Job Notes

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1 Preliminaries

Radius of convergence

2 Brain Teasers

Question 2.1 What is $\sqrt{2016} - \sqrt{2015}$?

Question 2.2 Separate any quadrilateral into two equal areas from one of its vertices. `http://activity.ntsec.gov.tw/activity/race-1/48/high/030414.pdf`

Question 2.3 1000 (with one bad) bottles of wine and 10 rats. How about 2 bad bottles? (**Goldman Sachs Strat**)

Answer 2.1 Binary coding.

Question 2.4 Say there are 4 people sit in a circle and numbered from 1 to 4. 1 has a gun, and he uses the gun to kill 2 and then hands it to 3; 3 then uses the gun to kill 4 and hands it to 1; 1 uses the gun to kill 3 and then only 1 survived, and the game ends. Who will survive if there are N people? (**Squarepoint Research Quant**)

3 Differential Equations

Question 3.1 Solve $f' = f$ and $g'' = g$. (**Goldman Sachs Strat**)

Answer 3.1

$$f(x) = Ce^x \tag{1}$$
$$g(x) = Ce^x + De^{-x} \tag{2}$$

Question 3.2 Solve $y'(x) + ay(x) = f(x)$ with $f(0) = x_0$. (**Barclays**)

4 Calculus

Question 4.1 Expand e^x using Taylor expansion. When does $\sum_{i=1}^{\infty} \frac{x^n}{n!}$ converge? (**Goldman Sachs Strat**)

Question 4.2 $\lim\limits_{x\to 0} \left[\frac{a^x+b^x}{2}\right]^{\frac{1}{x}}$ (**Barclays**)

Answer 4.1

$$f(x) = \left[\frac{a^x + b^x}{2}\right]^{\frac{1}{x}} \tag{3}$$
$$g(x) = ln(f(x)) = \frac{\ln \frac{a^x+b^x}{2}}{x} \tag{4}$$
$$\lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{\ln \frac{a^x+b^x}{2}}{x} \tag{5}$$
$$= \lim_{x\to 0} \frac{\frac{1}{\frac{a^x+b^x}{2}} \frac{(a^x \ln a + b^x \ln b)}{2}}{1} \tag{6}$$
$$= \lim_{x\to 0} \frac{(a^x \ln a + b^x \ln b)}{a^x + b^x} \tag{7}$$
$$= \frac{(\ln a + \ln b)}{2} \tag{8}$$
$$\lim_{x\to 0} f(x) = \exp(\lim_{x\to 0} g(x)) = \exp \frac{(\ln a + \ln b)}{2} = \sqrt{ab} \tag{9}$$

5 Linear Algebra

Question 5.1 Prove if $\mathbf{A}^T\mathbf{A} = \mathbf{A}^2$, then $\mathbf{A}$ is real symmetric ($\mathbf{A} = \mathbf{A}^T$).

Answer 5.1 Hint: to prove $\mathbf{B} = \mathbf{0}$, only need to show $\text{tr}(\mathbf{B}^T\mathbf{B}) = \mathbf{0}$. Hence,

$$\text{tr}((\mathbf{A} - \mathbf{A}^T)^T(\mathbf{A} - \mathbf{A}^T)) = \text{tr}(\mathbf{A}^T\mathbf{A} - \mathbf{A}^T\mathbf{A}^T - \mathbf{A}\mathbf{A} + \mathbf{A}\mathbf{A}^T) \tag{10}$$
$$= \text{tr}(\mathbf{A}^2 - (\mathbf{A}^T)^2 - \mathbf{A}^2 + \mathbf{A}\mathbf{A}^T) \tag{11}$$
$$= \text{tr}(-(\mathbf{A}^T)^2 + (\mathbf{A}^2)^T) \tag{12}$$
$$= \mathbf{0} \tag{13}$$

6 Probability and Statistics

Question 6.1 What is $\mathbb{E}_X[\Phi(aX+b)]$? X is a normal with expectation μ and variance σ^2. Φ is the standard normal CDF.

Answer 6.1 $\Phi\left(\frac{a\mu+b}{\sqrt{1+a^2\sigma^2}}\right)$.

Question 6.2 What are: 1) derivatives of $\phi(x)$, 2) integration of $\Phi(x)$?

Answer 6.2 $\phi'(x) = -x\phi(x)$, and $\int \Phi(x) = x\Phi(x) + \phi(x)$.

Question 6.3 What are the mean and variance of a log-normal distribution?

Answer 6.3 See book.

Question 6.4 Tell me all you know about the Gamma distribution.

Answer 6.4 See book.

Question 6.5 Two random variables X, Y, not sure if they are correlated. Find the minimum square error (MSE) estimator of $X|Y$ (**Cubist / Point72**).

Answer 6.5

$$\mathbb{E}[(\hat{X} - X)^2|Y] = \hat{X}^2 - 2\hat{X}\mathbb{E}(X)|Y \tag{14}$$

So the minimum square error estimator is $\hat{X} = \mathbb{E}(X|Y)$.

Question 6.6 Explain Cramér-Rao lower bound (CRLB) Theorem (**Cubist / Point72**)?

Answer 6.6 See book.

Question 6.7 N data points,

$$y_i = C + x_i \tag{15}$$

$E(N_i) = 0, Var(x_i) = \sigma_i^2$. Given y_i, what is the unbiased minimum variance linear estimator of C (**Cubist / Point72**)?

Answer 6.7

$$\hat{C} = \sum_{i=1}^{N} \beta_i y_i \tag{16}$$

Unbiasedness says $E(\hat{C}) = \sum_{i=1}^{N} \beta_i y_i = \sum_{i=1}^{N} \beta_i C = C$, or $\sum_{i=1}^{N} \beta_i = 1$. Minimum variance is to minimize the following objective function (Lagrange multiplier method):

$$\sum_{i=1}^{N} \beta_i^2 \sigma_i^2 + \lambda \left(\sum_{i=1}^{N} \beta_i - 1 \right) \tag{17}$$

which gives

$$\hat{\beta}_i = \frac{1/\sigma_i^2}{\sum_{i=1}^{N} \frac{1}{\sigma_i^2}} \tag{18}$$

Question 6.8 Two players roll a die. The first player wins if he gets a 1, otherwise the second player wins if he gets a 2, otherwise the game restarts. What is the probability that the first player wins (**Sameer**)?

Answer 6.8 6/11.

Question 6.9 Draw random numbers from a uniform distribution on $[0, 1]$. Draw again if the number is lower than the last one, and stop if higher. What is the expectation of draws (**Sameer**)?

Question 6.10 Two people take turns to flip a coin until it shows an "HT", and the one who flipped the "T" wins the game. What's the probability of winning if flip first? First, is this probability smaller or equal or bigger than 50%? (**Barclays**)

Answer 6.9 The probability is smaller than 50% since you need an "H" first. Use recursion. (**Barclays**)

Question 6.11 Generate two random variables X and Y, $Corr(X, Y) = \rho$, from two independent random variable $Z_1, Z2$ (Cholesky Decomposition). (**Barclays**)

Question 6.12 Toss a dice 12000 times, denote X as the number of times when get 6. $P(1800 < X < 2100)$? Central limit theorem. (**Barclays**)

Question 6.13 Roll a dice, you get the dollar amount equal to the number shown on the dice. You can stop anytime or roll again up to three times. What is your expected return? (**Barclays**)

Answer 6.10 Work backwards as a DP problem.

Question 6.14 Toss a coin, with probabiliy p the outcome is head (H), and $q = 1 - p$ a tail (T). What is the smallest expected number of tosses to obtain a patter P_n of length n? How about the k^{th} smallest?

Answer 6.11 For the first occurrence: shift the pattern P_n one at a time and collect all overlapping sub-patterns with P_n, then add the corresponding conditional expected numbers. Hence the more overlaps you have, the bigger the expected number of obtaining that pattern. For the follow-up occurrences, further add the marginal expected number of P_n.

Example 1: For pattern "HTHHTHH", the expected number of first occurrence is:

$$\begin{aligned} E(HTHHTHH) &= E(HTHH) + p^{-5}q^{-2} && (19) \\ &= E(H) + p^{-3}q^{-1} + p^{-5}q^{-2} && (20) \\ &= p^{-1} + p^{-3}q^{-1} + p^{-5}q^{-2} && (21) \end{aligned}$$

The expcted number of the $k^{th}(k \geq 2)$ occrences is

$$\begin{aligned} E((HTHHTHH)_k) &= E(HTHHTHH) + (k-1)p^{-5}q^{-2} && (22) \\ &= (p^{-1} + p^{-3}q^{-1} + p^{-5}q^{-2}) + (k-1)p^{-5}q^{-2} && (23) \end{aligned}$$

Example 2: For a fair coin ($p = 1/2$), the expected number of the first occurrence: $E(HH) = E(TT) = 6$, $E(HT) = E(TH) = 4$, $E(HHT) = E(THH) = 8$, $E(HTH) = E(THT) = 10$, $E(HHH) = E(TTT) = 14$, and etc.

Question 6.15 What is the expected number of (fair) coin tosses you need to have two heads in a row? (**Barclays**)

Answer 6.12 six, by Recursive function.

Question 6.16 Secretary problem.

Question 6.17 Describe the Central Limit Theorem. (**Goldman Sachs Strat**)

Question 6.18 What is $P(X = 0.5), if X \sim U[0, 1]$? Give an example of a combination of discrete and continuous random variable. (**Goldman Sachs Strat**)

Answer 6.13 $P(X = 0.5) = 0$. Toss a coin, if it's head, draw from a discrete distribution; otherwise if it's tail, draw from a continuous distribution.

Question 6.19 How to generate uniform distribution of [0,100] using one fair coin? How about using a biased coin? (**Goldman Sachs Strat**)

Answer 6.14 Since $2^7 = 128 > 100$, we can throw a fair coin 7 times and discard (any) 28 outcomes then use the remaining 100 outcomes to represent $[0, 100]$. Notice that regardless of p, HT and TH always have the same probability. So for a biased coin, we throw a coin twice but ignore HH and TT, and let say HT represent a *head* of a fair coin and TH a *tail* of a fair coin. Then this problem reduces to the fair coin version.

Question 6.20 Consider the following game. The player tosses a die once only. The payoff is 1 dollar for each dot on the upturned face. Assuming a fair die, at what level should you set the ticket price of this game? (**The Blue Book**)

Question 6.21 Suppose we play a game. I roll a die up to three times. Each time I roll, you can either take the number showing as dollars, or roll again. What is your expected winnings? (**The Blue Book**)

Question 6.22 Let's play a game. There are four sealed boxes. There is 100 pounds in one box and the others are empty. A player can pay X to open a box and take the contents as many times as they like. Assuming this is a fair game, what is the value of X? (**The Blue Book**)

Question 6.23 We play a game: I pick a number n from 1 to 100. If you guess correctly, I pay you $\$n$ and zero otherwise. How much would you pay to play this game? (**The Blue Book**)

Question 6.24 Suppose you have a fair coin. You start with a dollar, and if you toss a H, your position doubles, if you toss a T, your position halves. What is the expected value of the money you have if you toss the coin infinitely? (**The Blue Book**)

Question 6.25 Suppose we toss a fair coin, and let N denote the number of tosses until we get a head (including the final toss). What is $E(N)$ and $Var(N)$? (**The Blue Book**)

Question 6.26 We play a game, with a fair coin. The game stops when either two heads (H) or tails (T) appear consecutively. What is the expected time until the game stops? (**The Blue Book**)

Question 6.27 For a fair coin, what is the expected number of tosses to get three heads in a row? (**The Blue Book**)

Question 6.28 You toss a biased coin. What is the expected length of time until a head is tossed? For two consecutive heads? (**The Blue Book**)

Question 6.29 I have a bag containing nine ordinary coins and one double-headed one. I remove a coin and flip it three times. It comes up heads each time. What is the probability that it is the double-header? (**The Blue Book**)

Question 6.30 I take an ordinary-looking coin out of my pocket and flip it three times. Each time it is a head. What do you think is the probability that the next flip is also a head? What if I had flipped the coin 100 times and each flip was a head? (**The Blue Book**)

Question 6.31 You throw a fair coin one million times. What is the expected number of strings of 6 heads followed by 6 tails? (**The Blue Book**)

Question 6.32 Suppose you are throwing a dart at a circular board. What is your expected distance from the center? Make any necessary assumptions. Suppose you win a dollar if you hit 10 times in a row inside a radius of $R/2$, where R is the radius of the board. You have to pay 10 cents for every try. If you try 100 times, how much money would you have lost/made in expectation? Does your answer change if you are a professional and your probability of hitting inside R/2 is double of hitting outside $R/2$? (**The Blue Book**)

Question 6.33 If Y and Z are independent standard normal random variables, and $X = aY + bZ$, what are the expectation and variance of $X|Y$? (**Barclays**)
A: $E(X|Y) = aY$, $Var(X|Y) = b^2$.

Question 6.34 You and I each flip 3 fair coins, if we got same heads I pay you \$2, if different you pay me \$1. Will you play this game? (**SIG**)

Answer 6.15 Naive Way:

$$P(A=B=0) = \left[\binom{3}{0}\left(\frac{1}{2}\right)^3\right]^2 = \frac{1}{64} \tag{24}$$

$$P(A=B=1) = \left[\binom{3}{1}\left(\frac{1}{2}\right)^3\right]^2 = \frac{9}{64} \tag{25}$$

$$P(A=B=2) = \left[\binom{3}{2}\left(\frac{1}{2}\right)^3\right]^2 = \frac{9}{64} \tag{26}$$

$$P(A=B=3) = \left[\binom{3}{3}\left(\frac{1}{2}\right)^3\right]^2 = \frac{1}{64} \tag{27}$$

So

$$P(A=B) = \frac{1}{64} + \frac{9}{64} + \frac{9}{64} + \frac{1}{64} = \frac{5}{16} \tag{28}$$

Don't forget the case when both had no head (A=B=0).

Clever Way: Let both do their tossing and then let B turn over each of her coins. Then the event you are looking for is that exactly three out of six coins show heads. Since B's "trick" doesn't destroy any randomness or independency, the answer is

$$P(A=B) = \binom{6}{3}\left(\frac{1}{2}\right)^3 = \frac{5}{16} \tag{29}$$

Question 6.35 In a hospital there were 3 boys and some girls. A woman gave birth to a child in the hospital. A nurse picked up a child at random and was a boy. What is the probability that that woman gave birth to a boy? (**SIG**)

Answer 6.16 Naive Way:

$$\begin{aligned}
P(\text{Boy} \mid \text{Picked a Boy}) &= \frac{P(\text{Picked a Boy} \mid \text{Boy})P(\text{Boy})}{P(\text{Picked a Boy})} && (30)\\
&= \frac{P(\text{Picked a Boy} \mid \text{Boy})P(\text{Boy})}{P(\text{Picked a Boy} \mid \text{Boy})P(\text{Boy}) + P(\text{Picked a Boy} \mid \text{Girl})P(\text{Girl})} && (31)\\
&= \frac{\frac{4}{3+N+1}\cdot\frac{1}{2}}{\frac{4}{3+N+1}\cdot\frac{1}{2} + \frac{3}{3+N+1}\cdot\frac{1}{2}} && (32)\\
&= \frac{4}{4+3} = \frac{4}{7} && (33)
\end{aligned}$$

Question 6.36 A cup of water. You drink half, and I drink half of the rest, and you drink half of the rest, and let this process continue until the cup is empty. How much water did you drink? (**LinkedIn**)

Answer 6.17 Naive Way:

$$\frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots = \frac{\frac{1}{2}}{1-\frac{1}{4}} = \frac{2}{3} \tag{34}$$

Clever Way: The ratio is alway 2:1, so you had $\frac{2}{3}$.

Question 6.37 Dice With Increasing Number. Throw a fair dice three times, what is the probability that we obtain three numbers in strictly increasing order?

Answer 6.18 There are $6 \times 5 \times 4$ ways to pick three different numbers (for strictly increasing order), and in total there are $(6 \times 6 \times 6) \times 3!$ ways to pick arbitrary three numbers. So the probability is $\frac{6\times5\times4}{(6\times6\times6)\times3!} = \frac{5}{54}$.

Question 6.38 Airplane Seating Problem. 100 passengers are boarding an airplane with 100 seats. Everyone has a ticket with his seat number. These 100 passengers boards the airplane in order. However, the first passenger lost his ticket so he just take a random seat. For any subsequent passenger, he either sits on his own seat or, if the seat is taken, he takes a random empty seat. What's the probability that the last passenger would sit on his own seat? There is a very simple explanation for the result.

Question 6.39 The Balance. You have a balance and need to weigh objects. The weight of each object will be between 1 and 40 pounds inclusive and will be a round number. Whats the fewest number of weights that you need to be able to balance any of these objects?

Question 6.40 Burning Sticks. A stick burns out in one hour from one end to the other. How do you measure 45 minutes using two such sticks? Note that sticks are made of different material and the burning speed along different sections are different so you can't use the length of the burnt section to estimate time.

Question 6.41 3 3 8 8 Puzzle. Using the four numbers 3, 3, 8, and 8, and the usual four arithmetic operations (addition, subtraction, multiplication and division), can you make the number 24?

Answer 6.19 $8 \div (3 - 8 \div 3)$.

7 Finance

Question 7.1 What is implied volatility? (**Barclays**)

Question 7.2 Why you don't excise an American call option early? How about American put? (**Barclays**)

Question 7.3 Portfolio optimization. Say the client has 20 stocks, what information you need from the client to do optimization, what information you need to collect from Bloomberg (consider time horizon, the client's expected return), who to calculate variance-covariance matrix. Write down the minimized function and its constraint. (**Barclays**)

8 Financial Mathematics

Question 8.1 What is the distribution of the minimum of a Brownian motion in a given time frame $[0, t]$, i.e., $\min_{[0,t]} W(s), s \in [0, t]$?

Answer 8.1 Use passage time density of a Brownian motion.

Question 8.2 What is $\mathbb{E}[\int_0^t f(s)\,\mathrm{d}\,W_1(s) \int_0^t g(s)\,\mathrm{d}\,W_2(s)]$, where $\mathrm{d}\,W_1\,\mathrm{d}\,W_2 = \rho\,\mathrm{d}\,t$?

Answer 8.2 $\rho \int_0^t f(s)g(s)\,\mathrm{d}\,t$.

Question 8.3 GARCH model. (**Barclays**)

Question 8.4 What is the distribution of $\int W_t dW_t$? (**Barclays**)

Question 8.5 Is $Z_t = W_t^3 + 3tW_t$ a Martingale? (**Barclays**)

Question 8.6 Describe Ito's lemma. Derive the Black-Scholes equation. Solve

$$dr_t = a(b - r_t)dt + \sigma dW_t \tag{35}$$

But this is not a good model, since the interest rate r_t must not be negative. How to fix it? Add a $\sqrt{r_t}$ term. This is called the CIR modelCIR:

$$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}dW_t \tag{36}$$

Find the first and second moment of the CIR model. What is the distribution of r_t? In general, how would you find a distribution? Hint: the distribution of $\sqrt{r_t}$ is easy. (**Goldman Sachs Strat**)

Answer 8.3 The CIR model bounces back to $r_t = b$. That is, if it's high, the drift term drives it down; if it's low, the volatility part drives it up.

Question 8.7 CIR process, what is the expectation of X and its second moment. (**Barclays**)

Question 8.8 Compute $E(B_t^3|B_s),\ t > s$ (**Barclays**)

Question 8.9 If W_t denotes a Brownian motion, what is dW_t^n, where W_t^n is the n^{th} power of W_t? (**Barclays**)

Answer 8.4 use Ito's lemma.

Question 8.10 If $m = \int_0^1 W_t dt$, what is m^2? (**Barclays**)

Answer 8.5 1/3.

9 Option Pricing

9.1 Black-Scholes

Question 9.1 Derive the Black-Scholes equation for a stock, S. What boundary conditions are satisfied at $S = 0$ and $S = \infty$? (**The Blue Book**)

Answer 9.1 TBD

Question 9.2 Derive the Black-Scholes equation so that an undergrad can understand it. (**The Blue Book**)

Answer 9.2 TBD

Question 9.3 Explain the Black-Scholes equation. (**The Blue Book**)

Answer 9.3 TBD

Question 9.4 Suppose two assets in a Black-Scholes world have the same volatility but different drifts. How will the price of call options on them compare? Now suppose one of the assets undergoes downward jumps at random times. How will this affect option prices? (**The Blue Book**)

Answer 9.4 TBD

Question 9.5 Suppose an asset has a deterministic time dependent volatility. How would I price an option on it using the Black-Scholes theory? How would I hedge it? (**The Blue Book**)

Answer 9.5 TBD

Question 9.6 In the Black-Scholes world, price a European option with a payoff of $\max(S_T^2 - K, 0)$ at time T. (**The Blue Book**)

Answer 9.6 TBD

Question 9.7 Develop a formula for the price of a derivative paying $\max(S_T(S_T - K), 0)$ in the Black-Scholes model. (**The Blue Book**)

Answer 9.7 TBD

Question 9.8 Give me the price of a derivative which pays $\log(S_T)S_T$, you can assume that the Black-Scholes model is valid. How can we get the price more efficiently? (**The Blue Book**)

Answer 9.8 TBD

Question 9.9 Prove that the implied volatility of a put and the implied volatility of a call (with the same strike) are the same. (**The Blue Book**)

Answer 9.9 TBD

Question 9.10 Why drifts are not in the Black Scholes formula?

Answer 9.10 TBD

9.2 Option price properties

Question 9.11 Stock price is \$50 for the moment. Using B-S model we calculated the call option price \$5 using volatility 30%. What would be the price if the volatility is actually 35%? (**SIG**)

Answer 9.11 TBD

Question 9.12 Sketch the value of a vanilla call option as a function of spot. How will it evolve with time? (**The Blue Book**)

Answer 9.12 TBD

Question 9.13 Is it ever optimal to early exercise an American call option? What about a put option? (**The Blue Book**)

Answer 9.13 TBD

Question 9.14 In FX markets an option can be expressed as either a call or a put, explain. Related your answer to 9.13. (**The Blue Book**)

Answer 9.14 TBD

Question 9.15 Approximately how much would a one-month call option at-the-money with a million dollar notional and spot 1 be worth? (**The Blue Book**)

Answer 9.15 TBD

Question 9.16 Suppose a call option only pays off if spot never passes below a barrier B. Sketch the value as a function of spot. Now suppose the option only pays off if spot passes below B instead. Sketch the value of the option again. Relate the two graphs. (**The Blue Book**)

Answer 9.16 TBD

Question 9.17 What is meant by put-call parity? (**The Blue Book**)

Answer 9.17 TBD

Question 9.18 What happens to the price of a vanilla call option as volatility tends to infinity? (**The Blue Book**)

Answer 9.18 TBD

10 Machine Learning

11 Optimization

Question 11.1 What is Newton's method? (**Barclays**)

Question 11.2 Numerical method to find the square root of a number. Newton's method and its convergence, how to set the start point, how to do the test, consider the situation when the input is 0. (**Barclays**)

12 Programming

Question 12.1 In C++, what is a virtual function?

13 Algorithms and Data Structures

Question 13.1 Given an array of numbers, *nums*, return an array of numbers *products*, where $products[i]$ is the product of all $nums[j], j! = i$. Give an $O(N)$ algorithm using only multiplication. (**Squarepoint, Research Quant**)

Question 13.2 Write down a function to compute the sum of digits of an integer input. Write down a recursive version. (**Sameer**)

Question 13.3 Approximation of $cot(x)$, use Taylor expansion to expand the numerator and denominator separately. Draw the graph, what is the difference between the approximation and the "real" curve. How to modify it. (**Barclays**)

Question 13.4 Construct a phone-book using hash table, how to deal with collision (chaining, probing), how to check whether the manipulation is to insert a new value or to update the old value. (**Barclays**)

Question 13.5 Write a program to do matrix multiplication, given the matrix class (consider the extreme situation, throw exception). (**Barclays**)

Question 13.6 Give a list of coordinates $\{(x_1, y_1), \cdots, (x_n, y_n)\}$, sorted by x, write a program to interpolate y given x. Consider robostness (corner cases) and efficiency $O(\log n)$. (**Barclays**)

Question 13.7 Which data structure is not good for searching: binary-search tree, B-tree, heap, hash? (**Barclays**)