Cutland Computability Exercises

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1 Chapter 1 Exercises

Exercise 1.1 (2.2).

$$8, 4, 2, 0, 0, 0, \cdots$$
$$I_1$$
$$8, 4, 2, 0, 0, 0, \cdots$$
$$I_2$$
$$8, 5, 2, 0, 0, 0, \cdots$$
$$I_3$$
$$8, 5, 3, 0, 0, 0, \cdots$$
$$I_4, \text{don't jump}$$
$$8, 5, 3, 0, 0, 0, \cdots$$
$$I_5, \text{jump to 2}$$
$$8, 5, 3, 0, 0, 0, \cdots$$
$$I_2$$
$$8, 6, 3, 0, 0, 0, \cdots$$
$$I_3$$
$$8, 6, 4, 0, 0, 0, \cdots$$
$$I_4, \text{don't jump}$$
$$8, 6, 4, 0, 0, 0, \cdots$$
$$I_5, \text{jump to 2}$$
$$8, 6, 4, 0, 0, 0, \cdots$$

$$I_2$$

$$8, 7, 4, 0, 0, 0, \cdots$$

$$I_3$$

$$8, 7, 5, 0, 0, 0, \cdots$$

$$I_4, \text{don't jump}$$

$$8, 7, 5, 0, 0, 0, \cdots$$

$$I_5, \text{jump to 2}$$

$$8, 7, 5, 0, 0, 0, \cdots$$

$$I_2$$

$$8, 8, 5, 0, 0, 0, \cdots$$

$$I_3$$

$$8, 7, 6, 0, 0, 0, \cdots$$

$$I_4, \text{jump to 6}$$

$$8, 7, 6, 0, 0, 0, \cdots$$

$$I_6$$

$$6, 7, 6, 0, 0, 0, \cdots$$

Exercise 1.2 (2.3)**.** We proceed with a proof by induction. After executing each instruction, we want to show that $r_1 < r_2$ and the program does not terminate. Consider the first executed instruction, $J(1, 2, 6)$. Since $2 < 3$, we do not jump and $r_1 < r_2$. Suppose the program has not terminated after instruction n, and $r_1 < r_2$. Consider instruction $n + 1$. The instruction must be $S(2), S(3), J(1, 2, 6), J(1, 1, 2), T(3, 1)$.
Case 1: $S(2)$. Then we have that $r_1 < r_2 < r_2 + 1$. There is a next instruction, so this case is solved.
Case 2: $S(3)$. Since r_1, r_2 are not affected, it remains that $r_1 < r_2$. The next line is instruction 4, so the program does not terminate.
Case 3: $J(1, 2, 6)$. Since $r_1 < r_2$, by the induction hypothesis, we do not jump, and instead run instruction 5. It remains that $r_1 < r_2$
Case 4: $J(1, 1, 2)$. Since jumps do not affect our registers, $r_1 < r_2$ remains. In addition, we jump to instruction 2, so the program does not terminate.
Case 5: $T(3, 1)$. For the instruction to have fired, we must be on line 6. Since line 5 always jumps to line 2, it is the case that we must have executed $J(1, 2, 6)$ previously. However, $r_1 < r_2$ for the previous step, so this case is invalid.

Thus after executing $n+1$ instructions, the program does not terminate. Thus the program wil never terminate.

Exercise 1.3 (3.3.1a).

$$I_1 : J(1, 2, 5)$$

Checks if $x = 0$

$$I_2 : S(2)$$

If $x \neq 0$, add one

$$I_3 : T(2, 1)$$

Move 1 or 0 into the output register

Exercise 1.4 (3.3.1b).

$$I_1 : S(2)$$

$$I_2 : S(2)$$

$$I_3 : S(2)$$

$$I_4 : S(2)$$

$$I_5 : S(2)$$

$$I_6 : T(2, 1)$$

This one is self-explanatory

Exercise 1.5 (3.3.1c).

$$I_1 : J(1, 2, 3)$$

Check if $x = y$

$$I_2 : S(3)$$

If $x \neq y$, add one to register 3

$$I_3 : T(3, 1)$$

Move 0 or 1 to the output register

Exercise 1.6 (3.3.1d).

$$I_1 : J(1, 2, 9)$$

Check to see if $x = y$

$$I_2 : T(1, 3)$$

Move x to register 3

$$I_3 : T(2, 4)$$

Move y to register 4

$$I_4 : J(1, 2, 9)$$

$$I_5 : J(3, 4, 12)$$

Check to see if $y + k = x$ or $x + k = y$

$$I_6 : S(1)$$

$$I_7 : S(4)$$

Add one to x and y

$$I_8 : J(1, 1, 4)$$

Jump back to the checks

$$I_9 : T(5, 1)$$

$$I_{10} : J(1, 1, 14)$$

Put 0 in the output register

$$I_{12} : S(5)$$

$$I_{13} : T(5, 1)$$

Put 1 in the output register

Exercise 1.7 (3.3.1e)**.**

$$I_1 : J(1, 3, 6)$$

$$I_2 : S(2)$$

$$I_3 : S(3)$$

$$I_4 : S(3)$$

$$I_5 : S(3)$$

$$I_6 : T(2, 1)$$

Exercise 1.8 (3.3.1f)**.**

$$I_1 : J(1, 2, 6)$$

$$I_2 : S(2)$$

$$I_3 : S(3)$$

$$I_4 : S(3)$$

$$I_5 : J(1, 1, 1)$$

$$I_6 : T(3, 1)$$

Getting $2X$ into the output register

$$I_7 : Z(2)$$

$$I_8 : Z(3)$$

$$I_9 : S(2)$$

$$I_{10} : S(3)$$

$$I_{11} : S(3)$$

Setting up for $3k+1, 3k+2, 3k$

$$I_{12} : J(1,4,25)$$

$$I_{13} : S(2)$$

$$I_{14} : S(2)$$

$$I_{15} : S(2)$$

$$I_{16} : S(3)$$

$$I_{17} : S(3)$$

$$I_{18} : S(3)$$

$$I_{19} : S(4)$$

$$I_{20} : S(4)$$

$$I_{21} : S(4)$$

$$I_{22} : J(1,2,25)$$

$$I_{23} : J(1,2,25)$$

$$I_{23} : J(1,2,25)$$

If any are hit, it means the floor is found in register 4

$$I_{24} : J(1,1,13)$$

$$I_{25} : T(4,1)$$

Transfer register 4 to output

Exercise 1.9 (3.3.2)**.**

$$f_P^{(2)}(x,y) = \begin{cases} x-y & x>y \\ \text{undefined} & \text{o/w} \end{cases}$$

Exercise 1.10 (3.3.3)**.** We proceed via proof by induction. We want to prove that after executing n instructions, for all registers R_k, we have that either $r_k = m$ or $r_k = x + a$. Since there are no jump instructions, we execute the same number instructions no matter what the input is. Consider $n = 0$. We have that $r_1 = x = x + 0$ and $r_b = 0$, for $b \neq 1$. Suppose after executing n instructions, for all $k \in \mathbb{N}$, we have that $r_k = m_k$ or $r_k = x + a_k$, for some m_k and a_k. Consider the $n+1$th instruction.
Case 1: It is $Z(s)$ for some $s \in \mathbb{N}$. Then $r_s = 0$ and the rest of the registers remain the same. Thus this case is closed.

Case 2: It is $S(l)$ for some $l \in \mathbb{N}$. Then by the induction hypothesis, either $r_l = m_l$ or $r_l = x + a_l$. Then after applying the successor operation, $r_1 = m_l + 1$, still a constant or $r_l = x + (a_l + 1)$, still in the same form. The other registers remain the same.
Case 3: $T(p, o)$. Now $r_o = r_p$, where $r_p = m_p$ or $r_p = x + a_p$. Thus we have proven the induction step for all three cases.
Thus when the program terminates, the value at the output register is either m or $x + a$ for some $m, a \in \mathbb{N}$. Since the value of x was arbitrary, it must be the case that $f(x) = m$ for all $x \in \mathbb{N}$ or $f(x) = x + a$ for all $x \in \mathbb{N}$.

Exercise 1.11 (3.3.4).

$$I_1 : Z(n)$$
$$I_2 : J(n, m, 5)$$
$$I_3 : S(n)$$
$$I_4 : J(1, 1, 2)$$

Exercise 1.12 (4.3b).

$$I_1 : J(1, 2, 10)$$
$$I_2 : T(1, 3)$$
$$I_3 : T(2, 4)$$
$$I_4 : J(1, 2, 9)$$
$$I_5 : J(3, 4, 10)$$
$$I_6 : S(1)$$
$$I_7 : S(4)$$
$$I_8 : J(1, 1, 4)$$
$$I_9 : S(5)$$
$$I_{10} : T(5, 1)$$

Exercise 1.13 (4.3b).

$$I_1 : S(2)$$
$$I_2 : S(2)$$
$$I_3 : S(2)$$
$$I_4 : J(1, 2, 6)$$
$$I_5 : S(3)$$
$$I_6 : T(3, 1)$$

Exercise 1.14 (4.3c).

$$I_1 : S(2)$$
$$I_2 : J(1,2,10)$$
$$I_3 : J(1,3,9)$$
$$I_4 : S(2)$$
$$I_5 : S(2)$$
$$I_6 : S(3)$$
$$I_7 : S(3)$$
$$I_8 : J(1,1,2)$$
$$I_9 : S(4)$$
$$I_{10} : T(4,1)$$

Exercise 1.15 (5.2.1).

$$f \circ \alpha^{-1}(m) = \begin{cases} m & \text{if } m \text{ is even,} \\ -(m+1) & \text{if } m \text{ is odd.} \end{cases}$$

$$\alpha \circ f \circ \alpha^{-1}(m) = \begin{cases} 2m & \text{if } m \text{ is even,} \\ 2m+1 & \text{if } m \text{ is odd.} \end{cases}$$

The set of instructions is as follows:

$$I_1 : J(1,4,6)$$
$$I_2 : S(4)$$
$$I_3 : S(5)$$
$$I_4 : S(5)$$
$$I_5 : J(1,1,1)$$
$$I_6 : S(2)$$
$$I_7 : J(1,2,14)$$
$$I_8 : J(1,3,15)$$
$$I_9 : S(2)$$
$$I_{10} : S(2)$$
$$I_{11} : S(3)$$

$$I_{12} : S(3)$$

$$I_{13} : J(1, 1, 7)$$

$$I_{14} : S(5)$$

$$I_{15} : T(5, 1)$$

Exercise 1.16. (5.2.2)

$$f \circ \alpha^{-1}(m) = \begin{cases} 1 & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd.} \end{cases}$$

$$\alpha \circ f \circ \alpha^{-1}(m) = \begin{cases} 2 & \text{if } m \text{ is even,} \\ 1 & \text{if } m \text{ is odd.} \end{cases}$$

The set of instructions is as follows:

$$I_1 : S(2)$$

$$I_2 : J(1, 2, 10)$$

$$I_3 : J(1, 3, 9)$$

$$I_4 : S(2)$$

$$I_5 : S(2)$$

$$I_6 : S(3)$$

$$I_7 : S(3)$$

$$I_8 : J(1, 1, 2)$$

$$I_9 : S(4)$$

$$I_{10} : S(4)$$

$$I_{11} : T(4, 1)$$