Solutions: Compound Interest

 # 1: Annual Compounding:

$$\begin{aligned}
(1+0.04)^7 \times 15,000 &= \$19,739 \\
\Longrightarrow \text{ total interest} &= \$19,739 - \$15,000 \\
&= \$4739 \\
\text{interest on principal} &= 7 \times 0.04 \times \$15,000 \\
&= \$4200 \\
\text{interest on interest} &= \$4739 - \$4200 \\
&= \$539
\end{aligned}$$

Semiannual Compounding:

$$\begin{aligned}
\left(1+\frac{0.04}{2}\right)^{2\times 7} \times 15,000 &= \$19,792 \\
\Longrightarrow \text{ total interest} &= \$19,792 - \$15,000 \\
&= \$4792 \\
\text{interest on principal} &= 14 \times \frac{0.04}{2} \times \$15,000 \\
&= \$4200 \\
\text{interest on interest} &= \$4792 - \$4200 \\
&= \$592
\end{aligned}$$

Quarterly Compounding:

$$\begin{aligned}
\left(1+\frac{0.04}{4}\right)^{4\times 7} \times 15,000 &= \$19,819 \\
\Longrightarrow \text{ total interest} &= \$19,819 - \$15,000 \\
&= \$4819 \\
\text{interest on principal} &= 28 \times \frac{0.04}{4} \times \$15,000 \\
&= \$4200 \\
\text{interest on interest} &= \$4819 - \$4200 \\
&= \$619
\end{aligned}$$

Monthly Compounding:

$$\begin{aligned}
\left(1+\frac{0.04}{12}\right)^{12\times 7}\times 15,000 &= \$19,838 \\
\Longrightarrow \text{ total interest} &= \$19,838 - \$15,000 \\
&= \$4838 \\
\text{interest on principal} &= 84\times\frac{0.04}{12}\times \$15,000 \\
&= \$4200 \\
\text{interest on interest} &= \$4838 - \$4200 \\
&= \$638
\end{aligned}$$

 # 2: Let K be the unknown doubling time. Then we require

$$V(K) = (1+0.04)^K \times 10,000 \geq 20,000$$

We take logarithms:

$$\begin{aligned}
K\log(1.04) &\geq \log\left(\frac{20,000}{10,000}\right) \\
&= \log(2) \\
\Longrightarrow K &\geq \frac{\log(2)}{\log(1.04)} \\
&= 17.67
\end{aligned}$$

The doubling time is 18 years.

If r is the semiannually compounded rate which doubles an investment in 8 years, then \$100 invested at this rate will grow to \$2000 in 8 years, so

$$\begin{aligned}
\left(1+\frac{r}{2}\right)^{2\times 8}\times 100 &= 200 \\
\Longrightarrow 1+\frac{r}{2} &= [2]^{\frac{1}{16}} \\
\Longrightarrow r &= 2([2]^{\frac{1}{16}}-1) \\
&= 0.08854 \\
&= 8.85\%
\end{aligned}$$

 # 3: Let $V(K)$ be the value of an investment at either of the proposed interest rates at time t. Suppose the initial investment is $V(0) = \$1000$. Then we calculate $V(1)$ under the 2 proposed interest rates:

Annual compounding:

$$\begin{aligned} V(1) &= (1+0.06)\times \$1000 \\ &= \$1060 \end{aligned}$$

Monthly compounding:

$$\begin{aligned} V(1) &= \left(1+\frac{0.055}{12}\right)^{12} \times \$1000 \\ &= \$1056 \end{aligned}$$

So, we should choose the annually compounded rate.

 # 4: (a) Let r be the annually compounded interest rate. Then

$$\begin{aligned} (1+r)^4(5000) &= 6800 \\ \Longrightarrow (1+r)^4 &= \frac{6800}{5000} \\ \Longrightarrow 1+r &= \left[\frac{6800}{5000}\right]^{\frac{1}{4}} \\ \Longrightarrow r &= \left[\frac{6800}{5000}\right]^{\frac{1}{4}} - 1 \\ &= 0.079902 \\ &= 7.99\% \end{aligned}$$

(which is basically 8%).

(b) Let r be the semiannually compounded interest rate. Then r satisfies

$$\begin{aligned} \left(1+\frac{r}{2}\right)(10,000) &= 10,800 \\ \Longrightarrow 1+\frac{r}{2} &= \frac{10,800)}{10,000} \\ &= 1.08 \\ \Longrightarrow r &= 2(1.08-1) \\ &= 0.16 \\ &= 16\% \end{aligned}$$

 # 5: (a) (i) We will prove that an investment at an annually compounded interest rate r has value

$$V(K) = (1+r)^K V(0)$$

where $V(J)$ is the value of the investment after J years.
Base step (K=1): This is the case

$$V(1) = (1+r)V(0)$$

We established this in lecture, but will repeat the argument here. After 1 year the value of the investment is

$$\begin{aligned} V(1) &= \text{principal} + \text{interest} \\ &= V(0) + rV(0) \\ &= (1+r)V(0) \end{aligned}$$

Induction step: We assume we have established the identity for K (in fact, for all integers from 1 to K), and from that, we show that it holds for $K+1$. That means, we assume we have already proven that

$$V(K) = (1+r)^K V(0)$$

and from that we will deduce that

$$V(K+1) = (1+r)^{K+1} V(0)$$

We note that, under the rules of annual compounding, the effective principal after K years is $V(K)$ and this is the principal we use to calculate the interest charge for the next year. Therefore

$$\begin{aligned} V(K+1) &= K\text{-year principal} + \text{interest} \\ &= V(K) + rV(K) \\ &= (1+r)^K V(0) + r(1+r)^K V(0) \\ &= (1+r) \times (1+r)^K V(0) \\ &= (1+r)^{K+1} V(0) \end{aligned}$$

Note that in the second step we have used the induction hypothesis, that the identity has been established for K years.
(ii) We work through the proposed alternative argument. Whatever $V(K)$ might be, we may still apply the rules of annual compounding. By these rules, $V(K)$ is the principal used after K years for calculating the next year's interest charged, and so it follows that

$$\begin{aligned} V(K+1) &= \text{principal} + \text{interest} \\ &= V(K) + rV(K) \\ &= (1+r)V(K) \end{aligned}$$

This argument is valid for any $K \geq 1$. We work through the consequences for a few low values of K:

$$\begin{aligned} V(1) &= (1+r)V(0) \\ V(2) &= (1+r)V(1) \\ &= (1+r)(1+r)V(0) \\ &= (1+r)^2V(0) \\ V(3) &= (1+r)V(2) \\ &= (1+r)(1+r)^2V(0) \\ &= (1+r)^3V(0) \end{aligned}$$

It is not hard to convince oneself that this can be continued indefinitely to establish

$$V(K) = (1+r)^K V(0)$$

for any K.

(b) We will carry out a similar induction argument for semiannually compounded rates as for annual compounding. Let r now be a semiannually compounded interest rate. As we have noted, we now work with half year periods rather than whole years. We thus need to change notation somewhat from how the identity is stated in the problem statement to set up the induction properly. The stated identity is

$$V(T) = \left(1 + \frac{r}{2}\right)^{2T} V(0)$$

Since we are working with half-year periods, T will always be of the form

$$T = \frac{j}{2}$$

for an integer j. Making this substitution into the identity we have

$$V\left(\frac{j}{2}\right) = \left(1 + \frac{r}{2}\right)^{j} V(0)$$

We will now prove this by induction on j.

Base step ($j = 1$): We must show

$$V\left(\frac{1}{2}\right) = \left(1 + \frac{r}{2}\right) V(0)$$

This was also established in lecture, but we repeat the argument again. Under the rules of semiannual compounding, an interest rate of $\frac{r}{2}$ is levied on the initial principal for the first interest charge after the first half year period, so that

$$\begin{aligned} V\left(\frac{1}{2}\right) &= \text{principal} + \text{interest} \\ &= V(0) + \frac{r}{2}V(0) \\ &= \left(1+\frac{r}{2}\right)V(0) \end{aligned}$$

Induction step: We assume we have established the identity for any integer up to j and from this we will prove it for $j+1$. In other words, we assume

$$V\left(\frac{j}{2}\right) = \left(1+\frac{r}{2}\right)^j V(0)$$

and from this we deduce

$$V\left(\frac{j+1}{2}\right) = \left(1+\frac{r}{2}\right)^{j+1} V(0)$$

After j half year periods, the effective principal for the next interest calculation is $V(\frac{j}{2})$, so under the rules of semiannual compounding

$$\begin{aligned} V\left(\frac{j+1}{2}\right) &= \text{principal} + \text{interest} \\ &= V\left(\frac{j}{2}\right) + \frac{r}{2}V\left(\frac{j}{2}\right) \\ &= \left(1+\frac{r}{2}\right)^j V(0) + \frac{r}{2}\left(1+\frac{r}{2}\right)^j V(0) \\ &= \left(1+\frac{r}{2}\right) \times \left(1+\frac{r}{2}\right)^j V(0) \\ &= \left(1+\frac{r}{2}\right)^{j+1} V(0) \end{aligned}$$

The alternative argument is adapted similarly. Again, it is a matter of applying the semiannual compounding rules to the principal $V(\frac{j}{2})$ as of j half year periods into the investment so that

$$\begin{aligned} V\left(\frac{j+1}{2}\right) &= \text{principal} + \text{interest} \\ &= V\left(\frac{j}{2}\right) + \frac{r}{2}V\left(\frac{j}{2}\right) \\ &= \left(1+\frac{r}{2}\right)V\left(\frac{j}{2}\right) \end{aligned}$$

and applying this to the cases $V(\frac{1}{2})$, $V(0)$, $V(\frac{3}{2})$, etc.

(c) We can easily adapt the arguments from parts (a) and (b) to a general compounding frequency k, using the appropriate compounding rules. We work with periods of length $\frac{1}{k}$ (in years) and the identity that must be established

$$V(T) = \left(1 + \frac{r}{k}\right)^{kT} V(0)$$

is put into a form that induction can be applied to by the substitution

$$T = \frac{j}{k}$$

The rules for frequency k compounding are that after j periods of length $\frac{1}{k}$ the current investment value of $V(\frac{j}{k})$ is used for the next interest charge, and the interest rate of $\frac{r}{k}$ is levied on this principal, so that the updated principal for the next period is

$$\begin{aligned} V\left(\frac{j+1}{k}\right) &= \text{principal} + \text{interest} \\ &= V\left(\frac{j}{k}\right) + \frac{r}{k} V\left(\frac{j}{k}\right) \\ &= \left(1 + \frac{r}{k}\right) V\left(\frac{j}{k}\right) \end{aligned}$$

This is the crucial step used in either of the arguments to establish the general identity.

 # 6: (a) We proceed with the outlined argument. The value of an investment at time T under either interest rate is

$$\begin{aligned} V(T) &= \left(1 + \frac{r_k}{k}\right)^{kT} V(0) \\ &= \left(1 + \frac{r_\ell}{\ell}\right)^{\ell T} V(0) \end{aligned}$$

We equate the values $V(1)$ under either interest rate

$$\begin{aligned}
\left(1+\frac{r_k}{k}\right)^k &= \left(1+\frac{r_\ell}{\ell}\right)^\ell \\
\Longrightarrow 1+\frac{r_\ell}{\ell} &= \left(1+\frac{r_k}{k}\right)^{\frac{k}{\ell}} \\
\Longrightarrow \frac{r_\ell}{\ell} &= \left(1+\frac{r_k}{k}\right)^{\frac{k}{\ell}} - 1 \\
\Longrightarrow r_\ell &= \ell\left(1+\frac{r_k}{k}\right)^{\frac{k}{\ell}} - \ell
\end{aligned}$$

Now from the condition

$$\left(1+\frac{r_k}{k}\right)^k = \left(1+\frac{r_\ell}{\ell}\right)^\ell$$

equivalent to the relation between the rates, we can argue, for any valid time T

$$\begin{aligned}
V(T) &= \left(1+\frac{r_k}{k}\right)^{kT} V(0) \\
&= \left[\left(1+\frac{r_k}{k}\right)^k\right]^T V(0) \\
&= \left[\left(1+\frac{r_\ell}{\ell}\right)^\ell\right]^T V(0) \\
&= \left(1+\frac{r_\ell}{\ell}\right)^{\ell T} V(0)
\end{aligned}$$

which is the value of the investment at time T under the interest rate r_ℓ with frequency ℓ compounding.

(b) For each rate, we compute the equivalent annually compounded rate and compare it with the rate we are currently earning. For the semiannually compounded offer we have

$$\begin{aligned}
r_1 &= \left(1+\frac{r_2}{2}\right)^2 - 1 \\
&= \left(1+\frac{0.059}{2}\right)^2 - 1 \\
&= 0.05987 \\
&= 5.99\%
\end{aligned}$$

For the quarterly compounded offer:

$$
\begin{aligned}
r_1 &= \left(1+\frac{r_4}{4}\right)^2 - 1 \\
&= \left(1+\frac{0.0585}{4}\right)^4 - 1 \\
&= 0.05979 \\
&= 5.98\%
\end{aligned}
$$

For the monthly compounded offer:

$$
\begin{aligned}
r_1 &= \left(1+\frac{r_{12}}{12}\right)^{12} - 1 \\
&= \left(1+\frac{0.0584}{12}\right)^{12} - 1 \\
&= 0.05998881 \\
&= 6.0\%
\end{aligned}
$$

For the daily compounded offer:

$$
\begin{aligned}
r_1 &= \left(1+\frac{r_{365}}{365}\right)^{365} - 1 \\
&= \left(1+\frac{0.0583}{365}\right)^{365} - 1 \\
&= 0.060026 \\
&= 6.0\%
\end{aligned}
$$

Only the daily compounded rate implies an annually compounded rate that is higher than what is being earned now, so you should switch to the daily compounded offer but no other. Notice that the quarterly compounded offer rounds to 6%, but when sufficient accuracy is kept, it is less.

 # 7: (a) This follows from the rate conversion formula from the previous problem using $\ell = 1$ and noting that, as defined in the problem

$$\text{EAR} = r_1$$

and

$$\text{APR} = r_k$$

(b)

$$
\begin{aligned}
\text{effective quarterly rate} &= \frac{4\%}{4} = 1\% \\
\text{APR} &= 4\% \\
\text{EAR} &= \left(1 + \frac{\text{APR}}{4}\right)^4 - 1 \\
&= \left(1 + \frac{0.04}{4}\right)^4 - 1 \\
&= 0.04060 \\
&= 4.06\%
\end{aligned}
$$

(c)

$$
\begin{aligned}
\text{effective 6 month rate} &= \frac{17,000}{10,000} - 1 \\
&= 0.7 \\
&= 70\% \\
\text{APR} &= 2 \times 70\% \\
&= 140\% \\
\text{EAR} &= \left(1 + \frac{\text{APR}}{2}\right)^2 - 1 \\
&= \left(1 + \frac{1.4}{2}\right)^2 - 1 \\
&= 1.89 \\
&= 189\%
\end{aligned}
$$

(d) For monthly compounding we have

$$
\begin{aligned}
1 + \text{EAR} &= \left(1 + \frac{\text{APR}}{12}\right)^{12} \\
\implies \text{APR} &= 12(1 + \text{EAR})^{\frac{1}{12}} - 12
\end{aligned}
$$

So with the given EAR of 20%=0.2, we have

$$
\begin{aligned}
\text{APR} &= 12(1 + 0.2)^{\frac{1}{12}} - 12 \\
&= 0.18371 \\
&= 18.37\%
\end{aligned}
$$