$$
\begin{aligned}
1*2^1+2*2^2+...+k*2^k &= (k-1)2^{k+1}+2 && (1)\\
1*2^1+...+k*2^k+(k+1)2^{k+1} &= (k-1)2^{k+1}+2+(k+1)2^{k+1} && (2)\\
&= (k-1+k+1)2^{k+1}+2 && (3)\\
&= (2k)2^{k+1}+2 && (4)\\
&= k*2^{k+2}+2 && (5)\\
&= ((k+1)-1)2^{(k+1)+1} && (6)
\end{aligned}
$$

We can see that $P(k+1)$ holds true from the inductive step $P(k)$.

20.

For all integers $n > 6$, $P(n)$ is true, where $P(n) = 3^n < n!$.
Base case: $P(7)$

$$3^7 < 7! \tag{1}$$
$$2187 < 5040 \tag{2}$$

Inductive step:

$$3^k < k! \tag{1}$$
$$3^k * 3 < k! * 3 \tag{2}$$
$$3^{k+1} < k! * 3 < (k+1)! \qquad \text{When k} > 6 \tag{3}$$
$$3^{k+1} < (k+1)! \tag{4}$$

We can kind of cheat a little bit, as the statement $3^k * 3 < k! * 3$ is true. When $k > 6$, then $(k+1)! = (k+1)(k)(k-1)$ and $(k+1) > 3$.

36.

For all positive integers n, $P(n)$ is true, where $P(n) = 21 \mid (4^{n+1} + 5^{2n-1})$.
Base case: $P(1)$

$$21 \mid (4^{1+1} + 5^{2-1}) \tag{1}$$
$$21 \mid (16 + 5) \tag{2}$$
$$21 \mid 21 \tag{3}$$

Inductive step:

$$0 \equiv 4^{k+1} + 5^{2k-1} \bmod 21 \tag{1}$$
$$0 \equiv 4^{k+2} + 5^{2k+1} - 4^{k+1} - 5^{2k-1} \bmod 21 \tag{2}$$
$$0 \equiv 4 * 4^{k+1} + 25 * 5^{2k-1} - 4^{k+1} - 5^{2k-1} \bmod 21 \tag{3}$$
$$0 \equiv 3 * 4^{k+1} + 24 * 5^{2k-1} \bmod 21 \tag{4}$$

We can see that all of these equivalencies hold true, even at the last step. If we subtracted $4^{k+1}+5^{2k-1}$ from the equation 3 more times, then we will be left with $4^{k+1}+21*5^{2k-1}$ mod 21. This is again proved true, as $21*5^{2k-1} \equiv 5^{2k-1}$ mod 21.

50.

For all positive integers n, $P(n)$ is true, where $P(n) = \sum_{i=1}^{n} i = (n + \frac{1}{2})^2/2$
Base case: $P(1)$

$$1 = (1 + \frac{1}{2})^2/2 \tag{1}$$
$$1 = \frac{3}{2}^2/2 \tag{2}$$
$$1 = \frac{9}{4}/2 \tag{3}$$
$$1 \neq \frac{9}{8} \tag{4}$$

The base case fails at $n = 1$

54.

For all positive integers n, $P(n)$ is true, where $P(n) =$"A set of $n + 1$ positive integers (none exceeding $2n$) contains at least one integer in this set that divides another integer in the set."
Base case: $P(1)$

Let S be a set containing $1 + 1$ elements, and no element exceeding $2(1)$.
Then S contains $\{1, 1\}$, $\{1, 2\}$, $\{2, 1\}$, or $\{2, 2\}$.
We can see that, $1 \mid 1, 2$ and $2 \mid 2$

Therefore, the base case holds.
Inductive step:

Let S be a set containing $k + 1$ elements, and no element exceeding $2k$.

Assume $P(k)$ is true for the set S.

Let T also be the set S with an extra element.

Therefore T is a set with $k + 2$ elements, and no element exceeding $2(k + 1)$

We see that if T does not contain $2k + 1$ or $2k + 2$, then $P(k + 1)$ is true.

If T contains $2k + 1$, then from S we know that we can get a product of elements equal to $2k + 1$ from prime factorization.

If T contains $2k+2$, then from S we know that there is a number that divides $(2k + 2)/2$ as $(2k + 2)/2$ is now within the domain of S. Therefore there is also a number that divides $2k + 2$.