COL226: Programming Languages

II semester 2015-16

Tue 22 Mar 2016 Minor 2 60 minutes Max Marks 40

Answer only in the space provided on the question paper.

1. **[7 marks]** Construct an example to show that there exist λ-terms L and M such that if $L \longrightarrow_\beta^n M$ for some $n > 0$, then $L \not\longrightarrow_{||\beta}^1 M$

   Solution. Consider the terms $L \equiv_\alpha ((\mathsf{S}\ \mathsf{K})\ \mathsf{K})$ and $M \equiv_\alpha \mathsf{I}$. We know that $L \longrightarrow^* M$ in five steps of β-reduction. However this reduction <u>cannot</u> be done in a single step of $||\beta$-reduction.

   A simple rule (derived from the syntax of the λ-calculus) to identify β-redexes in well-formed λ-terms (by our syntax) is to just expand all the defined names and then look for substrings of the form "$(\lambda$".

   Even after expanding the combinators S and K there is only one β-redex available (expanding K does not produce any more β-redexes that may be used simultaneously in a $||\beta$-reduction).

$$\begin{array}{rl} & L \\ \equiv_\alpha & ((\lambda x[\lambda y[\lambda z[((x\ z)\ (y\ z))]]]\ \mathsf{K})\ \mathsf{K}) \\ \longrightarrow_{||\beta}^1 & (\lambda y[\lambda z[((\mathsf{K}\ z)\ (y\ z))]]\ \mathsf{K}) \\ \equiv_\alpha & (\lambda y[\lambda z[((\lambda u[\lambda v[u]]\ z)\ (y\ z))]]\ \mathsf{K}) \end{array}$$

   At this point notice that there are two redexes available and applying them both simultaneously we get

$$\begin{array}{rl} \longrightarrow_{||\beta}^1 & \lambda z[(\lambda v[z]\ (\mathsf{K}\ z))] \\ \equiv_\alpha & \lambda z[(\lambda v[z]\ (\lambda x[\lambda w[x]]\ z))] \end{array}$$

   Here again there are two redexes that may be applied simultaneously to yield

$$\begin{array}{rl} \longrightarrow_{||\beta}^1 & \lambda z[z] \\ \equiv_\alpha & \mathsf{I} \\ \equiv & M \end{array}$$

2. **[12 marks]** Let $\mathsf{C} \equiv_\alpha \lambda x\ y\ z[(x\ (y\ z))]$ be a combinator. Prove that for all u, v, w,

$$(\mathsf{C}\ u\ (\mathsf{C}\ v\ w)) =_\beta (\mathsf{C}\ (\mathsf{C}\ u\ v)\ w)$$

Solution. Notice that for any f and g

$$(\mathsf{C}\ f\ g) \longrightarrow^2_\beta \lambda z[(f\ (g\ z))] \tag{1}$$

Hence

$$\begin{array}{llcll}
 & (\mathsf{C}\ u\ (\mathsf{C}\ v\ w)) & | & & (\mathsf{C}\ (\mathsf{C}\ u\ v)\ w) \\
\longrightarrow^2_\beta & \lambda z[(u\ ((\mathsf{C}\ v\ w)\ z))] & | & \longrightarrow^2_\beta & \lambda z[((\mathsf{C}\ u\ v)\ (w\ z))] \\
\longrightarrow^2_\beta & \lambda z[(u\ (\lambda y[(v\ (w\ y))]\ z))] & | & \longrightarrow^2_\beta & \lambda z[(\lambda y[(u\ (v\ y))]\ (w\ z))] \\
\longrightarrow^1_\beta & \lambda z[(u\ (v\ (w\ z)))] & \equiv_\alpha & \longrightarrow^1_\beta & \lambda z[(u\ (v\ (w\ z)))]
\end{array}$$

3. **[3+8 = 11 marks]** Church defined the truth values by the combinators $\mathsf{true} \stackrel{df}{=} \lambda\ x\ y[x]$ and $\mathsf{false} \stackrel{df}{=} \lambda\ x\ y[y]$ and the *if-then-else operator* by the combinator $\mathsf{ite} \stackrel{df}{=} \lambda\ x\ y\ z[((x\ y)\ z)]$. Shannon showed that all boolean operations may be represented using only the *if-then-else operator* and the two constants *true* and *false*.

   (a) Use the above combinators to define the boolean operation *and*.

   (b) Prove that your definitions of *and* satisfies the truth-table for *and*

   Solution.

   See exercise 13 on page 15 of http://www.cse.iitd.ernet.in/~sak/courses/ilfp/lambda-annotated.pdf.pdf

4. **[10 marks]** Prove that if a binary relation on λ-terms is terminating and locally confluent then it is confluent.

Solution.

$\longrightarrow$ on Λ is given to be terminating and locally confluent. We need to show that it is confluent. That is for any L, we are given that

(a) there is no infinite sequence of reductions of L, i.e. every maximal sequence of reductions of L is of length n for some $n \geq 0$.

(b)
$$N_1 \ {}^1\!\!\longleftarrow L \longrightarrow^1 M_1 \Rightarrow \exists P : M_1 \longrightarrow^* P \ {}^*\!\!\longleftarrow N_1 \tag{2}$$

We need to show for any term L that

$$N \ {}^*\!\!\longleftarrow L \longrightarrow^* M \Rightarrow \exists S : M \longrightarrow^* S \ {}^*\!\!\longleftarrow N \tag{3}$$

Let L be any term. Consider the graph $G(L) = \langle \Gamma(L), \longrightarrow^1 \rangle$ such that $\Gamma(L) = \{M \mid L \longrightarrow^* M\}$. Since $\longrightarrow$ is a terminating reduction

Fact 0.1 *The graph $G(L)$ is acyclic for any term L.*

If $G(L)$ is not acyclic, there must be a cycle of length $k > 0$ such that $M_0 \longrightarrow^1 M_1 \longrightarrow^1 \cdots \longrightarrow^1 M_{k-1} \longrightarrow^1 M_0$ which implies there is also an infinite reduction sequence of the form $L \longrightarrow^* M_0 \longrightarrow^k M_0 \longrightarrow^k \cdots$ which is impossible.

Since there are only a finite number of sub-terms of L that may be reduced under $\longrightarrow$, for each L there is a maximum number $p \geq 0$, which is the length of the longest reduction sequence.

Fact 0.2 *For every $M \in \Gamma(L)$,*

(a) $G(M)$ is a sub-graph of $G(L)$ and

(b) For every $M \in \Gamma(L) - \{L\}$, the length of the longest reduction sequence of M is less than p.

We proceed by induction on p.

Basis. $p = 0$. Then $\Gamma(L) = \{L\}$ and there are no reductions possible, so it is trivially confluent.

Induction Hypothesis (IH).

> *For any L whose longest reduction sequence is of length k, $0 \leq k < p$, property (3) holds.*

Induction Step. Assume L is a term whose longest reduction sequence is of length $p > 0$. Also assume $N \ {}^*\!\!\longleftarrow L \longrightarrow^* M$ i.e. $\exists m, n \geq 0 : N \ {}^n\!\!\longleftarrow L \longrightarrow^m M$.

Case $m = 0$. If $m = 0$ then $M \equiv_\alpha L$ and hence $S \equiv_\alpha N$.

Case $n = 0$. Then $N \equiv_\alpha L$ and we have $S \equiv_\alpha M$.

Case $m, n > 0$. Then consider M_1 and N_1 such that

$$N \ {}^*\!\!\longleftarrow N_1 \ {}^1\!\!\longleftarrow L \longrightarrow^1 M_1 \longrightarrow^* M \tag{4}$$

By (2), $\exists P : M_1 \longrightarrow^* P \ {}^*\!\!\longleftarrow N_1$. Clearly $M_1, N_1, P \in \Gamma(L) - \{L\}$. Hence by fact 0.2, $G(M_1)$, $G(N_1)$ and $G(P)$ are all sub-graphs of $G(L)$ and all their reduction sequences are of length smaller than p. Hence by induction hypothesis, we get

$$P \ {}^*\!\!\longleftarrow M_1 \longrightarrow^* M \Rightarrow \exists Q : M \longrightarrow^* Q \ {}^*\!\!\longleftarrow P \tag{5}$$

and

$$N \ {}^*\!\!\longleftarrow N_1 \longrightarrow^* P \Rightarrow \exists R : P \longrightarrow^* R \ {}^*\!\!\longleftarrow N \tag{6}$$

But by (5) and (6) and the induction hypothesis we have

$$R \ {}^*\!\!\longleftarrow P \longrightarrow^* Q \Rightarrow \exists S : Q \longrightarrow^* S \ {}^*\!\!\longleftarrow R \tag{7}$$

Combining (7) with (4), (5) and (6) we get

$$N \ {}^*\!\!\longleftarrow L \longrightarrow^* M \Rightarrow \exists S : M \longrightarrow^* S \ {}^*\!\!\longleftarrow N \tag{8}$$