Problem Set: Compound Interest

 # 1: How much is a \$15,000 investment worth in 7 years if invested at an annually compounded rate of 4%? What if it is semannually compounded? Quarterly compounded? Monthly compounded? In all cases, calculate the total interest earned, the interest earned solely on the principal, and the interest earned on interest.

 # 2: How long does it take for \$10,000 to double if invested at an annually compounded rate of 4%. If it doubles in 8 years, what semiannually compounded rate was it invested at?

 # 3: Suppose you can choose between 2 investment products, one paying a 6% annually compounded interest rate and one paying a 5.5 % monthly compounded rate. Which should you choose?

 # 4: (a) Suppose a \$5000 investment is worth \$6800 4 years later. If it was invested at an annually compounded interest rate, what was the rate?

(b) If a \$10,000 investment is worth \$10,800 after 6 months, what semiannually compounded rate was it invested at?

 # 5: (a) Consider carefully the justification of the formula for the value of an investment made at an annually compounded interest rate r. We let $V(K)$ denote the value of the investment after K years, where $K = 0, 1, 2, \ldots$. The initial amount invested is then $V(0)$ and we may write the formula from lecture as

$$V(K) = (1 + r)^K V(0) \tag{1}$$

The justification we gave in lecture was intuitive and reasonably convincing, but it was still suggestive rather than watertight. For a more completely correct justification, here are 2 suggested ways to proceed.

(i) Induction: The general principle of mathematical induction is that if you have a mathematical statement that depends on a positive integer parameter k, which we denote

$$\text{statement}_k$$

then you can prove this statement for *all* values of k by proving the following 2 things:

1. statement_1;
2. that statement_k implies statement_{k+1}.

If you are comfortable with the idea of induction, then you can attempt to justify the formula for the value of an investment by induction, where statement$_k$ is equation (1), the investment value formula. To do this, start by writing down explicitly what statement$_1$ is and think carefully about what you have to do to prove it (hint: we did it in lecture). That's step 1 of induction, and then to complete step 2, again, start by writing down exactly what statement$_k$ and statement$_{k+1}$ are, and think about how to derive the latter from the former. Once again, the key is to look carefully at the argument from lecture, and apply it for general k (strictly speaking, we only did it for k =1 and 2).

(ii) As an alternative to induction, one can reason as follows. Leaving aside any formula for $V(K)$, the argument from lecture can be applied to show that

$$V(K+1) = (1+r)V(K)$$

for any value of K. From this one can immediately deduce the chain of propositions

$$\begin{aligned} V(1) &= (1+r)V(0) \\ V(2) &= (1+r)V(1) = (1+r)^2V(0) \\ V(3) &= (1+r)V(2) = (1+r)^3V(0) \end{aligned}$$

and from this it is not hard to convince oneself that

$$V(K) = (1+r)^K V(0)$$

(b) Use the reasoning from part (a) to justify the formula for the value of an investment at a semiannually compounded rate r, which, for reference, is

$$V(T) = \left(1+\frac{r}{2}\right)^{2T} F$$

Note one major difference from the annually compounded case is that one works with half year compounding periods, and that T can now be of the form $\frac{j}{2}$ for an integer j.

(c) Now consider the case of a general compounding frequency k, and adapt the arguments from parts (a) and (b) to justify it. You might want to warm up by trying a few more cases, like quarterly compounding, or monthly compounding, for which, explicit formula were given in lecture.

6: (a) Justify the formula from lecture relating 2 compound interest rates with different compounding frequencies

$$r_\ell = \ell\Big(1 + \frac{r_k}{k}\Big)^{\frac{k}{\ell}} - \ell$$

by carrying out the following steps:
(i) imposing the condition that any amount invested at either of the 2 rates will have grown to the same value in 1 year;
(ii) carrying out the algebra from the condition in (i) to express one of the rates in terms of the other;
(iii) show that if the condition from (ii) is satisfied, then the two rates imply the same investment growth for any investment period that is a whole multiple of both compounding periods.
(b) Suppose you currently have your life savings in a bank account earning 6%, annually compounded. Suppose the bank is offering to switch you to accounts earning 5.9% compounded semiannually, 5.85% compounded quarterly, 5.84% monthly, and 5.83% compounded daily. Should you take the bank's offer and switch to any of these accounts?

 # 7: In many domiciles (like the US) banks are required to report some kind of annualized interest rate number to their customers. If a customer is paying a compound interest rate r_k with compounding frequency k, the bank is required by law to report the rate r_k as the **annual percentage rate (APR)**. Note that the growth rate of an investment invested at r_k in one compounding period is

$$\frac{r_k}{k}$$

We can refer to this growth rate as the *effective rate* and note that this is an effective rate with period $\frac{1}{k}$. The point of these reporting laws is to make sure bank customers have an annualized rate reported to them, which the APR is and the effective rate is not, so as to be able to compare the interest rates from different banks.

Unfortunately, APR is a flawed quantity for this purpose because it does not take compounding into account. It would be more useful for comparision purposes to report the equivalent (in the sense of the previous problem) annually compounded rate, which in this context is referred to as the **Effective Annual Rate (EAR)**. It follows from the discussion on rate conversions from this lecture that the most convenient way to translate between APR and EAR is by using the conversion formulas.

(a) Show that the relationship between the APR and the EAR is

$$1 + \mathrm{EAR} = \left(1 + \frac{\mathrm{APR}}{k}\right)^k$$

(b) Suppose a bank is offering you a savings account with a quarterly compounded interest rate of 4%. What is the effective quarterly rate, the APR, and the EAR?

(c) Your local payday lender is offering to lend you \$10,000 for an emergency, but you must pay the lender back \$17,000 in 6 months. What is the effective 6 month rate, APR and EAR for this loan?

(d) If a bank wishes to charge an EAR of 20% on its main credit card product, what APR will it advertise to its clients (recall that interest on credit cards is charged monthly).