# Calculus

Xi Tan (tan19@purdue.edu)

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# Preface

This book reviews calculus, advanced calculus, real analysis, and functional Analysis. The main references to be used are [?] for calculus, [?] for advanced calculus, [?] for real analysis, and [?] for functional analysis. Other useful texts include: [?] and [?] for real analysis.

# Part I

# Calculus

# Chapter 1

# Infinite Sequences and Series

## 1.1 Convergence Tests

There are five common techniques to test whether or not an infinite series is convergent. But first of all, a necessary condition:

**Theorem 1.1.1** If the limit of the summand is undefined or nonzero, that is, $\lim\limits_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ must diverge.

**Theorem 1.1.2 Comparison Test.** If $\{a_n\}, \{b_n\} > 0$, and the limit $\lim\limits_{n\to\infty} \frac{a_n}{b_n}$ exists, is finite and is not zero, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

**Theorem 1.1.3 Integral Test.** Let $f : [1, \infty) \to \mathbf{R}_+$ be a positive and monotone decreasing function such that $f(n) = a_n$. Then the series $\{a_n\}$ converges if and only if the integral $\int_1^{\infty} f(x)dx$ converges.

**Theorem 1.1.4 Ratio Test.** Suppose there exists $r$ such that $\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| = r$. If $r < 1$, then the series converges. If $r > 1$, then the series diverges. If $r = 1$, the ratio test is inconclusive, and the series may converge or diverge.

**Theorem 1.1.5 Root Test.** Define $r = \limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}$. If $r < 1$, then the series converges. If $r > 1$, then the series diverges. If $r = 1$, the ratio test is inconclusive, and the series may converge or diverge.

**Theorem 1.1.6 Alternating Series Test.** If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n, (b_n > 0)$ satisfies

1. $b_{n+1} \leq b_n,$ for all $n$; and,
2. $\lim_{n\to\infty} b_n = 0$.

Then the series is convergent.

**Theorem 1.1.7** A series is said to be absolutely convergent if $\sum_{i=1}^{\infty} |a_n|$ converges. Every absolutely convergent series is convergent. But not all convergent series are absolutely convergent. A convergent series that is not absolutely convergent is called conditionally convergent.

# Chapter 2

# Vectors and the Geometry of Space

## 2.1 Lines in $\mathbb{R}^n$

**Definition 2.1.1** Given a vector $\mathbf{p} \in \mathbb{R}^n$ and a nonzero vector $\mathbf{v} \in \mathbb{R}^n$, the set of all points $\mathbf{y} \in \mathbb{R}^n$ such that

$$\mathbf{y} = t\mathbf{v} + \mathbf{p}, \quad t \in \mathbb{R} \tag{2.1}$$

is called the *line* through $\mathbf{p}$ in the direction of $\mathbf{v}$.

**Example 2.1.1** The shortest distance from a point $\mathbf{q} \in \mathbb{R}^n$ to a line $L$ with equation $\mathbf{y} = t\mathbf{v} + \mathbf{p}$ is

$$\left\| (\mathbf{q} - \mathbf{p}) - \frac{(\mathbf{q} - \mathbf{p})^T \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v} \right\| \tag{2.2}$$

## 2.2 Hyperplanes in $\mathbb{R}^n$

**Definition 2.2.1** Suppose $\mathbf{n}$ is a normal vector for a hyperplan $H$ through $\mathbf{p} \in \mathbb{R}^n$, then the normal equation for $H$ is

$$\mathbf{n}^T(\mathbf{y} - \mathbf{p}) = 0 \tag{2.3}$$

If $H$ is in $\mathbb{R}^3$, we can use cross-product $\times$ to obtain the normal vector given two vectors on the hyperplane.

**Example 2.2.1** The shortest distance from a point $\mathbf{q} \in \mathbb{R}^n$ to a hyperplane $H$ with equation $\mathbf{n}^T(\mathbf{y} - \mathbf{p}) = 0$ is

$$\left| \frac{\mathbf{n}^T(\mathbf{q} - \mathbf{p})}{\|\mathbf{n}\|} \right| \tag{2.4}$$

A hyperplane is a set satisfies $H = \{x : w^T x = b\}$. An equivalent form is $w^T(x - \frac{w}{\|w\|^2} b) = 0$, which suggests that the vector $w$ is perpendicular to the hyperplane, called a normal vector.

Particularly, since $\frac{w^T w}{\|w\|^2} b = b$, we know $x_0 = \frac{w}{\|w\|} \frac{b}{\|w\|}$ is on the hyperplane. The $x_0$ is actually the projection of the origin, since $w$ is orthogonal to the hyperplane and it is nothing but a scaled $w$ on the hyperplane. Therefore, the shortest distance (along the direction of $w$) from the origin to the hyperplane is given by $\frac{b}{\|w\|}$ (could be negative, which means $w$ is on the other side of the hyperplane).

In general, if a hyperplane is given by the equation $f(x) = w^T x - b = 0$, the distance from any arbitrary vector $p$ to the hyperplane $w^T x = b$ is given by

$$\frac{f(p)}{\|w\|} = \frac{w^T p - b}{\|w\|}, \quad \text{if } p \text{ is on the opposite side of the origin} \tag{2.5}$$

$$-\frac{f(p)}{\|w\|} = -\frac{w^T p - b}{\|w\|}, \quad \text{if } p \text{ is on the same side of the origin} \tag{2.6}$$

Particularly, when $p = 0$ (the origin), the above becomes $\frac{b}{\|w\|}$, which agrees with our previous result.

Proof: Let's prove the first case. Suppose there is a vector $x$ on the hyperplane, such that $p - x = d\frac{w}{\|w\|}$. Since $w$ is orthogonal to the hyperplane, the scalar $d$ is the distance we are after. Now, multiply both sides by $w^T$,

$w^T p - w^T x = dw^T \frac{w}{\|w\|}$

$w^T p - (w^T x - b) = d\frac{w^T w}{\|w\|} + b$

$w^T p = d\|w\| + b$

$d = \frac{w^T p - b}{\|w\|}$

The proof for the other case is similar.

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