MATH 51 Homework #8

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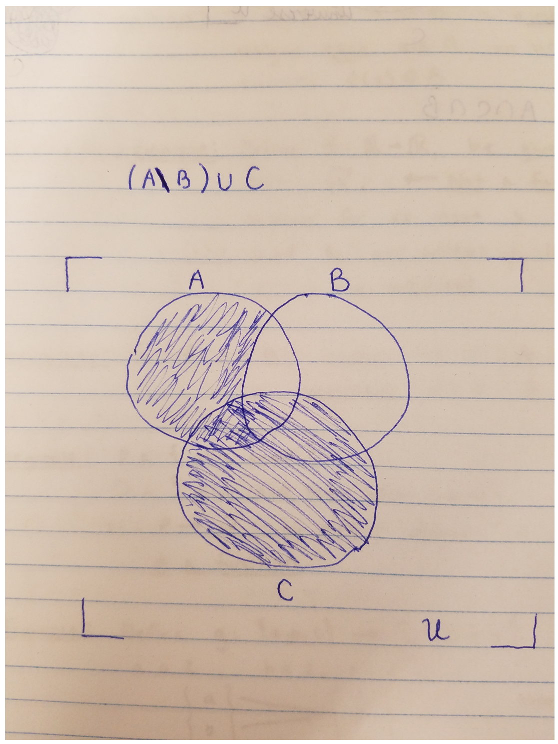
1. Let S and T be subsets of some universal set, U. Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

$$\text{Let } x \in \overline{A \cap B} \tag{1}$$
$$\neg[x \in A \cap B] \tag{2}$$
$$\neg[(x \in A) \wedge (x \in B)] \tag{3}$$
$$(x \notin A) \vee (x \notin B) \tag{4}$$
$$(x \in \overline{A}) \vee (x \in \overline{B}) \tag{5}$$
$$x \in \overline{A} \cup \overline{B} \tag{6}$$

2. (a) Using a Venn diagram, formulate another way to write $(A - B) \cup C$



(b) Prove that your formula is correct

$$= (A - B) \cup C \tag{1}$$
$$= (A \cap \overline{B}) \cup C \tag{2}$$
$$= (A \cup C) \cap (\overline{B} \cup C) \tag{3}$$

3. (a) Prove that $(A \times C) \cap (B \times D) = (A \cap B) \times (C \cap D)$

$$
\begin{align}
&= (A \times C) \cap (B \times D) \tag{1}\\
&= ((x, y) \in A \times C) \wedge ((x, y) \in B \times D) \tag{2}\\
&= (x \in A \wedge y \in C) \wedge (x \in B \wedge y \in D) \tag{3}\\
&= (x \in A \wedge x \in B) \wedge (y \in C \wedge y \in D) \tag{4}\\
&= (x \in A \cap B) \wedge (y \in C \cap D) \tag{5}\\
&= (x, y) \in (A \cap B) \times (C \cap D) \tag{6}
\end{align}
$$

(b) Prove that $(A \times C) \cup (B \times D) \subseteq (A \cup B) \times (C \cup D)$. Give an example of specific A, B, C, and D where equality does not hold.

The proof will be very similar as the proof above.

$$
\begin{align}
&= (A \times C) \cup (B \times D) \tag{1}\\
&= ((x, y) \in A \times C) \vee ((x, y) \in B \times D) \tag{2}\\
&= (x \in A \vee x \in B) \wedge (y \in C \vee y \in D) \tag{3}\\
&= (x \in A \cup B) \wedge (y \in C \cup D) \tag{4}\\
&= (x, y) \in (A \cup B) \times (C \cup D) \tag{5}
\end{align}
$$

The easiest counterexample for equality is have A and B be equal sets, and C and D to be equal sets.

$$
\begin{align}
&= (A \times C) \cup (B \times D) \tag{1}\\
&= \{(1, 2)\} \cup \{(1, 2)\} \tag{2}\\
&= \{(1, 2)\} \tag{3}
\end{align}
$$

$$
\begin{align}
&= (A \cup B) \times (C \cup D) \tag{4}\\
&= \{1, 2\} \times \{1, 2\} \tag{5}\\
&= \{(1, 1), (1, 2), (2, 1), (2, 2)\} \tag{6}
\end{align}
$$

We see from the two results, that they are not the same, but rather $(1, 2)$ is an element in the larger set.

4. (a) For any sets A, B, and C, prove that $A \subseteq B \cap C$ if and only if $A \subseteq B$ and $A \subseteq C$

$$--Case\ 1 \rightarrow\ -- \tag{1}$$
$$A \subseteq B \cap C \tag{2}$$
$$x \in A \rightarrow x \in B \cap C \tag{3}$$
$$x \in A \wedge x \in B \wedge x \in C \tag{4}$$
$$A \subseteq B \wedge A \subseteq C \tag{5}$$
$$--Case\ 2 \leftarrow\ -- \tag{6}$$
$$A \subseteq B \wedge A \subseteq C \tag{7}$$
$$x \in A \rightarrow x \in B \wedge x \in C \tag{8}$$
$$x \in B \cap C \tag{9}$$
$$A \subseteq B \cap C \tag{10}$$

(b) Recall that for a set $S, P(S)$ is its power set. Given two sets S and T, prove that $S \subseteq T$ if and only if $P(S) \subseteq P(T)$

$$--Case\ 1 \rightarrow\ -- \tag{1}$$
$$S \subseteq T \tag{2}$$
$$x \in S \subseteq T \tag{3}$$
$$x \in P(S) \wedge x \in P(T) \tag{4}$$
$$P(S) \subseteq P(T) \tag{5}$$
$$--Case\ 2 \leftarrow\ -- \tag{6}$$
$$P(S) \subseteq P(T) \tag{7}$$
$$S \in P(S) \wedge S \in P(T) \tag{8}$$
$$x \in S \rightarrow x \in P(S) \wedge x \in P(T) \tag{9}$$
$$x \in S \wedge x \in T \tag{10}$$
$$S \subseteq T \tag{11}$$

(c) Prove that $P(S \cap T) = P(S) \cap P(T)$ for any sets S and T

$$x \in P(S \cap T) \tag{1}$$
$$x \subseteq S \cap T \tag{2}$$
$$x \subseteq S \wedge x \subseteq T \tag{3}$$
$$x \in P(S) \wedge x \in P(T) \tag{4}$$
$$x \in P(S) \cap P(T) \tag{5}$$

(d) Give an example of sets S and T where $P(S \cup T) \neq P(S) \cup P(T)$

$S = \{1, 2\}$ and $T = \{2, 3\}$, therefore $S \cup T = \{1, 2, 3\}$.

$P(S \cup T) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$

$P(S) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\} \cup P(T) = \{\varnothing, \{2\}, \{3\}, \{2, 3\}\} = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}\}$

We're missing the set-element $\{1, 2, 3\}$ from the second part.