Quantitative Finance

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Preface

This book project, which consists of four subjects: Finance, Mathematics, Statistics, and Computer Science, is tailored specifically to prepare someone for a quant career. It originated from my general belief of the hierarchy of solving a problem — problems are solved at strategic, tactical, and operational levels.

Microeconomics and *Macroeconomics* explain the driving forces of capital markets, from a legislator's perspective. *Accounting* and *Corporate Finance* take a closer and necessary look at these forces, from a different angle. *Stochastic Calculus* and *Asset Pricing* provide with a set of tools and ideas that enables us to **strategically** model one of the central problems in Quantitative Finance.

Generally speaking, there are two paths to solve a quantitative finance problem at the **tactical** level: the mathematical way and the statistical way. There are only two pieces of math we need to know: *Analysis*, in particular measure-theoretical probability and differential equations; and *Linear Algebra*, with functional analysis in mind. Statistics, on the other hand, should start with *Statistical Experiment Design*, from which we learn how to collect data for statistical models. Next, the study of *Random Variables* and *Stochastic Processes* introduce the building blocks of the statistical "pillbox", with *Mathematical Statistics* the "scaffold". Once the "pillbox" is ready, we are equipped to tackle our problems using *Machine Learning*, which is essentially a collection of statistical models and optimization algorithms.

Computer Architecture and *Operating System* are respectively about the "hardware" and "software" of a single computer. The interaction of multiple computers is understood in *Computer Network*. Once we are comfortable with these concepts, we will be able to use *Data Structure and Algorithms* to solve problems at the **operational** level, and use *C++* and/or *Java* to implement our ideas.

I am aware that it can take a while, and even multiple advanced degrees, to finish this curriculum, but let's remember the motto from the Leipzig Gewandhaus Orchestra: "***Res severa est verum gaudium***".

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West Lafayette, IN
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