

**WORKSHOP 7 SOLUTIONS**

$$1. \quad (i) \quad [A|b] = \left[ \begin{array}{ccc|c} 2 & -1 & 0 & 3 \\ 4 & 3 & -1 & 1 \\ 0 & 3 & 1 & 0 \end{array} \right]$$

$$(ii) \quad [A|b] = \left[ \begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & -2 \\ 0 & 1 & -3 & 0 & 1 & -1 \\ 1 & -2 & 4 & 0 & 3 & 0 \end{array} \right]$$

2. (i) Yes it is in row-echelon form.

(ii) No it is not in row-echelon form, since rows of zeros must be along the bottom row.

$$3. \quad (i) \quad A = \left[ \begin{array}{cc} 2 & 1 \\ -4 & 0 \end{array} \right] \quad R_2 = R_2 + 2R_1 \quad \sim \left[ \begin{array}{cc} 2 & 1 \\ 0 & 2 \end{array} \right]$$

There are two non-zero rows, so rank = 2.

$$(ii) \quad B = \left[ \begin{array}{ccc} 2 & 3 & -1 \\ -8 & -7 & 6 \\ 6 & -1 & -7 \end{array} \right] \quad \begin{array}{l} R_2 = R_2 + 4R_1 \\ R_3 = R_3 - 3R_1 \end{array} \quad \sim \left[ \begin{array}{ccc} 2 & 3 & -1 \\ 0 & 5 & 2 \\ 0 & -10 & -4 \end{array} \right] \quad R_3 = R_3 + 2R_2$$

$$\sim \left[ \begin{array}{ccc} 2 & 3 & -1 \\ 0 & 5 & 2 \\ 0 & 0 & 0 \end{array} \right]$$

There are two non-zero rows, so rank = 2.

$$(iii) \quad C = \left[ \begin{array}{ccccc} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{array} \right] \quad \begin{array}{l} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 3R_1 \end{array} \quad \sim \left[ \begin{array}{ccccc} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & -6 & 4 & 14 & -20 \\ 0 & -9 & 6 & 5 & -6 \end{array} \right] \quad \begin{array}{l} R_3 = R_3 - 2R_2 \\ R_4 = R_4 - 3R_2 \end{array}$$

$$\sim \left[ \begin{array}{ccccc} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & -10 & 15 \end{array} \right] \quad R_4 = R_4 + \frac{5}{2}R_3 \quad \sim \left[ \begin{array}{ccccc} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

There are three non-zero rows, so rank = 3.

$$4. \quad (i) \quad \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array} \right] \quad \begin{array}{l} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 3R_1 \end{array} \quad \sim \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array} \right] \quad R_3 = -\frac{1}{5}R_3$$

$$\sim \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 1 & 2 & 4 \end{array} \right] \quad R_3 = 7R_3 + R_2 \quad \sim \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & 10 & 30 \end{array} \right]$$

$$r(A) = r(A|b) = n = 3 \Rightarrow \text{Unique solution.}$$

$$\text{Row 3: } 10x_3 = 30 \Rightarrow \boxed{x_3 = 3}$$

$$\text{Row 2: } -7x_2 - 4x_3 = 2 \Rightarrow -7x_2 - 12 = 2 \Rightarrow -7x_2 = 14 \Rightarrow \boxed{x_2 = -2}$$

$$\text{Row 1: } x_1 + 2x_2 + 3x_3 = 6 \Rightarrow x_1 - 4 + 9 = 6 \Rightarrow x_1 = 6 - 5 \Rightarrow \boxed{x_1 = 1}$$

$$\begin{aligned} \text{(ii)} \quad \left[ \begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{array} \right] R_3 = 2R_3 - 5R_1 & \sim \left[ \begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -1 & 4 & -3 \end{array} \right] R_3 = R_3 + R_2 \\ & \sim \left[ \begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5 \end{array} \right] \end{aligned}$$

$$r(A) = 2 < r(A|b) = 3 \Rightarrow \text{No solution.}$$

$$\begin{aligned} \text{(iii)} \quad \left[ \begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array} \right] R_2 = R_2 + R_1, R_3 = R_3 - 2R_1 & \sim \left[ \begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & -9 & 0 & -18 \end{array} \right] R_2 = \frac{1}{3}R_2, R_3 = R_3 + 3R_2 \\ & \sim \left[ \begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right] \end{aligned}$$

$$r(A) = 2 = r(A|b) < n = 3 \Rightarrow \text{Infinitely many solutions.}$$

Need  $n - r = 3 - 2 = 1$  parameter.

Let  $\boxed{x_3 = t}$ ,  $t \in \mathbb{R}$

$$\text{Row 2: } \boxed{x_2 = 2}$$

$$\begin{aligned} \text{Row 1: } 3x_1 + 5x_2 - 4x_3 = 7 & \Rightarrow 3x_1 + 10 - 4t = 7 \Rightarrow 3x_1 = 4t - 3 \Rightarrow \\ \boxed{x_1 = \frac{4}{3}t - 1} \end{aligned}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}t - 1 \\ 2 \\ t \end{bmatrix} = t \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}$$

5. Let  $x_1$  be the number of days Factory A is scheduled,  $x_2$  be the number of days Factory B is scheduled, and  $x_3$  be the number of days Factory C is scheduled.

The requirements for refrigerators, dishwashers and stoves can be written as:

$$\begin{aligned} 10x_1 + 20x_2 + 20x_3 &= 100 \\ 50x_1 + 40x_2 + 10x_3 &= 290 \\ 30x_1 + 10x_2 + 40x_3 &= 180 \end{aligned}$$

Now we solve the augmented matrix:

$$\begin{aligned} \left[ \begin{array}{ccc|c} 10 & 20 & 20 & 100 \\ 50 & 40 & 10 & 290 \\ 30 & 10 & 40 & 180 \end{array} \right] \begin{array}{l} R_1 = \frac{1}{10}R_1 \\ R_2 = \frac{1}{10}R_2 \\ R_3 = \frac{1}{10}R_3 \end{array} & \sim \left[ \begin{array}{ccc|c} 1 & 2 & 2 & 10 \\ 5 & 4 & 1 & 29 \\ 3 & 1 & 4 & 18 \end{array} \right] \begin{array}{l} R_2 = R_2 - 5R_1 \\ R_3 = R_3 - 3R_1 \end{array} \\ & \sim \left[ \begin{array}{ccc|c} 1 & 2 & 2 & 10 \\ 0 & -6 & -9 & -21 \\ 0 & -5 & -2 & -12 \end{array} \right] \begin{array}{l} R_2 = -\frac{1}{3}R_2 \\ R_3 = 6R_3 - 5R_2 \end{array} & \sim \left[ \begin{array}{ccc|c} 1 & 2 & 2 & 10 \\ 0 & 2 & 3 & 7 \\ 0 & 0 & 33 & 33 \end{array} \right] \end{aligned}$$

$r(A) = r(A|b) = n = 3 \Rightarrow$  Unique solution.

Row 3:  $33x_3 = 33 \Rightarrow \boxed{x_3 = 1}$

Row 2:  $2x_2 + 3x_3 = 7 \Rightarrow 2x_2 + 3 = 7 \Rightarrow 2x_2 = 4 \Rightarrow \boxed{x_2 = 2}$

Row 1:  $x_1 + 2x_2 + 2x_3 = 10 \Rightarrow x_1 + 4 + 2 = 10 \Rightarrow x_1 = 10 - 6 \Rightarrow \boxed{x_1 = 4}$

So Factory A should be scheduled for 4 days, Factory B for 2 days, and Factory C for 1 day.

6. Need to solve  $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

*i.e.*

$$c_1 + 2c_2 - c_3 = 0$$

$$c_2 + 4c_3 = 0$$

$$5c_1 + 8c_2 = 0$$

Now we solve the augmented matrix:

$$\begin{aligned} \left[ \begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 5 & 8 & 0 & 0 \end{array} \right] \begin{array}{l} R_3 = R_3 - 5R_1 \end{array} & \sim \left[ \begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -2 & 5 & 0 \end{array} \right] \begin{array}{l} R_3 = R_3 + 2R_2 \end{array} \\ & \sim \left[ \begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{array} \right] \end{aligned}$$

$r(A) = r(A|b) = n = 3 \Rightarrow$  Unique solution.

Row 3:  $13c_3 = 0 \Rightarrow \boxed{c_3 = 0}$

Row 2:  $c_2 + 4c_3 = 0 \Rightarrow c_2 + 4(0) = 0 \Rightarrow c_2 + 0 = 0 \Rightarrow \boxed{c_2 = 0}$

Row 1:  $c_1 + 2c_2 - c_3 = 0 \Rightarrow c_1 + 2(0) - 0 = 0 \Rightarrow c_1 + 0 = 0 \Rightarrow \boxed{c_1 = 0}$

$\therefore$  Since  $c_1 = c_2 = c_3 = 0$  they are linearly independent.