limsup and liminf

Xi Tan (tan19@purdue.edu)

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Definition 0.1

$$\liminf_n A_n = \cup_{n=1}^{\infty} \cap_{i=n}^{\infty} A_i = \{x \mid x \in A_i \text{ eventually}\} \tag{1}$$

The meaning of lim inf can be seen by re-writing the above definition as: $x \in \liminf_n A_n$ if $\exists n \in \mathbb{N}$, s.t. $\forall i \geq n$ and $i \in \mathbb{N}$, $x \in A_i$. Hence the elements in $\liminf_n A_n$ are in all but (the first) finitely many sets[1].

Definition 0.2

$$\limsup_n A_n = \cap_{n=1}^{\infty} \cup_{i=n}^{\infty} A_i = \{x \mid x \in A_i \text{ for infinitely many } i\} \tag{2}$$

lim sup can be best seen by examining its complement, according to the De Morgan's laws:

Proposition 0.1

$$(\limsup_n A_n)^c = \liminf_n A_n^c \tag{3}$$

Recall the definition of lim inf, the right hand side of the above can be re-wriiten as: $x \in \liminf_n A_n^c$ if $\exists n \in \mathbb{N}$, s.t. $\forall i \geq n$ and $i \in \mathbb{N}$, $x \notin A_i$. Hence the elements in $\liminf_n A_n^c$ are only in some of the first finitely many sets, and the elements in $\limsup_n A_n$ are therefore those elements in infinitely many sets.

[1] Note, the number of "first finitely many sets" may be different for different elements in lim inf.