MATH1019 Mid-Semester Test

Solutions and marking scheme

2018 Semester 1

The marking scheme includes marks of the following different types:

$\boxed{\textbf{1M}}$ — this indicates one *method* mark, where the mark is for applying a correct method. Note this is NOT for the accuracy of the calculation. Sometimes the method will be implicit rather than explicitly shown.

$\boxed{\textbf{1A}}$ — this indicates one *answer / accuracy* mark, for a correct answer or calculation. To get this mark, students must have this exact answer (in any equivalent form). Give these marks consequentially, following on from any earlier accuracy marks.

Half marks should generally only be given in certain circumstances (usually indicated in the marking scheme) where there are multiple elements in the calculation, and there is only a small numerical error in a small part of this.

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1. Consider the following system of equations:

$$\begin{aligned} x - 3y + 2z &= 1 \\ 2x - 5y + 6z &= 5 \\ -x + 5y + 2z &= 5 \end{aligned}$$

   (a) Write the system as an augmented matrix, and use the Gauss-Jordan method to manipulate the augmented matrix into reduced row echelon form.

   (b) State the number of solutions to the system. Find all solutions, or justify why there are none.

[8 marks]

$$\left[\begin{array}{rrr|r} 1 & -3 & 2 & 1 \\ 2 & -5 & 6 & 5 \\ -1 & 5 & 2 & 5 \end{array}\right] \sim \left[\begin{array}{lrr|r} 1 & -3 & 2 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \end{array}\right] \begin{array}{l} \\ (R_2 - 2R_1) \\ (R_3 + R_1) \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 8 & 10 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} (R_1 + 3R_2) \\ \\ (R_3 - 2R_2) \end{array}$$

Since the column corresponding to z is without a leading entry, we let $z = t,\ t \in \mathbb{R}$.
$x + 8z = 10 \ \Rightarrow \ x = 10 - 8t$
$y + 2z = 3 \ \Rightarrow \ y = 3 - 2t$

(A total of 10 marks for Question 1)

2. If k is a real constant, then we can define a system by

$$\begin{array}{rcrcrcr} 2x & - & 4y & + & 4z & = & 12 \\ 3x & + & y & - & 8z & = & 4 \\ -5x & + & 11y & + & kz & = & -32 \end{array}$$

(a) Write the system as an augmented matrix, and use Gaussian elimination to reduce the matrix to row-echelon form.

(b) Briefly justify for which value(s) of k, if any, the system has:

(i) a unique solution

(ii) no solution

(iii) infinitely many solutions

You do **not** have to solve the system in any of these cases.

(a)

$$\left[\begin{array}{ccc|c} 2 & -4 & 4 & 12 \\ 3 & 1 & -8 & 4 \\ -5 & 11 & k & -32 \end{array}\right] \quad R_1 = \tfrac{1}{2}R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 2 & 6 \\ 3 & 1 & -8 & 4 \\ -5 & 11 & k & -32 \end{array}\right] \quad \begin{array}{l} R_2 = R_2 - 3R_1 \\ R_3 = R_3 + 5R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 2 & 6 \\ 0 & 7 & -14 & -14 \\ 0 & 1 & k+10 & -2 \end{array}\right] \quad R_2 = \tfrac{1}{7}R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 2 & 6 \\ 0 & 1 & -2 & -2 \\ 0 & 1 & k+10 & -2 \end{array}\right] \quad R_3 = R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 2 & 6 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & k+12 & 0 \end{array}\right]$$

(b) For $k \neq -12$ the number of non-zero rows equals the number of variables, so there is a unique solution. For $k = -12$ the number of non-zero rows is less than the number of variables, so there are an infinite number of solutions.

[10 marks]

(A total of 10 marks for Question 2)

3. Consider the system of equations

$$\begin{aligned} x - 2y - 4z &= 1 \\ 2x - 3y - 6z &= 2 \\ -3x + 6y + 15z &= 3 \end{aligned}$$

(a) Express the system in the form $A\underset{\sim}{\boldsymbol{x}} = \underset{\sim}{\boldsymbol{b}}$, where A is the matrix of coefficients, $\underset{\sim}{\boldsymbol{x}}$ is the column vector of variables, and $\underset{\sim}{\boldsymbol{b}}$ is the column vector of constants.

(b) Use the inverse of A to solve the system of equations, or justify why there are no solutions.

[10 marks]

(a) $\begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. So $A = \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix}$

(b)

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 2 & -3 & -6 & 0 & 1 & 0 \\ -3 & 6 & 15 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 3 & 3 & 0 & 1 \end{array}\right] \begin{array}{l} \\ (R_2 - 2R_1) \\ (R_3 + 3R_1) \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} \end{array}\right] \begin{array}{l} (R_1 + 2R_2) \\ \\ (\frac{1}{3}R_3) \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & 0 \\ 0 & 1 & 0 & -4 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} \end{array}\right] (R_2 - 2R_3)$$

So $A^{-1} = \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix}$

$$X = A^{-1}B = \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3+4+0 \\ -4+2-2 \\ 1+0+1 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}.$$

So $x = 1,\ y = -4,\ z = 2$

(A total of 10 marks for Question 3)

4. Calculate the rank and determinant of each of the following matrices

(a) $A = \begin{bmatrix} 1 & -3 & 7 \\ -3 & 9 & -21 \\ 2 & -6 & 14 \end{bmatrix}$

$$\det(A) = 1\begin{vmatrix} 9 & -21 \\ -6 & 14 \end{vmatrix} - (-3)\begin{vmatrix} -3 & -21 \\ 2 & 14 \end{vmatrix} + 7\begin{vmatrix} -3 & 9 \\ 2 & -6 \end{vmatrix}$$
$$= 1\Big(9\times 14 - (-21)\times(-6)\Big) + 3\Big(-3\times 14 - (-21)\times 2\Big) + 7\Big((-3)\times(-6) - 9\times 2\Big)$$
$$= (126-126)+3(-42+42)+7(18-18) \quad = 0$$

(b) $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

rank is 3, det is -8

[2 marks]

(A total of 10 marks for Question 4)

5. Use Cramer's rule to solve the following system:

$$\begin{aligned} 3x_1 - 2x_2 &= 6 \\ -5x_1 + 4x_2 &= 8 \end{aligned}$$

[7 marks]

$$A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}, \ \underset{\sim}{\boldsymbol{b}} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}, \ A_1 = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix}, \ A_2 = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix}$$

$$\det(A) = (3)(4) - (-2)(-5) = 12 - 10 = 2$$

$$\det(A_1) = (6)(4) - (-2)(8) = 24 + 16 = 40$$

$$\det(A_2) = (3)(8) - (6)(-5) = 24 + 30 = 54$$

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{20}{2} = 20$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{54}{2} = 27$$

(A total of 7 marks for Question 5)

END OF TEST