Investigate $p(\mu, \sigma^2 | y_{new}, y_{old})$

STA 695: In-class Discussion

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Univariate Normal with unknown mean and variance

Likelihood

$$p(y \mid \mu, \sigma^2) = (\sigma^2)^{-n/2} \exp\left[-\frac{1}{2\sigma^2}\left\{(n-1)s^2 + n(\bar{y}-\mu)^2\right\}\right]$$

Prior

$$p(\mu, \sigma^2) = p(\sigma^2)p(\mu|\sigma^2) = \text{inv-gamma}(\nu_0/2, \sigma_0^2\nu_0/2)\text{N}(\mu_0, \sigma^2/\kappa_0)$$

Univariate Normal with unknown mean and variance

Posterior

$$p(\mu, \sigma^2|y) = p(\sigma^2|y)p(\mu|\sigma^2, y) = \text{inv-gamma}(\nu_n/2, \sigma_n^2\nu_n/2)\mathsf{N}(\mu_n, \sigma^2/\kappa_n)$$
$$= \text{inverse-}\chi^2(\nu_n, \sigma_n^2)\mathsf{N}(\mu_n, \sigma^2/\kappa_n)$$

where

$$\begin{aligned}
\mu_n &= \frac{\kappa_0}{\kappa_0 + n}\mu_0 + \frac{n}{\kappa_0 + n}\bar{y} \\
\kappa_n &= \kappa_0 + n \\
\nu_n &= \nu_0 + n \\
\nu_n\sigma_n^2 &= \nu_0\sigma_0^2 + (n-1)s^2 + \frac{\kappa_0 n}{\kappa_0 + n}(\bar{y} - \mu_0)^2.
\end{aligned}$$

Investigate $p(\mu, \sigma^2 | y_{new}, y_{old})$

Suppose we have two data sets $y_{old} = (y_1^1, \ldots, y_{n_1}^1)$ and $y_{new} = (y_1^2, \ldots, y_{n_2}^2)$, with sample size n_1 and n_2, respectively.

$$\bar{y}_1 = \frac{1}{n_1} \sum y_i^1, \qquad s_1^2 = \frac{1}{n_1 - 1} \sum (y_i^1 - \bar{y}_1)^2,$$
$$\bar{y}_2 = \frac{1}{n_2} \sum y_i^2, \qquad s_2^2 = \frac{1}{n_2 - 1} \sum (y_i^2 - \bar{y}_2)^2.$$

With conjugate prior

$$p(\mu, \sigma^2) = p(\sigma^2) p(\mu | \sigma^2) = \text{inv-gamma}(\nu_0/2, \sigma_0^2 \nu_0/2) \mathrm{N}(\mu_0, \sigma^2/\kappa_0)$$

Want to find

$$p(\mu, \sigma^2 | y_{new}, y_{old})$$

Approach One

Combine y_{new} and y_{old} as y and calculate posterior directly.
From previous results, we have

$$
\begin{aligned}
p(\mu, \sigma^2 | y_{new}, y_{old}) &= p(\mu, \sigma^2 | y) \\
&= \text{inv-gamma}(\nu_n/2, \sigma_n^2 \nu_n/2) \mathrm{N}(\mu_n, \sigma^2/\kappa_n)
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_n &= \frac{\kappa_0}{\kappa_0 + n}\mu_0 + \frac{n}{\kappa_0 + n}\bar{y} \\
\kappa_n &= \kappa_0 + n \\
\nu_n &= \nu_0 + n \\
\nu_n \sigma_n^2 &= \nu_0 \sigma_0^2 + (n-1)s^2 + \frac{\kappa_0 n}{\kappa_0 + n}(\bar{y} - \mu_0)^2.
\end{aligned}
$$

Approach One

Since

$$\begin{aligned}
\bar{y} &= \frac{n_1}{n_1+n_2}\bar{y}_1 + \frac{n_2}{n_1+n_2}\bar{y}_2 \\
(n-1)s^2 &= \sum_{i=1}^{n}(y_i - \bar{y})^2 \\
&= \sum_{i=1}^{n_1}(y_i^1 - \bar{y}_1 + \bar{y}_1 - \bar{y})^2 + \sum_{i=1}^{n_2}(y_i^2 - \bar{y}_2 + \bar{y}_2 - \bar{y})^2 \\
&= (n_1-1)s_1^2 + (n_2-1)^2 s_2^2 + n_1(\bar{y}_1 - \bar{y})^2 + n_2(\bar{y}_2 - \bar{y})^2 \\
&= (n_1-1)s_1^2 + (n_2-1)^2 s_2^2 + \frac{n_1 n_2}{n_1+n_2}(\bar{y}_1 - \bar{y}_2)^2
\end{aligned}$$

Approach One

We have

$$
\begin{aligned}
\mu_n =& \frac{\kappa_0}{\kappa_0 + n_1 + n_2}\mu_0 + \frac{n_1}{\kappa_0 + n_1 + n_2}\bar{y}_1 + \frac{n_2}{\kappa_0 + n_1 + n_2}\bar{y}_2 \\
\kappa_n =& \kappa_0 + n_1 + n_2 \\
\nu_n =& \nu_0 + n_1 + n_2 \\
\nu_n\sigma_n^2 =& \nu_0\sigma_0^2 + (n-1)s^2 + \frac{\kappa_0 n}{\kappa_0 + n}(\bar{y} - \mu_0)^2 \\
=& \nu_0\sigma_0^2 + (n_1 - 1)s_1^2 + (n_2 - 1)s_2^2 \\
& + \frac{(\kappa_0 + n_2)n_1}{\kappa_0 + n_1 + n_2}(\bar{y}_1 - \mu_0)^2 + \frac{(\kappa_0 + n_1)n_2}{\kappa_0 + n_1 + n_2}(\bar{y}_2 - \mu_0)^2 \\
& - \frac{2n_1 n_2}{\kappa_0 + n_1 + n_2}(\bar{y}_1 - \mu_0)(\bar{y}_2 - \mu_0).
\end{aligned}
$$

Approach Two

Use the posterior $p(\mu, \sigma^2|y_{old})$ to calculate $p(\mu, \sigma^2|y_{old}, y_{new})$. We have known that

$$p(\theta|y_{old}, y_{new}) = \frac{p(y_{new}|\theta)p(\theta|y_{old})}{\int p(y_{new}|\theta)p(\theta|y_{old})d\theta} \propto p(y_{new}|\theta)p(\theta|y_{old})$$

$$p(y_{new} \mid \mu, \sigma^2) = (\sigma^2)^{-n/2} \exp\left[-\frac{1}{2\sigma^2}\left\{(n-1)s_2^2 + n(\bar{y}_2 - \mu)^2\right\}\right]$$

$$p(\mu, \sigma^2|y_{old}) = \text{inv-gamma}(\nu_1/2, \sigma_1^2\nu_1/2)\text{N}(\mu_1, \sigma^2/\kappa_1)$$

where

$$\mu_1 = \frac{\kappa_0}{\kappa_0 + n_1}\mu_0 + \frac{n_1}{\kappa_0 + n_1}\bar{y}_1, \kappa_1 = \kappa_0 + n_1, \nu_1 = \nu_0 + n_1$$

$$\nu_1\sigma_1^2 = \nu_0\sigma_0^2 + (n_1 - 1)s_1^2 + \frac{\kappa_0 n_1}{\kappa_0 + n_1}(\bar{y}_1 - \mu_0)^2.$$

Approach Two

Because the conjugacy, we can regard $p(\mu, \sigma^2 | y_{old})$ as a new prior for y_{old} and use the results for posterior distribution. Then

$$p(\mu, \sigma^2 | y_{new}, y_{old}) = \text{inv-gamma}(\tilde{\nu}_n/2, \tilde{\sigma}_n^2 \tilde{\nu}_n/2) \text{N}(\tilde{\mu}_n, \sigma^2/\tilde{\kappa}_n)$$

where

$$\begin{aligned}
\tilde{\mu}_n &= \frac{\kappa_1}{\kappa_1 + n_2}\mu_1 + \frac{n_2}{\kappa_1 + n_2}\bar{y}_2 \\
\tilde{\kappa}_n &= \kappa_1 + n_2 \\
\tilde{\nu}_n &= \nu_1 + n_2 \\
\tilde{\nu}_n \tilde{\sigma}_n^2 &= \nu_1 \sigma_1^2 + (n_2 - 1)s_2^2 + \frac{\kappa_1 n_2}{\kappa_1 + n_2}(\bar{y}_2 - \mu_1)^2.
\end{aligned}$$

Approach Two

$$
\begin{aligned}
\tilde{\mu}_n =& \frac{\kappa_0}{\kappa_0 + n_1 + n_2}\mu_0 + \frac{n_1}{\kappa_0 + n_1 + n_2}\bar{y}_1 + \frac{n_2}{\kappa_0 + n_1 + n_2}\bar{y}_2 = \mu_n \\
\tilde{\kappa}_n =& \kappa_0 + n_1 + n_2 = \kappa_n \\
\tilde{\nu}_n =& \nu_0 + n_1 + n_2 = \nu_n
\end{aligned}
$$

Approach Two

$$
\begin{aligned}
\tilde{\nu}_n\tilde{\sigma}_n^2 =& \nu_0\sigma_0^2 + (n_1-1)s_1^2 + \frac{\kappa_0 n_1}{\kappa_0+n_1}(\bar{y}_1-\mu_0)^2 + (n_2-1)s_2^2 \\
&+ \frac{\kappa_1 n_2}{\kappa_1+n_2}(\bar{y}_2-\mu_1)^2 \\
=& \nu_0\sigma_0^2 + (n_1-1)s_1^2 + (n_2-1)s_2^2 + \frac{\kappa_0 n_1}{\kappa_0+n_1}(\bar{y}_1-\mu_0)^2 \\
&+ \frac{n_2}{(\kappa_0+n_1+n_2)(\kappa_0+n_1)}[(\kappa_0+n_1)(\bar{y}_2-\mu_0) - n_1(\bar{y}_1-\mu_0)]^2 \\
=& \nu_0\sigma_0^2 + (n_1-1)s_1^2 + (n_2-1)s_2^2 \\
&+ \frac{(\kappa_0+n_2)n_1}{\kappa_0+n_1+n_2}(\bar{y}_1-\mu_0)^2 + \frac{(\kappa_0+n_1)n_2}{\kappa_0+n_1+n_2}(\bar{y}_2-\mu_0)^2 \\
&- \frac{2n_1n_2}{\kappa_0+n_1+n_2}(\bar{y}_1-\mu_0)(\bar{y}_2-\mu_0) = \mu_n\sigma_n^2.
\end{aligned}
$$