MA538: Probability Theory I

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Contents

Preface

TBD

1 Probability Space and Measure

1.1 Algebra of Sets

1.1.1 Set Operations

Given two sets $A, B \in \Omega$, there are four basic binary operations on sets:

1. *Union*: $A \cup B = \{x : x \in A \text{ or } x \in B\}$
2. *Intersection*: $A \cap B = \{x : x \in A \text{ and } x \in B\}$
3. *Set difference*: $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$
4. *Set complement*: $A^c = \Omega \setminus A$

Set complement is the "strongest" operation, because if a collection of sets $\mathscr{A}$ is closed under complement, and if it is also closed under any one of the other three operations, $\mathscr{A}$ is closed under the rest two. That is seen from,

$$\text{If closed under } \textit{union} \begin{cases} A \cap B = (A^c \cup B^c)^c \\ A \setminus B = A \cap B^c = (A^c \cup B)^c \end{cases} \tag{1}$$

$$\text{If closed under } \textit{intersection} \begin{cases} A \cup B = (A^c \cap B^c)^c \\ A \setminus B = A \cap B^c \end{cases} \tag{2}$$

$$\text{If closed under } difference \begin{cases} A \cap B = A \setminus (A \setminus B) \\ A \cup B = (A^c \cap B^c)^c = (A^c \setminus (A^c \setminus B^c))^c \end{cases} \tag{3}$$

The difference operation is the second "strongest" operation, in that if a collection of sets $\mathscr{A}$ is closed under difference, it is closed under intersection, that is seen from,

$$A \cap B = A \setminus (A \setminus B) \tag{4}$$

The third "strongest" operation is the union, which can not be implied from difference or intersection, or their combination.

The "weakest" operation is the intersection, which can be implied from the difference.

To summarize, the "strongest" pair would be the complement plus any one more, which implies everything else; the second "strongest" would be the difference plus the union, which implies the intersection but not the complement (Ω may not be in the collection).

1.1.2 Class of Set Collection

Definition 1.1 Given a set Ω, a non-empty collection $\mathscr{P} \subset 2^\Omega$ is called a *π-system* iff:

$$\forall A, B \in \mathscr{P},\ A \cap B \in \mathscr{P}$$

Notice, this has the weakest requirement.

Definition 1.2 Given a set Ω, a non-empty collection $\mathscr{Q} \subset 2^\Omega$ is called a *semiring* iff:

$$\forall A, B \in \mathscr{Q} \text{ and } A \supset B$$
$$\exists C_k \subset \mathscr{Q}, s.t., A \setminus B = \bigcup_{k=1}^n C_k$$

Definition 1.3 Given a set Ω, a non-empty collection $\mathscr{R} \subset 2^\Omega$ is called a *ring* iff:

$$\forall A, B \in \mathscr{R},\ A \cup B \in \mathscr{R} \text{ and } B \setminus A \in \mathscr{R}$$

There are two points need to mention: the empty set is in a ring, since $A \setminus A = \emptyset$; $\mathscr{A}$ is also closed under intersection (the reverse need not be true).

Definition 1.4 A *ring* $\mathscr{A}$ is called an *field* iff $\Omega \in \mathscr{A}$.

So a field is closed under all *finite* combination of set operations.

Definition 1.5 An *field* is called a σ-*field* if for any sequence A_n of sets in $\mathscr{A}$, $\cup_{n\geq 1} A_n \in \mathscr{A}$.

$\nexists$

2 Integration Theory

3 Random Variables

4 Law of Large Numbers

5 Central Limit Theorem

6 Some Tricks

6.1 Prove by Contraposition

6.2 Construct Finer Partition

Given twofinite partitions $\{A_n\}$ and $\{B_m\}$, a finer partition can be constructed as

$$(\cup_{i=1}^n A_i) \bigcap (\cup_{j=1}^m B_j) = \bigcup (\cap_{i=1}^n \cap_{j=1}^m A_i B_j)$$

6.3 Prove Equality

To prove two numerical quantities are equal $X = Y$, often times we can do this by showing $X \leq Y$ and $X \geq Y$. Similarly, to prove two sets are equal $E = F$, we can show $E \subset F$ and $E \subset F$.

6.4 An Epsilon of Room

If one has to show that $X \leq Y$, try proving that $X \leq Y + \epsilon, \forall \epsilon > 0$. This trick combines well with the "Prove Equality" trick.

In a similar spirit, if one needs to show that a quantity X vanishes, try showing that $|X| \leq \epsilon, \forall \epsilon > 0$.

If one wants to show that a sequence x_n of real numbers converges to zero, try showing that $\limsup_{n \to \infty} |x_n| \leq \epsilon, \forall \epsilon > 0$

One caveat: for finite x, and any $\epsilon > 0$, it is true that $x + \epsilon > x$ and $x - \epsilon < x$, but it is not true when x is equal to $+\infty$ or $-\infty$.