4.6

15] $y'' + y = \cos^2 x$

Let y be the general solution —

$$y = y_c + y_p$$

for y_c $\quad y'' + y = 0$ — (11)

Let

$y = e^{sx}$, $y' = s e^{sx}$, $y'' = s^2 e^{sx}$

from (11) $\quad s^2 e^{sx} + s e^{sx} \cdot 0 + e^{sx} = 0$

$$\Rightarrow e^{sx} \underbrace{(s^2+1)}_{\text{characteristic eq}^n} = 0 \quad \Big| \; e^{sx} \neq 0 \;\; \text{So } s^n$$

So, $\quad s^2 + 1 = 0$

$$\Rightarrow s = \frac{-0 \pm \sqrt{0^2 - 4\cdot1\cdot1}}{2\cdot1}$$

$$= \frac{\sqrt{4-4}}{2}$$

$$= \frac{2i}{2} = \pm i \qquad \alpha = 0,\ \beta = 1$$

$-1 = i^2$

$i =$

$$y_c = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

$$= e^0 (c_1 \cos 1x + c_2 \cos 1x)$$

$$y_c = c_1 \cos x + c_2 \sin x \leftarrow \text{general solution}$$

$\therefore y_1 = \cos x, \quad y_2 = \sin x$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x = 1$$

$$u_1' = \frac{W_1}{W} = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{1}$$

$$= \begin{vmatrix} 0 & \sin x \\ \cos^2 x & \cos x \end{vmatrix}$$

$$= -\cos^2 x \sin x$$

$$u_2' = \int \frac{W_2}{W} = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{1}$$

$$= \begin{vmatrix} \cos x & 0 \\ -\sin x & \cos^2 x \end{vmatrix}$$

$$= \cos^3 x = \cos x \cos^2 x$$

$$= \cos x \, (1 - \sin^2 x)$$

$$u_1 = \int u_1' \, dx$$

$$= \int -\cos^2 x \sin x \, dx$$

$$= \int z^2 \, dz$$

$$= \frac{z^3}{3} = \frac{\cos^3 x}{3}.$$

let, $z = \cos x$

$$\frac{dz}{dx} = -\sin x$$

$$\therefore dz = -\sin x \, dx$$

$u_2 = \int u_2' \, dx$

$= \int \cos x \, (1-\sin^2 x) \, dx$

$= \int (1-z^2) \, dz$

$= z - \frac{z^3}{3}$

$= \sin x - \frac{\sin^3 x}{3}.$

let, $z = \sin x$

$\frac{dz}{dx} = \cos x$

$\therefore dz = \cos x \, dx$

$\therefore y = y_c + y_p$

$= c_1 \cos x + c_2 \sin x + \frac{\cos^3 x}{3} \cos x + \left(\sin x - \frac{\sin^3 x}{3}\right) \sin x$

$= c_1 \cos x + c_2 \sin x + \frac{\cos^4 x}{3} + \sin^2 x + \frac{\sin^4 x}{3}.$

Ans

9) $y'' - 4y = \frac{e^{2x}}{x}$

let y be the general solution,

$y = y_c + y_p$

for y_c, $y'' - 4y = 0$ —— (11)

$y = e^{\delta x}$, $y' = \delta e^{\delta x}$, $y'' = \delta^2 e^{\delta x}$

putting this into (11)

$\delta^2 e^{\delta x} - 4 e^{\delta x} = 0$

$e^{\delta x}(\delta^2 - 4) = 0$

$\Rightarrow$ $e^{\delta x}$ can't be 0.

So, $\delta^2 - 4 = 0.$

$\therefore\ \delta^2 - 4 = 0$

$(\delta-2)(\delta+2) = 0$

$\therefore\ \delta_1 = 2\ ,\ \delta_2 = -2$

$\therefore\ y_1 = e^{2x}\ ,\ y_2 = x \cdot e^{-2x}$

$\therefore\ y_c = c_1 e^{2x} + c_2 e^{-2x}.$

$\therefore\ W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2'' \end{vmatrix}$

$= \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -2e^0 - 2e^0$

$= -2-2 = -4.$

$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y'' \end{vmatrix} = \begin{vmatrix} 0 & e^{-2x} \\ \frac{e^{2x}}{x} & -2e^{-2x} \end{vmatrix} =$

$= -\frac{e^0}{x} = \frac{-1}{x}$

$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1'' & f(x) \end{vmatrix} = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & \frac{e^{2x}}{x} \end{vmatrix} = \frac{1}{x} e^{4x}$

$\therefore\ u_1' = \frac{W_1}{W} = \frac{-1}{x} \cdot \frac{-1}{4} = \frac{1}{4x}$

$u_2' = \frac{W_2}{W} = \frac{e^{4x}}{x} \cdot \frac{-1}{4} = -\frac{e^{4x}}{4x}$

$\therefore\ u_1 = \int \frac{1}{4x}\,dx = \frac{1}{4}\ln x$

$u_2 = -\frac{1}{4}\int_{x_0}^{x} \frac{e^{4t}}{t}\,dt$

$\therefore y = \cancel{c}\, y_c + u_1 y_1 + u_2 y_2 \leftarrow$ put the value here

10/ $y'' - 9y = \dfrac{9x}{e^{3x}}$

$y = y_c + y_p$

for y_c, $y'' - 9y = 0$

$y = e^{\delta x}$, $y' = \delta e^{\delta x}$, $y'' = \delta^2 e^{\delta x}$

$\delta^2 e^{\delta x} - 9\cancel{\delta} e^{\delta x} = 0$

$e^{\delta x}(\delta^2 - 9) = 0$

$\therefore \delta^2 - 9 = 0$

$\therefore \delta_1 = 3, \delta_2 = -3$

$\therefore y_c = c_1 e^{3x} + c_2 e^{-3x}.$

$W = \Big|$ Similar to previous one.