Question 3a

Step 1:

At step 1 we observe a reading for Z of 2. Let $t = 1$. We solve for the MAP as follows:

$$\bar{bel}(X_t = 1) = P(X_t = 1|X_o = 1) + P(X_t = 1|X_o = 2) + P(X_t = 1|X_o = 3) \tag{1}$$
$$= P(X_t = 1|X_o = 1)bel(X_o = 1) + P(X_t = 1|X_o = 2)bel(X_o = 2) + P(X_t = 1|X_o = 3)bel(X_o = 3) \tag{2}$$
$$= (0.2)(1/3) + (0.2)(1/3) + (0.4)(1/3) = 0.26 \tag{3}$$
$$bel(X_t = 1) = \mathrm{D}P(Z_t = 2|X_t = 1)\bar{bel}(X_t = 1) = \mathrm{D} \times 0.1 \times 0.26 = \mathrm{D}0.026 \tag{4}$$

$$\bar{bel}(X_t = 2) = P(X_t = 2|X_o = 1) + P(X_t = 2|X_o = 2) + P(X_t = 2|X_o = 3) \tag{5}$$
$$= P(X_t = 2|X_o = 1)bel(X_o = 1) + P(X_t = 2|X_o = 2)bel(X_o = 2) + P(X_t = 2|X_o = 3)bel(X_o = 3) \tag{6}$$
$$= (0.3)(1/3) + (0.7)(1/3) + (0.2)(1/3) = 0.3\dot{9} \tag{7}$$
$$bel(X_t = 2) = \mathrm{D}P(Z_t = 2|X_t = 2)\bar{bel}(X_t = 2) = \mathrm{D} \times 0.7 \times 0.39 = \mathrm{D}0.272\dot{9} \tag{8}$$

$$\bar{bel}(X_t = 3) = P(X_t = 4|X_o = 1) + P(X_t = 4|X_o = 2) + P(X_t = 4|X_o = 3) \tag{9}$$
$$= P(X_t = 3|X_o = 1)bel(X_o = 1) + P(X_t = 3|X_o = 2)bel(X_o = 2) + P(X_t = 3|X_o = 3)bel(X_o = 3) \tag{10}$$
$$= (0.5)(1/3) + (0.1)(1/3) + (0.4)(1/3) = 0.\dot{3} \tag{11}$$
$$bel(X_t = 3) = \mathrm{D}P(Z_t = 2|X_t = 3)\bar{bel}(X_t = 3) = \mathrm{D} \times 0.3 \times 0.3 = \mathrm{D}0.0\dot{9} \tag{12}$$

Therefore:

$$\mathrm{D} = (0.0\dot{9} + 0.026 + 0.272\dot{9})^{-1} = 0.398 \tag{13}$$

This implies:

$$bel(X_1 = 1) \approx 0.065, bel(X_1 = 2) \approx 0.684, bel(X_1 = 3) \approx 0.25 \tag{14}$$

The MAP is therefore $X_1 = 2$.

Step 2:

At step 2 we observe a reading for Z of 3. Let $t = 2$. We solve for the MAP as follows:

$$\bar{bel}(X_t = 1) = P(X_t = 1|X_1 = 1) + P(X_t = 1|X_1 = 2) + P(X_t = 1|X_1 = 3) \tag{15}$$
$$= P(X_t = 1|X_1 = 1)bel(X_1 = 1) + P(X_t = 1|X_1 = 2)bel(X_1 = 2) + P(X_t = 1|X_1 = 3)bel(X_1 = 3) \tag{16}$$

$$= (0.2)(0.065) + (0.2)(0.684) + (0.4)(0.25) = 0.2498 \tag{17}$$
$$bel(X_t = 1) = \mathrm{D}P(Z_t = 2|X_t = 1)\bar{bel}(X_t = 1) = \mathrm{D} \times 0.1 \times 0.2498 = \mathrm{D}0.02498 \tag{18}$$

$$\bar{bel}(X_t = 2) = P(X_t = 2|X_1 = 1) + P(X_t = 2|X_1 = 2) + P(X_t = 2|X_1 = 3) \tag{19}$$
$$= P(X_t = 2|X_1 = 1)bel(X_1 = 1) + P(X_t = 2|X_1 = 2)bel(X_1 = 2) + P(X_t = 2|X_1 = 3)bel(X_1 = 3) \tag{20}$$
$$= (0.3)(0.065) + (0.7)(0.684) + (0.2)(0.25) = 0.5483 \tag{21}$$

$$bel(X_t = 2) = \mathrm{D}P(Z_t = 2|X_t = 2)\bar{bel}(X_t = 2) = \mathrm{D} \times 0.7 \times 0.5483 = \mathrm{D}0.38381 \tag{22}$$

$$\bar{bel}(X_t = 3) = P(X_t = 3|X_1 = 1) + P(X_t = 3|X_1 = 2) + P(X_t = 3|X_1 = 3) \tag{23}$$

$$= P(X_t = 3|X_1 = 1)bel(X_1 = 1) + P(X_t = 3|X_1 = 2)bel(X_1 = 2) + P(X_t = 3|X_1 = 3)bel(X_1 = 3) \tag{24}$$

$$= (0.5)(0.065) + (0.1)(0.684) + (0.4)(0.25) = 0.2 \tag{25}$$

$$bel(X_t = 3) = \eta P(Z_t = 3|X_t = 2)\bar{bel}(X_t = 3) = \eta \times 0.2 \times 0.2 = \eta 0.04 \tag{26}$$

The normalization constant is thus: \$ (0.02498 + 0.38381 + 0.04)ˆ{-1} = 2.22821 \$ So the beliefs are:

$$bel(X_2 = 1) \approx 0.055, bel(X_2 = 2) \approx 0.85, bel(X_2 = 3) \approx 0.089 \tag{27}$$

The MAP is therefore $X_2 = 2$.

Step 3:

At step 3 we observe a reading for Z of 1. Let $t = 3$. We solve for the MAP as follows:

$$\bar{bel}(X_t = 1) = P(X_t = 1|X_2 = 1) + P(X_t = 1|X_2 = 2) + P(X_t = 1|X_2 = 3) \tag{28}$$

$$= P(X_t = 1|X_2 = 1)bel(X_2 = 1) + P(X_t = 1|X_2 = 2)bel(X_2 = 2) + P(X_t = 1|X_2 = 3)bel(X_2 = 3) \tag{29}$$

$$= (0.2)(0.055) + (0.2)(0.85) + (0.4)(0.089) = 0.2166 \tag{30}$$

$$bel(X_t = 1) = \eta P(Z_t = 1|X_t = 1)\bar{bel}(X_t = 1) = \eta \times 0.6 \times 0.2166 = \eta 0.12996 \tag{31}$$

$$\bar{bel}(X_t = 2) = P(X_t = 2|X_2 = 1) + P(X_t = 2|X_2 = 2) + P(X_t = 2|X_2 = 3) \tag{32}$$

$$= P(X_t = 2|X_2 = 1)bel(X_2 = 1) + P(X_t = 2|X_2 = 2)bel(X_2 = 2) + P(X_t = 2|X_2 = 3)bel(X_2 = 3) \tag{33}$$

$$= (0.2)(0.055) + (0.7)(0.85) + (0.2)(0.089) = 0.6238 \tag{34}$$

$$bel(X_t = 2) = \eta P(Z_t = 1|X_t = 2)\bar{bel}(X_t = 2) = \eta \times 0.1 \times 0.6238 = \eta 0.06238 \tag{35}$$

$$\bar{bel}(X_t = 3) = P(X_t = 3|X_2 = 1) + P(X_t = 3|X_2 = 2) + P(X_t = 3|X_2 = 3) \tag{36}$$

$$= P(X_t = 3|X_2 = 1)bel(X_2 = 1) + P(X_t = 3|X_2 = 2)bel(X_2 = 2) + P(X_t = 3|X_2 = 3)bel(X_2 = 3) \tag{37}$$

$$= (0.5)(0.055) + (0.1)(0.85) + (0.4)(0.089) = 0.1481 \tag{38}$$

$$bel(X_t = 2) = \eta P(Z_t = 1|X_t = 3)\bar{bel}(X_t = 2) = \eta \times 0.2 \times 0.1481 = \eta 0.02962 \tag{39}$$

The normalization constant is

$$(0.12996 + 0.06238 + 0.02962)^{-1} = 4.50532 \tag{40}$$

. So the beliefs are:

$$bel(X_3 = 1) \approx 0.5855, bel(X_3 = 2) \approx 0.281, bel(X_3 = 3) \approx 0.133 \tag{41}$$

The MAP is therefore $X_3 = 1$.

Therefore,

$$arg_{X_t} max P(X_t|Z^t), t \in \{1, 2, 3\} = \{2, 2, 1\} \tag{42}$$

Question 3b

For the forward step:[1]

$$\alpha_0(1) = P(Z_0 = 2|X_0 = 1)P(X_0 = 1) = \text{Ŋ}_0(0.1)(1/3) = 0.09 \tag{43}$$
$$\alpha_0(2) = P(Z_0 = 2|X_0 = 2)P(X_0 = 2) = \text{Ŋ}_0(0.7)(1/3) = 0.64 \tag{44}$$
$$\alpha_0(3) = P(Z_0 = 2|X_0 = 3)P(X_0 = 3) = \text{Ŋ}(0.3)(1/3) = 0.27 \tag{45}$$

Where

$$\text{Ŋ}_0 = (\alpha_0(1) + \alpha_0(2) + \alpha_0(3))^{-1} = 2.72727 \tag{46}$$

$$\alpha_1(1) = P(Z_1 = 3|X_1 = 1)[P(X_1 = 1|X_0 = 1)\alpha_0(1) + P(X_1 = 1|X_0 = 2)\alpha_0(2) + P(X_1 = 1|X_0 = 3)\alpha_0(3)] \tag{47}$$
$$= (0.6)[(0.2)(0.09) + (0.2)(0.64) + (0.4)(0.27)] = \text{Ŋ}_1 0.15 = 0.41 \tag{48}$$

$$\alpha_1(2) = P(Z_1 = 3|X_1 = 2)[P(X_1 = 2|X_0 = 1)\alpha_0(1) + P(X_1 = 2|X_0 = 2)\alpha_0(2) + P(X_1 = 2|X_0 = 3)\alpha_0(3)] \tag{49}$$
$$= (0.2)[(0.3)(0.09) + (0.7)(0.64) + (0.2)(0.27)] = \text{Ŋ}_1 0.105 = 0.29 \tag{50}$$

$$\alpha_1(3) = P(Z_1 = 3|X_1 = 3)[P(X_1 = 3|X_0 = 1)\alpha_0(1) + P(X_1 = 3|X_0 = 2)\alpha_0(2) + P(X_1 = 3|X_0 = 3)\alpha_0(3)] \tag{51}$$
$$= (0.5)[(0.5)(0.09) + (0.1)(0.64) + (0.4)(0.27)] = \text{Ŋ}_1 0.108 = 0.30 \tag{52}$$

Where

$$\text{Ŋ}_1 = (\alpha_1(1) + \alpha_1(2) + \alpha_1(3))^{-1} = 2.75482 \tag{53}$$

$$\alpha_2(1) = P(Z_2 = 1|X_2 = 1)[P(X_2 = 1|X_1 = 1)\alpha_1(1) + P(X_2 = 1|X_1 = 2)\alpha_1(2) + P(X_2 = 1|X_1 = 3)\alpha_1(3)] \tag{54}$$
$$= (0.6)[(0.2)(0.41) + (0.2)(0.29) + (0.4)(0.3)] = \text{Ŋ}_2 0.156 = 0.63 \tag{55}$$

$$\alpha_2(2) = P(Z_2 = 1|X_2 = 2)[P(X_2 = 2|X_1 = 1)\alpha_1(1) + P(X_2 = 2|X_1 = 2)\alpha_1(2) + P(X_2 = 2|X_1 = 3)\alpha_1(3)] \tag{56}$$
$$= (0.1)[(0.3)(0.41) + (0.7)(0.29) + (0.2)(0.3)] = \text{Ŋ}_2 0.0386 = 0.16 \tag{57}$$

$$\alpha_2(3) = P(Z_2 = 1|X_2 = 3)[P(X_2 = 3|X_1 = 1)\alpha_1(1) + P(X_2 = 3|X_1 = 2)\alpha_1(2) + P(X_2 = 3|X_1 = 3)\alpha_1(3)] \tag{58}$$
$$= (0.2)[(0.3)(0.41) + (0.1)(0.29) + (0.4)(0.3)] = \text{Ŋ}_2 0.0544 = 0.22 \tag{59}$$

Where we calculate the normalization constant as above,

$$\text{Ŋ}_2 = 4.01606 \tag{60}$$

For the backward step, define:

$$\beta_2(1) = 1, \beta_2(2) = 1, \beta_2(3) = 1 \tag{61}$$

This can be normalized to sum to 1 by multiplying it by a normalization constant

$$\text{Ŋ}_2 = 1/3 \tag{62}$$

With this as our starting point, we calculate:

$$\begin{aligned} \beta_1(1) = {} & P(Z_2 = 1|X_2 = 1)P(X_2 = 1|X_1 = 1)\beta_2(1) + P(Z_2 = 1|X_2 = 2)P(X_2 = 2|X_1 = 1)\beta_2(2) \\ & + P(Z_2 = 1|X_2 = 3)P(X_2 = 3|X_1 = 1)\beta_2(3) \end{aligned} \tag{63}$$

[1] I round to two significant digits.

$$= (0.6)(0.2)(1/3) + (0.1)(0.2)(1/3) + (0.2)(0.5)(1/3) = \text{Ŋ}_1 0.08 = 0.31 \tag{64}$$

$$\begin{aligned}
\beta_1(2) &= P(Z_2 = 1|X_2 = 1)P(X_2 = 1|X_1 = 2)\beta_2(1) + P(Z_2 = 1|X_2 = 2)P(X_2 = 2|X_1 = 2)\beta_2(2) \\
&\quad + P(Z_2 = 1|X_2 = 3)P(X_2 = 3|X_1 = 2)\beta_2(3) \\
&= (0.6)(0.2)(1/3) + (0.1)(0.7)(1/3) + (0.2)(0.1)(1/3) = \text{Ŋ}_1 0.07 = 0.27
\end{aligned} \tag{65}$$

$$\begin{aligned}
\beta_1(3) &= P(Z_2 = 1|X_2 = 1)P(X_2 = 1|X_1 = 3)\beta_2(1) + P(Z_2 = 1|X_2 = 2)P(X_2 = 2|X_1 = 3)\beta_2(2) \\
&\quad + P(Z_2 = 1|X_2 = 3)P(X_2 = 3|X_1 = 3)\beta_2(3) \\
&= (0.6)(0.4)(1/3) + (0.1)(0.2)(1/3) + (0.2)(0.4)(1/3) = \text{Ŋ}_1 0.11 = 0.42
\end{aligned} \tag{66}$$

Where

$$\text{Ŋ}_2 = 3.84615 \tag{67}$$

$$\begin{aligned}
\beta_0(1) &= P(Z_1 = 1|X_1 = 1)P(X_1 = 1|X_0 = 1)\beta_1(1) + P(Z_1 = 1|X_1 = 2)P(X_1 = 2|X_0 = 1)\beta_1(2) \\
&\quad + P(Z_1 = 1|X_1 = 3)P(X_1 = 3|X_0 = 1)\beta_1(3) \\
&= (0.3)(0.2)(0.31) + (0.2)(0.3)(0.27) + (0.5)(0.5)(0.42) = \text{Ŋ}_2 0.1398 = 0.40
\end{aligned} \tag{68}$$

$$\begin{aligned}
\beta_0(2) &= P(Z_1 = 1|X_1 = 1)P(X_1 = 1|X_0 = 2)\beta_1(1) + P(Z_1 = 1|X_1 = 2)P(X_1 = 2|X_0 = 2)\beta_1(2) \\
&\quad + P(Z_1 = 1|X_1 = 3)P(X_1 = 3|X_0 = 2)\beta_1(3) \\
&= (0.3)(0.2)(0.31) + (0.2)(0.7)(0.27) + (0.5)(0.1)(0.42) = \text{Ŋ}_2 0.0774 = 0.22
\end{aligned} \tag{69}$$

$$\begin{aligned}
\beta_0(3) &= P(Z_1 = 1|X_1 = 1)P(X_1 = 1|X_0 = 3)\beta_1(1) + P(Z_1 = 1|X_1 = 2)P(X_1 = 2|X_0 = 3)\beta_1(2) \\
&\quad + P(Z_1 = 1|X_1 = 3)P(X_1 = 3|X_0 = 3)\beta_1(3) \\
&= (0.3)(0.4)(0.31) + (0.2)(0.2)(0.27) + (0.5)(0.4)(0.42) = \text{Ŋ}_2 0.132 = 0.38
\end{aligned} \tag{70}$$

Where

$$\text{Ŋ}_2 = 2.86369$$

From these we calculate $ P(Zˆ{T})$ as

$$P(Z^T) = \sum_{X_T} \alpha(X_T)\beta(X_T) = \sum_{X_T} \alpha(X_T)$$

because

$$\beta(X_T) = 1$$

$$\sum_{X_t=0}^{2} \alpha(X_t) = \alpha(X_0 = 2) + \alpha(X_1 = 3) + \alpha(X_2 = 1) = 0.64 + 0.3 + 0.63 = 1.57 \tag{71}$$

Which we use to calculate the probability of each state at a given point in time.

$$P(X_t|Z^T) = \frac{\alpha(X_t)\beta(X_t)}{P(Z^T)} \tag{72}$$

For Step 0:

$$P(X_0 = 1|Z^T) = \frac{\alpha_0(1)\beta_0(1)}{P(Z^T)} = \frac{0.09 \times 0.40}{1.57} = 0.023 \tag{73}$$

$$P(X_0 = 2|Z^T) = \frac{\alpha_0(2)\beta_0(2)}{P(Z^T)} = \frac{0.64 \times 0.22}{1.57} = 0.089 \tag{74}$$

$$P(X_0 = 3|Z^T) = \frac{\alpha_0(3)\beta_0(3)}{P(Z^T)} = \frac{0.27 \times 0.38}{1.57} = 0.065 \tag{75}$$

Once again, we normalize and this produces probabilities of:

$$P(X_0 = 1|Z^T) = 0.13, P(X_0 = 2|Z^T) = 0.5, P(X_0 = 3|Z^T) = 0.37$$

So the MAP estimate for $t = 0$ is state 1. Repeating this process for steps 1 and 2, we obtain the following estimates:

$$\begin{gathered} P(X_1 = 1|Z^T) = \mathrm{D}0.081 = 0.12, P(X_1 = 2|Z^T) = \mathrm{D}0.5 = 0.75, P(X_1 = 3|Z^T) = \mathrm{D}0.08 = 0.12 \\ P(X_2 = 1|Z^T) = \mathrm{D}0.13 = 0.62, P(X_2 = 1|Z^T) = \mathrm{D}0.034 = 0.16, P(X_2 = 2|Z^T) = \mathrm{D}0.047 = 0.22 \end{gathered}$$

Therefore,

$$arg_{X_t} max P(X_t|Z^T) = \{1, 2, 1\} \tag{76}$$

Question 3c

First we solve for the $\delta_0(i), i \in \{1, 2, 3\}$:

$$\begin{aligned} \delta_0(1) &= P(Z_0 = 2|X_0 = 1)P(X_0 = 1) = 0.1 \times (1/3) \\ \delta_0(2) &= P(Z_0 = 2|X_0 = 2)P(X_0 = 2) = 0.7 \times (1/3) \\ \delta_0(3) &= P(Z_0 = 2|X_0 = 3)P(X_0 = 3) = 0.3 \times (1/3) \end{aligned}$$

We then evaluate $\delta_1(i), i \in \{1, 2, 3\}$:

$$\begin{aligned} \delta_1(1) = P(Z_1 = 3|X_1 = 1)&argmax(priors), \\ priors = \{&P(X_1 = 1|X_0 = 1)\delta_0(1) = (0.2)(\delta_0(1)), \\ &P(X_1 = 1|X_0 = 2)\delta_0(2) = (0.2)(\delta_0(2)), \\ &P(X_1 = 1|X_0 = 3)\delta_0(3) = (0.4)(\delta_0(3))\} \end{aligned} \tag{77}$$

$$\begin{aligned} \delta_1(2) = P(Z_1 = 3|X_1 = 2)&argmax(priors), \\ priors = \{&P(X_1 = 2|X_0 = 1)\delta_0(1) = (0.3)(\delta_0(1)), \\ &P(X_1 = 2|X_0 = 2)\delta_0(2) = (0.7)(\delta_0(2)), \\ &P(X_1 = 2|X_0 = 3)\delta_0(3) = (0.2)(\delta_0(3))\} \end{aligned} \tag{78}$$

$$\begin{aligned} \delta_1(3) = P(Z_1 = 3|X_1 = 3)&argmax(priors), \\ priors = \{&P(X_1 = 3|X_0 = 1)\delta_0(1) = (0.5)(\delta_0(1)), \\ &P(X_1 = 3|X_0 = 2)\delta_0(2) = (0.1)(\delta_0(2)), \\ &P(X_1 = 3|X_0 = 3)\delta_0(3) = (0.4)(\delta_0(3))\} \end{aligned} \tag{79}$$

Next, we repeat this procedure for time step 2:

$$\begin{aligned} \delta_2(1) = P(Z_2 = 1|X_1 = 1)&argmax(priors), \\ priors = \{&P(X_1 = 1|X_0 = 1)\delta_1(1) = (0.2)(\delta_1(1)), \\ &P(X_1 = 1|X_0 = 2)\delta_1(2) = (0.2)(\delta_1(2)), \\ &P(X_1 = 1|X_0 = 3)\delta_1(3) = (0.4)(\delta_1(3))\} \end{aligned} \tag{80}$$

$$
\begin{aligned}
\delta_2(2) = P(Z_2 = 1|X_1 = 2)&argmax(priors), \\
&priors = \{P(X_1 = 2|X_0 = 1)\delta_1(1) = (0.2)(\delta_1(1)), \\
&P(X_1 = 2|X_0 = 2)\delta_1(2) = (0.7)(\delta_1(2)), \\
&P(X_1 = 2|X_0 = 3)\delta_1(3) = (0.2)(\delta_1(3))\}
\end{aligned} \tag{81}
$$

$$
\begin{aligned}
\delta_2(3) = P(Z_2 = 1|X_1 = 3)&argmax(priors), \\
&priors = \{P(X_1 = 3|X_0 = 1)\delta_1(1) = (0.5)(\delta_1(1)), \\
&P(X_1 = 3|X_0 = 2)\delta_1(2) = (0.1)(\delta_1(2)), \\
&P(X_1 = 3|X_0 = 3)\delta_1(3) = (0.4)(\delta_1(3))\}
\end{aligned} \tag{82}
$$

The MAP sequence we obtain is $\{X_0 = 3, X_1 = 3, X_2 = 1\}$. This compares to the filtering sequence $\{X_0 = 2, X_1 = 3, X_2 = 1\}$ and the smoothing sequence $\{X_0 = 1, X_1 = 2, X_2 = 1\}$.

Question 5a

I worked briefly (30 mins) with Joe Denby and Ben Pick.

Question 5b

12 as of 9pm friday