Solutions: Continuous Compounding

 # 1: Under continuous compounding at a rate of 6% for 9 years the investment value is

$$\begin{aligned} V &= e^{(0.06)9}(5000) \\ &= \$8580 \end{aligned}$$

 # 2: We need to find the investment time T such that

$$\begin{aligned} e^{(0.02)T}(9000) &= 18,000 \\ \Longrightarrow e^{(0.02)T} &= \frac{18,000}{9000} \\ &= 2 \end{aligned}$$

By taking logarithms we have

$$\begin{aligned} (0.02)T &= \log(2) \\ \Longrightarrow T &= \frac{\log(2)}{0.02} \\ &= 34.7 \end{aligned}$$

So it takes 34.7 years to double at this continuously compounded interest rate.

If the investment is worth $15,000 in 8 years, the continuously compounded investment rate r satisfies

$$\begin{aligned} e^{r\cdot 8}(9000) &= 15,000 \\ \Longrightarrow e^{r\cdot 8} &= \frac{15,000}{9000} \\ \Longrightarrow r\cdot 8 &= \log\Big(\frac{15,000}{9000}\Big) \\ \Longrightarrow r &= \frac{1}{8}\log\Big(\frac{15,000}{9000}\Big) \\ &= 0.06385 \\ &= 6.39\% \end{aligned}$$

 # 3: \$1000 invested at an annually compounded interest rate of 6% for 5 years will grow to

$$(1+0.06)^5(1000) = \$1338$$

The equivalent semiannually compounded rate r_s must sastisfy

$$\begin{aligned}
(1+\frac{r_2}{2})^{2\cdot 5}(1000) &= 1338 \\
\Longrightarrow (1+\frac{r_s}{2})^{10} &= \frac{1338}{1000} \\
\Longrightarrow r_s &= 2\Big[\Big(\frac{1338}{1000}\Big)^{\frac{1}{10}} - 1\Big] \\
&= 0.05904 \\
&= 5.90\%
\end{aligned}$$

The equivalent monthly compounded rate r_m satisfies

$$\begin{aligned}
\Big(1+\frac{r_m}{12}\Big)^{12\cdot 5}(1000) &= 1338 \\
\Longrightarrow \Big(1+\frac{r_m}{12}\Big)^{60} &= \frac{1338}{1000} \\
\Longrightarrow r_m &= 12\Big[\Big(\frac{1338}{1000}\Big)^{\frac{1}{60}} - 1\Big] \\
&= 0.05837 \\
&= 5.84\%
\end{aligned}$$

The equievalent continuously compounded rate r_c satisfies

$$\begin{aligned}
e^{r_c\cdot 5}(1000) &= 1338 \\
\Longrightarrow e^{r_c\cdot 5} &= \frac{1338}{1000} \\
\Longrightarrow r_c\cdot 5 &= \log\Big(\frac{1338}{1000}\Big) \\
r_c &= \frac{1}{5}\log\Big(\frac{1338}{1000}\Big) \\
&= 0.0582 \\
&= 5.82\%
\end{aligned}$$

4: We have, for $V(T)$ the value of an investment after T years

$$\begin{aligned} V(1) &= e^r V(0) \\ &= \Big(1+\frac{r_k}{k}\Big)^k V(0) \\ \Longrightarrow e^r &= \Big(1+\frac{r_k}{k}\Big)^k \end{aligned}$$

By taking logarithms we have

$$\begin{aligned} r &= \log\Big[\Big(1+\frac{r_k}{k}\Big)^k\Big] \\ &= k\log\Big(1+\frac{r_k}{k}\Big) \end{aligned}$$

Alternatively, we may take kth roots:

$$\begin{aligned} e^{\frac{r}{k}} &= 1+\frac{r_k}{k} \\ \Longrightarrow \frac{r_k}{k} &= e^{\frac{r}{k}}-1 \\ \Longrightarrow r_k &= k(e^{\frac{r}{k}}-1) \end{aligned}$$

5: Given the described circumstances, we take the APR as a continuously compounded rate r and the EAR as an annually compounded rate r_1, and the solution to this problem is an application of the conversion formula derived in the previous problem:

$$\begin{aligned} \text{EAR} &= e^{\text{APR}}-1 \\ &= e^{0.035}-1 \\ &= 0.035619 \\ &= 3.56\% \end{aligned}$$

6: (a) We will take the APR as some fixed rate. EAR then varies as the compounding frequency increases. It follows from the relationship between APR and EAR (from the compound interest problem set) that

$$1+\text{EAR} = \Big(1+\frac{\text{APR}}{k}\Big)^k$$

If we take the limit as k gets large, we have

$$\begin{aligned} 1 + \mathrm{EAR} &= \lim_{k\to\infty}\left(1 + \frac{\mathrm{APR}}{k}\right)^k \\ &= e^{\mathrm{APR}} \end{aligned}$$

by the same calculus limit as was used to derive continuously compounded rates. So, as the compounding frequency goes to infinity, the relationship between the EAR and the fixed rate APR becomes the relationship between the EAR and a continuously compounded APR.

(b) A continuously compounded interest rate corresponds to the limit as the compounding frequency goes to infinity. The EAR is a measure of the overall growth of the investment independent of the compounding frequency. As compounding is the process of charging interest on interest already charged, the intuitive expectation is that the higher the compounding frequency the greater the ultimate return, and thus the EAR should be more. From this logic the largest possible EAR for a given APR should be that given when the APR is continuously compounded. So, for a fixed APR, the EAR from continuous compounding

$$\mathrm{EAR} = e^{\mathrm{APR}} - 1$$

should be the largest possible. Then, for APR $= 5\%$ the largest possible EAR should be

$$e^{0.05} - 1 = 0.051271 = 5.1271\%$$

One should check a few cases to make sure this is true.