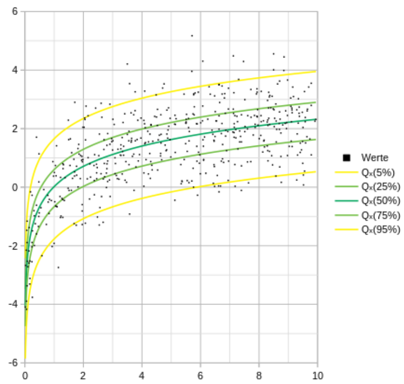
# Quantile regression 分位数回归

**Quantile regression** is a type of regression analysis used in statistics and econometrics. Whereas the method of least squares estimates the conditional *mean* of the response variable across values of the predictor variables, quantile regression estimates the conditional *median* (or other *quantiles*) of the response variable. Quantile regression is an extension of linear regression used when the conditions of linear regression are not met.

> 分位数回归是统计学和计量经济学中使用的一种回归分析。最小二乘法估计跨预测变量值的响应变量的条件均值，而分位数回归估计响应变量的条件中位数（或其他分位数）。分位数回归是线性回归的扩展，在不满足线性回归的条件时使用。

## Advantages and applications

## 优势与应用




Example for quantile regression

> 分位数回归示例

One advantage of quantile regression relative to ordinary least squares regression is that the quantile regression estimates are more robust against outliers in the response measurements. However, the main attraction of quantile regression goes beyond this and is advantageous when conditional quantile functions are of interest. Different measures of central tendency and statistical dispersion can be used to more comprehensively analyze the relationship between variables.[1]

> 分位数回归相对于普通最小二乘回归的优势之一是分位数回归估计对于响应测量中的异常值更加稳健。然而，分位数回归的主要吸引力超出了这一点，并且在对条件分位数函数感兴趣时具有优势。集中趋势和统计离散度的不同度量可以更全面地分析变量之间的关系。[1]

In ecology, quantile regression has been proposed and used as a way to discover more useful predictive relationships between variables in cases where there is no relationship or only a weak relationship between the means of such variables. The need for and success of quantile regression in ecology has been attributed to the complexity of interactions between different factors leading to data with unequal variation of one variable for different ranges of another variable.[2]

> 在生态学中，分位数回归已被提出并用作发现变量之间更有用的预测关系的方法，在这些变量的均值之间没有关系或只有微弱关系的情况下。生态学中分位数回归的必要性和成功归因于不同因素之间相互作用的复杂性，导致数据的一个变量对于另一个变量的不同范围具有不相等的变化。[2]

Another application of quantile regression is in the areas of growth charts, where percentile curves are commonly used to screen for abnormal growth.[3][4]

> 分位数回归的另一个应用是在生长图表领域，其中百分位数曲线通常用于筛查异常生长。[3] [4]

## History 历史

The idea of estimating a median regression slope, a major theorem about minimizing sum of the absolute deviances and a geometrical algorithm for constructing median regression was proposed in 1760 by Ruđer Josip Bošković, a Jesuit Catholic priest from Dubrovnik.[1]:4[5] He was interested in the ellipticity of the earth, building on Isaac Newton's suggestion that its rotation could cause it to bulge at the equator with a corresponding flattening at the poles.[6] He finally produced the first geometric procedure for determining the equator of a rotating planet from three observations of a surface feature. More importantly for quantile regression, he was able to develop the first evidence of the least absolute criterion and preceded the least squares introduced by Legendre in 1805 by fifty years.[7]

> 估计中值回归斜率的想法、关于最小化绝对偏差和的主要定理以及构建中值回归的几何算法于 1760 年由来自杜布罗夫尼克的耶稣会天主教神父 Ruđer Josip Bošković 提出。 [1] :4 [5] 他对地球的椭圆率很感兴趣，建立在艾萨克牛顿的建议之上，即地球的自转会导致它在赤道凸起，并在两极相应地变平。 [6] 他终于提出了第一个几何程序，用于根据对表面特征的三个观察来确定旋转行星的赤道。更重要的是，对于分位数回归，他能够开发出最小绝对标准的第一个证据，并且比勒让德于 1805 年引入的最小二乘法早了 50 年。 [7]

Other thinkers began building upon Bošković's idea such as Pierre-Simon Laplace, who developed the so-called "methode de situation." This led to Francis Edgeworth's plural median[8] - a geometric approach to median regression - and is recognized as the precursor of the simplex method.[7] The works of Bošković, Laplace, and Edgeworth were recognized as a prelude to Roger Koenker's contributions to quantile regression.

> 其他思想家开始在 Bošković 的想法的基础上发展，例如 Pierre-Simon Laplace，他开发了所谓的“情境方法”。这导致了 Francis Edgeworth 的复数中值 [8] ——一种中值回归的几何方法——并被公认为单纯形法的先驱。 [7] Bošković、Laplace 和 Edgeworth 的作品被认为是 Roger Koenker 对分位数回归所做贡献的序曲。

Median regression computations for larger data sets are quite tedious compared to the least squares method, for which reason it has historically generated a lack of popularity among statisticians, until the widespread adoption of computers in the latter part of the 20th century.

> 与最小二乘法相比，较大数据集的中值回归计算相当繁琐，因此在历史上它在统计学家中不受欢迎，直到 20 世纪后期计算机被广泛采用。

# Quantiles 分位数

Quantile regression expresses the conditional quantiles of a dependent variable as a linear function of the explanatory variables. Crucial to the practicality of quantile regression is that the quantiles can be expressed as the solution of a minimization problem, as we will show in this section before discussing conditional quantiles in the next section.

> 分位数回归将因变量的条件分位数表示为解释变量的线性函数。分位数回归实用性的关键是分位数可以表示为最小化问题的解，正如我们将在下一节讨论条件分位数之前在本节中展示的那样。

## Quantile of a random variable

> ## 随机变量的分位数

Let $Y$ be a real-valued random variable with cumulative distribution function $F_Y(y) = P(Y \leq y)$. The $\tau$ th quantile of Y is given by

> 令 $Y$ 为具有累积分布函数 $F_Y(y) = P(Y \leq y)$ 的实值随机变量。 Y 的第 $\tau$ 分位数由下式给出

$$q_Y(\tau) = F_Y^{-1}(\tau) = \inf\{y : F_Y(y) \geq \tau\}$$

where $\tau \in (0,1)$. $\tau \in (0,1)$. 在哪里

Define the loss function as $\rho_\tau(m) = m(\tau - \mathbb{I}_{(m<0)})$, where $\mathbb{I}$ is an indicator function. A specific quantile can be found by minimizing the expected loss of $Y - u$ with respect to $u$:[1](pp. 5–6):

> 定义损失函数为 $\rho_\tau(m) = m(\tau - \mathbb{I}_{(m<0)})$ ，其中 $\mathbb{I}$ 为指示函数。可以通过最小化 $Y - u$ 相对于 $u$ 的预期

> 损失来找到特定的分位数：[1]（第 5–6 页）：

$$q_Y(\tau) = \arg\min_u E(\rho_\tau(Y-u)) = \arg\min_u \left\{(\tau-1)\int_{-\infty}^{u}(y-u)dF_Y(y) + \tau\int_{u}^{\infty}(y-u)dF_Y(y)\right\}.$$

This can be shown by computing the derivative of the expected loss via an application of the Leibniz integral rule, setting it to 0, and letting $q_\tau$ be the solution of

> 这可以通过应用莱布尼茨积分规则计算预期损失的导数来显示，将其设置为 0，并让 $q_\tau$ 成为解决方案

$$0 = (1-\tau)\int_{-\infty}^{q_\tau} dF_Y(y) - \tau\int_{q_\tau}^{\infty} dF_Y(y).$$

This equation reduces to

> 这个等式简化为

$$0 = F_Y(q_\tau) - \tau,$$

and then to 然后到

$$F_Y(q_\tau) = \tau.$$

If the solution $q_\tau$ is not unique, then we have to take the smallest such solution to obtain the $\tau$th quantile of the random variable $Y$.

> 如果解 $q_\tau$ 不是唯一的，那么我们必须取最小的这样的解来获得随机变量 Y 的第 $\tau$ th 分位数。

### Example 例子

Let $Y$ be a discrete random variable that takes values $y_i = i$ with $i = 1, 2, \ldots, 9$ with equal probabilities. The task is to find the median of Y, and hence the value $\tau = 0.5$ is chosen. Then the expected loss of $Y - u$ is

> 设 $Y$ 是一个离散随机变量，它以相等的概率取值 $y_i = i$ 和 $i = 1, 2, \ldots, 9$ 。任务是找到 Y 的中位数，因此选择值 $\tau = 0.5$ 。那么 $Y - u$ 的预期损失是

$$L(u) = E(\rho_\tau(Y-u)) = \frac{(\tau-1)}{9}\sum_{y_i<u}(y_i-u) + \frac{\tau}{9}\sum_{y_i\geq u}(y_i-u) = \frac{0.5}{9}\left(-\sum_{y_i<u}(y_i-u) + \sum_{y_i\geq u}(y_i-u)\right).$$

Since $0.5/9$ is a constant, it can be taken out of the expected loss function (this is only true if $\tau = 0.5$). Then, at $u$=3,

> 由于 $0.5/9$ 是一个常量，所以可以从预期的损失函数中取出（只有 $\tau = 0.5$ 才成立）。然后，在 u=3 时，

$$L(3) \propto \sum_{i=1}^{2} -(i-3) + \sum_{i=3}^{9}(i-3) = [(2+1) + (0+1+2+\ldots+6)] = 24.$$

Suppose that $u$ is increased by 1 unit. Then the expected loss will be changed by $(3) - (6) = -3$ on changing $u$ to 4. If, $u$=5, the expected loss is

> 假设 u 增加 1 个单位。然后将u更改为4时预期损失将改变 $(3) - (6) = -3$ 。如果u = 5，则预期损失为

$$L(5) \propto \sum_{i=1}^{4} i + \sum_{i=0}^{4} i = 20,$$

and any change in $u$ will increase the expected loss. Thus $u$=5 is the median. The Table below shows the expected loss (divided by $0.5/9$) for different values of $u$.

> u 的任何变化都会增加预期损失。因此 u=5 是中位数。下表显示了不同 u 值的预期损失（除以 $0.5/9$ ）。

| *u* 你 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| Expected loss 预期损失 | 36 | 29 | 24 | 21 | 20 | 21 | 24 | 29 | 36 |

### Intuition 直觉

Consider $\tau = 0.5$ and let $q$ be an initial guess for $q_\tau$. The expected loss evaluated at $q$ is

> 考虑 $\tau = 0.5$ 并让 q 为 $q_\tau$ 的初始猜测。在 q 评估的预期损失是

$$L(q) = -0.5 \int_{-\infty}^{q} (y-q) dF_Y(y) + 0.5 \int_{q}^{\infty} (y-q) dF_Y(y).$$

In order to minimize the expected loss, we move the value of $q$ a little bit to see whether the expected loss will rise or fall. Suppose we increase $q$ by 1 unit. Then the change of expected loss would be

> 为了最小化预期损失，我们稍微移动一下q的值，看看预期损失会上升还是下降。假设我们将 q 增加 1 个单位。那么预期损失的变化将是

$$\int_{-\infty}^{q} 1 dF_Y(y) - \int_{q}^{\infty} 1 dF_Y(y).$$

The first term of the equation is $F_Y(q)$ and second term of the equation is $1 - F_Y(q)$. Therefore, the change of expected loss function is negative if and only if $F_Y(q) < 0.5$, that is if and only if $q$ is smaller than the median. Similarly, if we reduce $q$ by 1 unit, the change of expected loss function is negative if and only if $q$ is larger than the median.

> 等式的第一项是 $F_Y(q)$ ，等式的第二项是 $1 - F_Y(q)$ 。因此，当且仅当 $F_Y(q) < 0.5$ ，即当且仅当q小于中位数时，期望损失函数的变化为负。类似地，如果我们将 q 减少 1 个单位，当且仅当 q 大于中位数时，预期损失函数的变化为负。

In order to minimize the expected loss function, we would increase (decrease) $L(q)$ if $q$ is smaller (larger) than the median, until $q$ reaches the median. The idea behind the minimization is to count the number of points (weighted with the density) that are larger or smaller than $q$ and then move $q$ to a point where $q$ is larger than $100\tau\%$ of the points.

> 为了最小化预期的损失函数，如果 q 小于（大于）中位数，我们将增加（减少）L(q)，直到 q 达到中位数。最小化背后的想法是计算大于或小于 q 的点数（用密度加权），然后将 q 移动到 q 大于 $100\tau$ % 的点。

## Sample quantile 样本分位数

The $\tau$ sample quantile can be obtained by using an importance sampling estimate and solving the following minimization problem

> $\tau$ 样本分位数可以通过使用重要性抽样估计并解决以下最小化问题来获得

$$\hat{q}_\tau = \arg\min_{q\in\mathbb{R}} \sum_{i=1}^{n} \rho_\tau(y_i - q),$$
$$= \arg\min_{q\in\mathbb{R}} \left[(\tau-1)\sum_{y_i<q}(y_i-q) + \tau\sum_{y_i\geq q}(y_i-q)\right],$$
$$= \arg\min_{q\in\mathbb{R}} \left[(\tau-1)\sum_{y_i<q}(y_i-q) + \tau\sum_{y_i\geq q}(y_i-q)\right],$$

where the function $\rho_\tau$ is the tilted absolute value function. The intuition is the same as for the population quantile.

> 其中函数 $\rho_\tau$ 是倾斜的绝对值函数。直觉与人口分位数相同。

## Conditional quantile and quantile regression

> ## 条件分位数和分位数回归

The $\tau$th conditional quantile of $Y$ given $X$ is the $\tau$th quantile of the Conditional probability distribution of $Y$ given $X$,

> 给定 $X$ 的 $Y$ 的第 $\tau$ 个条件分位数是给定 $X$ 的 $Y$ 的条件概率分布的第 $\tau$ 个分位数，

$$Q_{Y|X}(\tau) = \inf\left\{y : F_{Y|X}(y) \geq \tau\right\}.$$

We use a capital $Q$ to denote the conditional quantile to indicate that it is a random variable.

> 我们使用大写的 $Q$ 来表示条件分位数，以表明它是一个随机变量。

In quantile regression for the $\tau$th quantile we make the assumption that the $\tau$th conditional quantile is given as a linear function of the explanatory variables:

> 在第 $\tau$ th 分位数的分位数回归中，我们假设第 $\tau$ th 条件分位数作为解释变量的线性函数给出：

$$Q_{Y|X}(\tau) = X\beta_\tau.$$

Given the distribution function of $Y$, $\beta_\tau$ can be obtained by solving

> 给定 $Y$ 的分布函数， $\beta_\tau$ 可以通过求解

$$\beta_\tau = \arg\min_{\beta\in\mathbb{R}^k} E(\rho_\tau(Y - X\beta)).$$

Solving the sample analog gives the estimator of $\beta$.

> 求解样本模拟给出了 $\beta$ 的估计量。

$$\hat{\beta}_\tau = \arg\min_{\beta\in\mathbb{R}^k} \sum_{i=1}^{n}(\rho_\tau(Y_i - X_i\beta)).$$

Note that when $\tau = 0.5$, the loss function $\rho_\tau$ is proportional to the absolute value function, and thus median regression is the same as linear regression by least absolute deviations.

> 请注意，当 $\tau = 0.5$ 时，损失函数 $\rho_\tau$ 与绝对值函数成正比，因此中值回归与最小绝对偏差的线性回归相同。

# Computation of estimates for regression parameters

# 计算回归参数的估计值

The mathematical forms arising from quantile regression are distinct from those arising in the method of least squares. The method of least squares leads to a consideration of problems in an inner product space, involving projection onto subspaces, and thus the problem of minimizing the squared errors can be reduced to a problem in numerical linear algebra. Quantile regression does not have this structure, and instead the minimization problem can be reformulated as a linear programming problem

> 分位数回归产生的数学形式与最小二乘法产生的数学形式不同。最小二乘法导致考虑内积空间中的问题，涉及到子空间的投影，因此最小化平方误差的问题可以简化为数值线性代数中的问题。分位数回归没有这种结构，相反，最小化问题可以重新表述为线性规划问题

$$\min_{\beta, u^+, u^- \in \mathbb{R}^k \times \mathbb{R}^{2n}_+} \left\{ \tau 1_n' u^+ + (1-\tau) 1_n' u^- | X\beta + u^+ - u^- = Y \right\},$$

where 在哪里

$$u_j^+ = \max(u_j, 0), \quad u_j^- = -\min(u_j, 0). \quad u_j^+ = \max(u_j, 0), u_j^- = -\min(u_j, 0).$$

Simplex methods[1]:181 or interior point methods[1]:190 can be applied to solve the linear programming problem.

> 可以应用单纯形法 [1] :181 或内点法 [1] :190 来解决线性规划问题。

## Asymptotic properties 渐近性质

For $\tau \in (0,1)$, under some regularity conditions, $\hat{\beta}_\tau$ is asymptotically normal:

> 对于 $\tau \in (0,1)$，在某些正则条件下，$\hat{\beta}_\tau$ 是渐近正态的：

$$\sqrt{n}(\hat{\beta}_\tau - \beta_\tau) \xrightarrow{d} N(0, \tau(1-\tau) D^{-1} \Omega_x D^{-1}),$$

where 在哪里

$$D = E(f_Y(X\beta) XX') \text{ and } \Omega_x = E(X'X). \quad D = E(f_Y(X\beta) XX') \text{ 和 } \Omega_x = E(X'X).$$

Direct estimation of the asymptotic variance-covariance matrix is not always satisfactory. Inference for quantile regression parameters can be made with the regression rank-score tests or with the bootstrap methods.[9]

> 渐近方差-协方差矩阵的直接估计并不总是令人满意。分位数回归参数的推断可以通过回归等级得分测试或引导方法进行。[9]

## Equivariance 等方差

See invariant estimator for background on invariance or see equivariance.

> 有关不变性的背景，请参见不变估计器或参见等方差。

### Scale equivariance 尺度等方差

For any $a > 0$ and $\tau \in [0, 1]$

> 对于任何 $a > 0$ 和 $\tau \in [0, 1]$

$$\hat{\beta}(\tau; aY, X) = a\hat{\beta}(\tau; Y, X),$$
$$\hat{\beta}(\tau; -aY, X) = -a\hat{\beta}(1 - \tau; Y, X).$$

### Shift equivariance 移位等方差

For any $\gamma \in R^k$ and $\tau \in [0, 1]$

> 对于任何 $\gamma \in R^k$ 和 $\tau \in [0, 1]$

$$\hat{\beta}(\tau; Y + X\gamma, X) = \hat{\beta}(\tau; Y, X) + \gamma.$$

### Equivariance to reparameterization of design

> ### 设计重新参数化的等效性

Let $A$ be any $p \times p$ nonsingular matrix and $\tau \in [0, 1]$

> 令 $A$ 为任意 $p \times p$ 非奇异矩阵， $\tau \in [0, 1]$

$$\hat{\beta}(\tau; Y, XA) = A^{-1}\hat{\beta}(\tau; Y, X).$$

### Invariance to monotone transformations

> ### 单调变换的不变性

If $h$ is a nondecreasing function on $\mathbb{R}$, the following invariance property applies:

> 如果 $h$ 是 $\mathbb{R}$ 上的非递减函数，则适用以下不变性：

$$h(Q_{Y|X}(\tau)) \equiv Q_{h(Y)|X}(\tau).$$

Example (1): 例子（一）：

If $W = \exp(Y)$ and $Q_{Y|X}(\tau) = X\beta_\tau$, then $Q_{W|X}(\tau) = \exp(X\beta_\tau)$. The mean regression does not have the same property since $\mathrm{E}(\ln(Y)) \neq \ln(\mathrm{E}(Y))$.

> 如果 $W = \exp(Y)$ 和 $Q_{Y|X}(\tau) = X\beta_\tau$ ，则 $Q_{W|X}(\tau) = \exp(X\beta_\tau)$ 。自 $\mathrm{E}(\ln(Y)) \neq \ln(\mathrm{E}(Y))$. 以来，均值回归没有相同的属性

## Bayesian methods for quantile regression

> ## 分位数回归的贝叶斯方法

Because quantile regression does not normally assume a parametric likelihood for the conditional distributions of Y|X, the Bayesian methods work with a working likelihood. A convenient choice is the asymmetric Laplacian likelihood,[10] because the mode of the resulting posterior under a flat prior is the usual quantile regression estimates. The posterior inference, however, must be interpreted with care. Yang, Wang and He[11] provided a posterior variance adjustment for valid inference. In addition, Yang and He[12] showed that one can have asymptotically valid posterior inference if the working likelihood is chosen to be the empirical likelihood.

> 由于分位数回归通常不假设 Y|X 的条件分布的参数似然，因此贝叶斯方法使用工作似然。一个方便的选择是非对称拉普拉斯似然， [10] ，因为在平坦先验下得到的后验模型是通常的分位数回归估计。然而，必须谨慎解释后验推理。 Yang、Wang 和 He [11] 为有效推理提供了后验方差调整。此外，Yang 和 He [12] 表明，如果选择工作似然作为经验似然，则可以进行渐近有效的后验推理。

# Machine learning methods for quantile regression

> # 分位数回归的机器学习方法

Beyond simple linear regression, there are several machine learning methods that can be extended to quantile regression. A switch from the squared error to the tilted absolute value loss function allows gradient descent-based learning algorithms to learn a specified quantile instead of the mean. It means that we can apply all neural network and deep learning algorithms to quantile regression,[13][14] which is then referred to as nonparametric quantile regression.[15] Tree-based learning algorithms are also available for quantile regression (see, e.g., Quantile Regression Forests,[16] as a simple generalization of Random Forests).

> 除了简单的线性回归之外，还有几种机器学习方法可以扩展到分位数回归。从平方误差到倾斜绝对值损失函数的切换允许基于梯度下降的学习算法学习指定的分位数而不是均值。这意味着我们可以将所有的神经网络和深度学习算法应用于分位数回归， [13] [14] ，这就是所谓的非参数分位数回归。 [15] 基于树的学习算法也可用于分位数回归（参见，例如分位数回归森林， [16] 作为随机森林的简单概括）。

# Censored quantile regression

> # 截尾分位数回归

If the response variable is subject to censoring, the conditional mean is not identifiable without additional distributional assumptions, but the conditional quantile is often identifiable. For recent work on censored quantile regression, see: Portnoy[17] and Wang and Wang[18]

> 如果响应变量受到删失，则条件均值在没有附加分布假设的情况下无法识别，但条件分位数通常是可识别的。有关删失分位数回归的最新工作，请参阅：Portnoy [17] 和 Wang and Wang [18]

Example (2): 例子（二）：

Let $Y^c = \max(0, Y)$ and $Q_{Y|X} = X\beta_\tau$. Then $Q_{Y^c|X}(\tau) = \max(0, X\beta_\tau)$. This is the censored quantile regression model: estimated values can be obtained without making any distributional assumptions, but at the cost of computational difficulty,[19] some of which can be avoided by using a simple three step censored quantile regression procedure as an approximation.[20]

> 让 $Y^c = \max(0, Y)$ 和 $Q_{Y|X} = X\beta_\tau$ 。然后 $Q_{Y^c|X}(\tau) = \max(0, X\beta_\tau)$ 。这是截尾分位数回归模型：可以在不做任何分布假设的情况下获得估计值，但是以计算困难为代价， [19] 通过使用简单的三步截尾分位数回归程序作为近似值可以避免其中的一些困难。 [20]

For random censoring on the response variables, the censored quantile regression of Portnoy (2003)[17] provides consistent estimates of all identifiable quantile functions based on reweighting each censored point appropriately.

> 对于响应变量的随机删失，Portnoy (2003) [17] 的删失分位数回归基于适当重新加权每个删失点提供了所有可识别分位数函数的一致估计。

## Heteroscedastic Errors 异方差误差

The quantile regression loss needs to be adapted in the presence of heteroscedastic errors in order to be efficient.[21]

> 为了提高效率，需要在存在异方差误差的情况下调整分位数回归损失。[21]

## Implementations 实现

Numerous statistical software packages include implementations of quantile regression:

> 许多统计软件包包括分位数回归的实现：

- Matlab function `quantreg`[22]

  > Matlab函数 `quantreg` [22]

- gretl has the `quantreg` command.[23]

  > gretl 有 `quantreg` 命令。[23]

- R offers several packages that implement quantile regression, most notably `quantreg` by Roger Koenker,[24] but also `gbm`,[25] `quantregForest`,[26] `qrnn`[27] and `qgam`[28]

  > R 提供了几个实现分位数回归的包，最著名的是 Roger Koenker 的 `quantreg` 、 [24] ，还有 `gbm` 、 [25] 、 `quantregForest` 、 [26] 、 `qrnn` 、 [27] 和 `qgam` 、 [28] 。

- Python, via `Scikit-garden`[29] and `statsmodels`[30]

  > Python，来自 `Scikit-garden` [29] 和 `statsmodels` [30]

- SAS through `proc quantreg` (ver. 9.2)[31] and `proc quantselect` (ver. 9.3).[32]

  > SAS 通过 `proc quantreg` （9.2 版） [31] 和 `proc quantselect` （9.3 版） 。[32]

- Stata, via the `qreg` command.[33][34]

  > Stata，通过 `qreg` 命令。[33] [34]

- Vowpal Wabbit, via `--loss_function quantile`.[35]

  > Vowpal Wabbit，来自 `--loss_function quantile` 。[35]

- Mathematica package `QuantileRegression.m`[36] hosted at the MathematicaForPrediction project at GitHub.

  > Mathematica 包 `QuantileRegression.m` [36] 托管在 GitHub 的 MathematicaForPrediction 项目中。

- Wolfram Language function `QuantileRegression`[37] hosted at Wolfram Function Repository.

  > Wolfram 语言函数 `QuantileRegression` [37] 托管在 Wolfram 函数库中。

# See also 也可以看看

- Least-absolute-deviations regression

> 最小绝对偏差回归

# References 参考

1. Koenker, Roger (2005). *Quantile Regression* (https://archive.org/details/quantileregressi00koen_757). Cambridge University Press. pp. 146 (https://archive.org/details/quantileregressi00koen_757/page/n151)–7. ISBN 978-0-521-60827-5.

> 罗杰·科恩克 (2005)。分位数回归。剑桥大学出版社。第 146–7 页。国际标准书号 978–0–521–60827–5。

2. Cade, Brian S.; Noon, Barry R. (2003). "A gentle introduction to quantile regression for ecologists" (http://www.econ.uiuc.edu/~roger/research/rq/QReco.pdf) (PDF). *Frontiers in Ecology and the Environment*. **1** (8): 412–420. doi:10.2307/3868138 (https://doi.org/10.2307%2F3868138). JSTOR 3868138 (https://www.jstor.org/stable/3868138).

> 凯德，布赖恩 S.；中午，巴里 R. (2003)。“生态学家对分位数回归的简单介绍”(PDF)。生态学与环境前沿。 1 (8): 412—420。内政部：10.2307/3868138。 JSTOR 3868138。

3. Wei, Y.; Pere, A.; Koenker, R.; He, X. (2006). "Quantile Regression Methods for Reference Growth Charts". *Statistics in Medicine*. **25** (8): 1369–1382. doi:10.1002/sim.2271 (https://doi.org/10.1002%2Fsim.2271). PMID 16143984 (https://pubmed.ncbi.nlm.nih.gov/16143984). S2CID 7830193 (https://api.semanticscholar.org/CorpusID:7830193).

> 魏，Y。神父，A.；科恩克，R.；他，X. (2006)。“参考增长图表的分位数回归方法”。医学统计。 25 (8): 1369—1382。 doi：10.1002/sim.2271。 PMID 16143984。S2CID 7830193。

4. Wei, Y.; He, X. (2006). "Conditional Growth Charts (with discussions)". *Annals of Statistics*. **34** (5): 2069–2097 and 2126–2131. arXiv:math/0702634 (https://arxiv.org/abs/math/0702634). doi:10.1214/009053606000000623 (https://doi.org/10.1214%2F009053606000000623). S2CID 88516697 (https://api.semanticscholar.org/CorpusID:88516697).

> 魏，Y。他，X. (2006)。“条件增长图表（带讨论）”。统计年鉴。 34 (5): 2069—2097 和 2126—2131。 arXiv：数学/0702634。内政部：10.1214/009053606000000623。 S2CID 88516697。

5. Stigler, S. (1984). "Boscovich, Simpson and a 1760 manuscript note on fitting a linear relation". *Biometrika*. **71** (3): 615–620. doi:10.1093/biomet/71.3.615 (https://doi.org/10.1093%2Fbiomet%2F71.3.615).

> Stigler, S. (1984)。“Boscovich、Simpson 和 1760 年关于拟合线性关系的手稿注释”。生物识别。 71 (3): 615—620。 doi：10.1093/biomet/71.3.615。

6. Koenker, Roger (2005). *Quantile Regression* (https://archive.org/details/quantileregressi00koen_757). Cambridge: Cambridge University Press. pp. 2 (https://archive.org/details/quantileregressi00koen_757/page/n7). ISBN 9780521845731.

> 罗杰·科恩克 (2005)。分位数回归。剑桥：剑桥大学出版社。第 2 页。ISBN 9780521845731。

7. Furno, Marilena; Vistocco, Domenico (2018). *Quantile Regression: Estimation and Simulation*. Hoboken, NJ: John Wiley & Sons. pp. xv. ISBN 9781119975281.

> 弗诺，玛丽莲娜；多梅尼科·维斯托科 (2018)。分位数回归：估计和模拟。新泽西州霍博肯：John Wiley & Sons。第十五页国际标准书号 9781119975281。

8. Koenker, Roger (August 1998). "Galton, Edgeworth, Frisch, and prospects for quantile regression in economics" (http://www.econ.uiuc.edu/~roger/research/galton/Galton.pdf) (PDF). *UIUC.edu*. Retrieved August 22, 2018.

> Koenker, Roger（1998 年 8 月）。“Galton、Edgeworth、Frisch 和经济学中分位数回归的前景”(PDF)。 UIUC.edu。检索 August 22, 2018。

9. Kocherginsky, M.; He, X.; Mu, Y. (2005). "Practical Confidence Intervals for Regression Quantiles". *Journal of Computational and Graphical Statistics*. **14** (1): 41–55. doi:10.1198/106186005X27563 (https://doi.org/10.1198%2F106186005X27563). S2CID 120598656 (https://api.semanticscholar.org/CorpusID:120598656).

> Kocherginsky, M.；他，X。 Mu, Y. (2005)。“回归分位数的实用置信区间”。计算和图形统计杂志。 14 (1): 41–55。内政部：10.1198/106186005X27563。 S2CID 120598656。

10. Kozumi, H.; Kobayashi, G. (2011). "Gibbs sampling methods for Bayesian quantile regression" (https://www.b.kobe-u.ac.jp/papers_files/2009_02.pdf) (PDF). *Journal of Statistical Computation and Simulation*. **81** (11): 1565–1578. doi:10.1080/00949655.2010.496117 (https://doi.org/10.1080%2F00949655.2010.496117). S2CID 44015988 (https://api.semanticscholar.org/CorpusID:44015988).

> Kozumi, H.；小林，G.（2011 年）。“用于贝叶斯分位数回归的吉布斯抽样方法”(PDF)。统计计算与模拟杂志。 81 (11): 1565–1578。内政部：10.1080/00949655.2010.496117。 S2CID 44015988。

11. Yang, Y.; Wang, H.X.; He, X. (2016). "Posterior Inference in Bayesian Quantile Regression with Asymmetric Laplace Likelihood". *International Statistical Review*. **84** (3): 327–344. doi:10.1111/insr.12114 (https://doi.org/10.1111%2Finsr.12114). hdl:2027.42/135059 (https://hdl.handle.net/2027.42%2F135059). S2CID 14947362 (https://api.semanticscholar.org/CorpusID:14947362).

> 杨，Y。王，H.X.；他，X. (2016)。“具有不对称拉普拉斯似然的贝叶斯分位数回归中的后验推理”。国际统计评论。 84 (3): 327–344。内政部：10.1111/insr.12114。高密度脂蛋白：2027.42/135059。S2CID 14947362。

12. Yang, Y.; He, X. (2010). "Bayesian empirical likelihood for quantile regression". *Annals of Statistics*. **40** (2): 1102–1131. arXiv:1207.5378 (https://arxiv.org/abs/1207.5378). doi:10.1214/12-AOS1005 (https://doi.org/10.1214%2F12-AOS1005). S2CID 88519086 (https://api.semanticscholar.org/CorpusID:88519086).

> 杨，Y。他，X. (2010)。“分位数回归的贝叶斯经验似然”。统计年鉴。 40 (2): 1102–1131。 arXiv：1207.5378。内政部：10.1214/12-AOS1005。 S2CID 88519086。

13. Petneházi, Gábor (2019-08-21). "QCNN: Quantile Convolutional Neural Network". arXiv:1908.07978 (https://arxiv.org/abs/1908.07978) [cs.LG (https://arxiv.org/archive/cs.LG)].

> Petneházi, Gábor (2019-08-21)。“QCNN：分位数卷积神经网络”。 arXiv：1908.07978 [cs.LG]。

14. Rodrigues, Filipe; Pereira, Francisco C. (2018-08-27). "Beyond expectation: Deep joint mean and quantile regression for spatio-temporal problems". arXiv:1808.08798 (https://arxiv.org/abs/1808.08798) [stat (https://arxiv.org/archive/stat)].

> 罗德里格斯，菲利普； Pereira, Francisco C. (2018-08-27)。“超出预期：时空问题的深度联合均值和分位数回归”。 arXiv：1808.08798 [统计]。

15. Nonparametric Quantile Regression: Non-Crossing Constraints and Conformal Prediction by Wenlu Tang, Guohao Shen, Yuanyuan Lin, Jian Huang, https://arxiv.org/pdf/2210.10161.pdf

> 非参数分位数回归：非交叉约束和共形预测作者：Wenlu Tang、Guohao Shen、Yuanyuan Lin、Jian Huang，https://arxiv.org/pdf/2210.10161.pdf

16. Meinshausen, Nicolai (2006). "Quantile Regression Forests" (http://www.jmlr.org/papers/volume7/meinshausen06a/meinshausen06a.pdf) (PDF). *Journal of Machine Learning Research*. **7** (6): 983–999.

> Meinshausen, Nicolai (2006)。“分位数回归森林”(PDF)。机器学习研究杂志。 7 (6): 983–999。

17. Portnoy, S. L. (2003). "Censored Regression Quantiles". *Journal of the American Statistical Association*. **98** (464): 1001–1012. doi:10.1198/016214503000000954 (https://doi.org/10.1198%2F016214503000000954). S2CID 120674851 (https://api.semanticscholar.org/CorpusID:120674851).

> Portnoy, S. L. (2003)。“截尾回归分位数”。美国统计协会杂志。 98 (464)：1001—1012。内政部：10.1198/016214503000000954。 S2CID 120674851。

18. Wang, H.; Wang, L. (2009). "Locally Weighted Censored Quantile Regression". *Journal of the American Statistical Association*. **104** (487): 1117–1128. CiteSeerX 10.1.1.504.796 (https://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.504.796). doi:10.1198/jasa.2009.tm08230 (https://doi.org/10.1198%2Fjasa.2009.tm08230). S2CID 34494316 (https://api.semanticscholar.org/CorpusID:34494316).

> 王，H。王 L. (2009)。“局部加权截尾分位数回归”。美国统计协会杂志。 104 (487)：1117—1128。 CiteSeerX 10.1.1.504.796。 doi：10.1198/jasa.2009.tm08230。 S2CID 34494316。

19. Powell, James L. (1986). "Censored Regression Quantiles". *Journal of Econometrics*. **32** (1): 143–155. doi:10.1016/0304-4076(86)90016-3 (https://doi.org/10.1016%2F0304-4076%2886%2990016-3).

> 鲍威尔，詹姆斯 L. (1986)。“截尾回归分位数”。计量经济学杂志。 32 (1): 143—155。内政部：10.1016/0304–4076(86)90016–3。

20. Chernozhukov, Victor; Hong, Han (2002). "Three-Step Censored Quantile Regression and Extramarital Affairs". *J. Amer. Statist. Assoc.* **97** (459): 872–882. doi:10.1198/016214502388618663 (https://doi.org/10.1198%2F016214502388618663). S2CID 1410755 (https://api.semanticscholar.org/CorpusID:1410755).

> 切尔诺朱科夫，维克多；韩洪 (2002)。“三步截尾分位数回归和婚外情”。 J.阿梅尔。国家主义者。协会 97 (459)：872—882。内政部：10.1198/016214502388618663。 S2CID 1410755。

21. Efficient Quantile Regression for Heteroscedastic Models by, Yoonsuh Jung, Yoonkyung Lee, Steven N. MacEachern, https://www.tandfonline.com/doi/abs/10.1080/00949655.2014.967244?journalCode=gscs20

> 异方差模型的有效分位数回归，作者：Yoonsuh Jung、Yoonkyung Lee、Steven N. MacEachern, https://www.tandfonline.com/doi/abs/10.1080/00949655.2014.967244?journalCode=gscs20

22. "quantreg(x,y,tau,order,Nboot) - File Exchange - MATLAB Central" (http://www.mathworks.com/matlabcentral/fileexchange/32115-quantreg-x-y-tau-order-nboot-). *www.mathworks.com*. Retrieved 2016-02-01.

> “quantreg(x,y,tau,order,Nboot) – 文件交换 – MATLAB Central”。 www.mathworks.com。检索到 2016–02–01 。

23. "Gretl Command Reference" (http://ricardo.ecn.wfu.edu/pub//gretl/manual/en/gretl-ref.pdf) (PDF). April 2017.

> “Gretl 命令参考”(PDF)。 2017 年 4 月。

24. "quantreg: Quantile Regression" (https://cran.r-project.org/web/packages/quantreg/index.html). *R Project*. 2018-12-18.

> “quantreg：分位数回归”。 R 项目。 2018–12–18。

25. "gbm: Generalized Boosted Regression Models" (https://cran.r-project.org/web/packages/gbm/). *R Project*. 2019-01-14.

> “gbm：广义增强回归模型”。 R 项目。 2019–01–14。

26. "quantregForest: Quantile Regression Forests" (https://cran.r-project.org/web/packages/quantregForest/index.html). *R Project*. 2017-12-19.

> “quantregForest：分位数回归森林”。 R 项目。 2017–12–19。

27. "qrnn: Quantile Regression Neural Networks" (https://cran.r-project.org/web/packages/qrnn/index.html). *R Project*. 2018-06-26.

    > "qrnn：分位数回归神经网络"。 R 项目。 2018-06-26.

28. "qgam: Smooth Additive Quantile Regression Models" (https://cran.r-project.org/web/packages/qgam/index.html). *R Project*. 2019-05-23.

    > "qgam：平滑加性分位数回归模型"。 R 项目。 2019-05-23.

29. "Quantile Regression Forests" (https://scikit-garden.github.io/examples/QuantileRegressionForests/). *Scikit-garden*. Retrieved 3 January 2019.

    > "分位数回归森林"。 Scikit 花园。检索到 3 January 2019。

30. "Statsmodels: Quantile Regression" (https://www.statsmodels.org/dev/examples/notebooks/generated/quantile_regression.html). *Statsmodels*. Retrieved 15 November 2019.

    > "统计模型：分位数回归"。统计模型。检索到 15 November 2019。

31. "An Introduction to Quantile Regression and the QUANTREG Procedure" (http://www2.sas.com/proceedings/sugi30/213-30.pdf) (PDF). *SAS Support*.

    > "分位数回归和 QUANTREG 程序简介"(PDF)。 SAS 支持。

32. "The QUANTSELECT Procedure" (https://support.sas.com/rnd/app/stat/procedures/quantselect.html). *SAS Support*.

    > "QUANTSELECT 程序"。 SAS 支持。

33. "qreg — Quantile regression" (https://www.stata.com/manuals15/rqreg.pdf) (PDF). *Stata Manual*.

    > "qreg — 分位数回归"(PDF)。斯塔塔手册。

34. Cameron, A. Colin; Trivedi, Pravin K. (2010). "Quantile Regression" (https://books.google.com/books?id=UkKQRAAACAAJ&pg=PA211). *Microeconometrics Using Stata* (Revised ed.). College Station: Stata Press. pp. 211–234. ISBN 978-1-59718-073-3.

    > 卡梅伦，A. 科林； Trivedi, Pravin K. (2010)。 "分位数回归"。使用 Stata 的微观计量经济学（修订版）。大学城：Stata 出版社。第 211–234 页。国际标准书号 978-1-59718-073-3。

35. "JohnLangford/vowpal_wabbit" (https://github.com/JohnLangford/vowpal_wabbit/wiki/Loss-functions). *GitHub*. Retrieved 2016-07-09.

    > "JohnLangford/vowpal_wabbit" GitHub。检索到 2016-07-09 。

36. "QuantileRegression.m" (https://github.com/antononcube/MathematicaForPrediction/blob/master/QuantileRegression.m). *MathematicaForPrediction*. Retrieved 3 January 2019.

    > "分位数回归.m"。数学预测。检索到 3 January 2019。

37. "QuantileRegression" (https://resources.wolframcloud.com/FunctionRepository/resources/QuantileRegression). *Wolfram Function Repository*. Retrieved 14 September 2022.

    > "分位数回归"。 Wolfram 函数库。检索到 14 September 2022。


## Further reading 延伸阅读


- Angrist, Joshua D.; Pischke, Jörn-Steffen (2009). "Quantile Regression" (https://books.google.com/books?id=ztXL21Xd8v8C&pg=PA269). *Mostly Harmless Econometrics: An Empiricist's Companion*.

Princeton University Press. pp. 269–291. ISBN 978-0-691-12034-8.

> 安格里斯特，约书亚 D.； Pischke, Jörn-Steffen (2009)。“分位数回归”。基本无害的计量经济学：经验主义者的伴侣。普林斯顿大学出版社。第 269–291 页。国际标准书号 978-0-691-12034-8。

- Koenker, Roger (2005). *Quantile Regression* (https://books.google.com/books?id=hdkt7V4NXsgC). Cambridge University Press. ISBN 978-0-521-60827-5.

> 罗杰·科恩克 (2005)。分位数回归。剑桥大学出版社。国际标准书号 978-0-521-60827-5。

---