Solutions: Investment Returns

 # 1: The value of the investment at the investment horizon is

$$\begin{aligned} V(10) &= (1+\frac{0.07}{2})^{2(10)}(50,000) \\ &= \$98,489 \end{aligned}$$

Now we calculate the return measures. The P&L is

$$\begin{aligned} P\&L &= \$98,489 - \$50,000 \\ &= \$48,489 \end{aligned}$$

The gross return:

$$\begin{aligned} \text{gross return} &= \frac{98,489}{50,000} \\ &= 1.99 \end{aligned}$$

and the return:

$$\begin{aligned} \text{return} &= \text{gross return} - 1 \\ &= 0.99 \\ &= 99\% \end{aligned}$$

For the annualized return, you might want to try to guess what it is before you calculate it. Here is the calculation:

$$\begin{aligned} \text{annualized return} &= \left(\frac{98,489}{50,000}\right)^{\frac{1}{10}} - 1 \\ &= 0.071182 \\ &= 7.12\% \end{aligned}$$

2: We have $V(0)$ =\$40,000 and $V(3)$ =\$105,000. We thus calculate

$$
\begin{aligned}
P\&L &= V(3) - V(0) \\
&= \$105,000 - \$40,000 \\
&= \$65,000 \\
\text{gross return} &= \frac{V(3)}{V(0)} \\
&= \frac{105,000}{40,000} \\
&= 2.625 \\
\text{return} &= \text{gross return} - 1 \\
&= 1.625 \\
&= 162.5\% \\
\text{annualized return} &= \left[\frac{V(3)}{V(0}\right]^{\frac{1}{3}} - 1 \\
&= [2.625]^{1/3} - 1 \\
&= 0.37946 \\
&= 3.79\%
\end{aligned}
$$

3: (a) (i) We denote by ρ the annualized return over N years. We will denote by ρ_j for $j = 1, \ldots N$ the annual return over the jth year, so that

$$
1 + \rho_j = \frac{V(j)}{V(j-1)}
$$

We have

$$
\begin{aligned}
(1+\rho)^N &= \frac{V(N)}{V(0)} \\
&= \frac{V(1)}{V(0)} \cdot \frac{V(2)}{V(1)} \cdot \cdots \frac{V(N)}{V(N-1)}
\end{aligned}
$$

where we have multiplied and divided by $V(1)$, $V(2)$, up to $V(N-1)$. We now take logarithms:

$$
\begin{aligned}
N\log(1+\rho) &= \log\Big(\frac{V(1)}{V(0)}\cdot\frac{V(2)}{V(1)}\cdot\dots\cdot\frac{V(N)}{V(N-1)}\Big) \\
&= \log\Big(\frac{V(1)}{V(0)}\Big)+\log\Big(\frac{V(2)}{V(1)}\Big)+\dots+\log\Big(\frac{V(N)}{V(N-1)}\Big) \\
&= \log(1+\rho_1)+\log(1+\rho_2)+\dots+\log(1+\rho_N) \\
\Longrightarrow \log(1+\rho) &= \frac{1}{N}\Big[\log(1+\rho_1)+\log(1+\rho_2)+\dots+\log(1+\rho_N)\Big]
\end{aligned}
$$

Now applying the suggested Taylor expansion $\log(1+x)\approx x$ for all of the logarithms appearing in the equation gives

$$\rho \approx \frac{\rho_1+\rho_2+\dots\rho_N}{N}$$

which explicitly shows that the N year annualized return is approximately an arithmetic average of the successive 1 year returns.

(ii) The 1 year gross return implied by the annualized return ρ is simply $1+\rho$, so the above identity can be written

$$
\begin{aligned}
(1+\rho)^N &= \frac{V(1)}{V(0)}\cdot\frac{V(2)}{V(1)}\cdot\dots\cdot\frac{V(N)}{V(N-1)} \\
\Longrightarrow 1+\rho &= \Big[\frac{V(1)}{V(0)}\cdot\frac{V(2)}{V(1)}\cdot\dots\cdot\frac{V(N)}{V(N-1)}\Big]^{\frac{1}{N}}
\end{aligned}
$$

The right hand side of this identity is simply the geometric mean of the 1 year gross returns over the investment period. Note that this second identity is exact whereas the previous equation showing ρ as an arithmetic average of 1 year returns was only approximate.

(b) We can apply the same arguments as in (a) to the "semiannualized" return ρ defined by

$$(1+\rho)^{2N}=\frac{V(N)}{V(0)}$$

The only difference from (a) is that we factor the overall gross return

$$\frac{V(N)}{V(0)}$$

into "half year" gross returns

$$\frac{V(\frac{1}{2})}{V(0)}, \frac{V(0)}{V(\frac{1}{2})}, \dots \frac{V(N)}{V(N-\frac{1}{2})}$$

Otherwise, the arguments are the same. If we define half year returns by

$$1+\rho_{\frac{1}{2}} = \frac{V(\frac{1}{2})}{V(0)}, 1+\rho_1 = \frac{V(1)}{V(\frac{1}{2})}$$

and so on, we get

$$\rho \approx \frac{\rho_{\frac{1}{2}} + \rho_1 + \dots \rho_N}{2N}$$

and

$$1+\rho = \Big[\frac{V(\frac{1}{2})}{V(0)} \cdot \frac{V(1)}{V(\frac{1}{2})} \cdot \dots \cdot \frac{V(N)}{V(N-\frac{1}{2})}\Big]^{\frac{1}{2N}}$$

(c) The arguments are the same as for parts (a) and (b). We define either the quarterly averaged or monthly averaged returns ρ by

$$(1+\rho)^{4N} = \frac{V(N)}{V(0)}$$

or

$$(1+\rho)^{12N} = \frac{V(N)}{V(0)}$$

respectively. We factor the right hand side into either quarterly or monthly gross returns and by the same arguments as used before we can write the return as both approximate arithmetic averages and exact geometric averages of the quarterly or monthly returns.

 # 4: Let $V(N)$ be the value of an investment made at the annually compounded interest rate r so that

$$V(N) = (1+r)^N V(0)$$

We calculate the semiannualized return R over N years, that is, over $2N$ half years:

$$\begin{aligned}
(1+R)^{2N} &= \frac{V(N)}{V(0)} \\
&= \frac{(1+r)^N V(0)}{V(0)} \\
&= (1+r)^N \\
\Longrightarrow (1+R)^2 &= 1+r \\
\Longrightarrow 1+R &= \sqrt{1+r} \\
\Longrightarrow R &= \sqrt{1+r}-1
\end{aligned}$$

(b) If r is a semiannually compounded interest rate an investment has a value

$$V(N) = \left(1+\frac{r}{2}\right)^{2N} V(0)$$

after N years, and

$$\begin{aligned}
(1+R)^{2N} &= \frac{V(N)}{V(0)} \\
&= \frac{(1+\frac{r}{2})^{2N} V(0)}{V(0)} \\
&= \left(1+\frac{r}{2}\right)^{2N} \\
\Longrightarrow (1+R)^2 &= \left(1+\frac{r}{2}\right)^2 \\
\Longrightarrow 1+R &= 1+\frac{r}{2} \\
\Longrightarrow R &= \frac{r}{2}
\end{aligned}$$

The annualized return U satisfies

$$\begin{aligned}
(1+U)^N &= \frac{V(N)}{V(0)} \\
&= \frac{(1+\frac{r}{2})^{2N}V(0)}{V(0)} \\
&= \left(1+\frac{r}{2}\right)^{2N} \\
\Longrightarrow\ 1+U &= \left(1+\frac{r}{2}\right)^{2} \\
\Longrightarrow\ \ &= 1+r+\frac{r^2}{4} \\
\Longrightarrow\ U &= r+\frac{r^2}{4}
\end{aligned}$$

 # 5: (a) We consider investing \$10,000 for 1 year at an interest rate of 17%, with annual, semiannual, quarterly, and monthly compounding:

annual compounding:
$V(1) = (1.17) \times \$10,000 = \$11,700$
semiannual compounding:
$V(1) = \left(1+\frac{0.17}{2}\right)^2 \times \$10,000 = \$11,772$
quarterly compounding:
$V(1) = \left(1+\frac{0.17}{4}\right)^4 \times \$10,000 = \$11,811$
monthly compounding:
$V(1) = \left(1+\frac{0.17}{12}\right)^{12} \times \$10,000 = \$11,839$
daily compounding:
$V(1) = \left(1+\frac{0.17}{365}\right)^{365} \times \$10,000 = \$11,853$
continuous compounding:
$V(1) = e^{0.17} \times \$10,000 = \$11,853$

The annualized returns can be read off without calculation (can you see why?):

$$
\begin{aligned}
\text{annual compounding} &: 17\% \\
\text{semiannual compounding} &: 17.72\% \\
\text{quarterly compounding} &: 18.11\% \\
\text{monthly compounding} &: 18.39\% \\
\text{daily compounding} &: 18.53\% \\
\text{continuous compounding} &: 18.53\%
\end{aligned}
$$

Notice that once we are compounding every day we have basically reached the annualized return of continuous compounding (the slight difference is within roundoff error).

The reason annualized return increases as the compounding frequency increases is that the higher the compounding frequency, the more times interest is levied on interest, so the more interest is ultimately received by the end of the year.

 # 6: (a) With daily compounding we have for the value of an investment at this APR

$$V(1) = \left(1 + \frac{\text{APR}}{365}\right)^{365} V(0)$$

so that

$$
\begin{aligned}
1 + \text{annualized return} &= \frac{V(1)}{V(0)} \\
&= \left(1 + \frac{\text{APR}}{365}\right)^{365}
\end{aligned}
$$

Thus, for an annualized return of 6% we have

$$
\begin{aligned}
1.06 &= \left(1 + \frac{\text{APR}}{365}\right)^{365} \\
\Longrightarrow \text{APR} &= 365[(1.06)^{1/365} - 1] \\
&= 0.058273 \\
&= 5.83\%
\end{aligned}
$$

From our previous results relating EAR and APR we have

$$\begin{aligned}
\text{EAR} &= \left(1+\frac{\text{APR}}{365}\right)^{365}-1 \\
&= \left(1+\frac{0.058273}{365}\right)^{365}-1 \\
&= 0.059999 \\
&= 6\%
\end{aligned}$$

which is, of course, the annualized return we targeted when we chose this APR. In fact, the EAR will always coincide with the annualized return, for any compounding convention, which is an immediate consequence of the general formula

$$1+\text{EAR} = \left(1+\frac{\text{APR}}{k}\right)^{k}$$

(b) Annually compounded case:

$$\begin{aligned}
1+\text{EAR} &= 1+\text{APR} \\
\Longrightarrow \text{EAR} &= \text{APR} \\
&= 9\%
\end{aligned}$$

Semiannually compounded case:

$$\begin{aligned}
1+\text{EAR} &= \left(1+\frac{\text{APR}}{2}\right)^{2} \\
&= \left(1+\frac{0.09}{2}\right)^{2} \\
&= 1.09202 \\
\Longrightarrow \text{EAR} &= 0.09202 \\
&= 9.20\%
\end{aligned}$$

Monthly compounded case:

$$\begin{aligned}
1+\text{EAR} &= \left(1+\frac{\text{APR}}{12}\right)^{12} \\
&= \left(1+\frac{0.09}{12}\right)^{12} \\
&= 1.09380 \\
\Longrightarrow \text{EAR} &= 0.09380 \\
&= 9.38\%
\end{aligned}$$

Now we consider \$25,000 invested for 5 years at each of these APRs. The annually compounded case:

$$\begin{aligned} V(5) &= (1.09)^5(25,000) \\ &= \$38,460 \end{aligned}$$

The annualized return is:

$$\begin{aligned} r &= \left[\frac{38,460}{25,000}\right]^{\frac{1}{5}} - 1 \\ &= 0.09000 \\ &= 9.00\% \end{aligned}$$

The semiannually compounded case:

$$\begin{aligned} V(5) &= (1 + \frac{0.09}{2})^{10}(25,000) \\ &= \$38,824 \end{aligned}$$

The annualized return is:

$$\begin{aligned} r &= \left[\frac{38,824}{25,000}\right]^{\frac{1}{5}} - 1 \\ &= 0.09202 \\ &= 9.20\% \end{aligned}$$

The monthly compounded case:

$$\begin{aligned} V(5) &= (1 + \frac{0.09}{12})^{60}(25,000) \\ &= \$39,142 \end{aligned}$$

The annualized return is:

$$\begin{aligned} r &= \left[\frac{39,142}{25,000}\right]^{\frac{1}{5}} - 1 \\ &= 0.09381 \\ &= 9.38\% \end{aligned}$$

These all agree with our earlier theoretical calculation.