Part I

Topics

Chapter 1

Minors and Cofactors

1.1 Definition

Definition 1.1.1 General definition of a minor.

Let $\mathbf{A}$ be an $m \times n$ matrix and k an integer with $0 < k \leq \min m, n$. A $k \times k$ minor of $\mathbf{A}$ is the determinant of a $k \times k$ matrix obtained from $\mathbf{A}$ by deleting $m-k$ rows and $n-k$ columns. For such a matrix there are a total of $\binom{m}{k} \cdot \binom{n}{k}$ minors of size $k \times k$.

Definition 1.1.2 First minors and cofactors.

If A is a square matrix, then the minor of the entry in the i-th row and j-th column (also called the (i,j) minor, or a first minor, is the determinant of the submatrix formed by deleting the i-th row and j-th column. This number is often denoted M_{ij}. The (i,j) cofactor is obtained by multiplying the minor by $(-1)^{i+j}$.

Example 1.1.1 To illustrate these definitions, consider the following 3 by 3 matrix,

$$\begin{bmatrix} 1 & 4 & 7 \\ 3 & 0 & 5 \\ -1 & 9 & 11 \end{bmatrix} \tag{1.1}$$

To compute the minor M_{23} and the cofactor C_{23}, we find the determinant of the above matrix with row 2 and column 3 removed.

$$M_{2,3} = \det \begin{bmatrix} 1 & 4 & \square \\ \square & \square & \square \\ -1 & 9 & \square \end{bmatrix} = \det \begin{bmatrix} 1 & 4 \\ -1 & 9 \end{bmatrix} = (9 - (-4)) = 13$$

So the cofactor of the (2,3) entry is $C_{23} = (-1)^{2+3}(M_{23}) = -13$.

An important application of cofactors is the **Laplace's formula** for the expansion of determinants.

$$\det(\mathbf{A}) = \sum_{i=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij} \tag{1.2}$$

If $k \neq i$, we see that

$$\sum_{j=1}^{n} a_{kj} C_{ij} = 0 \tag{1.3}$$

Similarly, if $k \neq j$

$$\sum_{i=1}^{n} a_{ik} C_{ij} = 0 \tag{1.4}$$

This is essentially the determinant of a matrix with the k-th row the same as the i-th row, or the k-th column the same as the j-th column, which is zero.

1.2 The Cramer's Rule and the Adjugate Matrix

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ \vdots & \vdots & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n & = & b_n \end{array} \tag{1.5}$$

If we multiply the above by the row vector of cofactors of the 1^{st} column, $[C_{11}, C_{21}, \cdots, C_{n1}]$, we obtain

$$[\det(\mathbf{A}), 0, \cdots, 0] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = [C_{11}, C_{21}, \cdots, C_{n1}]\mathbf{b} \tag{1.6}$$

The left hand side used Equation 1.4. The right hand side is nothing but the determinant of a matrix with the first column replaced by $\mathbf{b}$.

Similarly, we can multiply the linear system by the row vector of cofactors of the $2^{nd}, 3^{rd}, \cdots, n^{th}$, and we obtain

$$\det(\mathbf{A})\mathbf{x} = \begin{bmatrix} C_{11} & \cdots & C_{n1} \\ C_{12} & \cdots & C_{n2} \\ \vdots & & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix} \mathbf{b} \tag{1.7}$$

which gives us

$$\det(\mathbf{A}) = \begin{bmatrix} C_{11} & \cdots & C_{1n} \\ C_{21} & \cdots & C_{2n} \\ \vdots & & \vdots \\ C_{n1} & \cdots & C_{nn} \end{bmatrix}^T \mathbf{A} \tag{1.8}$$

The matrix on the right

$$\mathrm{adj}(\mathbf{A}) = \mathbf{C}^T = \begin{bmatrix} C_{11} & \cdots & C_{1n} \\ C_{21} & \cdots & C_{2n} \\ \vdots & & \vdots \\ C_{n1} & \cdots & C_{nn} \end{bmatrix}^T \tag{1.9}$$

is called the adjugate matrix of $\mathbf{A}$, which is the transpose of the cofactor matrix $\mathbf{C}$.