Math 135 Lecture 2 Notes

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1 Introduction to the Theory of Sets

We will formalize set theory and prove theorems based on the constructions from the axioms. The theory we will be developing is called ZFC (Zermelo-Fraenkel Set Theory with Choice). ZFC is a first-order logical theory expressed in the language of set theory called

$$\mathscr{L}(\epsilon)$$

ϵ is the only non-logical symbol, called a binary relation, "$a \in b$".

Where a and b are variables, we can combine atomic formula, logical formula, and quantifiers to form our language. Atomic formula include $a = b$ and $a \in b$. Logical formula include $a \implies b$, $\neg a$, $a \wedge b$, $a \vee b$, $a \iff b$. Quantifiers are $\exists x \varphi$ and $\forall x \varphi$.

An $\mathscr{L}(\epsilon)$-structure is a nonempty set V together with a set $\epsilon \subset V \times V$. For $a, b \in V$

$$(V, \epsilon^V) \models a \in b \iff (a, b) \in \epsilon^V$$

Example 1.1 (The universe of sets)**.**

$$(\mathbb{V}, \epsilon)$$

Theorem 1.1 (ZFC cannot prove that there is a model of ZFC)**.** *i.e. a set V and a set $\epsilon^V \subseteq V \times V$ s.t every axiom of ZFC is true.*

Theorem 1.2 (Godel's Incompleteness Theorem for ZFC)**.** *If ZFC can prove that ZFC is consistent then ZFC is inconsistent*

1.1 Introducing the Axioms

Axiom 1.1 (Axiom of Extensionality)**.** *This axiom defines what it means for two sets to be the same. The equality operator is taken to be primitive in first-order logic and we need to define how it relates to the other symbols.*

$$\forall A \forall B [A = B \iff \forall t [t \in A \iff t \in B]]$$

Example 1.2 (An example of a structure)**.**

$$V = \{0,1\}$$

$$\epsilon^V = \{(0,0),(0,1)\}$$

$$V \models 0 \in 1$$

$$V \models 0 \in 0$$

$(V, \epsilon^V) \not\models$ the axiom of extensionality.

$$V \models \neg(0 = 1)$$

but

$$V \models \forall t[t \in 0 \iff t \in 1]$$

Axiom 1.2 (Empty Set Axiom)**.** *There exists an empty set.*

$$\exists x \forall t \neg[t \in x]$$

We can abbreviate this with the following

$$\exists x \forall t (t \notin x)$$

Definition 1.1 (Defining the $\notin$ symbol)**.** We want to define formula that aren't strictly part of our language. Let us define the negation of the set membership.

$$\Delta_{\notin} : \forall x \forall y [x \notin y \iff \neg(x \in y)] \in \mathscr{L}(\epsilon, \not\epsilon)$$

Proposition 1.1. *For any $\mathscr{L}(\epsilon)$ structure (M, ϵ^M), there is unique expansion $(M, \epsilon^M, /\epsilon^M) \models \Delta_{\notin}$*

Remark 1.1 (Introducing Symbols)**.** We will want to introduce many new symbols throughout the course of this class. e.g. $\subseteq, \mathcal{P}\mathbb{A}$. We will need to define new symbols logically each time we introduce them and can do so hierarchically with previously defined symbols.

Example 1.3 (A structure consistent with the Axioms)**.**

$$V = \{0\}$$

$$\epsilon^V = \emptyset$$

$(V, \epsilon^V) \models$ Empty Set and Extensionality Axioms

Proposition 1.2 (Uniqueness of the Empty Set)**.** *There is exactly one empty set.*

Proof. By the Empty Set Axiom, there is at least one empty set. If x and y are two empty sets. i.e. $(\forall t) t \notin x$ and $(\forall t) t \notin y$, then $\forall t(t \in x \iff t \in y)$. By the Axiom of Extensionality, $x = y$ $\square$

Definition 1.2 (Empty Set Formula).

$$\Delta_{\emptyset} : \forall x[x = \emptyset \iff \forall t(t \notin x)] \in \mathscr{L}(\epsilon, \not\epsilon, \emptyset)$$

where the ϵ and $\not\epsilon$ are binary relation symbols and the $\emptyset$ symbol is called a constant symbol.

Definition 1.3 (Subset Formula).

$$\Delta_{\subseteq} : \forall x \forall y[x \subseteq y \iff \forall t[t \in x \implies t \in y]]$$

Axiom 1.3 (Power Set Axiom).

$$\forall x \exists z \forall t[t \in z \iff t \subseteq x]$$

Proposition 1.3 (Uniqueness of the Power Set). $\forall x$ *there is a unique power set of* x

Definition 1.4 (Power Set Operation).

$$\Delta_{\mathcal{P}} : \forall x \forall z[z = \mathcal{P}x \iff \forall t[t \in z \iff t \subseteq x]]$$

$\mathcal{P}$ is a unary function symbol.

Example 1.4 (What Sets can we build so far). $\emptyset, \mathcal{P}(\emptyset) = \{\emptyset\}, \mathcal{P}\mathcal{P}\emptyset = \{\emptyset, \{\emptyset\}\}$, etc We can keep iterating powersets. We will need to add more axioms. For example, we cannot prove the existence of $\{\{\emptyset\}\}$ with the axioms we have so far.

Axiom 1.4 (Pair Set Axiom).

$$\forall x \forall y \exists z \forall t[t \in z \iff (t = x \vee t = y)]$$

Definition 1.5 (Pair Set Operation).

$$\Delta_{\{,\}} : \forall x \forall t \forall z[z = \{x, y\} \iff \forall t[t \in z \iff t = z \vee t = y]]$$

Example 1.5 (Singleton Set).

$$x = y$$

$$\{x, y\} = \{x, x\} = \{x\}$$

Proposition 1.4 (Existence of the Singleton).

$$\forall x \exists z \forall t[t \in z \iff t = x]$$

Proof. We can apply the Pair Set Axiom with $y = x$ $\square$

Definition 1.6 (Singleton Operation)**.**

$$\Delta_{\{\}} = \forall x \forall y [z = \{x\} \iff \forall t [t \in z \iff t = x]]$$

Example 1.6 (Triplet Set)**.** Can we create a set with three items? We can attempt to apply the pair set axiom to $\{a\}$ and $\{b, c\}$, but unfortunately we will end up with $\{\{a\}, \{b, c\}\}$. We will need to define the union.

Axiom 1.5 (Binary Union Axiom (Not ZFC))**.**

$$\forall x \forall y \exists z \forall t [t \in z \iff (t \in x \lor t \in y)]$$

Definition 1.7 (Binary Union Operation)**.**

$$\Delta_{\cup} : \forall x \forall y \forall z [z = x \cup y \iff \forall t [t \in z \iff (t \in x \lor t \in y)]]$$

Example 1.7 (Triplet Set Reprise)**.** Now with the binary union axiom, we can apply the operation to $\{a\}$ and $\{b, c\}$, which were found by the pair set axiom. Using the Binary Union Axiom,

$$\{a\} \cup \{b, c\} = \{a, b, c\}$$