Math 135 Lecture 5 Notes

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1 More on Relations and Functions

Example 1.1 (Function Example).

$$F : \mathbb{R} \to \mathbb{R}$$

$$x \mapsto x^2$$

$$F = \{t \in \mathbb{R} \times \mathbb{R} : \exists x, x \in R, t = \langle x, x^2 \rangle\}$$

$$\text{ran}F = \{x \in \mathbb{R} : x \geq 0\}$$

Let R, S, A be sets.

Definition 1.1 (Converse Relation).

$$R^{-1} := \{t \in \text{ran}R \times \text{dom}R : \exists a \exists b \exists t, t = \langle b, a \rangle, \langle a, b \rangle \in R\}$$

Essentially, in our (relation) set, we swap every pair of ordered pairs. This should be like an inverse, with the operation of composition.

Definition 1.2 (Composition of relations).

$$R \circ S := \{\langle a, c \rangle : \exists \langle a, b \rangle \in S, \exists \langle b, c \rangle \in R\}$$

$$R \circ S := \{t \in \text{dom}S \times \text{ran}R : \exists a \exists b \exists c, t = \langle a, c \rangle, \langle a, b \rangle \in S, \langle b, c \rangle \in R\}$$

We want this to work similarly to how we think of function composition.

Proposition 1.1. *If F, G are functions, then so is $F \circ G$.*

Proof. Every composition is a relation by definition, so $F \circ G$ is a relation. Let $\langle a, b \rangle \in F \circ G$. Similarly, let $\langle a, c \rangle \in F \circ G$. Unraveling the definitions gives us the existence of some d, e such that $\langle a, d \rangle \in G$ and $\langle d, b \rangle \in F$, as well as $\langle a, e \rangle \in G$, and $\langle e, d \rangle \in F$. Since G is a function, $d = e$. Thus $\langle d, b \rangle \in F$ and $\langle e, c \rangle = \langle d, c \rangle \in F$. Since F is a function, we have that $b = c$. This concludes the proof. $\square$

Remark 1.1 (What is $\text{dom}(F \circ G)$?).

$$\text{dom}(F \circ G) = \{t \in \text{dom}G : \exists c, \langle t, c \rangle \in F \circ G\}$$

$$\text{dom}(F \circ G) = \{t \in \text{dom}G : \exists b \exists c, \langle t, b \rangle \in G \wedge \langle b, c \rangle \in F\}$$

Definition 1.3 (Relation/Function Restriction). If we want to restrict R to A, we denote this as $R \restriction A$

$$R \restriction A = \{t \in R : \exists a \exists b, a \in A, t = \langle a, b \rangle, t \in R\}$$

We just want to make sure the first coordinate comes from A.

Definition 1.4 (Image of a function/relation). The image of A under R is denoted as

$$R[\![A]\!] = \text{ran}(R \restriction A)$$

Definition 1.5 (Function Application).

$$f(a) = b \iff f \text{ is a function} \wedge \langle a, b \rangle \in f$$

Remark 1.2. We reserve the parentheses notation only for the latter notation. If we wrote $R(A)$, this would indicate that the set A is an element of the domain of R. In the bracket notation, we want to think of A as a subset of the domain of R. This extends the way we think about a function to something on the powerset.

Definition 1.6 (Preimage). We take a look everything in some set which might be in the codomain and we look at every set which maps to this set. The preimage of a set A under R is

$$\{t : a \in A \wedge \langle t, a \rangle \in R\}$$

We can formally write this as

$$R^{-1}[\![A]\!]$$

Proposition 1.2.

$$domR \restriction A = (domR) \cap A$$

Proposition 1.3.

$$dom(R \circ S) = S^{-1}[\![domR]\!]$$

Proof.

$$a \in \text{dom}(R \circ S) \iff$$

$$\exists c, \langle a, c \rangle \in R \circ S \iff$$

$$\exists b, \langle a, b \rangle \in S \wedge \langle b, c \rangle \in R \iff$$

$$\exists b, b \in \text{dom}R, \langle b, a \rangle \in S^{-1} \iff$$

$$a \in S^{-1}[\![\text{dom}R]\!]$$

$\square$

Remark 1.3. We keep using the inverse notation, but the inverse doesn't mean much without an identity. So what is the identity supposed to do? It should take any element $x \in X$ and return x.

Definition 1.7 (Identity Function)**.** For any set X,

$$I_X = \{t \in X \times X : \exists x, x \in X, t = \langle x, x \rangle\}$$

Example 1.2 (Identity Function on the Empty Set)**.**

$$I_\emptyset = \emptyset$$

$$\text{dom}\emptyset = \text{ran}\emptyset = \emptyset$$

$$\emptyset^{-1} = \emptyset$$

Proposition 1.4 (Composition of relation and its converse)**.** *What will $R^{-1} \circ R$ be? Will it be I_{domR}? We know $I_{domR} \subseteq R^{-1} \circ R$*

Proof.

$$t \in I_{\text{dom}R} \iff$$

$$\exists b, b \in \text{dom}R \wedge t = \langle b, b \rangle$$

Since $\exists a$ such that $\langle b, a \rangle \in R$ and $\langle b, a \rangle \in R^{-1}$. Then by definition, $\langle b, b \rangle \in R^{-1} \circ R$. $\square$

Example 1.3 ($I_{\text{dom}R} \neq R^{-1} \circ R$)**.** Let X be any set with at least two elements. For example,

$$X = \{\emptyset, \{\emptyset\}\}$$

Consider the function $F = X \times \{b\}$. Computing the converse yields $F^{-1} = \{b\} \times X$.

$$F^{-1} \circ F = \{\langle x, z \rangle : \langle x, y \rangle \in F, \langle y, z \rangle \in F^{-1}\} = X \times X$$

Definition 1.8 (Single Rooted Relations)**.** A relation R is called single rooted iff

$$\forall a \forall b \forall c [\langle a, c \rangle \in R \wedge \langle b, c \rangle \in R \implies a = b]$$

Remark 1.4. A relation R is single rooted $\iff$ R^{-1} is a function

Definition 1.9 (Injective)**.** A function F is injective if F is single-rooted. We also call injective functions one-to-one.

Proposition 1.5. *If R is a single-rooted relation, then*

$$R^{-1} \circ R = I_{domR}$$

Proof. Suppose $\langle a, c\rangle \in R^{-1} \circ R$. Then there exists c such that $\langle a, b\rangle \in R$ and $\langle b, c\rangle \in R^{-1}$. If $\langle b, c\rangle \in R^{-1}$, it follows that $\langle c, b\rangle \in R$. By the definition of single-rooted, $a = c$. Thus $\langle a, c\rangle = \langle a, a\rangle \in I_{\mathrm{dom}R}$. We proved the other inclusion above. $\square$

Theorem 1.1 (1-1 Functions)**.** *Let $F : X \to Y$ be a function. Then TFAE*

- *F is one-to-one*
- $\exists g, g : Z \to X \wedge g \circ F = I_X$
- *F^{-1} is a function and $F^{-1} \circ F = I_X$*