Cutland Computability Exercises

Liam Tan

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1 Chapter 2 Exercises

Exercise 1.1 (3.4.1a)**.** The zero function is computable. Claim:

$$f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \ldots + x_n$$

is computable for $n \in \omega$ such that $n \geqslant 1$ Base Case(s): We showed $f_2(x, y) = x + y$ was computable earlier in the book. It is also easy to show that $f_1(x) = x$ is computable.
Induction Step: Assume that $f_n(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \ldots + x_n$ is computable. Consider $g(x_1, \ldots, x_{n+1}) = x_1 + \ldots + x_{n+1}$.

$$g(x) = f_n(U_1^{n+1}(x), U_2^{n+1}(x), \ldots, U_{n-1}^{n+1}(x), a(x))$$

$$a(x) = a(x_1, \ldots, x_{n+1}) = x_n + x_{n+1}$$

Let us prove $a(x)$ is computable.

$$a(x) = f_n(0, 0, \ldots, 0, U_n^{n+1}(x), U_{n+1}^{n+1}(x))$$

By Theorem 3.1, since f_n is computable, the zero function is computable, and the projection function is computable, $a(x)$ must be computable. By the same theorem, since $a(x)$ is computable and the projection function is computable, g must be computable.
Consider

$$m(x) = f_m(1, 1, \ldots, 1)$$

Since f_m is computable and 1 is computable (simply the successor function), m is computable.

Exercise 1.2 (3.4.1b)**.**

$$I(x) = x$$

is clearly computable.

$$g(x) = f_m(I(x), \ldots, I(x)) = x + \ldots + x = mx$$

We used the lemma proven in the previous part

Exercise 1.3 (3.4.2).

$$h(x) \simeq f(I(x), m(x))$$

Since we showed m and I are computable, and we know f is computable, by theorem 3.1, h must be computable.

Exercise 1.4 (3.4.3). We showed that

$$f(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Consider the following function:

$$h(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{o/w} \end{cases}$$

We construct g with the following algorithm:

$$I_1 : S(3)$$

$$I_2 : J(1,2,5)$$
$$I_3 : T(2,1)$$
$$I_4 : J(1,1,6)$$
$$I_5 : T(3,1)$$

Consider the function

$$M(x,y) \simeq h(f^*(x,y))$$
$$f^*(x,y) = f(g(x), U_2^n(x,y))$$

Since g and the projection function are computable, f^* is computable by Theorem 3.1. Then since h is computable and f^* is computable, then $M(x,y)$ is computable.

Putting it all together

$$f^*(x,y) = \begin{cases} 0 & \text{if } g(x) = y \\ 1 & \text{if } g(x) \neq y \end{cases}$$

$$M(x,y) \simeq h(f^*(x,y)) = \begin{cases} 1 & \text{if } g(x) = y \\ 0 & \text{if } g(x) \neq y \end{cases}$$

Exercise 1.5 (4.16.1a).

$$f(x,z) = \Pi_{u<z} x = x^z$$

$$g(x,z) = a_z f(x,z) = a_z x^z$$

$$h(x) = \sum_{z \leqslant n} g(x,z)$$

Exercise 1.6 (4.16.1b).

$$\lfloor \sqrt{x} \rfloor = (\mu z < x(z^2 > x)) \dot{-} 1$$

Exercise 1.7 (4.16.1c).

$$LCM(x, y) = \mu z < xy(\mathrm{div}(x, z) \text{ and } \mathrm{div}(y, z))$$

Exercise 1.8 (4.16.1d).

$$HCF(x, y) = qt(LCM(x, y), xy)$$

Exercise 1.9 (4.16.1e).

$$\sum_{z<x} pr(z)div(z, x)$$

Exercise 1.10 (4.16.1f).

$$\sum_{z<x} \bar{sg}(|1 - HCF(x, z)|)$$

Exercise 1.11 (4.16.3).

$$g(0) = 6$$

$$g(x + 1) = 2^{(g(x))_2} 3^{(g(x))_1 + (g(x))_2}$$

$$f(x) = (g(x))_1$$

Exercise 1.12 (4.16.4a).

$$\bar{sg}(div(2, x))$$

Exercise 1.13 (4.16.4b).

$$\bar{sg}(|1 - \sum_{z \leqslant x} pr(z)div(z, x)|)$$

Exercise 1.14 (4.16.4c).

$$\exists z < x(\bar{sg}(|z^3 - x|))$$

Exercise 1.15 (5.4.1).

$$\mu z(\bar{\mathrm{sg}}(|y - f(z)|))$$

Exercise 1.16 (5.4.2).

$$\mu z(p(z) - a)$$

Exercise 1.17 (5.4.2).

$$\mu z(\mathrm{sg}(z)\bar{\mathrm{sg}}(|zy - x|))$$