Lecture 6. Relations

Ref.: K H Rosen Chapter 7

Binary Relation

- A ***binary relation*** **R from a set A to a set B is a subset R $\subseteq$ A×B.**

E.g. Let A = { a, b, c}

B = {1, 2, 3, 4}

R is defined by the ordered pairs

{(a, 1), (a, 2), (c, 4)}

aRb denotes that $(a,b) \in R$ while a$\not R$b means $(a,b) \notin R$

When aRb, a is said to be related to b by R.

Relation on a Set

- **A binary relation R *on a set A* is a relation from *A* to *A*.**

E.g. Let A = { a, b, c}

R={(a, a), (a, b), (c, c)}

Question: How many binary relations are there on a set A with n elements?

Representations of Relations

Connection Matrices

Let R be a relation from set $A = \{a_1, a_2, . . . , a_m\}$ to $B = \{b_1, b_2, . . . , b_n\}$.

A *m*n connection matrix M* for R is defined by

$$M_{ij} = 1 \quad \text{if } \langle a_i, b_j \rangle \text{ is in } R;$$
$$= 0 \quad \text{otherwise.}$$

Example

- assume the rows are labeled with the elements of A and the columns are labeled with the elements of B.

Let A = {a, b, c}, B = {e, f, g, h} and
R = {<a,e>, <c, g>}.
Then the connection matrix M for R is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Representations of Relations

Graphical representation

Definition: A *Directed graph* or a *Digraph* D from A to B is a collection of *vertices* and a collection of *edges* . If there is an ordered pair e = <x, y> in *R* then there is an *arc* or *edge* from x to y in D. The elements x and y are called the *initial* and *terminal* vertices of the edge e.

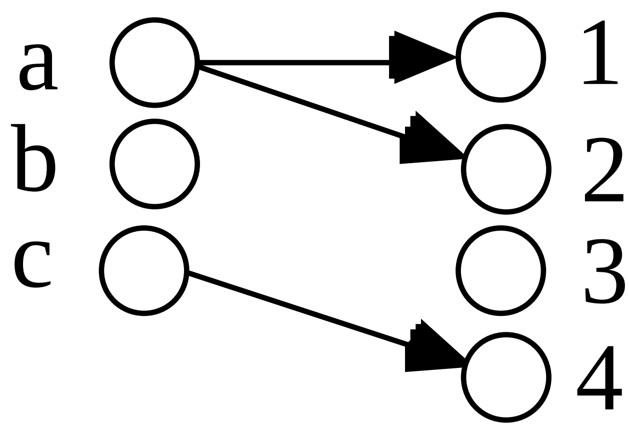
Example

Let A = { a, b, c} and B = {1, 2, 3, 4}

R is defined by the ordered pairs or edges

{<a, 1>, <a, 2>, <c, 4>}

R can be represented by the digraph D:

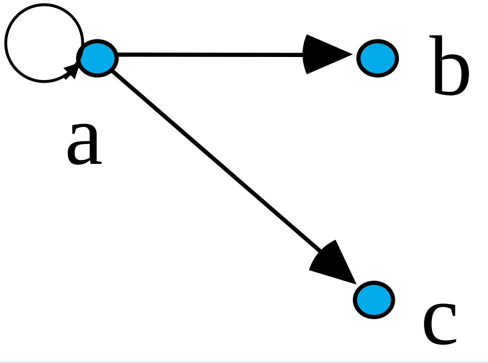


Example

For a relation R on a set A = {a, b, c}

R = {<a, a>, <a, b>, <a, c>}.

Then a digraph representation of R is:



An arc of the form <x, x> on a digraph is called a *loop*.

Special Properties of Binary Relations

Given: a Universe U and a binary relation R on a subset A of U

- **R is *reflexive* iff $(a,a) \in R$ for every element $a \in A$**

A relation R on a set A is reflexive if every element of A is related to itself.

E.g. Is the "Divides" relation on the set of positive integers reflexive?
Yes, since $a|a$ for all positive integer a.

Special Properties of Binary Relations

- **R is *symmetric* iff $(a,b) \in R$ whenever $(b,a) \in R$ for every $a, b \in A$.**
- **R is called *antisymmetric* such that $(a,b) \in R$ and $(b,a) \in R$ only if $a=b$ for every $a, b \in A$.**

Symmetric and antisymmetric are not opposite since a relation can have both or lack both of them.
(Can you think of some examples?)

Special Properties of Binary Relations

- **R is *transitive* iff whenever $(a,b)\in R$ and $(b,c)\in R$, then $(a,c)\in R$, for every $a, b, c \in A$.**

This is the most difficult one to check. Algorithms have been developed to check this.

Relation properties in Matrix

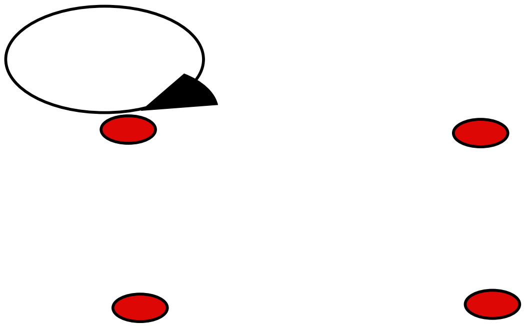
Let R be a binary relation on a set A and let M be its connection matrix. Then

- **R is *reflexive* iff $M_{ii} = 1$ for all i.**
- **R is *symmetric* iff M is a symmetric matrix: $M = M^T$**
- **R is *antisymetric* iff $M_{ij} = 0$ or $M_{ji} = 0$ for all $i \neq j$.**

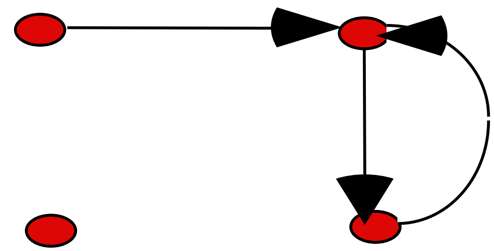
Relation properties in Digraph

- A relation is **reflexive** iff there is a loop at **every** vertex of its digraph.
- A relation is **symmetric** iff for every edge between distinct vertices in its digraph there is an edge in the opposite direction.
- A relation is **antisymmetric** iff there are never two edges in the opposite directions between distinct vertices.
- A relation is **transitive** iff whenever there is a edge from vertex x to y and from y to z, there is a edge from x to z.

Example



A

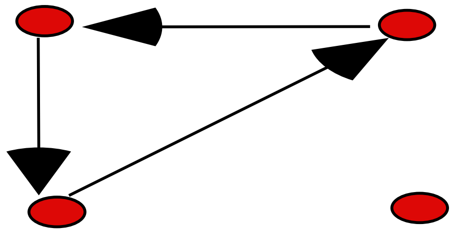


B

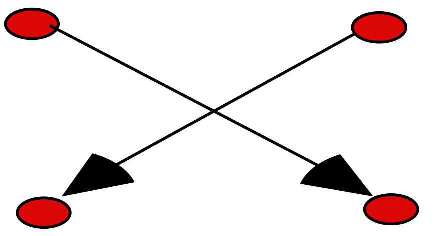
A: not reflexive
symmetric
antisymmetric
transitive

B: not reflexive
not symmetric
not antisymmetric
not transitive

Example



C



D

C: not reflexive
not symmetric
antisymmetric
not transitive

D: not reflexive
not symmetric
antisymmetric
transitive

Combining Relations

Since Relations from A to B are subset of A×B, two relations from A to B can be combined in any way two sets can be combined.

- Union $R_1 \cup R_2$
- Intersection $R_1 \cap R_2$
- Difference $R_1 - R_2$
- Compliment $\overline{R_1}$

Combining Relations: Example

Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$

The Relations $R_1=\{(1,1), (2, 2), (3,3)\}$ and $R_2 =\{(1,1), (1, 2), (1,3), (1,4)\}$ can be combined to obtained as

- $R_1 \cup R_2 = \{(1,1), (1, 2), (1,3), (1,4), (2, 2), (3,3)\}$
- $R_1 \cap R_2 = \{(1,1)\}$
- $R_1 - R_2 = \{(2, 2), (3,3)\}$
- $R_2 - R_1 = \{(1, 2), (1,3), (1,4)\}$

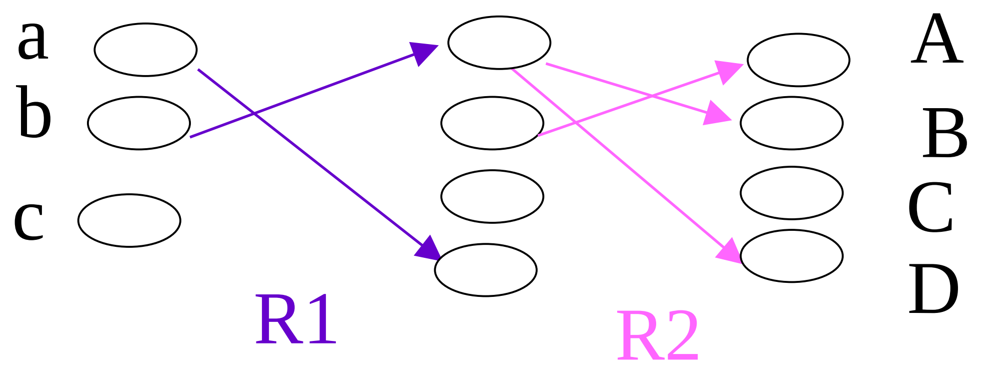
Composition of Relations

Suppose R1 is a relation from set A to B, R2 is a relation from set B to C. Then the composition of R2 with R1, denoted R2oR1 is the relation from A to C:

If (a,b) is a member of R1 and (b,c) is a member of R2 then (a,c) is a member of R2oR1.

- For (a, c) to be in the composite relation R2oR1 there must exist a b in B . . . .
- We read them right to left as in functions.

Example



R2oR1 = {(b, D), (b, B)}

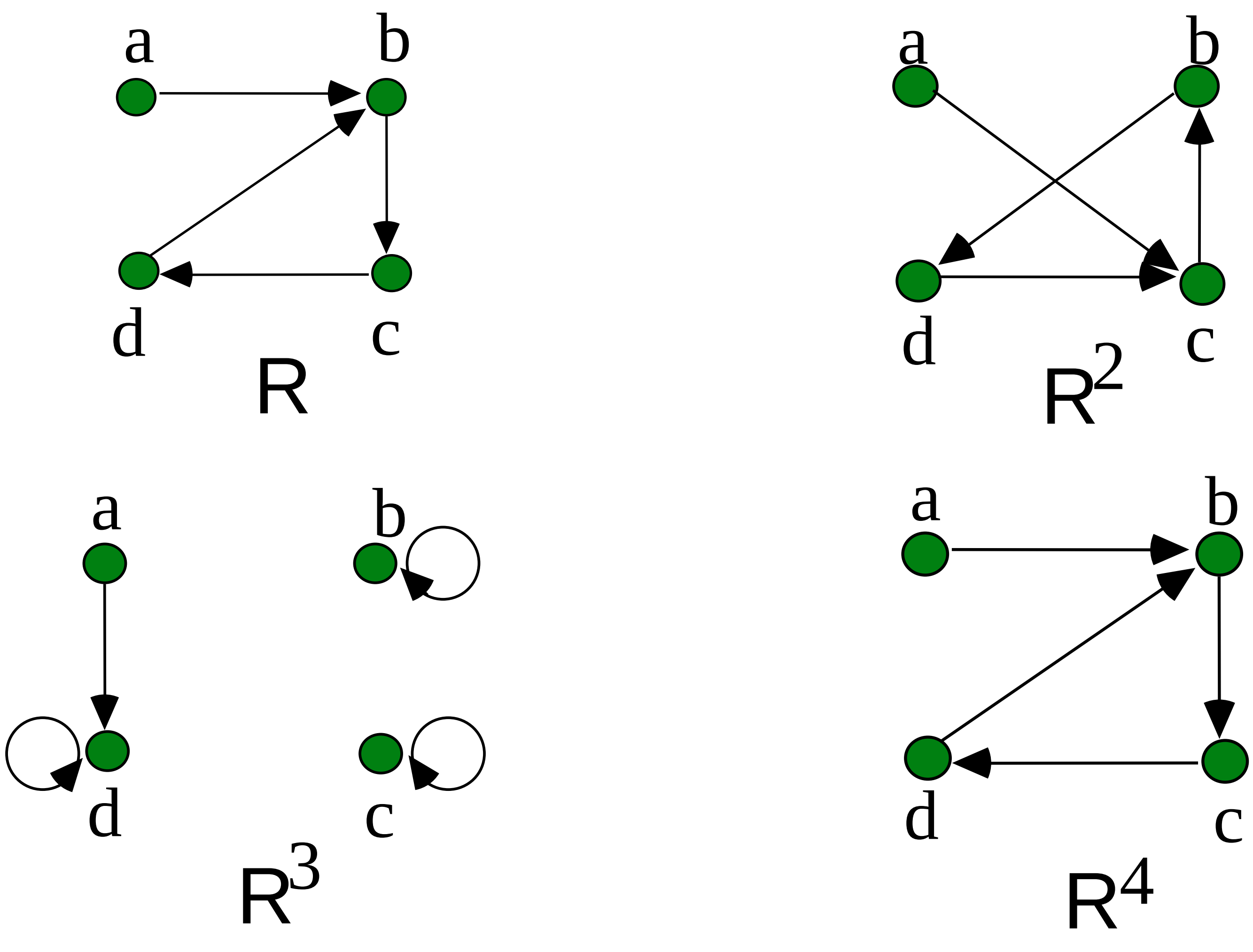
Powers of a Relation

Let R be a binary relation on A. Then R^n: the power of R are defined inductively by

***Basis*: $R^1 = R$**

***Induction*: $R^{n+1} = R^n \circ R$**

Example



Inverse of R

Let R be a relation on A. Then R^{-1} or the *inverse* of R is the relation

$$R^{-1} = \{ (y,x) \mid (x,y) \in R\}$$

Example:

Let $A = \{1, 2, 3, 4\}$

The Relation $R = \{(1,1), (1, 2), (1,3),(1,4)\}$

$$R^{-1} = \{ (1,1),(2,1),(3,1), (4,1)\}$$

This relation is sometimes denoted as R^T or R^c and called the *converse* of R.

n-ary Relation

Let $A_1, A_2, \ldots, A_n$ be sets. An *n-ary* relation on these sets is a subset of $A_1 \times A_2 \times \ldots \times A_n$. The sets $A_1, A_2, \ldots, A_n$ are called the *domains* of the relation, and *n* is called its degree.

E.g.

Let R be the relation consisting of a 4-tuples (Student, ID, School, Age) representing student particulars in Curtin. If 21-year-old John with ID 1234567 is a student in CS, (John, 1234567, CS, 21) belongs to R.

Equivalence Relations

Definition: A relation *R* on a set *A* is an *equivalence relation* iff *R* is reflexive, symmetric and transitive.

Examples

- Let R be a relation on the strings of English letters such that aRb iff $l(a)=l(b)$ where $l(x)$ is the length of the string x.

  R is an equivalence relation since:
  - aRa Reflexive
  - $aRb \rightarrow bRa$ Symmetric
  - $aRb \wedge bRc \rightarrow aRc$ Transitive

Examples

- Let R be a relation on the set of real numbers such that aRb iff a-b is an integer.

  R is an equivalence relation since:
  - a-a=0 Reflexive
  - if a-b=k, b-a=-k (integer) Symmetric
  - if a-b=i and b-c=j, a-c=(a-b)+(b-c)=i+j Transitive

Examples

- Let m be a positive integer greater than 1. Show that $R = \{(a,b) \mid a \equiv b \pmod{m}\}$ is an equivalence relation on the integer set.

  R is called the congruence modulo m. (This is the most widely used equivalence relation.)

Examples

$a \equiv b \pmod m$ iff m divides $a-b$

Hence,

$a-a=0$ divisible by m. $\therefore a=a \pmod m$. Reflexive.

If $a=b \pmod m$, $a-b=km$ for certain k.

$b-a=(-k)m$. $\therefore b=a \pmod m$. Symmetric.

If $a=b \pmod m$ and $b=c \pmod m$

$a-b=im$ and $b-c=jm$ for certain i, j.

$a-c=(a-b)+(b-c)=im+jm=(i+j)m$

$\therefore a=c \pmod m$. Transitive.

Equivalence class

Definition: Let R be an equivalence relation on A. The set of all elements that are related to an element a in A is called the equivalence class of a.

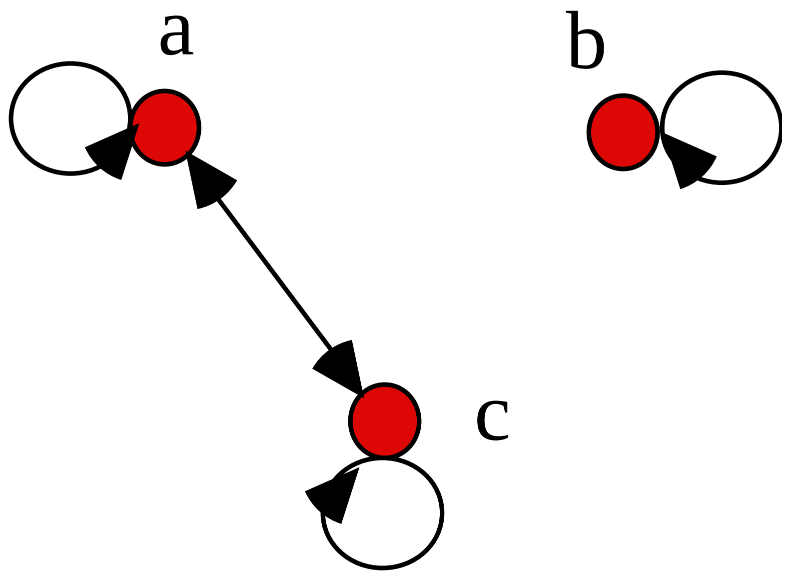
The equivalence class of a with respect to R is denoted as $[a]_R$.

When only one relation is under consideration, the subscript R can be deleted and write it as [a].

$$[a] = \{x \mid (a, x) \in R\}$$

a is called a representative of this class.

Example



[a] = {a, c}, [c] = {a, c}, [b] = {b}.

Partition

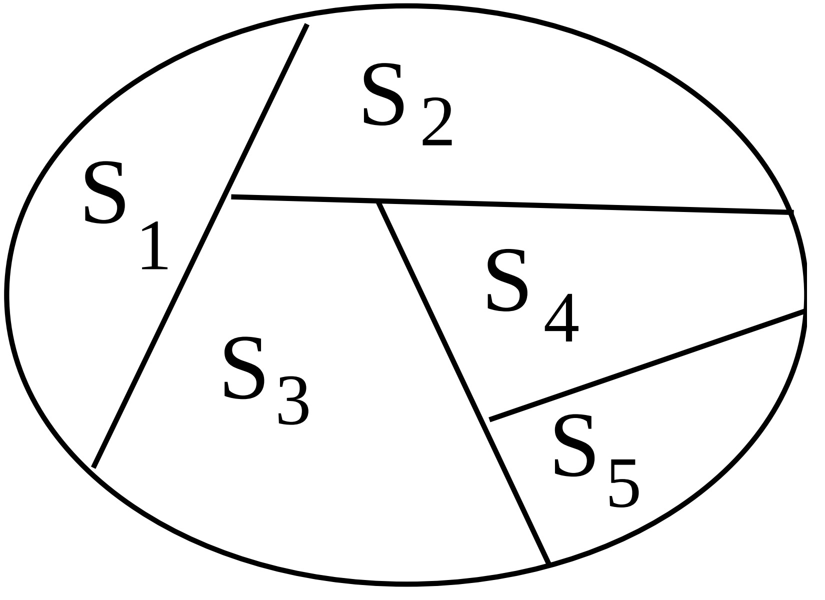
Definition: Let $S_1, S_2, \ldots, S_n$ be a collection of subsets of A. Then the collection forms a *partition* of A if the subsets are nonempty, disjoint and *exhaust* A:

$$S_i \neq \emptyset$$

$$S_i \cap S_j = \emptyset \text{ if } i \neq j$$

$$\cup\, S_i = A$$

Partition: Example

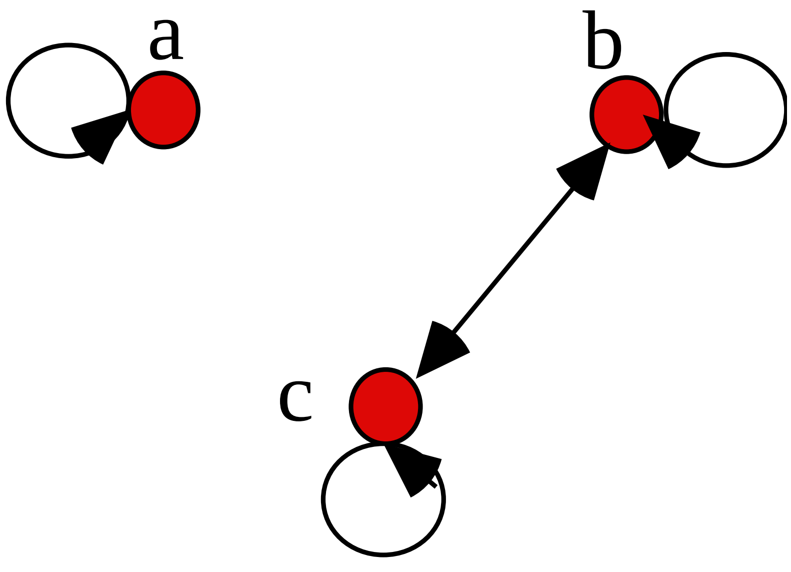


A

Partition

Theorem: Let R be an equivalence relation on a set A. Then the equivalence classes of R form a partition of A. Conversely, given a partition $\{S_i \mid i \in I\}$ of the set A, there is an equivalence relation R that has the sets S_i as its equivalence classes.

Example



$A=[a] \cup [b] = [a] \cup [c] = \{a\} \cup \{b,c\}$

Example

What are the sets in the partition of integers arising from congruence modulo 4?

The four congruence classes, $[0]_4$, $[1]_4$, $[2]_4$, $[3]_4$ form a partition of integers.

They are the sets:

$[0]_4 = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$

$[1]_4 = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$

$[2]_4 = \{\ldots, -6, -2, 2, 6, 10, \ldots\}$

$[3]_4 = \{\ldots, -5, -1, 3, 7, 11, \ldots\}$

Partial Order

Definition: Let *R* be a relation on A. Then R is a *partial order* iff *R* is reflexive, <u>antisymmetric</u> and transitive.
***(A, R)* is called a partially ordered set or a *poset*.**

Note: It is <u>not</u> required that every two elements be related under a partial order. That's the *partial* part of it.

Example

Show that the divisibility relation "|" is a partial ordering on the set of positive integers.

"|" is reflexive since for any $a \in Z^+$, $a|a$;

"|" is antisymmetric since for any $a,b \in Z^+$,

$a|b \wedge b|a \rightarrow a=b$;

"|" is transitive since for any $a, b, c \in Z^+$,

$a|b \wedge b|c \rightarrow a|c$;

Hence $(Z^+, |)$ is a poset.

Example

Show that relation "$\subseteq$" is a partial ordering on the power set of a set S.

"$\subseteq$" is reflexive since for any $A \in P(S)$, $A \subseteq A$;

"$\subseteq$" is antisymmetric since for any $A, B \in P(S)$,

$$A \subseteq B \wedge B \subseteq A \rightarrow A = B;$$

"$\subseteq$" is transitive since for any $A, B, C \in P(S)$,

$$A \subseteq B \wedge B \subseteq C \rightarrow A \subseteq C;$$

Hence $(P(S), \subseteq)$ is a poset.

Notation $\leq$

In a poset (A, R), the notation $a \leq b$ is often used to denote that aRb. Hence poset is often written as $(A, \leq)$.

Note that the symbol "$\leq$" is used to denote the relation in any poset, not just the "less than or equals" relation.

The notation $a < b$ denotes that $a \leq b$ but $a \neq b$. It's also read as "a is less than b" or "b is greater than a".

Notation $\leq$

Definition: The elements *a* and *b* of a poset (A, $\leq$) are called comparable if either $a \leq b$ or $b \leq a$. Otherwise a and b are called incomparable.

E.g.

In the poset ($P(Z)$, $\subseteq$), {1,2} and {1,3} are not comparable.

In the poset (Z^+, |), 3 and 9 are comparable, while 5 and 7 are not comparable.

Total Order

Definition: If every two elements in a poset (A, ≤) are comparable, A is called a *total ordered* or *linearly ordered* or *simply ordered set, and* ≤ is called a total order or linear order or simple order.

A totally ordered set is also called a *chain*.

Example

- $(Z, \leq)$ is a poset. In this case either $a \leq b$ or $b \leq a$ so two elements are always related. Hence, $\leq$ is a total order and $(Z, \leq)$ is a chain.
- If S is a set then $(P(S), \subseteq)$ is a poset. For any two set A, B $\in$ P(S), it may not be the case that $A \subseteq B$ or $B \subseteq A$. Hence, $(P(S), \subseteq)$ is not totally ordered.
- $(Z^+, |)$ is a poset which is not a chain.

Summary

- Binary relation
- Relation on a set
- Combining and composition of relations
- Power and Inverse of relation
- Special properties of relation:
  - reflexive
  - symmetric
  - antisymmetric
  - transitive

Summary

- **Equivalence relations and classes**
- **Partitions**
- **Partial order and Total order**