Solutions to Methods in Winter 2004

1. (a) We will consider the least square (LE) estimation which minimizes

$$\ell(\beta) = \sum_{i=1}^{n}(Y_i - \beta)^2.$$

Since

$$\frac{\partial \ell}{\partial \beta} = \frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_i - \beta X_i)X_i$$

we have

$$\hat{\beta} = \frac{\sum_{i=1}^{n}X_i Y_i}{\sum_{i=1}^{n}X_i^2}$$

and so

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \hat{\beta}X_i)^2.$$

(b) Note that $E(Y_i) = \beta X_i$. We have

$$E(\hat{\beta}) = \frac{\sum_{i=1}^{n}X_i E(Y_i)}{\sum_{i=1}^{n}X_i^2} = \beta$$

and

$$V(\hat{\beta}) = V\left[\frac{\sum_{i=1}^{n}X_i E(Y_i)}{\sum_{i=1}^{n}X_i^2}\right] = \frac{1}{(\sum_{i=1}^{n}X_i^2)^2}\sum_{i=1}^{n}X_i E(Y_i) = \frac{\sigma^2}{\sum_{i=1}^{n}X_i^2}.$$

(c) The estimate of σ^2 is

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - X_i\frac{\sum_{i=1}^{n}X_i Y_i}{\sum_{i=1}^{n}X_i^2})^2.$$

It has $n-1$ degrees of freedom

(d) If $V(Y_i) = X_i\sigma^2$, we need to consider the generalized least square (GLS) estimation. It minimizes

$$\ell(\beta) = \sum_{i=1}^{n}\frac{(Y_i - \beta)^2}{X_i}$$

and its derivatives is

$$\frac{\partial \ell}{\partial \beta} = \frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_i - \beta X_i).$$

This gives us

$$\hat{\beta} = \frac{\bar{Y}}{\bar{X}}.$$

2. (a) The factor effects ANOVA model with all the interaction effects is

$$Y_{ijkl} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} + \epsilon_{ijkl}$$

where $\epsilon_{ijkl} \sim^{i.i.d.} N(0, \sigma^2)$, $i = 1, 2, 3, 4$, $j = 1, 2$, $k = 1, 2, 3, 4$ and $l = 1, 2, 3$, and

$$\sum_{i=1}^{4} \alpha_i = \sum_{j=1}^{2} \beta_j = \sum_{k=1}^{4} \gamma_k,$$

$$\sum_{i=1}^{4}(\alpha\beta)_{ij} = \sum_{j=1}^{2}(\alpha\beta)_{ij} = \sum_{i=1}^{4}(\alpha\gamma)_{ik} = \sum_{k=1}^{4}(\alpha\gamma)_{ik} = \sum_{j=1}^{2}(\beta\gamma)_{ik} = \sum_{k=1}^{4}(\beta\gamma)_{jk},$$

and

$$\sum_{i=1}^{4}(\alpha\beta\gamma)_{ijk} = \sum_{i=j}^{2}(\alpha\beta\gamma)_{ijk} = \sum_{k=1}^{4}(\alpha\beta\gamma)_{ijk} = 0.$$

(b) The factor effects ANOVA model for Z_{ikl} is

$$Z_{ikl} = \mu' + \alpha'_i + \gamma'_k + (\alpha\gamma)'_{ik} + \epsilon'_{ikl}$$

where $i, k = 1, 2, 3, 4$, $\ell = 1, 2, 3$, and

$$\begin{aligned}
\mu' &= \beta_1 - \beta_2 = 2\beta_1, \\
\alpha'_i &= (\alpha\beta)_{i1} - (\alpha\beta)_{i2} = 2(\alpha\beta)_{i1}, \\
\gamma'_k &= (\beta\gamma)'_{1k} - (\beta\gamma)'_{2k} = 2(\beta\gamma)_{1k}, \\
(\alpha\gamma)'_{ik} &= (\alpha\beta\gamma)_{i1k} - (\alpha\beta\gamma)_{i2k} = 2(\alpha\beta\gamma)_{i1k}, \\
\epsilon'_{ijk} &= \epsilon_{i1kl} - \epsilon_{i2kl}.
\end{aligned}$$

Then $\epsilon'_{ijk} \sim^{i.i.d.} N(0, 2\sigma^2)$.

(c) It shows that the three-factor interaction effect is not significant. Because the two straight lines show the interaction effects are all 0, then we can say $(\alpha\beta\gamma)_{i1k} = 0$ for all i and k. Then, we can say that $(\alpha\beta\gamma)_{ijk} = 0$ for all i, j and k.

(d) The ANOVA table is

Source	Df	Sum Sq	Mean Sq	F value	$Pr(> F)$
A	3	0.04063	0.01354	0.1776	0.9107936
C	3	1.75229	0.58410	7.6603	0.0005382
A:C	9	0.50021	0.05558	0.7289	0.6794016
Residuals	32	2.44000	0.07625		

3. This is a typical question for grouping in GLM. Two models are fitted differently. One is binomial model and the other is Bernoulli model. Let n_i be the number of individuals

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in group i and let n_{iy} be the number of individuals who went to colloge. Denote δ_{ij} be the indicator of entering collage for person j in district i. Then, $n_{iy} = \sum_{i=1}^{n_i} \delta_{ij}$. Let x_i be the dollars spent per students in district $i$and I be the total number of district. Suppose the model is

$$\log(\frac{\pi(x)}{1 - \pi(x)}) = \alpha + \beta x_i \Leftrightarrow \pi(x) = \frac{e^{\alpha+\beta x}}{1 + e^{\alpha+\beta x}}.$$

For binomial model, the likelihood is

$$L_1 = \prod_{i=1}^{I} \binom{n_i}{n_{iy}} \left[\frac{e^{\alpha+\beta x}}{1 + e^{\alpha+\beta x}}\right]^{n_{iy}} \left[\frac{1}{1 + e^{\alpha+\beta x}}\right]^{n_i - n_{iy}}$$

and for Bernoullin model, the likelihood is

$$L_2 = \prod_{i=1}^{I} \prod_{j=1}^{n_i} \left[\frac{e^{\alpha+\beta x}}{1 + e^{\alpha+\beta x}}\right]^{\delta_{ij}} \left[\frac{1}{1 + e^{\alpha+\beta x}}\right]^{1-\delta_{ij}} = \prod_{i=1}^{I} \left[\frac{e^{\alpha+\beta x}}{1 + e^{\alpha+\beta x}}\right]^{n_{iy}} \left[\frac{1}{1 + e^{\alpha+\beta x}}\right]^{n_i - n_{iy}}.$$

(a) They will get the same MLE since L_1/L_2 is a constant.

(b) They will get the same standard error. The standard error is estimated from Fisher Information Matrix. They will have the same second derivative and so they will have the same Fisher Information Matrix.

4. (a) Let M be the matrix with all one on the first column, X on the second column, 0 and 1 on the first ten and the second ten on the third column and 0 on the product of the second column and the third column. Then, the matrix expression of the model is
$$Y = X\beta + \epsilon$$
where Y is a 20-dimensional vector, $\beta = (\mu, \alpha, \gamma, (\alpha\gamma))$, μ is the intercept, β is the slope of X for method A, γ is the difference of the intercept between the two methods, $(\alpha\gamma)$ is the difference of the slopes between the two methods.

(b) The estimated line for method A is

$$Y = 16.4226 + 0.9754X$$

and the estimated line for method B is

$$Y = 16.4226 + 0.9754X - 1.3139 - 0.2536X = 15.1087 - 0.7218X.$$

The interaction effect is not significant at level 0.05. Thus, the different of the slopes is not significant.

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(c) Let μ' be the intercept, α' be the slope, γ' be the difference of the intercept and $(\alpha\gamma)$ be the different of the slope in ht model by using X'. Then,

$$\mu' = \hat{\mu} + \hat{\beta}\bar{X} \Rightarrow \hat{\mu}' = 16.4226 + 15.55(0.9754) = 31.5901$$

$$\alpha' = \alpha \Rightarrow \hat{\alpha}' = 0.9754$$

$$\gamma' = \gamma + (\alpha\gamma)\bar{X} \Rightarrow \hat{\gamma}' = -1.3139 - 0.2536(15.55) = -5.2574$$

$$(\alpha\gamma)' = (\alpha\gamma) \Rightarrow \hat{(\alpha\gamma)}' = -0.2536.$$

(d) The estimated value of the method main effect is -5.2574. Its variance is

$$\hat{Var}(\hat{\gamma}') = 9.80305 - 2(15.55)(0.54277) + 15.55^2(0.03584) = 1.5891.$$

Thus, its standard error is $\sqrt{1.5891} = 1.2606$. The t-value is $-5.2574/1.2606 = -4.1706$. Therefore, it is significant at level 0.05.

5. The probability mass function of X_1 is

$$P[X_1 = m] = \frac{(2\pi\lambda_1 tr)^m}{m!}e^{-2\pi\lambda_1 tr}$$

and the probability mass function of X_2 is

$$P[X_2 = n] = \frac{(\pi\lambda_2 tr^2)^n}{n!}e^{-\pi\lambda_2 tr^2}$$

Therefore, the log-likelihood function of r is

$$\ell(r) = -\log(m!n!) - 2\pi\lambda_1 tr - \pi\lambda_2 tr^2 + m\log(2\pi\lambda_1 tr) + n\log(\pi\lambda_2 tr^2).$$

Then,

$$\ell'(r) = -2\pi\lambda_1 t - 2\pi\lambda_2 tr + \frac{m}{r} + \frac{2n}{r},$$

and

$$\ell''(r) = -2\pi\lambda_2 t - \frac{m + 2n}{r^2}.$$

(a) The ML estimator of r can be solved from $\ell(r) = 0$. Then, we have

$$2\pi\lambda_2 tr^2 + 2\pi\lambda_1 tr - (m + 2n) = 0 \Rightarrow r = \frac{-2\pi\lambda_1 t \pm \sqrt{4\pi^2\lambda_1^2 t^2 - 8\pi\lambda_2 t(m + 2n)}}{4\pi\lambda_2 t}$$

$$\Rightarrow r_1 = -12.433; \ r_2 = 4.433.$$

Thus, the ML estimator is $\hat{r} = 4.433$. The Fisher Infromation is

$$I(r) = E[-\ell''(r)] = 2\pi\lambda_2 t + \frac{E(X_1) + 2E(X_2)}{r^2} = 4\pi\lambda_2 t + \frac{2\pi\lambda_1 t}{r}.$$

Thus, we have

$$I(\hat{r}) = 38.25.$$

The asymptotic variance of MLE of r is

$$Var(\hat{r}) = \frac{1}{38.25} = 0.02614.$$

Thus, the 95% confidence interval of r is

$$[4.433 - 1.96\sqrt{0.02614}, 4.433 + 1.96\sqrt{0.02614}] = [4.116, 4.750].$$

(b) The standard error of the MLE is

$$\sigma = [4\pi\lambda_2 t + \frac{2\pi\lambda_1 t}{r}]^{-\frac{1}{2}} \leq \frac{r}{10}$$

Then, we have

$$4\pi\lambda_2 tr^2 + 2\pi\lambda_1 tr - 100 \geq 0 \Rightarrow 47.18t^2 + 44.57t - 100 \geq 0 \Rightarrow t \geq 2.003$$

At least t shoudl be 2.003.

6. (a) The model assumption is
$$Y = \mu_i + \alpha_j + \epsilon_{ijk}$$
where $i = 1, 2$, $j = 1, 2, \cdots, 10$, $k = 1, \cdots, n_{ij}$, where $\alpha_j \sim^{i.i.d.} N(0, \sigma_\alpha^2)$ and $\epsilon_{ijk} \sim N(0, \sigma^2)$ and α_j and ϵ_{ijk} are independent. The estimated values are: $\hat{\mu}_1 = 16.1262$, $\hat{\mu}_2 = 16.1262 - 2.0440 = 14.0822$, $\hat{\sigma}_\alpha = 0.9398$, $\hat{\sigma} = 1.1576$.

(b) The predicted value is 16.1262 with variance

$$0.6425^2 = 0.2068.$$

The variance of the predicted value is

$$0.2068 + 0.9398^2 + 1.1576^2 = 2.4301.$$

Thus, the 95% confidence interval of the predicted value is

$$[16.1262 - 1.96 \times \sqrt{2.4301}, 16.1262 + 1.96 \times \sqrt{2.4301}] = [13.07, 19.18].$$

(c) It is impossible to study the diet-sow interaction effect because the sows are different within two diet methods.

7. Let δ_i be denoted by sensoring, that is $\delta_i = 1$ if not dropoff and $\delta_i = 0$ if dropoff.

(a) Let λ_j denote the parameter in group j. Then, we have

$$\hat{\lambda}_j = \frac{\sum_{i=1}^n \delta_i}{\sum_{i=1}^n t_i}.$$

The null hypothesis is $H_0 : \lambda_1 = \lambda_2$ and the alternative hypothesis is $H_A : \lambda_1 \neq \lambda_2$. Then, based on the null hypothesis, we have $\lambda = 0.0339$ and based on the alternative hypothesis we have $\hat{\lambda}_1 = 0.02222$ and $\hat{\lambda}_2 = 0.05606$. The likelihood function for group j is

$$L(\lambda_j) = \prod_{i=1}^n (\lambda_j)^{\delta_i}[e^{-\lambda_j t_i}].$$

The loglikelihood ratio is

$$\Lambda = 2.96.$$

Based on χ_1^2 distribution, it is not significant.

(b) For group 1, we have $S(20) = 1 - 6/15 = 0.6$, $S(90) = 1 - 12/15 = 0.2$ and $S(95) = 0.2$.

8. (a) The log-likelihood function of $Y_1, \cdots, Y_n$ is

$$\ell(\alpha) = \sum_{i=1}^n [Y_i \log \lambda_i - \lambda_i - \log(Y_i!)] = \sum_{i=1}^n [Y_i(\alpha + \beta X_i) - e^{\alpha + \beta X_i} - \log(Y_i!)].$$

Then,

$$\ell'(\alpha) = \sum_{i=1}^n [Y_i - e^{\alpha} e^{\beta X_i}] \Rightarrow \hat{\alpha} = \log \frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n e^{\beta X_i}}.$$

(b) Because $\ell''(\alpha) = -e^{\alpha} \sum_{i=1}^n e^{\beta X_i}$, we have

$$I(\alpha) = e^{\alpha} \sum_{i=1}^n e^{\beta x_i}.$$

Thus, the standard error of the MLE is

$$\sigma(\hat{\alpha}) = \frac{1}{\sqrt{I(\hat{\alpha})}} = \frac{1}{\sqrt{\sum_{i=1}^n Y_i}}.$$

(c) Use the test-statistic $\hat{\alpha}/\sigma(\hat{\alpha})$ to test $\alpha = 0$. if

$$|\frac{\hat{\alpha}}{\sigma(\hat{\alpha})}| > z_{\frac{\alpha}{2}},$$

then reject $\alpha = 0$.

(d) The negative value of 2 times the log-likelihood ratio is

$$\Lambda = -2(\ell(0) - \ell(\hat{\alpha})) = 2[\sum_{i=1}^{n} Y_i \hat{\alpha} - e^{\hat{\alpha}} \sum_{i=1}^{n} e^{\beta X_i} + \sum_{i=1}^{n} e^{\beta X_i}]$$

$$= 2\sum_{i=1}^{n}[Y_i \hat{\alpha} + e^{\beta X_i} - e^{\beta X_i} \frac{\sum_{i=1}^{n} Y_i}{\sum_{i=1}^{n} e^{\beta X_i}}]$$

$$= 2(\hat{\alpha} - 1)\sum_{i=1}^{n} Y_i + 2\sum_{i=1}^{n} e^{\beta X_i}$$

$$= 2(e^{-\alpha} - 1 + \hat{\alpha})\sum_{i=1}^{n} Y_i$$

$$\approx \frac{\hat{\alpha}^2}{(\sum_{i=1}^{n} Y_i)}.$$

since

$$(e^{-\alpha} - 1 + \hat{\alpha}) = \frac{\hat{\alpha}^2}{2}$$

when $\hat{\alpha}$ is small. Thus, the two tests are asymptotically equivalent.