Advanced Calculus

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Contents

Chapter 1

The Real and Complex Number Systems

1.1 Introduction

1.1.1 Definitions

Definition 1.1.1 If A is any set (whose elements may be numbers or any other objects), we write $x \in A$ to indicate that x is a memeber (or an element) of A. If x is not a memeber of A, we write: $x \notin A$.

Definition 1.1.2 Throughtout Chap. 1, the set of all rational numbers will be denoted by Q.

Definition 1.1.3 Suppose S is an ordered set, $E \subset S$, and E is bounded above. Suppose there exists an $\alpha \in S$ with the following properties:

(i) α is an upper bound of E.

(ii) If $\gamma < \alpha$ then γ is not an upper bound of E.

Then α is called the *least upper bound* of E [that there is at most one such α is clear from (ii)] or the *supremum* of E, and we write

$$\alpha = \sup E.$$

The *greatest lower bound*, or *infimum*, of a set E which is bounded below is defined in the same manner: The statement

$$\alpha = \inf E$$

means that α is a lower bound of E and that no β with $\beta > \alpha$ is a lower bound of E.

Definition 1.1.4 The extended real number system consists of the real field R and two symbols, $+\infty$ and $-\infty$. We preserve the original order in R, and define

$$-\infty < x < +\infty$$

for every $x \in R$.

Chapter 2

Basic Topology

2.1 Finite, Countable, and Uncountable Sets

2.1.1 Definitions

Definition 2.1.1 Consider two sets A and B, whose elements may be any objects whatsoever, and suppose that with each element x of A there is associated, in some manner, an element of B, which we denote by $f(x)$. Then f is said to be a *function* from A to B (or a *mapping* of A into B). The set A is called the *domain* of f (we also say f is defined on A), and the elments $f(x)$ are called the *values* of f. The set of all values of f is called the *range* of f.

Definition 2.1.2 Let A and B be two sets and let f be a mapping of A into B. If $E \subset A$, $f(E)$ is defined to be the set of all elements $f(x)$, for $x \in E$. We call $f(E)$ the *image* of E under f. In this notation, $f(A)$ is the range of f. It is clear that $f(A) \subset B$. If $f(A) = B$, we say that f maps A *onto* B. (Note that, according to this usage, *onto* is more specific thatn *into*.)

Definition 2.1.3 If $E \subset B$ (E is not necessarily a subset of $f(A)$), $f^{-1}(E)$ denotes the set of all $x \in A$ such that $f(x) \in E$. We call $f^{-1}(E)$ the *inverse image* of E under f. If $y \in B$, $f^{-1}(y)$ is the set of all $x \in A$ such that $f(x) = y$. If, for each $y \in B$, $f^{-1}(y)$ consists of at most one element of A, then f is said to be a 1-1 (*one-to-one*) mapping of A into B. This may also be expressed as follows. f is a 1-1 mapping of A into B provided that $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2, x_1 \in A, x_2 \in A$.

Definition 2.1.4 If there exists a 1-1 mapping of A *onto* B, we say that A and B can be put in *1-1 correspondence*, or that A and B have the same *cardinal number*, or, briefly, that A and B are *equivalent*, and we write $A \sim B$. This relation clearly has the following properties:

It is reflexive: $A \sim A$

It is symmetric: If $A \sim B$, then $B \sim A$

It is transitive: If $A \sim B$ and $B \sim C$, then $A \sim C$

Any relation with theses three properties is called an *equivalence relation*.

Definition 2.1.5 For any positive integer n, let J_n be the set whose elements are the integers $1, 2, \cdots, n$; let J be the set consisting of all positive integers. For any set A, we say:

(a) A is *finite* if $A \sim J_n$ for some n (the empty set is also considered to be finite).

(b) A is *infinite* if A is not finite.

(c) A is *countable* if $A \sim J$.

(d) A is *uncountable* if A is neither finite nor countable.

(e) A is *at most countable* if A is finite or countable.

Countable sets are sometimes called *enumerable* or *denumerable*.

Definition 2.1.6 By a *sequence*, we mean a function f defined on the set J of all positive integers. If $f(n) = x_n$, for $n \in J$, it is customary to denote the sequence f by the symbol $\{x_n\}$, or sometimes by $x_1, x_2, x_3, \cdots$. The values of f, that is, the elements x_n, are called the *terms* of the sequence. If A is a set and if $x_n \in A$ for all $n \in J$, then $\{x_n\}$ is said to be a *sequence in A*, or a *sequence of elements of A*.

Definition 2.1.7 Let A and Ω be sets, and suppose that with each element α of A there is associated a subset of Ω which we denote by E_α. The set whose elements are the sets E_α will be denoted by $\{E_\alpha\}$. Instead of speaking of sets of sets, we shall sometimes speak of a collection of sets, or a family of sets. The *union* of the sets E_α is defined to be the set S such that $x \in S$ if and only if $x \in E_\alpha$ for at least one $\alpha \in A$. We use the notation

$$S = \bigcup_{\alpha \in A} E_\alpha.$$

The *intersection* of the sets E_α is defined to be the set P such that $x \in P$ if and only if $x \in E_\alpha$ for every $\alpha \in A$. We use the notation

$$P = \bigcap_{\alpha \in A} E_\alpha.$$

2.1.2 Theorems

Theorem 2.1.1 A is infinite if and only if A is equivalent to one of its proper subsets.

Theorem 2.1.2 Every infinite subset of a countable set A is countable.

Theorem 2.1.3 Let $\{E_n\}, n = 1, 2, 3, \cdots$, be a sequence of countable sets, and put

$$S = \bigcup_{n=1}^{\infty} E_n.$$

Then S is countable.

Theorem 2.1.4 Let A be a countable set, and let B_n be the set of all n-tuples $(a_1, \cdots, a_n)$, where $a_k \in A(k = 1, \cdots, n)$, and the elements $a_1, \cdots, a_n$ need not be distinct. Then B_n is countable.

Corollary 2.1.1 The set of all rational numbers is countable.

Theorem 2.1.5 Let A be the set of all sequences whose elements are the digits 0 and 1. This set A is uncountable.

2.2 Metric Spaces

2.2.1 Definitions

Definition 2.2.1 A set X, whose elements we shall call *points*, is said to be a *metric space* if with any two points p and q of X there is associated a real number $d(p, q)$, called the *distance* from p to q, such that

(a) $d(p, q) > 0$ if $p \neq q$; $d(p, q) = 0$;

(b) $d(p, q) = d(q, p)$;

(c) $d(p, q) \leq d(p, r) + d(r, q)$, for any $r \in X$.

Any function with these three properties is called a *distance function*, or a *metric*.

Definition 2.2.2

(a) By the *segment* (a, b) we mean the set of all real numbers x such that $a < x < b$.

(b) By the *interval* $[a, b]$ we mean the set of all real numbers x such that $a \leq x \leq b$.

(c) Occasionally we shall also encounter "half-open intervals" $[a, b)$ and $(a, b]$; the first consist of all x such that $a \leq x < b$, the second of all x such that $a < x \leq b$.

(d) If $a_i < b_i$ for $i = 1, \cdots, k$, the set of all points $\mathbf{x} = (x_1, \cdots, x_k)$ in R^k whose coordinates satisfy the inequalities $a_i \leq x_i \leq b_i (1 \leq i \leq k)$ is called a *k-cell*.

(e) If $\mathbf{x} \in R^k$ and $r > 0$, the *open* (or *closed*) *ball* B with center at $\mathbf{x}$ and radius r is defined to be the set of all $y \in R^k$ such that $|\mathbf{y} - \mathbf{x}| < r$ (or $|\mathbf{y} - \mathbf{x}| < r$).

Definition 2.2.3 We call a set $E \subset R^k$ *convex* if

$$\lambda \mathbf{x} + (1-\lambda)\mathbf{y} \in E$$

whenever $\mathbf{x} \in E, \mathbf{y} \in E$, and $0 < \lambda < 1$.

Definition 2.2.4 Let X be a metric space. All points and sets mentioned below are understood to be elements and subsets of X.

(a) A *neighborhood* of p is a set $N_r(p)$ consisting of all q such that $d(p,q) < r$, for some $r > 0$. The number r is called the *radius* of $N_r(p)$.

(b) A point p is a limit point of the set E if every neighborhood of p contains a point $q \neq p$ such that $q \in E$.

(c) If $p \in E$ and p is not a limit point of E, then p is called an *isolated point* An equivalent deff: There exsits a neighborhood of p such that the only element <u>in E</u> it contains is p itself. of E.

(d) E is *closed* if every limit point of E is a point of E.

(e) A point p is an *interior* point of E if there is a neighborhood N of p such that $N \subset E$.

(f) E is *open* if every point of E is an interior point of E.

(g) The *complement* of E (denoted by E^c) is the set of all points $p \in X$ such that $p \notin E$.

(h) E is *perfect* if E is closed and if every point of E is a limit point of E.

(i) E is *bounded* if there is a real number M and a point $q \in X$ such that $d(p,q) < M$ for all $p \in E$.

(j) E is *dense* in X if every point of X is a limit point of E, or a point of E (or both).

Definition 2.2.5 If X is a metric space, if $E \subset X$, and if E' denotes the set of all limit points of E in X, then the *closure* of E is the set $\bar{E} = E \cup E'$.

2.2.2 Theorems

Theorem 2.2.1

(a) Balls are convex.

(b) K-cells are convex.

Theorem 2.2.2 Every neighborhood is an open set.

Theorem 2.2.3 If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.

Corollary 2.2.1 A finite point set has no limit points.

Theorem 2.2.4 Let $\{E_n\}$ be a (finite or infinite) collection of sets E_n. Then

$$\left(\bigcup_{\alpha} E_\alpha\right)^c = \bigcap_{\alpha} (E_\alpha^c) \,.$$

Theorem 2.2.5 A set F is closed if and only if its complement is open.

Theorem 2.2.6

(a) For any collection $\{G_n\}$ of open sets, $\bigcup_n G_n$ is open.

(b) For any collection $\{F_n\}$ of closed sets, $\bigcap_n F_n$ is closed.

(c) For any finite collection $G_1, \cdots, G_n$ of open sets, $\bigcap_{i=1}^n G_i$ is open.

(d) For any finite collection $F_1, \cdots, F_n$ of closed sets, $\bigcup_{i=1}^n F_i$ is closed.

Theorem 2.2.7 If X is a metric space and $E \subset X$, then

(a) $\bar{E}$ is closed,

(b) $E = \bar{E}$ if and only if E is closed,

(c) $\bar{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.

By (a) and (c), $\bar{E}$ is the smallest closed subset of X that contains E,

Theorem 2.2.8 Let E be a nonempty set of real numbers which is bounded above. Let $y = supE$. Then $y \in \bar{E}$. Hence $y \in E$ if E is closed.

Theorem 2.2.9 Suppose $Y \subset X$. A subset E of Y is open relative to Y is and ony if $E = Y \cap G$ for some open subset G of X.

2.3 Compact Sets

2.3.1 Definitions

Definition 2.3.1 By an *open cover* of a set E in a metric space X we mean a collection $\{G_\alpha\}$ of open subsets of X such that $E \subset \bigcup_\alpha G_\alpha$.

Definition 2.3.2 A subset K of a metric space X is said to be *compact* if every open cover of K contains a *finite* subcover. It is clear that every finite set is compact.

2.3.2 Theorems

Theorem 2.3.1 Suppose $K \subset Y \subset X$. Then K is compact relative to X if and only if K is compact relative to Y. Every metric space X is an open subset of itself, and is a closed subset of itself.

Theorem 2.3.2 Compact subsets of metric spaces are closed.

Theorem 2.3.3 Cloased subsets of compact sets are compact.

Theorem 2.3.4 If F is closed and K is compact, the n$F \cap K$ is compact.

Theorem 2.3.5 If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\cap K_\alpha$ is nonempty.

Theorem 2.3.6 If E is an infinite subset of a compact set K, then E has a limit point in K.

Theorem 2.3.7 If $\{I_n\}$ is a sequence of intervals in R^1, such that $I_n \supset I_{n+1}$ $(n = 1, 2, 3, \cdots)$, then $\cap_{n=1}^{\infty} I_n$ is not empty.

Theorem 2.3.8 Let k be a positive integer. If $\{I_n\}$ is a sequence of k-cells such that $I_n \supset I_{n+1}$ $(n = 1, 2, 3, \cdots)$, then $\cap_{n=1}^{i} nftyI_n$ is not empty.

Theorem 2.3.9 Every k-cell is compact.

Theorem 2.3.10 If a set E in R^k has one of the following three properties, then it has the other two:

(a) E is closed and bounded.

(b) E is compact.

(c) Every infinite subset of E has a limit point in E.

Theorem 2.3.11 Every bounded infinite subset of R^k has a limit point in R^k.

2.4 Perfect Sets

2.4.1 Theorems

Theorem 2.4.1 Let P be a nonempty perfect set in R^k. Then P is uncountable.

Corollary 2.4.1 Every interval $[a, b]$ $(a < b)$ is uncountable. In particular, the set of all real numbers is uncountable.

2.5 Connected Sets

2.5.1 Definitions

Definition 2.5.1 Two subsets A and B of a metric space X are said to be *separated* if both $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty, i.e., if no point of Alies in the closure of B and no point of Blies in the closure of A. A set $E \subset X$ is siad to be *connected* if E is *not* a union of two nonempty separated sets. Separated sets are of course disjoint, but disjoint sets need not be sparated.

2.5.2 Theorems

Theorem 2.5.1 A subset E of the ral line R^1 is connected if and only if it has the following property: If $x \in E$, $y \in E$, and $x < z < y$, then $z \in E$.

Chapter 3

Numerical Sequences and Series

3.1 Convergent Sequences

3.1.1 Definitions

Definition 3.1.1 A sequence $\{p_n\}$ in a metric space X is said to *converge* If $\{p_n\}$ does not converge, it is said to *diverge*. if there is a point $p \in X$ with the following property: For every $\epsilon > 0$ there is an integer N such that $n \geq N$ implies that $d(p_n, p) < \epsilon$. (Here d denotes the distance in X.)

Definition 3.1.2 The sequence $\{p_n\}$ is said to be *bounded* if its range is bounded.

3.1.2 Thorems

Theorem 3.1.1 Let $\{p_n\}$ be a sequence in a metric space X.

(a) $\{p_n\}$ converges to $p \in X$ if and only if every neighborhood of p contains p_n for all but finitely many n.

(b) If $p \in X, p' \in X$, and if $\{p_n\}$ converges to p and to p', then $p' = p$.

(c) If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.

(d) If $E \subset X$ and if p is a limit point A point p is a limit point of a set E if and only if there is a sequence $\{p_n\}$ of distinct points of E converging to p. of E, then there is a sequence $\{p_n\}$ in E such that $p = \lim_{n\to\infty} p_n$

Theorem 3.1.2 Suppose $\{s_n\}, \{t_n\}$ are complex sequences, and $\lim_{n\to\infty}\{s_n\} = s$ and $\lim\{t_n\} = t$. Then,

(a) $\lim_{n\to\infty}(s_n + t_n) = s + t$;

(b) $\lim_{n\to\infty}(cs_n) = cs, \lim_{n\to\infty}(c + s_n) = c + s$, for all number c;

(c) $\lim s_n t_n = st$;

(d) $\lim \frac{1}{s_n} = \frac{1}{s}$, provided $s_n \neq 0(n = 1, 2, 3, \cdots)$, and $s \neq 0$.

Theorem 3.1.3

(a) Suppose $\mathbf{x}_n \in R^k(n = 1, 2, 3, \cdots)$ and

$$\mathbf{x}_n = (\alpha_{1,n}, \cdots, \alpha_{k,n}.$$

Then $\{\mathbf{x}_n\}$ converges to $\mathbf{x} = (\alpha_1, \cdots, \alpha_k)$ if and only if

$$\lim_{n\to\infty} \alpha_{j,n} = \alpha_j.$$

(b) Suppose $\{\mathbf{x}_n\}, \{\mathbf{y}_n\}$ are sequences in R^k, $\{\beta_n\}$ is a sequence of real numbers, and $\mathbf{x}_n \to \mathbf{x}, \mathbf{y}_n \to \mathbf{y}, \beta_n \to \beta$. Then

$$\lim_{n\to\infty}(\mathbf{x}_n + \mathbf{y}_n) = \mathbf{x} + \mathbf{y}, \lim_{n\to\infty}(\mathbf{x}_n \cdot \mathbf{y}_n) = \mathbf{x} \cdot \mathbf{y}, \lim_{n\to\infty} \beta_n \mathbf{x}_n = \beta \mathbf{x}.$$

3.2 Subsequences

3.2.1 Definitions

Definition 3.2.1 Given a sequence $\{p_n\}$, consider a sequence $\{n_k\}$ of positive integers, such that $n_1 < n_2 < n_3 < \cdots$. Then the sequence $\{p_{n_i}\}$ is called a *subsequence* of $\{p_n\}$. If $\{p_{n_i}\}$ converges, its limit is called a *subsequential limit* of $\{p_n\}$.

3.2.2 Theorems

Theorem 3.2.1 $\{p_n\}$ converges to p if and only if every subsequence of $\{p_n\}$ converges to p.

Theorem 3.2.2

(a) If $\{p_n\}$ is a sequence in a compact metric space X, then some subsequence of $\{p_n\}$ converges to a point of X.

(b) Every bounded sequence in R^k contains a convergent subsequence.

Theorem 3.2.3 The subsequential limits of a sequence $\{p_n\}$ in a metric space X form a closed subset of X.

3.3 Cauchy Sequences

3.3.1 Definitions

Definition 3.3.1 A sequence $\{p_n\}$ in a metric space X is said to be a *Cauchy sequence* if for every $\epsilon > 0$ there is an integer N such that $d(p_n, p_m) < \epsilon$ if $n \geq N$ and $m \geq N$.

Definition 3.3.2 Let E be a nonempty subset of a metric space X, and let S be the set of all real numbers of the form $d(p, q)$, with $p \in E$ and $q \in E$. The sup of S is called the *diameter* of E.
If $\{p_n\}$ is a sequence in X and if E_N consists of the points $p_N, p_{N+1}, p_{N+2}, \cdots$, it is clear from the two preceding deffs that $\{p_n\}$ is a *Cauchy sequence if and only if*

$$\lim_{N\to\infty} \text{diam } E_N = 0.$$

Definition 3.3.3 A metric space in which every Cauchy sequence converges is said to be *complete*.

Definition 3.3.4 A sequence $\{s_n\}$ of real numbers is said to be

(a) *monotonically* increasing if $s_n \leq s_{n+1} (n = 1, 2, 3, \cdots)$;

(b) *monotonically* decreasing if $s_n \geq s_{n+1} (n = 1, 2, 3, \cdots)$;

3.3.2 Theorems

Theorem 3.3.1

(a) If $\bar{E}$ is the closure of a set E in a metric space X, then

$$\text{diam } \bar{E} = \text{diam } E.$$

(b) If K_n is a sequence of compact sets in X such that $K_n \supset K_{n+1} (n = 1, 2, 3, \cdots)$ and if

$$\lim_{n\to\infty} \text{diam } K_n = 0,$$

then $\bigcap_1^\infty K_n$ consists of exactly one point.

Theorem 3.3.2

(a) In any metric space X, every convergent sequence is a Cauchy sequence.

(b) If X is a compact metric space and if $\{p_n\}$ is a Cauchy sequence in X, then $\{p_n\}$ converges to some point of X.

(c) in R^k, every Cauchy sequence converges.

The fact that a sequence converges in R^k if and only it is a Cauchy sequence is usually called the *Cauchy criterion* for convergence.
This thm says that *all compact metric spaces and all Euclidean spaces are complete*. It implies also that *every closed subset of E of a complete metric space X is complete*.

Theorem 3.3.3 Suppose $\{s_n\}$ is monotonic. Then $\{s_n\}$ converges if and only if it is bounded.

3.4 Upper and Lower Limits

3.4.1 Definitions

Definition 3.4.1 Let $\{s_n\}$ be a sequence of real numbers with the following property: For every real M there is an interger N such that $n \geq N$ implies $s_n \geq M$. We then write

$$s_n \to +\infty.$$

Similarly, if for every real M there is an integer N such that $n \geq N$ implies $s_n \leq M$, we write

$$s_n \to -\infty.$$

Definition 3.4.2 Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of numbers x (in the extended real number system) such that $s_{n_k} \to x$ for some subsequence $\{s_{n_k}\}$. This set E contains all subsequential limits as defined in Definition 3.2.1, plus possibly the numbers $+\infty, -\infty$.
We now recall Definition 1.1.3 and 1.1.4 and put

$$s^* = \sup E,$$

$$s_* = \inf E.$$

The numbers s^*, s_* are called the *upper* and *lower limits* of $\{s_n\}$; we use the notation

$$\limsup_{n\to\infty} s_n = s^*, \quad \liminf_{n\to\infty} s_n = s_*$$

3.4.2 Theorems

Theorem 3.4.1 Let$\{s_n\}$ be a sequence of real numbers. Let E and s^* have the same meaning as in Definition 3.4.2. Then s^* has the following two properties:

(a) $s^* \in E$

(b) If $x > s^*$, there is an integer N such that $n \geq N$ implies $s_n < x$.

Moreover, s^* is the only number with the properties (a) and (b).
Of course, an analogous result is true for s_*.

Theorem 3.4.2 If $s_n \leq t_n$ for $n \geq N$, where N is fixed, then

$$\liminf_{n\to\infty} s_n \leq \liminf_{n\to\infty} t_n,$$

$$\limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} t_n,$$

3.5 Some Special Sequences

3.5.1 Theorems

Theorem 3.5.1

(a) If $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$.

(b) If $p > 0$, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$.

(c) $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

(d) If $p > 0$ and α is real, then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$.

(e) If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.

3.6 Series

3.6.1 Definitions

Definition 3.6.1 Given a sequence $\{a_n\}$, we use the notation

$$\sum_{n=p}^{q} a_n \quad (p \leq q)$$

to denote the sum $a_p + a_{p+1} + \cdots + a_q$. With $\{a_n\}$ we associate a sequence $\{s_n\}$, where

$$s_n = \sum_{k=1}^{n} a_k.$$

For $\{s_n\}$ we also use the symbolic expression

$$a_1 + a_2 + a_3 + \cdots$$

or, more concisely,

$$\sum_{n=1}^{\infty} a_n.$$

The above symbol we call an *infinite series*, or just a *series*. The numbers $\{s_n\}$ are called the *partial sums* of the series. If $\{s_n\}$ converges to s, we say that the series *converges*, and write

$$\sum_{n=1}^{\infty} a_n = s.$$

The number s is called the sum of the series; but it should be clearly understood that *s is the limit of a sequence of sums*, and is not obtained simply by addition. If $\{s_n\}$ diverges, the series is said to diverge.

3.6.2 Theorems

Theorem 3.6.1 $\sum a_n$ converges if and only if for every $\epsilon > 0$ there is an integer N such that

$$|\sum_{k=m}^{m} a_k| \le \epsilon$$

if $m \ge n \ge N$.

Theorem 3.6.2 If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Theorem 3.6.3 A series of nonnegative terms converges if and only if its partial sums form a bounded sequence.

Theorem 3.6.4

(a) If $|a_n| \le c_n$ for $n \ge N_0$, where N_0 is some fixed integer, and if $\sum c_n$ converges, then $\sum a_n$ converges.

(b) If $a_n \ge d_n \ge 0$ for $n \ge N_0$, and if $\sum d_n$ diverges, then $\sum a_n$ diverges.

3.7 Series of Nonnegative Terms

3.7.1 Theorems

Theorem 3.7.1 If $0 \le x < 1$, then

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

If $x \ge 1$, the series diverges.

Theorem 3.7.2 Suppose $a_1 \ge a_2 \ge a_3 \ge \cdots \ge 0$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots$$

converges.

Theorem 3.7.3 $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$.

Theorem 3.7.4 If $p > 1$,

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$$

converges; if $p \le 1$, the series diverges.

3.8 The Number e

3.8.1 Definitions

Definition 3.8.1

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

3.8.2 Theorems

Theorem 3.8.1

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Theorem 3.8.2 e is irrational.

3.9 The Root and Ratio Tests

3.9.1 Theorems

Theorem 3.9.1 (Root Test) Given $\sum a_n$, put $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$.
Then

(a) if $\alpha < 1$, $\sum a_n$ converges;

(b) if $\alpha > 1$, $\sum a_n$ diverges;

(c) if $\alpha = 1$, the test gives no information.

Theorem 3.9.2 (Ratio Test) The series $\sum a_n$

(a) converges if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$,

(b) diverges if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for all $n \geq n_0$, where n_0 is some fixed integer.

Theorem 3.9.3 For any sequence $\{c_n\}$ of positive numbers,

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n\to\infty} \sqrt[n]{c_n},$$

$$\limsup_{n\to\infty} \sqrt[n]{c_n} \leq \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}.$$

3.10 Power Series

3.10.1 Definitions

Definition 3.10.1 Given a sequence $\{c_n\}$ of complex numbers, the series

$$\sum_{n=0}^{\infty} c_n z^n$$

is called a *power series*. The numbers $\{c_n\}$ are called the *coefficients* of the series; z is a complex number.

3.10.2 Theorems

Theorem 3.10.1 Given the power series $\sum c_n z^n$, put

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|}, \quad R = \frac{1}{\alpha}.$$

(if $\alpha = 0, R = +\infty$; if $\alpha = +\infty, R = 0$.) Then $\sum c_n z^n$ converges if $|z| < R$, and diverges if $|z| > R$.

3.11 Summation by Parts

3.11.1 Theorems

Theorem 3.11.1 Given two sequences $\{a_n\}, \{b_n\}$, put

$$A_n = \sum_{k=0}^{n} a_k$$

if $n \geq 0$; put $A_{-1} = 0$. Then, if $0 \leq p \leq q$, we have

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p.$$

Theorem 3.11.2 Suppose

(a) the partial sums A_n of $\sum a_n$ form a bounded sequences;

(b) $b_0 \geq b_1 \geq b_2 \geq \cdots$;

(c) $\lim_{n\to\infty} b_n = 0$.

Theorem 3.11.3 Suppose

(a) $|c_1| \geq |c_2| \geq |c_3| \geq \cdots$;

(b) $c_{2m-1} \geq 0, c_{2m} \leq 0 \quad (m = 1, 2, 3, \cdots)$;

(c) $\lim_{n\to\infty} c_n = 0$.

Then $\sum c_n$ converges.

Theorem 3.11.4 Suppose the radius of convergence of $\sum c_n z^n$ is 1, and suppose $c_0 \geq c_1 \geq c_2 \geq \cdots$, $\lim_{n\to\infty} c_n = 0$. Then $\sum c_n z^n$ converges at every point on the circle $|z| = 1$, except possibly at $z = 1$.

3.12 Absolute Convergence

3.12.1 Definitions

Definition 3.12.1 The series $\sum a_n$ is said to *converge absolutely* if the series $\sum |a_n|$ converges.

Definition 3.12.2 If $\sum a_n$ converges, but $\sum |a_n|$ diverges, we say that $\sum a_n$ converges *nonabsolutely*.

3.12.2 Theorems

Theorem 3.12.1 If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

3.13 Addition and Multiplication of Series

3.13.1 Definitions

Definition 3.13.1 Given $\sum a_n$ and $\sum b_n$, we put

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \quad (n = 0, 1, 2, \cdots)$$

and call $\sum c_n$ the *product* of the two given series.

3.13.2 Theorems

Theorem 3.13.1 If $\sum a_n = A$, and $\sum b_n = B$, then $\sum(a_n + b_n) = A + B$, and $\sum c a_n = cA$, for any fixed c.

Theorem 3.13.2 Suppose

(a) $\sum_{n=0}^{\infty} a_n$ converges absolutely,

(b) $\sum_{n=0}^{\infty} a_n = A$,

(c) $\sum_{n=0}^{\infty} b_n = B$,

(d) $c_n = \sum_{k=0}^{n} a_k b_{n-k} \quad (n = 0, 1, 2, \cdots)$.

Then

$$\sum_{n=0}^{\infty} c_n = AB.$$

That is, the product of two convergent series converges, and to the right value, if at least one of the two series converges absolutely.

Theorem 3.13.3 If the series $\sum a_n, \sum b_n, \sum c_n$ converge to A, B, C, and $c_n = a_0 b_n + \cdots + a_n b_0$ then $C = AB$.

3.14 Rearrangements

3.14.1 Definitions

Definition 3.14.1 Let $\{k_n\}, n = 1, 2, 3, \cdots$, be a sequence in which every positive integer appears once and only once (that is, $\{k_n\}$ is a 1-1 function from J onto J, in the notation of Definition 2.1.2). Putting

$$a'_n = a_{k_n} \quad (n = 1, 2, 3, \cdots),$$

we say that $\sum a'_n$ is a *rearrangement* of $\sum a_n$.

3.14.2 Theorems

Theorem 3.14.1 Let $\sum a_n$ be a series of real numbers which converges, but not absolutely. Suppose

$$-\infty \leq \alpha \leq \beta \leq \infty.$$

Then there exist a rearrangement $\sum a'_m$ with partial sums s'_n such that

$$\liminf_{n\to\infty} s'_n = \alpha, \quad \limsup_{n\to\infty} s'_n = \beta.$$

Theorem 3.14.2 If $\sum a_n$ is a series of complex numbers which converges absolutely, then every rearrangement of $\sum a_n$ converges, and they all converges to the same sum.

Chapter 4

Continuity

4.1 Limits of Functions

4.1.1 Definitions

Definition 4.1.1 Let X and Y be metric spaces; suppose $E \subset X$, f maps E into Y, and p is a limit point of E. The deff does not say anything about $f(p)$. We write $f(x) \to q$ as $x \to p$, or

$$\lim_{x \to p} f(x) = q$$

if there is a point $q \in Y$ with the following property: For every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$d_Y(f(x), q) < \epsilon$$

for all points $x \in E$ for which

$$0 < d_X(x, p) < \delta.$$

4.1.2 Theorems

Theorem 4.1.1 Let $X, Y, E, f,$ and p be as in Definition 4.1.1. Then

$$\lim_{x \to p} f(x) = q$$

if and only if

$$\lim_{n \to \infty} f(p_n) = q$$

for every sequence $\{p_n\}$ in E such that

$$p_n \neq p, \quad \lim_{n \to \infty} p_n = p.$$

Corollary 4.1.1 If f has a limit at p, this limit is unique.

Theorem 4.1.2 Suppose $E \subset X$, a metric space, p is a limit point of E, f and g are complex functions on E, and

$$\lim_{x \to p} f(x) = A, \ \lim_{x \to p} g(x) = B.$$

Then

(a) $\lim_{x \to p}(f+g)(x) = A + B$;

(b) $\lim_{x \to p}(fg)(x) = AB$;

(c) $\lim_{x \to p}(\frac{f}{g})(x) = \frac{A}{B}, \ if \ B \neq 0.$

4.2 Continuous Functions

4.2.1 Definitions

Definition 4.2.1 Suppose X and Y are metric spaces, $E \subset X$, $p \in E$, and f maps E into Y. Then f is said to be *continuous at p* if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\delta_Y(f(x), f(p)) < \epsilon$$

for all points $x \in E$ for which $d_X(x, p) < \delta$.

4.2.2 Theorems

Theorem 4.2.1 In the situation given in Definition 4.2.1, assume also that p is a limit point If p is an isolated point of E, then every function f which has E as its domain of defintion is continuous at p. of E. Then f is continuous at p if and only if $\lim_{x \to p} f(x) = f(p)$.

Theorem 4.2.2 Suppose X, Y, Z are metric spaces, $E \subset X$, f maps E into Y, g maps the range of f, $f(E)$, into Z, and h is the mapping of E into Z defined by

$$h(x) = g(f(x)) \quad (x \in E).$$

If f is continuous at a point $p \in E$ and if g is continuous at the point $f(p)$, then h is continuous at p.

Theorem 4.2.3 A mapping f of a metric space X into a metric space Y is onctinuous on X if and only if $f^{-1}(V)$ is open in X for every open set V in Y.

Corollary 4.2.1 A mapping f of a metric space X into a metric space Y is continuous if and only if $f^{-1}(C)$ is closed in X for every closed set C in Y.

Theorem 4.2.4 Let f and g be complex continuous functions on a metric space X. Then $f + g, fg,$ and f/g are continuous on X.

Theorem 4.2.5

(a) Let $f_1, \cdots, f_k$ be real functions on a metric space X, and let $\mathbf{f}$ be the mapping of X into R^k defined by

$$\mathbf{f}(x) = (f_1(x), \cdots, f_k(x)) \quad (x \in X);$$

then $\mathbf{f}$ is continuous if and only if each of the functions $f_1, \cdots, f_k$ is continuous.

(b) if $\mathbf{f}$ and $\mathbf{g}$ are continuous mappings of X into R^k, then $\mathbf{f} + \mathbf{g}$ and $\mathbf{f} \cdot \mathbf{g}$ are continuous on X.

4.3 Continuity and Compactness

4.3.1 Definitions

Definition 4.3.1 A mapping $\mathbf{f}$ of a set E into R^k is said to be *bounded* if there is a real number M such that $|\mathbf{f}(x)| \leq M$ for all $x \in E$.

Definition 4.3.2 Let f be a mapping of a metric space X into a metric space Y. We say that f is *uniformly continuous* on X if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$d_Y(f(p), f(q)) < \epsilon$$

for all p and q in X for which $d_X(p, q) < \delta$.

4.3.2 Theorems

Theorem 4.3.1 Suppose f is a continuous mapping of a compact metric space X into a metric space Y. Then $f(X)$ is compact.

Theorem 4.3.2 If $\mathbf{f}$ is a continuous mapping of a compact metric space X into R^k, then $\mathbf{f}(X)$ is closed and bounded. Thus, $\mathbf{f}$ is bounded.

Theorem 4.3.3 Suppose f is a continuous real function on a compact metric space X, and

$$M = \sup_{p \in X} f(p), \quad m = \inf_{p \in X} f(p).$$

Then there exist points $p, q \in X$ such that $f(p) = M$ and $f(q) = m$. This is to say, f attains its maximum (at p) and its minimum (at q).

Theorem 4.3.4 Suppose f is a continuous 1-1 mapping of a compact metric space X onto a metric space Y. Then the inverse mapping f^{-1} defined on Y by

$$f^{-1}(f(x)) = x \quad (x \in X)$$

is a continuous mapping of Y onto X.

Theorem 4.3.5 Let f be a continuous mapping of a compact metric space X into a metric space Y. Then f is uniformly continuous on X.

Theorem 4.3.6 Let E be a noncompact set in R^1. Then

(a) there exists a continuous function on E which is not bounded;

(b) there exists a continuous and bounded function on E which has no maximum. If, in addition, E is bounded, then

(c) there exists a continuous function on E which is not uniformly continuous.

4.4 Continuity and Connectedness

4.4.1 Theorems

Theorem 4.4.1 If f is a continuous mapping of a metric space X into a metric space Y, and if E is a connected subset of X, then $f(E)$ is connected.

Theorem 4.4.2 Let f be a continuous real function on the interval $[a, b]$. If $f(a) < f(b)$ and if c is a number such that $f(a) < c < f(b)$, then there exists a point $x \in (a, b)$ such that $f(x) = c$.

4.5 Discontinuities

4.5.1 Definitions

Definition 4.5.1 Let f be defined on (a, b). Consider any point x such that $a \leq x < b$. We write

$$f(x+) = q$$

if $f(t_n) \to q$ as $n \to \infty$, for all sequences $\{t_n\}$ in (x, b) such that $t_n \to x$. To obtain the deff of $f(x-)$, for $a < x \leq b$, we restrict ourselves to sequences $\{t_n\}$ in (a, x). It is clear that any point x of (a, b), $\lim_{t \to x} f(t)$ exists if and only if

$$f(x+) = f(x-) = \lim_{t \to x} f(t).$$

Definition 4.5.2 Let f be defined on (a, b). If f is discontinuous at a point x, and if $f(x+)$ and $f(x-)$ exist, then f is said to have a discontinuity of the *first kind*, or a *simple discontinuity* at x. There are two ways in which a function can have a simple discontinuity: either $f(x+) \neq f(x-)$, in which case the value $f(x)$ is immaterial, or $f(x+) = f(x-) \neq f(x)$. Otherwise the discontinuity is said to be of the *second kind*.

4.6 Monotonic Functions

4.6.1 Definitions

Definition 4.6.1 Let f be real on (a, b). Then f is said to be *monotonically increasing* on (a, b) if $a < x < y < b$ implies $f(x) \leq f(y)$. If the last inequality is reversed, we obtain the deff of a *monotonically decreasing* function. The class of monotonic functions consists of both the increasing and the deceasing functions.

4.6.2 Theorems

Theorem 4.6.1 Let f be monotonically increasing on (a, b). Then $f(x+)$ and $f(x-)$ exist at every point of x of (a, b). More precisely,

$$\sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x<t<b} f(t).$$

Furthermore, if $a < x < y < b$, then

$$f(x+) \leq f(y-).$$

Analogous results evidently hod for monotonically decreasing functions.

Corollary 4.6.1 Monotonic functions have no discontinuities of the second kind. Compare with Corollary 5.3.1

Theorem 4.6.2 Let f be monotonic on (a, b). Then the set of points of (a, b) at which f is discontinuous is at most countable.

4.7 Infinite Limits and Limits at Infinity

4.7.1 Definitions

Definition 4.7.1 For any real c, the set of real numbers x such that $x > c$ is called a neighborhood of $+\infty$ and is written $(c, +\infty)$. Similarly, the set $(-\infty, c)$ is a neighborhood of $-\infty$.

Definition 4.7.2 Let f be a real function defined on $E \subset R$. We say that

$$f(t) \to A \text{ as } t \to x,$$

where A and x are in the extended real number system, if for every neighborhood U of A there is a neighborhood V of x such that $V \cap E$ is not empty, and such that $f(t) \in U$ for all $t \in V \cap E, t \neq x$.

4.7.2 Theorems

Theorem 4.7.1 Let f and g be defined on $E \subset R$. Suppose

$$f(t) \to A, \quad g(t) \to B \text{ as } t \to x.$$

Then

(a) $f(t) \to A'$ implies $A' = A$.

(b) $(f+g)(t) \to A+B$,

(c) $(fg)(t) \to AB$,

(d) $(f/g)(t) \to A/B$,

provided the right member of (b), (c), and (d) are defined.

Chapter 5

Differentiation

5.1 The Derivative of a Real Function

5.1.1 Definitions

Definition 5.1.1 Let f be defined (and real-valued) on $[a, b]$. For any $x \in [a, b]$ form the quotient

$$\phi(t) = \frac{f(t) - f(x)}{t - x}(a < t < b, t \neq x),$$

and define

$$f'(x) = \lim_{t \to x} \phi(t),$$

provided this limit exists in accordance with Definition 4.1.1. We thus associate with the function f a function f' whose domain is the set of points x at which the limit exists; f' is called the *derivative* of f. If f' is defined at a point x, we say that f is *differentiable* at x. If f' is defined at every point of a set $E \subset [a, b]$, we say that f is differentiable on E.

5.1.2 Theorems

Theorem 5.1.1 Let f be defined on $[a, b]$. If f is differentiable at a point $x \in [a, b]$, then f is continuous at x. Prove by using the fact that limit of a product is the product of limits.

Theorem 5.1.2 Suppose f and g are defined on $[a, b]$ and are differentiable at a point $x \in [a, b]$. Then $f + g, fg,$ and f/g are differentiable at x, and

(a) $(f + g)'(x) = f'(x) + g'(x)$;

(b) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$;

(c) $(\frac{f}{g})'(x) = \frac{g(x)f'(x) - g'(x)f(x)}{g^2(x)}$

In (c), we assume of course that $g(x) \neq 0$.

Theorem 5.1.3 Suppose f is continuous on $[a,b]$, $f'(x)$ exists at some point $x \in [a,b]$, g is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$. If

$$h(t) = g(f(t)) \quad (a \leq t \leq b),$$

then h is differentiable at x, and

$$h'(x) = g'(f(x))f'(x).$$

5.2 Mean Value Theorems

5.2.1 Definitions

Definition 5.2.1 Let f be a real function defined on a metric space X. We say that f has a *local maximum* at a point $p \in X$ if there exists $\delta > 0$ such that $f(q) \leq f(p)$ for all $q \in X$ with $d(p,q) < \delta$.

5.2.2 Theorems

Theorem 5.2.1 Let f be defined on $[a,b]$; if f has a local maximum at a point $x \in (a,b)$, and if $f'(x)$ exists, then $f'(x) = 0$. Prove by showing the left-hand and right-hand derivatives

Theorem 5.2.2 If f and g are continuous real functions on $[a,b]$ which are differentiable in (a,b), then there is a point $x \in (a,b)$ at which

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x).$$

Note that differentiability is not required at the endpoints.

Theorem 5.2.3 If f is a real continuous function on $[a,b]$ which is differentiable in (a,b), then there is a point $x \in (a,b)$ at which

$$f(b) - f(a) = (b-a)f'(x).$$

Theorem 5.2.4 Suppose f is differentiable in (a,b).

(a) If $f'(x) \geq 0$ for all $x \in (a,b)$, then f is monotonically increasing.

(b) If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant.

(c) If $f'(x) \leq 0$ for all $x \in (a,b)$, then f is monotonically decreasing.

5.3 The Continuity of Derivatives

5.3.1 Theorems

Theorem 5.3.1 Suppose f is a real differentiable function on $[a, b]$ and suppose $f'(a) < \lambda < f'(b)$. Then there is a point $x \in (a, b)$ such that $f'(x) = \lambda$.

Corollary 5.3.1 If f is differentiable on $[a, b]$, then f' cannot have any simple discontinuities on $[a, b]$. Compare with Corollary 4.6.1

5.4 L'Hospital's Rule

5.4.1 Theorems

Theorem 5.4.1 Suppose f and g are real and differentiable in (a, b), and $g'(x) \neq 0$ for all $x \in (a, b)$, where $-\infty \leq a < b \leq +\infty$. Suppose

$$\frac{f'(x)}{g(x)} \to A \; as \; x \to a.$$

If

$$f(x) \to 0 \; and \; g(x) \to 0 \; as \; x \to a,$$

or if

$$g(x) \to +\infty \; as \; x \to a,$$

then

$$\frac{f(x)}{g(x)} \to A \; as \; x \to a.$$

5.5 Derivatives of Higher Order

5.5.1 Definitions

Definition 5.5.1 If f has a derivative f' on an interval, and if f' is itself differentiable, we denote the derivative of f' b f'' and call f'' the second derivative of f. Continuing in this manner , we obtain functions

$$f, f', f'', f^{(3)}, \cdots, f^{(n)},$$

each of which is the derivative of the preceding one. $f^{(n)}$ is called the nth derivative, or the derivative of order n, of f.

5.6 Taylor's Theorem

5.6.1 Theorems

Theorem 5.6.1 Suppose f is a real function on $[a, b]$, n is a positive integer, $f^{(n-1)}$ is continuous on $[a, b]$, $f^{(n)}(t)$ exists for every $t \in (a, b)$. Let α, β be distincet points of $[a, b]$, and define

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k.$$

Then there exists a point x between α and β such that

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n.$$

5.7 Differentiation of Vector-valued Functions

5.7.1 Theorems

Theorem 5.7.1 Suppose $\mathbf{f}$ is a continuous mapping of $[a, b]$ into R^k and $\mathbf{f}$ is differentiable in (a, b). Then there exists $x \in (a, b)$ such that

$$|\mathbf{f}(b) - \mathbf{f}(a)| \leq (b-a)|\mathbf{f}'(x)|.$$

Chapter 6

The Riemann-Stieltjes Integral

6.1 Definition and Existence of the Integral

6.1.1 Definitions

Definition 6.1.1 We say that the partition $P*$ is a *refinement* of P if $P* \supset P$ (that is, if every point of P is a point of $P*$). Given two partitions, P_1 and P_2, we say that $P*$ is their *common refinement* if $P* = P_1 \cup P_2$.

6.1.2 Theorems

Theorem 6.1.1 If P^* is a refinement of P, then

$$L(P, f, \alpha) \leq L(P^*, f, \alpha)$$

and

$$U(P^*, f, \alpha) \leq U(P, f, \alpha).$$

Theorem 6.1.2 $\underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha$

Theorem 6.1.3 $f \in \mathbf{R}(\alpha)$ on $[a, b]$ if and only if for every $\epsilon > 0$ there exists a partition P such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon.$$

Theorem 6.1.4

(a) If Theorem 6.1.3 holds for some P and some ϵ, then Theorem 6.1.3 holds (with the same ϵ) for every refinement of P.

(b) If Theorem 6.1.3 holds for $P = \{x_0, \cdots, x_n\}$ and if s_i, t_i are arbitrary points in $[x_{i-1}, x_i]$, then

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)|\Delta\alpha_i < \epsilon.$$

(c) If $f \in \mathbf{R}(\alpha)$ and the hypotheses of (b) hold, then

$$|\sum_{i=1}^{n} |f(s_i) - f(t_i)|\Delta\alpha_i - \int_a^b f d\alpha| < \epsilon.$$

Theorem 6.1.5 If f is continuous on $[a, b]$ then $f \in \mathbf{R}(\alpha)$ on $[a, b]$.

Theorem 6.1.6 If f is monotonic on $[a, b]$, and if α is continuous on $[a, b]$, then $f \in \mathbf{R}(\alpha)$. (We still assume, of course, that α is monotonic.)

Theorem 6.1.7 Suppose f is bonded on $[a, b]$, f has only finitely many points of discontinuity on $[a, b]$, and α is continuous at every point at which f is discontinuous. Then $f \in \mathbf{R}(\alpha)$.

Theorem 6.1.8 Suppose $f \in \mathbf{R}(\alpha)$ on $[a, b]$, $m \leq f \leq M$, ϕ is continuous on $[m, M]$, and $h(x) = \phi(f(x))$ on $[a, b]$. Then $h \in \mathbf{R}(\alpha)$ on $[a, b]$.

6.2 Properties of the Integral

6.2.1 Definitions

Definition 6.2.1 The *unit step function* I is defined by

$$I(x) = \begin{cases} 0 & (x \leq 0) \\ 1 & (x > 0) \end{cases}$$

6.2.2 Theorems

Theorem 6.2.1 If $f \in \mathbf{R}(\alpha)$ and $g \in \mathbf{R}(\alpha)$ on $[a, b]$, then

(a) $fg \in \mathbf{R}(\alpha)$;

(b) $|f| \in \mathbf{R}(\alpha)$ and $\left|\int_a^b f d\alpha\right| \leq \int_a^b |f| d\alpha$.

Theorem 6.2.2 If $a < s < b$, f is bounded on $[a, b]$, f is continuous at s, and $\alpha(x) = I(x - s)$, then

$$\int_a^b f d\alpha = f(s).$$

Theorem 6.2.3 Suppose $c_n \geq 0$ for $1, 2, 3, \cdots$, $\sum c_n$ converges, $\{s_n\}$ is a sequence of distinct points in (a, b), and

$$\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n).$$

Let f be continuous on $[a, b]$. Then

$$\int_a^b f d\alpha = \sum_{n=1}^{\infty} c_n f(s_n).$$

Theorem 6.2.4 Assume α increases monotonically and $\alpha' \in \mathbf{R}$ on $[a, b]$. Let f be a bounded real function on $[a, b]$. Then $f \in \mathbf{R}(\alpha)$ if and only if $f\alpha' \in \mathbf{R}$. In that case,

$$\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x)dx.$$

Theorem 6.2.5 Suppose ϕ is a strictly increasing continuous function that maps an interval $[A, B]$ onto $[a, b]$. Suppose α is monotonnically increasing on $[a, b]$ and $f \in \mathbf{R}(\alpha)$ on $[a, b]$. Define β and g on $[A, B]$ by

$$\beta(y) = \alpha(\phi(y)),\ g(y) = f(\phi(y)).$$

Then $g \in \mathbf{R}(\beta)$ and

$$\int_A^B g d\beta = \int_a^b f d\alpha.$$

6.3 Integration and Differentiation

6.3.1 Theorems

Theorem 6.3.1 Let $f \in \mathbf{R}$ on $[a, b]$. For $a \leq x \leq b$, put

$$F(x) = \int_a^x f(t)dt.$$

Then F is continuous on $[a, b]$; furthermore, if f is continuous at a point x_0 of $[a, b]$, then F is differentiable at x_0, and

$$F'(x_0) = f(x_0).$$

Theorem 6.3.2 If $f \in \mathbf{R}$ on $[a,b]$ and if there is a differentiable function F on $[a,b]$ such that $F' = f$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Theorem 6.3.3 Suppose F and G are differentiable functions on $[a,b]$, $F' = f \in \mathbf{R}$, and $G' = g \in \mathbf{R}$. Then

$$\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx.$$

Chapter 7

Sequences and Series of Functions

7.1 Discussion of Main Problem

7.1.1 Definitions

Definition 7.1.1 Suppose $\{f_n\}$, $n = 1, 2, 3, \cdots$, is a sequence of functions defined on a set E, and suppose that the sequence of numbers $\{f_n(x)$ converges for every $x \in E$. We can then define a function f by

$$f(x) = \lim_{n\to\infty} f_n(x) \quad (x \in E).$$

Under these circumstances we say that $\{f_n\}$ converges on E and that f is the *limit*, or the *limit function*, of $\{f_n\}$. Sometimes we shall use a more descriptive terminology and shall say that "$\{f_n\}$ converges to f *pointwise* on E" if the above holds. Similarly, if $\sum f_n(x)$ converges for every $x \in E$, and if we define

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad (x \in E),$$

the function f is called the *sum* of the series $\sum f_n$.

7.2 Uniform Convergence

7.2.1 Definitions

Definition 7.2.1 We say that a sequence of functions $\{f_n\}, n = 1, 2, 3, \cdots$, converges *uniformly* on E to a function f if for every $\epsilon > 0$ there is an integer N such that $n \geq N$ implies

$$|f_n(x) - f(x)| \leq \epsilon$$

for all $x \in E$.

7.2.2 Theorems

Theorem 7.2.1 Suppose K is compact, and

(a) $\{f_n\}$ is a sequence of continuous functions on K,

(b) $\{f_n\}$ converges pointwise to a continuous function f on K,

(c) $f_n(x) \geq f_{n+1}(x)$ for all $x \in K, n = 1, 2, 3, \cdots$.

Then $f_n \to f$ uniformly on K.

Theorem 7.2.2 Supose

$$\lim_{n\to\infty} f_n(x) = f(x) \quad (x \in E).$$

Put

$$M_n = \sup_{x\in E} |f_n(x) - f(x)|.$$

Then $f_n \to f$ uniformly on E if and only if $M_n \to 0$ as $n \to \infty$.

Theorem 7.2.3 Suppose $\{f_n\}$ is a sequence of functions defined on E, and suppose

$$|f_n(x)| \leq M_n \quad (x \in E, n = 1, 2, 3, \cdots).$$

Then $\sum f_n$ converges uniformly on E if $\sum M_n$ converges.

7.3 Uniform Convergence and Continuity

7.3.1 Definitions

Definition 7.3.1 If X is a metric space, $\mathbf{C}(X)$ will denote the set of all complex-valued, continuous, bounded functions with domain X. We associate with each $f \in \mathbf{C}(X)$ its supreme norm

$$\|f\| = \sup_{x\in X} |f(x)|.$$

We also define the distance between $f \in \mathbf{C}(X)$ and $g \in \mathbf{C}(X)$ to be $\|f - g\|$.

7.3.2 Theorems

Theorem 7.3.1 Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of E, and suppose that

$$\lim_{t\in x} f_n(t) = A_n \quad (n = 1, 2, 3, \cdots).$$

Then $\{A_n\}$ converges, and

$$\lim_{t\in x} f(t) = \lim_{n\to\infty} A_n.$$

In other words, the conclusion is that

$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t).$$

Theorem 7.3.2 If $\{f_n\}$ is a sequence of continuous functions on E, and if $f_n \to f$ uniformly on E, then f is continuous on E.

Theorem 7.3.3 Suppose K is compact, and

(a) $\{f_n\}$ is a sequence of continuous functions on K,

(b) $\{f_n\}$ converges pointwise to a continuous function f on K,

(c) $f_n(x) \geq f_{n+1}(x)$ for all $x \in K, n = 1, 2, 3, \cdots$

Then $f_n \to f$ uniformly on K.

Theorem 7.3.4 The above metric makes $\mathbf{C}(X)$ into a complete metric space.

7.4 Uniform Convergence and Integration

7.4.1 Theorems

Theorem 7.4.1 Let α be monotonically increasing on $[a, b]$. Suppose $f_n \in \mathbf{R}(\alpha)$ on $[a, b]$, for $n = 1, 2, 3, \cdots$, and suppose $f_n \to f$ uniformly on $[a, b]$. Then $f \in \mathbf{R}(\alpha)$ on $[a, b]$, and

$$\int_a^b f d\alpha = \lim_{n\to\infty}\int_a^b f_n d\alpha.$$

(The existence of the limit is part of the conclusion.)

Corollary 7.4.1 If $f_n \in \mathbf{R}(\alpha)$ on $[a, b]$ and if

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad (a \leq x \leq b),$$

the series converging uniformly on $[a, b]$, then

$$\int_a^b f d\alpha = \sum_{n=1}^{\infty}\int_a^b f_n d\alpha.$$

In other words, the series may be integrated term by term.

7.5 Uniform Convergence and Differentiation

7.5.1 Theorems

Theorem 7.5.1 Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a, b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a, b]$. If $\{f'_n\}$ converges uniformly on $[a, b]$, then $\{f_n\}$ converges uniformly on $[a, b]$, to a function f, and

$$f'(x) = \lim_{n\to\infty} f'_n(x) \quad (a \leq x \leq b).$$

Theorem 7.5.2 There exists a real continuous function on the real line which is nowhere differentiable.

7.6 Equicontinuous Families of Functions

7.6.1 Definitions

Definition 7.6.1 Let $\{f_n\}$ be a sequence of functions defined on a set E. We say that $\{f_n\}$ is *pointwise bounded* on E if the sequence $\{f_n(x)\}$ is bounded for every $x \in E$, that is, if there exists a finite-valued function ϕ defined on E such that

$$|f_n(x)| < \phi(x) \quad (x \in E, n = 1, 2, 3, \cdots).$$

We say that $\{f_n\}$ is *uniformly bounded* on E if there exists a number M such that

$$|f_n(x)| < M \quad (x \in E, n = 1, 2, 3, \cdots).$$

Definition 7.6.2 A family $\mathbf{F}$ of complex functions f defined on a set E in a metric space X is said to be *equicontinuous* on E if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon$$

whenever $d(x, y) < \delta, x \in E, y \in E,\ and\ f \in \mathbf{F}$.

7.6.2 Theorems

Theorem 7.6.1 If $\{f_n\}$ is a pointwise bounded sequence of complex functions on a countable set E, then $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}\}$ converges for every $x\ inE$.

Theorem 7.6.2 If K is a compact metric space, if $f_n \in \mathbf{C}(K)$ for $n = 1, 2, 3, \cdots$, and if $\{f_n\}$ converges uniformly on K, then $\{f_n\}$ is equicontinuous on K.

Theorem 7.6.3 If K is compact, if $f_n \in \mathbf{C}(K)$ for $n = 1, 2, 3, \cdots$, and if $\{f_n\}$ is pointwise bounded and equicontinuous on K, then

(a) $\{f_n\}$ is uniformly bounded on K,

(b) $\{f_n\}$ contains a uniformly convergent subsequence.

7.7 The Stone-Weierstrass Theorem

7.7.1 Theorems

Theorem 7.7.1 If f is a continuous complex function on $[a, b]$, there exists a sequence of polynomials P_n such that

$$\lim_{n\to\infty} P_n(x) = f(x)$$

uniformly on $[a, b]$. If f is real, the P_n may be taken real.

Corollary 7.7.1 For every interval $[-a, a]$ there is a sequence of real polynomials P_n such that $P_n(0) = 0$ and such that

$$\lim_{n\to\infty} P_n(x) = |x|$$

uniformly on $[-a, a]$.

Chapter 8

Some Special Functions

Chapter 9

Functions of Several Variables

9.1 The Contraction Principle

9.1.1 Definitions

Definition 9.1.1 Let X be a metric space, with metric d. If ϕ maps X into X and if there is a number $c < 1$ such that

$$d(\phi(x), \phi(y)) \leq c\, d(x, y)$$

for all $x, y \in X$, then ϕ is said to be a *contraction* of X into X. If f is a contraction mapping then it is also a continuous mapping. The reverse is not true.

9.1.2 Theorems

Theorem 9.1.1 If X is a complete metric space, and if ϕ is a contraction of X into X, then there exists one and only one $x \in X$ such that $\phi(x) = x$.

Chapter 10

Exercises

10.1 Concept Questions

Problem 10.1.1 A sequence $\{a_n\}$ converges if and only if it is bounded.
- FALSE. $\{sin(n)\}$ is bounded but not convergent. However, if a sequence converges, then it is bounded. See Theorem **??**