Math 135 Lecture 6 Notes

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1 Functions and the Axiom of Choice

1.1 Functions

Theorem 1.1 (1-1 Functions). *Let $F : X \to Y$ be a function. Then TFAE*

- *F is one-to-one*
- $\exists g, g : Z \to X \wedge g \circ F = I_X$
- *F^{-1} is a function and $F^{-1} \circ F = I_X$*

Proof. We have shown 1 implies 3 in the previous lecture. 3 implies 2 is trivial. Let us prove 2 implies 1.
Supppose there exists $g : Z \to X$ such that $g \circ f = I_X$. Let $a, b \in \text{dom} f = X$. Suppose $f(a) = f(b)$. Apply g to both sides, to get $g(f(a)) = a = b = g(f(b))$. □

Definition 1.1 (Onto). A function $f : X \to Y$ is onto Y if and only if $\text{ran} f = Y$. Similarly, if $\exists g : Y \to X$ such that $f \circ g = I_Y$, then f is onto Y.

Proposition 1.1. *Let R be a relation and let R^{-1} be its converse relation. Then*

$$R \circ R^{-1} \supseteq I_{ranR}$$

Proof. Let $t \in I_{\text{ran}R}$. Then $\exists a \in \text{ran}R$ such that $t = \langle a, a \rangle$. Thus $\exists b$ such that $\langle b, a \rangle \in R$. This implies $\langle a, b \rangle \in R^{-1}$. From that we can conclude that $\langle a, a \rangle \in R \circ R^{-1}$ □

Proposition 1.2. *Let $f : X \to Y$ be a one-to-one function. Then*

$$dom(f \circ f^{-1}) = ranf$$

Proof. We have already shown $I_{\text{ran}f} \subseteq f \circ f^{-1}$. Let $b \in \text{ran}f$. Then $\exists a$ such that $\langle a, b \rangle \in f$, so $\langle b, a \rangle \in f^{-1}$. Thus $\langle b, b \rangle \in f \circ f^{-1}$. Thus $b \in \text{dom}(f \circ f^{-1})$.
Let $b \in \text{dom}(f \circ f^{-1})$. Then $\exists c \exists a$ such that $\langle b, c \rangle \in f^{-1}$ and $\langle c, a \rangle \in f$. Then we know that $\langle c, b \rangle \in f$. Thus $b \in \text{ran}f$ □

Proposition 1.3. *If $f : X \to Y$ is a one-to-one function, then $f \circ f^{-1}$ a function and $f \circ f^{-1} = I_{ranf}$.*

Proof. We showed that $f \circ f^{-1} \supseteq I_{\text{ran}f}$ and that $\text{dom}(f \circ f^{-1}) = \text{ran}f$. Let $t \in f \circ f^{-1}$. Then $\exists a \in \text{ran}f$ and $\exists a, c$ such that $t = \langle a, c \rangle$ and $\langle a, b \rangle \in f^{-1}$ and $\langle b, c \rangle \in f$. It follows that $\langle b, a \rangle \in f$. Since f is a function, we have that $a = c$, so $t = \langle a, a \rangle \in I_{\text{ran}f}$. $\square$

Example 1.1.

$$f : \mathbb{C} \to \mathbb{C}$$

$$z \mapsto z^2$$

Does there exists $g : \mathbb{C} \to \mathbb{C}$ such that $f \circ g = I_{\mathbb{C}}$? f^{-1} is not a function. For every complex number z, we can write $z = re^{i\theta}$ for $r \geq 0$ and $0 \leq \theta \leq 2\pi$.

$$g(z) = \sqrt{r}e^{i\frac{\theta}{2}}$$

$$f(g(z)) = (\sqrt{r}e^{i\frac{\theta}{2}})^2 = |r|e^{i\theta}$$

1.2 Intro to the Axiom of Choice

Axiom 1.1 (Axiom of Choice)**.** *For every relation R, there exists a function F such that $F \subseteq R$ and $domF = domR$.*

Remark 1.1. Start with the relation R. For each $a \in \text{dom}R$.

$$R[\![\{a\}]\!] = \{b : \langle a, b \rangle \in R\} \neq \emptyset$$

"Define" $F(a) = b$ by choosing some $b \in R[\![\{a\}]\!]$.

Theorem 1.2 (Axiom of Choice Implies Right Inverses)**.** *$f : X \to Y$ is onto Y if and only if $\exists g : Y \to X$ such that $f \circ g = I_Y$*

Proof. $\Longleftarrow$ was completed earlier today.
$\Longrightarrow$ By the axiom of choice there exists $g \subseteq f^{-1}$ a function with the same domain, Y. $\text{ran}g \subseteq \text{ran}f^{-1} = \text{dom}f = X$. Thus $g : Y \to X$ and $g \subseteq f^{-1}$. We conclude that $\text{dom}(f \circ g) = Y$. We know $f \circ g \subseteq f \circ f^{-1} \subseteq I_Y$. If $t \in f \circ f^{-1}$, then $\exists a \exists b$ such that $\langle a, b \rangle = t \in f \circ f^{-1}$ and $\exists c$ such that $\langle a, c \rangle \in f^{-1}$ and $\langle c, b \rangle \in f$. Thus $a = b$ and $t = \langle a, a \rangle$ Since $\text{dom}(f \circ g) = Y = \text{dom}(I_Y)$ and $f \circ g \subseteq I_Y$, we have that

$$f \circ g = I_Y$$

$\square$

Definition 1.2 (Functions from X to Y)**.**

$$^XY = \{f \in \mathcal{P}(X \times Y) : f : X \to Y\}$$

Example 1.2. Let Y be any set

$$^{\emptyset}\emptyset = \{f \in P(\emptyset \times \emptyset) : f : \emptyset \to \emptyset\} = \{\emptyset\}$$

$$^{\emptyset}Y = \{f \in P(\emptyset \times Y) : f : \emptyset \to Y\} = \{\emptyset\}$$

$$^{Y}\emptyset = \{f \in P(Y \times \emptyset) : f : Y \to \emptyset\} = \emptyset$$

Example 1.3. Let $X = \{1, 2\}$ and let $Y = \{1, 2, 3\}$ Calculate ^{X}Y and its cardinality.

Axiom 1.2 (Axiom of Choice formulation 2)**.** *Let I, Y be any set. Denote*

$$\bigtimes_{i \in I} X_i = \{f : f \text{ is a function such that } dom f = I \land \forall i \in I, f(i) \in X_i\}$$

The axiom of choice states that for $f : I \to Y$ such that $f(i) \neq \emptyset$ for all $i \in I$.

$$\bigtimes_{i \in I} f(i) \neq \emptyset$$