Lecture 5

Vectors

Vectors

We generally distinguish between scalars and vectors.

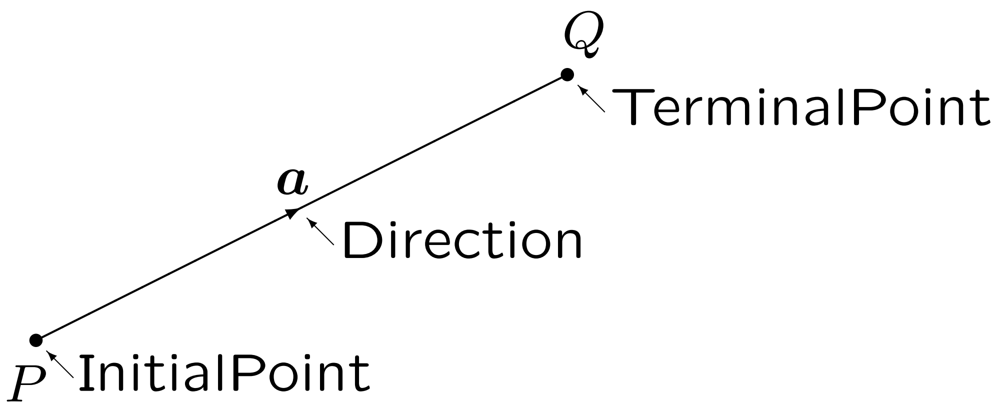
A scalar quantity is characterized only by magnitude. *e.g.* mass, temperature, time, speed etc

A vector quantity is characterized by both magnitude and direction. *e.g.* force, velocity etc

For scalars, we use a plain (usually lower case) letter such as 'a'.

For vectors, in typewritten text we use a boldface (and again usually lower case) letter such as '$\boldsymbol{a}$'. In handwritten text, there we usually underline the letter with a tilde, *i.e.* $\underset{\sim}{a}$.

Geometric representation:



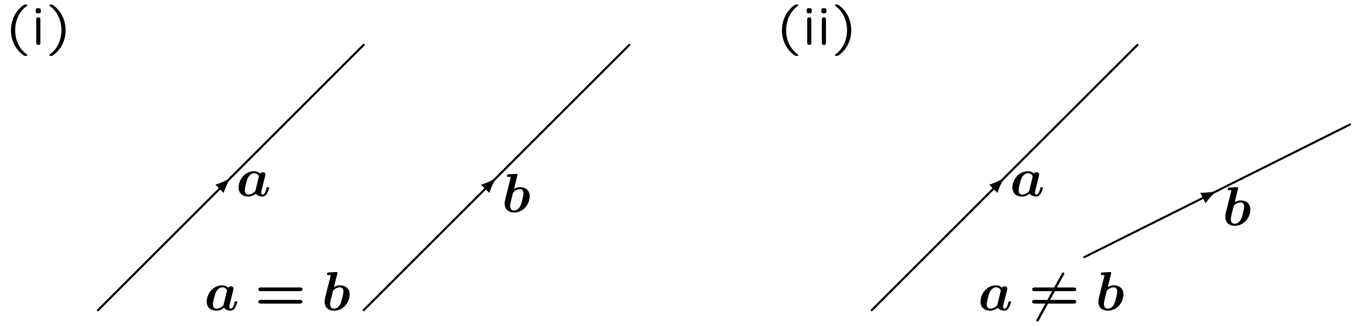
$\boldsymbol{a}$ is also written as $\overrightarrow{PQ}$.

The magnitude or length of a vector $\boldsymbol{a}$ is denoted by $||\boldsymbol{a}||$ or $|\boldsymbol{a}|$.

If $||\boldsymbol{u}|| = 1$ for a given vector $\boldsymbol{u}$, then we say that $\boldsymbol{u}$ is a unit vector.

The zero vector, denoted by $\mathbf{0}$, is defined to be a vector which has no particular direction and zero length.

We say $\boldsymbol{a} = \boldsymbol{b}$ if they have the same length and direction.

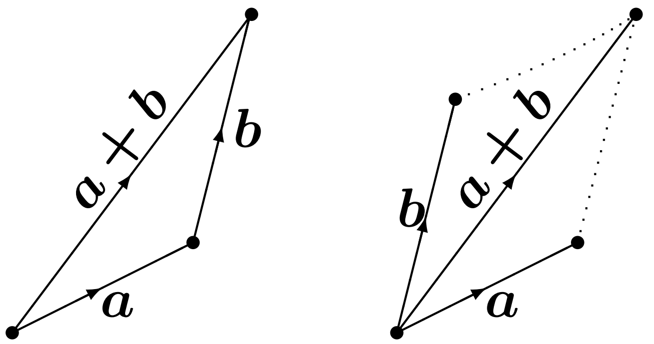


Vector Operations

The sum of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is written as

$$\boldsymbol{a} + \boldsymbol{b}$$

and we can use either the triangle law or parallelogram law:



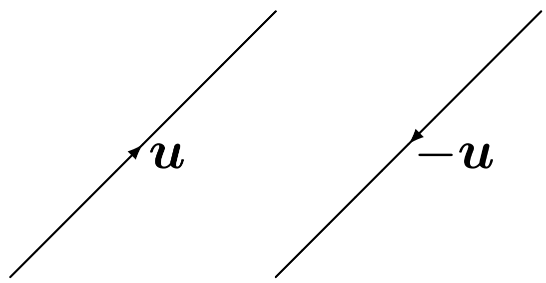
Special rule for addition with the zero vector: for any $\boldsymbol{u}$,

$$\boldsymbol{u} + \boldsymbol{0} = \boldsymbol{0} + \boldsymbol{u} = \boldsymbol{u}.$$

Also, it is clear from the workings of the parallelogram law that

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$$

The negative of a vector $\boldsymbol{u}$ is denoted by $-\boldsymbol{u}$ and it is simply the vector which has the same length as $\boldsymbol{u}$ but is opposite in direction, *i.e.*



Scalar Multiplication. Given a scalar c and a vector $\boldsymbol{a}$, the scalar multiple $c\boldsymbol{a}$ is a vector such that

(i) it has the same direction as $\boldsymbol{a}$ if $c > 0$;

(ii) its direction is opposite to that of $\boldsymbol{a}$ if $c < 0$.

In either case, the length of $c\boldsymbol{a}$ is $|c|$ times the length of $\boldsymbol{a}$, *i.e.* $||c\boldsymbol{a}|| = |c|\,||\boldsymbol{a}||$.

Note that $(0)\boldsymbol{u} = \boldsymbol{0}$ for any vector $\boldsymbol{u}$.

For any vector $\boldsymbol{a} \neq \boldsymbol{0}$, the unit vector in the direction of $\boldsymbol{a}$ is denoted by $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{a}} = \dfrac{\boldsymbol{a}}{||\boldsymbol{a}||}$.

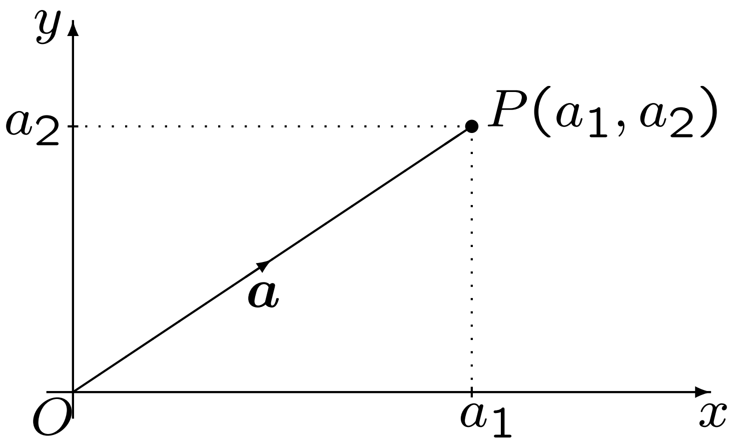
Finally, notice that $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel if and only if $\boldsymbol{a} = c\boldsymbol{b}$ for some scalar c.

Analytic Representation

Consider the Cartesian plane with origin O. For any $P(a_1, a_2)$, we can write down its position vector as

$$\overrightarrow{OP} = [a_1, a_2] = \boldsymbol{a}.$$

Note how we use square brackets when writing out the vector.



Next we look at the addition of two vectors $\boldsymbol{a} = [a_1, a_2]$ and $\boldsymbol{b} = [b_1, b_2]$.

We have,

$$\boldsymbol{a} + \boldsymbol{b} = [a_1 + b_1, a_2 + b_2].$$

Similarly, for two vectors $\boldsymbol{a} = [a_1, a_2, a_3]$ and $\boldsymbol{b} = [b_1, b_2, b_3]$ in 3 space,

$$\boldsymbol{a} + \boldsymbol{b} = [a_1 + b_1, a_2 + b_2, a_3 + b_3].$$

Suppose we have $\boldsymbol{a} = [a_1, a_2]$. We find that

$$c\boldsymbol{a} = [ca_1, ca_2].$$

Similarly, for $\boldsymbol{a} = [a_1, a_2, a_3]$ in 3 space,

$$c\boldsymbol{a} = [ca_1, ca_2, ca_3].$$

Given $\boldsymbol{a} = [a_1, a_2]$, we have

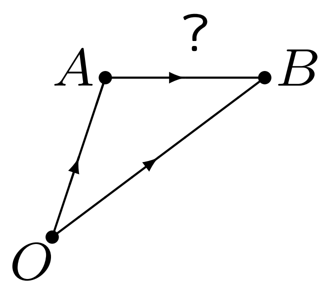
$$||\boldsymbol{a}|| = \sqrt{a_1^2 + a_2^2}.$$

Similarly, for $\boldsymbol{a} = [a_1, a_2, a_3]$ in 3 space,

$$||\boldsymbol{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Ex: Find $\boldsymbol{a}+2\boldsymbol{b}$, $||\boldsymbol{a}||$ and $\hat{\boldsymbol{a}}$ if $\boldsymbol{a}=[2,-1,-2]$ and $\boldsymbol{b}=[1,4,-2]$.

What are the components of a vector joining two points $A(a_1,a_2)$ and $B(b_1,b_2)$ in the plane?



$$\overrightarrow{AB}=[b_1,b_2]-[a_1,a_2]=[b_1-a_1,b_2-a_2],$$

In 3 space, the same applies: for points $A(a_1,a_2,a_3)$ and $B(b_1,b_2,b_3)$,

$$\overrightarrow{AB}=[b_1-a_1,b_2-a_2,b_3-a_3].$$

Ex: Given $A(4,-1,0)$ and $B(1,3,-2)$, find $\overrightarrow{BA}$.

Standard Unit Basis Vectors

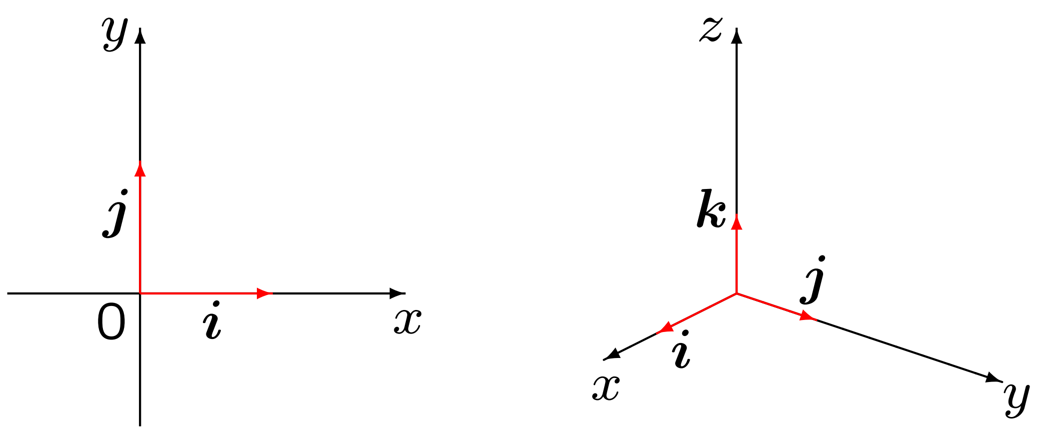
The vectors

$$\boldsymbol{i} = [1, 0] \qquad \text{and} \qquad \boldsymbol{j} = [0, 1]$$

in 2 space and

$$\boldsymbol{i} = [1, 0, 0], \quad \boldsymbol{j} = [0, 1, 0] \quad \text{and} \quad \boldsymbol{k} = [0, 0, 1]$$

in 3 space are known as the standard unit basis vectors.



Clearly, for any $\boldsymbol{a} = [a_1, a_2, a_3]$ in 3 space,

$$\begin{aligned}
\boldsymbol{a} &= [a_1, a_2, a_3] \\
&= [a_1, 0, 0] + [0, a_2, 0] + [0, 0, a_3] \\
&= a_1[1, 0, 0] + a_2[0, 1, 0] + a_3[0, 0, 1] \\
&= a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k},
\end{aligned}$$

e.g. $[3, 2, -4] = 3\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k}$.

Similarly, for any $\boldsymbol{a} = [a_1, a_2]$ in 2 space,

$$\boldsymbol{a} = [a_1, a_2] = a_1\boldsymbol{i} + a_2\boldsymbol{j}.$$

Note how the component notation and the standard unit basis vector notation is interchangeable.

Ex: Given $\boldsymbol{a} = -2\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i}+\boldsymbol{j}-4\boldsymbol{k}$, find $\boldsymbol{a} - 2\boldsymbol{b}$ and $||\boldsymbol{b}||$.

The Dot Product

The dot product (or scalar product) of $\boldsymbol{a} = [a_1, a_2, a_3]$ and $\boldsymbol{b} = [b_1, b_2, b_3]$ is

$$\boxed{\boldsymbol{a}.\boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3}$$

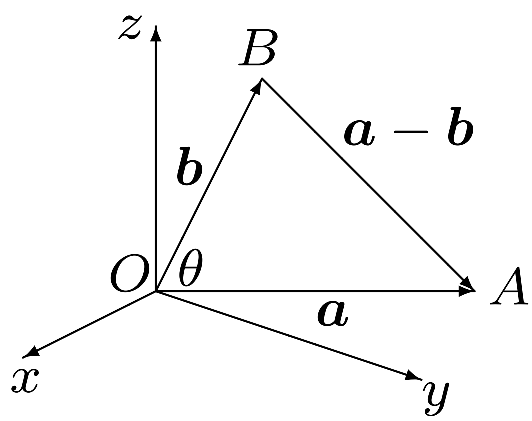
e.g. $[1, 2, 3].[-2, 0, 1] = (1)(-2) + (2)(0) + (3)(1) = 1$.

Note the following:

(i) $\boldsymbol{a}.\boldsymbol{b}$ is clearly a scalar quantity and $\boldsymbol{a}.\boldsymbol{b} = \boldsymbol{b}.\boldsymbol{a}$.

(ii) For vectors in 2 space, if $\boldsymbol{a} = [a_1, a_2]$, $\boldsymbol{b} = [b_1, b_2]$, then $\boldsymbol{a}.\boldsymbol{b} = a_1b_1 + a_2b_2$.

Consider $\boldsymbol{a}$ and $\boldsymbol{b}$ with an angle θ between them, $0^\circ \le \theta \le 180^\circ$. Suppose we locate both vectors with their initial points at the origin.



$$\boxed{\boldsymbol{a}.\boldsymbol{b} = ||\boldsymbol{a}||\,||\boldsymbol{b}||\cos\theta}$$

i.e. the dot product between two vectors (be it in 2 space or in 3 space) is equal to the product of the lengths times the cosine of the angle between them.

It follows that if $\boldsymbol{a} \neq \mathbf{0}$ and $\boldsymbol{b} \neq \mathbf{0}$, then

$$\boxed{\cos\theta = \frac{\boldsymbol{a}.\boldsymbol{b}}{||\boldsymbol{a}||\,||\boldsymbol{b}||}}$$

<u>**Ex:**</u> Find the angle between the vectors $\boldsymbol{a} = [3, 2, -1]$ and $\boldsymbol{b} = [-2, 2, 4]$.

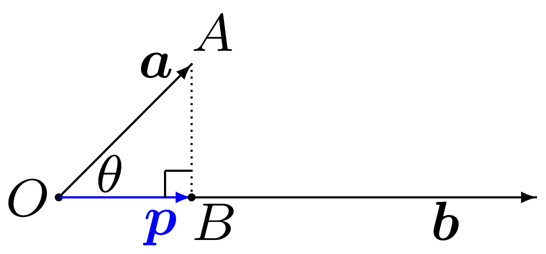
We say that $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal (or perpendicular) if the angle between them is 90°. In this case,

$$\frac{\boldsymbol{a}.\boldsymbol{b}}{||\boldsymbol{a}||\,||\boldsymbol{b}||} = \cos(90^\circ) = 0 \;\longrightarrow\; \boldsymbol{a}.\boldsymbol{b} = 0.$$

In fact, the only way in which to vectors can be orthogonal is if $\boldsymbol{a}.\boldsymbol{b} = 0$.

e.g. [1,-2,3].[2,1,0]=0, so vectors are orthogonal.

Projection and Component of a Vector



The scalar projection of $\boldsymbol{a}$ on $\boldsymbol{b}$ is $p = ||\overrightarrow{OB}||$,

$$\boxed{\text{scalar projection } = p = \boldsymbol{a}.\widehat{\boldsymbol{b}}}$$

where $\widehat{\boldsymbol{b}} = \dfrac{\boldsymbol{b}}{||\boldsymbol{b}||}$ is the unit vector in the direction of $\boldsymbol{b}$.

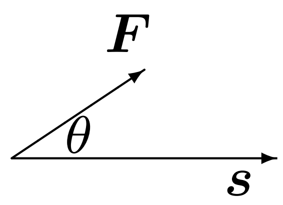
The vector projection of $\boldsymbol{a}$ on $\boldsymbol{b}$ is then just $\boldsymbol{p}$, *i.e.*

$$\boxed{\text{vector projection } = \boldsymbol{p} = p\widehat{\boldsymbol{b}},}$$

where p is the scalar projection defined above and $\widehat{\boldsymbol{b}}$ is the unit vector in the direction of $\boldsymbol{b}$.

Ex: Find the vector projection of $\boldsymbol{b} = [3, 1, 1]$ on $\boldsymbol{a} = [4, -2, -1]$.

Work Done by a Force



$$\begin{aligned} \text{Work} &= \text{force} \times \text{displacement} \\ &= \text{magnitude of } \boldsymbol{F} \text{ in direction of } \boldsymbol{s} \times ||\boldsymbol{s}|| \\ &= \text{scalar projection of } \boldsymbol{F} \text{ on } \boldsymbol{s} \times ||\boldsymbol{s}|| \\ &= ||\boldsymbol{F}|| \cos\theta ||\boldsymbol{s}|| \\ &= \boldsymbol{F}.\boldsymbol{s} \end{aligned}$$

$$\boxed{\mathit{Work} = \boldsymbol{F}.\boldsymbol{s}}$$

Ex: A force $\boldsymbol{F} = 5\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$ is applied to an object which moves it from the point $A(1,1,1)$ to the point $B(5,-1,2)$. Determine the work done by the force $\boldsymbol{F}$.