10: Introduction to Bayesian Computation

10/30/19

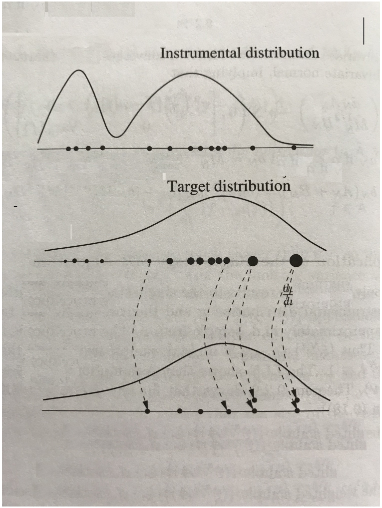
Adding Resampling

Importance Sampling gives you weighted draws $(\theta^1, w(\theta^1)), (\theta^2, w(\theta^2)), \ldots$

You can draw from these, with replacement. At the expense of more variance, it will give you unweighted draws from your target distribution: $\tilde{\theta}^1, \tilde{\theta}^2, \ldots$

This is known as **factored sampling** or **importance sampling with resampling** or **sampling importance resampling** (SIR).

Adding Resampling



Original, unedited image is from
`https://www.springer.com/us/book/9780387402642`

SIR

Stage 1: do importance sampling to get $\{\theta^i, w(\theta^i)\}_{i=1}^S$.

Stage 2: for $i = 1, \ldots, S$, select

$$P[\tilde{\theta}^i = \theta^j \mid \theta^1, w(\theta^1), \ldots, \theta^S, w(\theta^S)] = w(\theta^j).$$

Example 2 revisited

```
y <- 2 # fake data
log_unnorm_weight <- function(theta){
  # can ignore sqrt(2pi) because it will cancel out
  -.5*(y - theta)^2 }
num_samples <- 10000
theta_draws <- rt(num_samples , 1)
lunws <- log_unnorm_weight(theta_draws)
# note: prob arg automatically normalizes
random_indexes <- sample(x = num_samples,
                         size = num_samples,
                         replace = T,
                         prob = exp(lunws))
sort(random_indexes) # see there are repeats!
resampled_draws <- theta_draws[random_indexes]
hist(resampled_draws) # can't do this unless we resample
```

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  $\theta^1, \ldots, \theta^S \sim p_1(\theta|y)$
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  Importance sampling with $g(\theta) = q_1(\theta|y)$, $\tilde{w}(\theta) = q_2(\theta|y)/q_1(\theta|y)$, $w(\theta^s) = \tilde{w}(\theta^s)/\sum_{l=1}^{S} \tilde{w}(\theta^l)$

$$E_2(h(\theta)|y) = \sum_{i=1}^{S} h(\theta^s) w(\theta^s)$$

Going sequential

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sequential monte carlo or **particle filtering** methods are basically doing SIR over and over again.

At "time" $t-1$ you just resampled, so you have draws $\tilde{\theta}^1_{t-1}, \tilde{\theta}^2_{t-1}, \ldots, \tilde{\theta}^S_{t-1}$, and you want to turn them into draws for the next "time" period:

$$\tilde{\theta}^1_t, \tilde{\theta}^2_t, \ldots, \tilde{\theta}^S_t.$$

Examples of sequences of distributions

Data annealing[1]

$$p(\theta), p(\theta \mid y_1), p(\theta \mid y_{1:2}), \dots, p(\theta \mid y_{1:n}),$$

Temperature annealing[2]

$$p(y \mid \theta)^{a_0} p(\theta), p(y \mid \theta)^{a_1} p(\theta), \dots p(y \mid \theta)^{a_n} p(\theta)$$

with $0 = a_0 < a_1 < \cdots < a_n = 1$.

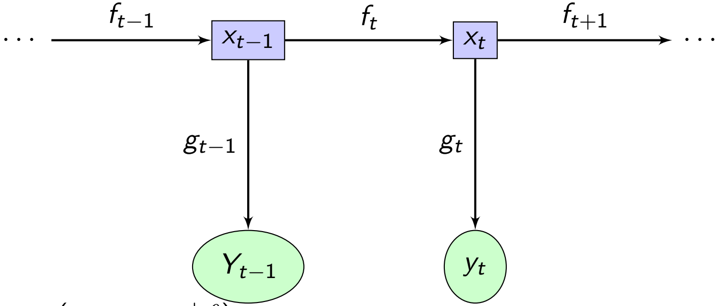
filtering and smoothing in state space models

$$p(x_1 \mid y_1, \theta), \dots, p(x_n \mid y_{1:n}, \theta)$$

[1] Chopin, *A Sequential Particle Filter Method for Static Models*.

[2] Neal, "Annealed Importance Sampling".

state space models



$$p(x_{1:T}, y_{1:T} \mid \theta)$$
$$= g(y_1 \mid x_1, \theta) f(x_1 \mid \theta) \prod_{t=2}^{T} g(y_t \mid x_t, \theta) f(x_t \mid x_{t-1}, \theta)$$

Example: filtering in state space models

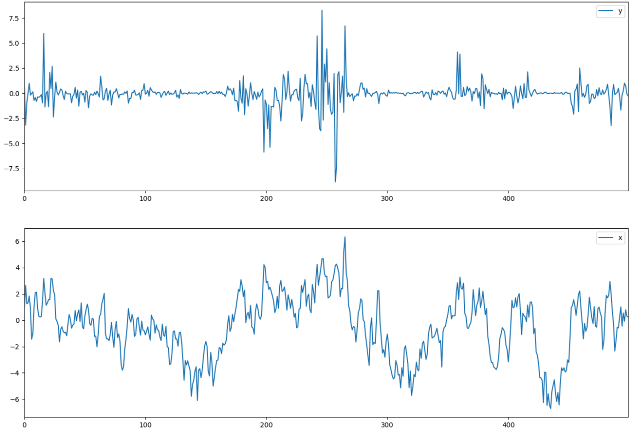
Here's an example of a state space model. y_t is a univariate time series, and x_t is a hidden/unobserved/latent time series.

$$y_t = \exp(x_t/2)\epsilon_t \tag{1}$$
$$x_t = c + \phi x_{t-1} + v_t \tag{2}$$

We sometimes refer to (1) as $g(y_t \mid x_t, \theta)$ or the observation equation, and (2) as the state transition equation or $f(x_t \mid x_{t-1}, \theta)$.

Example: filtering in state space models

$y_{1:t}$ observed, $x_{1:t}$ hidden. Goal: $p(x_t \mid y_{1:t})$ in real-time.



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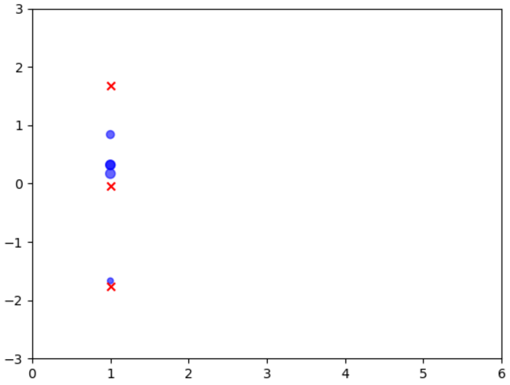
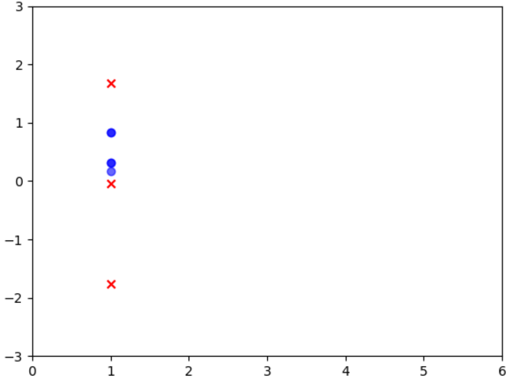


Going sequential



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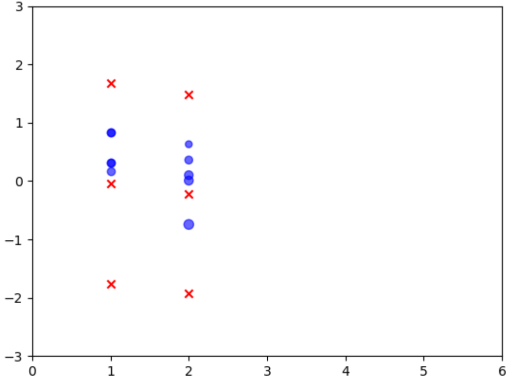

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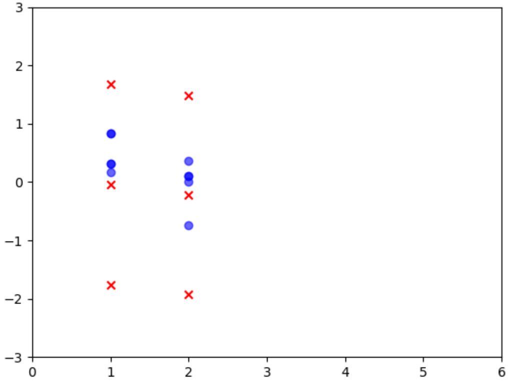

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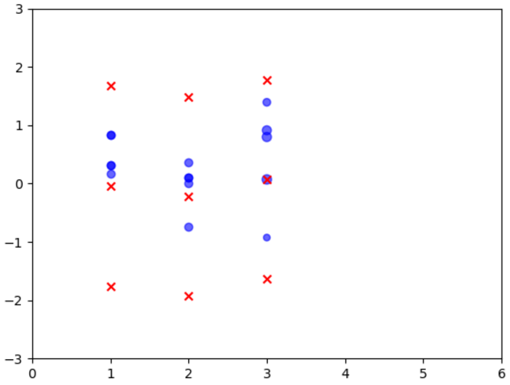

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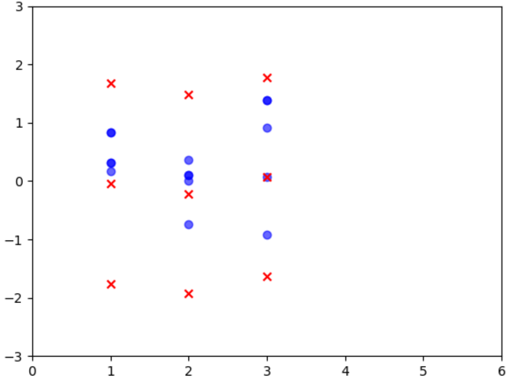

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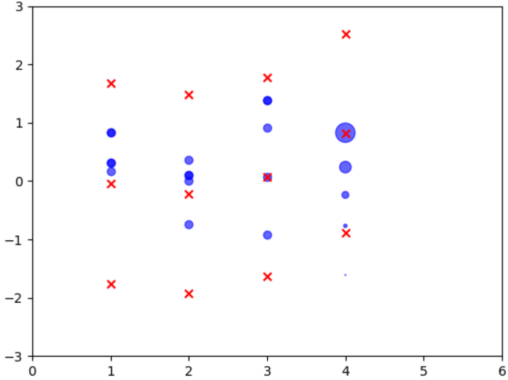

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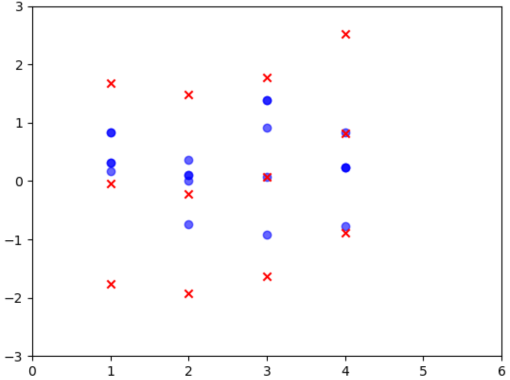

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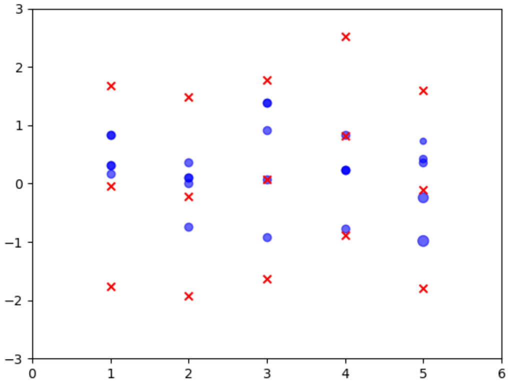

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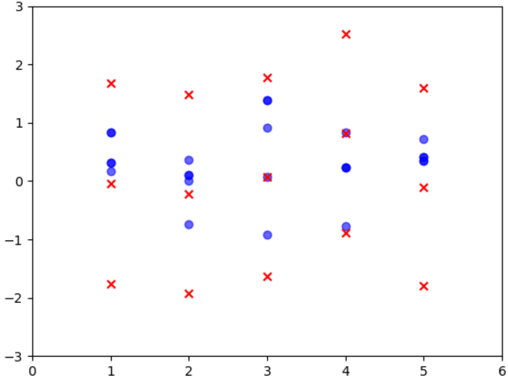

Filtering recursions

Drop dependence on θ from the notation...

$$
\begin{aligned}
p(x_{1:t}|y_{1:t}) &= C_t^{-1} p(x_t, y_t \mid x_{t-1}) p(x_{1:t-1} \mid y_{1:t-1}) \\
&= C_t^{-1} \frac{p(x_t, y_t \mid x_{t-1})}{q_t(x_t \mid y_t, x_{t-1})} \times \\
&\qquad\qquad q_t(x_t \mid y_t, x_{t-1}) p(x_{1:t-1} \mid y_{1:t-1}) \\
&= C_t^{-1} \frac{g(y_t|x_t) f(x_t|x_{t-1})}{q_t(x_t|x_{t-1}, y_t)} \times \\
&\qquad\qquad q_t(x_t \mid x_{t-1}, y_t) p(x_{1:t-1} \mid y_{1:t-1})
\end{aligned}
$$

Repeat through time:

1. start with samples from $p(x_{1:t-1} \mid y_{1:t-1})$
2. mutate/propogate/extend using $q_t(x_t \mid x_{t-1}, y_t)$
3. adjust weights by multiplying by $\frac{g(y_t|x_t)f(x_t|x_{t-1})}{q_t(x_t|x_{t-1},y_t)}$
4. resample, giving you particles distributed as $p(x_{1:t} \mid y_{1:t})$

Chopin, Nicolas. *A Sequential Particle Filter Method for Static Models*. 2000.

Neal, Radford M. "Annealed Importance Sampling". In: *Statistics and Computing* 11.2 (Apr. 2001), pp. 125–139. ISSN: 0960-3174. DOI: 10.1023/A:1008923215028. URL: https://doi.org/10.1023/A:1008923215028.