Lecture 4. Mathematical Induction

Ref.: K H Rosen Section 3.3

What is Mathematical Induction?

"Mathematical Induction is an extremely important proof technique that we use to prove results about a large variety of discrete objects."

What is Mathematical Induction?

E.g. What is the formula for the sum of the first *n* positive odd integers?

The sums of the first n positive odd integers for *n*=1,2,3,4,5…

n=1   1=1,

n=2   1+3=4,

n=3   1+3+5=9,

n=4   1+3+5+7=16,

n=5   1+3+5+7+9=25

What is Mathematical Induction?

Sum of the first n odd positive integers = n^2

Needs to be proven!

The sums of the first *n* odd positive integers for *n*=1,3,5,7,9…

n=1 1=1,

n=2 1+3=4,

n=3 1+3+5=9,

n=4 1+3+5+7=16,

n=5 1+3+5+7+9=25

What is Mathematical Induction?

"Mathematical Induction is NOT a tool for discovering formulae or theorems."

Inductive Proof

Prove propositions of the form "$\forall n$ P(n)"

The inductive proof consists of two steps:

- ***Basis Step***

  The proposition P(... wn to be true

- ***Inductive Step***

  The implication P(k) → P(k+1) is shown to be true for every positive integer k

Inductive Hypothesis

When both steps are complete, we have proved that "$\forall n$ P(n)" is true.

As a rule of inference...

Mathematical Induction can be stated as a rule of inference...

$$[P(1) \wedge \forall k(P(k) \rightarrow P(k+1))] \rightarrow \forall n P(n)$$

Basis

Hypothesis

Inductive step

***We do not use (assume) that P(k)* is true for all positive integers.**

Instead,

If $P(k)$ is true then $P(k+1)$ is also true.

Domino Example:

Infinite row of Dominoes labelled *1,2,3…n*

Let $P(n)$ denote "domino n is knocked over".

The first domino is knocked over, thus P(1) = true

When the k^{th} domino falls, it knocks down the $(k+1)^{th}$ domino ($P(k) \rightarrow P(k+1)$)

Hence all dominoes are knocked down

$\forall n P(n)$ is true

Why is Mathematical Induction Valid?

The Well Ordering Property

"Every non-empty set of non-negative integers has a least element."

Why is Mathematical Induction Valid?

Show P(n) must be true for all positive integers. (Proof by contradiction)

We know that $P(1)$ is true and $\forall k\, P(k) \rightarrow P(k+1)$

Assume that...

- ***There is at least one positive integer for which P(n) is false.***
- ***The set S for which P(n) is false is non empty.***

According to the well ordering property, *S* has a least element k_0: P(k_0) is false and P(k_0-1) is true

P(k_0) is true!

Contradiction!

Note...

- **Identifying P(k) is often the hardest part!**
- **Write down the assertion *P(k+1)*!**
- **Manipulate the assertion *P(k+1)* so that you can apply the induction hypothesis *P(k)*. If you do not apply the induction hypothesis somewhere, it is <u>not</u> a valid induction proof.**

Example (classical)

Prove:

Basis step: $P(0)$ is true since $0 = \frac{0(0+1)}{2}$

Inductive step: $\forall k\, P(k) \rightarrow P(k+1)$

$$P(k) \equiv \sum_{i=0}^{k} i = \frac{k(k+1)}{2}$$

$$\sum_{i=0}^{k+1} i = 0 + 1 + 2 + \ldots + k + (k+1) = \left(\sum_{i=0}^{k} i \right) + (k+1)$$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)(k+2)}{2} \equiv P(k+1)$$

Example

Use Mathematical Induction to prove that the sum of the first n odd positive integers is n^2

$P(n) \equiv$ sum of first n odd positive integers is n^2

The sums of the first n positive odd integers for n=1,2,3,4,5…

1=1,

1+3=4,

1+3+5=9,

1+3+5+7=16,

P(k): $1+3+5+ \ldots +(2k-1)=k^2$

$1+3+5+ \ldots +(2k-1)+(2k+1)= k^2+2k+1=(k+1)^2 \equiv P(k+1)$

Example

Use Mathematical Induction to prove that n^3-n is divisible by 3 whenever n is a positive integer.

$P(n) \equiv n^3-n$ is divisible by 3

Basis step: **$P(1)$ is true since $1^3-1 = 0$**

***Inductive step:* $\forall k\ P(k) \rightarrow P(k+1)$**

$(k+1)^3-(k+1)=k^3+3k^2+3k+1-k-1= k^3-k+3k^2+3k$

Thus P(k+1) is true.

Thus $\forall$n P(n).

Example

Use Mathematical Induction to prove that
$4+10+16+\ldots+(6n-2) = n(3n+1)$ for all integer $n \geq 1$.

$P(n) \equiv 4+10+16+\ldots+(6n-2) = n(3n+1)$

Basis step: $P(1)$ is true since $4 = 1*(3*1+1)$

Inductive step: $\forall k\ P(k) \rightarrow P(k+1)$

Assume $4+10+16+\ldots+(6k-2) = k(3k+1)$

$4+10+16+\ldots+(6k-2) + (6*(k+1)-2)$

$= k(3k+1) + (6*(k+1)-2) = k(3k+1)+6k+4=3k^2+7k+4$

$(k+1)*(3(k+1)+1) = (k+1)*(3k+4)=3k^2+7k+4$

Thus P(k+1) is true.

Thus $\forall n\ P(n)$.

Example

Use Mathematical Induction to prove that if $S_0=a$ and $S_n=2S_{n-1}+b$, then $S_n=2^n a+(2^n-1)b$ for every nonnegative integer n. .

$P(n) \equiv$ if $S_0=a$ and $S_n=2S_{n-1}+b$, then $S_n=2^n a+(2^n-1)b$

Basis step: $P(0)$ is true since

$$S_0=2^{0*}a+(2^0-1)*b = a$$

Example

Use Mathematical Induction to prove that if $S_0=a$ and $S_n=2S_{n-1}+b$, then $S_n=2^n a+(2^n-1)b$ for every nonnegative integer n. .

Inductive step: $\forall k$ $P(k) \rightarrow P(k+1)$

Assume that $S_k=2^k a+(2^k-1)b$

$$S_{k+1} = 2*S_k+b = 2(2^k a+(2^k-1)b) + b$$

$$= 2^{k+1}a+2^{k+1}b-2b+b = 2^{k+1}a+(2^{k+1}-1)b$$

Thus P(k+1) is true.

Thus $\forall$n P(n).

Strong mathematical induction

- ***Basis Step*** **. Prove $P(1)$ is true**
- ***Inductive Step.***

Prove $\forall k\ ([P(1) \wedge P(2) \wedge \ldots \wedge P(k)] \rightarrow P(k+1))$

When both of these steps are proven we have proved $\forall n\ P(n)$

Example

Every integer greater than 1 can be written as a product of primes ($\forall n\, P(n)$ with $P(n)$: If $n>1$ then n can be written as a product of primes).

P(2) is true.

Hypothesis : $P(1) \land P(2) \land \ldots \land P(k)$

Inductive step: $P(k+1)$?

1. **Case1: $k+1$ is a prime – done.**
2. **Case2: $k+1$ is a composite. Then $k+1=a*b$ with $a<k+1$ and $b<k+1$. Then a and b can be written as product of primes. Thus $k+1=a*b$ is a product of primes.** **Proven.**

Example: All horses are the same colour!

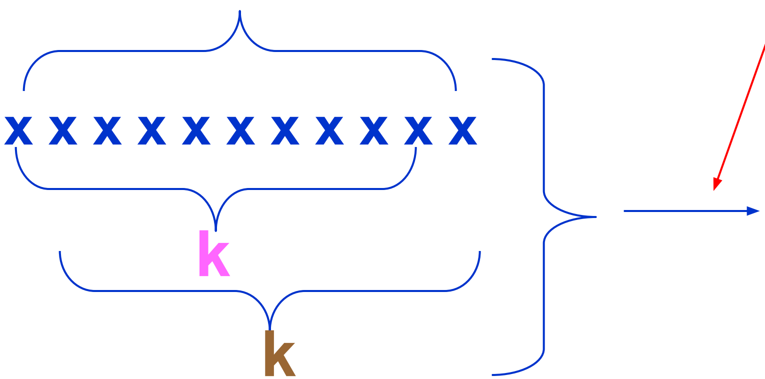
Proof (induction on the size of sets of horses of the same colour):

Basis: P(1) is true.

Inductive step: If in all sets of horses of size k all horses have the same colour then it is also true in all sets of horses of size n+1.

Proof of inductive step:

k+1 horses



All k+1 horses have the same colour. Thus $\forall n\, P(n)$.

What's wrong here?

Summary

- Mathematical Induction is used to prove propositions of the form $\forall n\, P(n)$
  - *Basis Step*: Prove $P(1)$ is true
  - *Inductive Step:*

    Prove $\forall k(P(k) \rightarrow P(k+1))$
- When both of these steps are proven we have proved $\forall n\, P(n)$

Summary

- **Strong mathematical Induction**
  - ***Basis Step*: Prove *P*(1) is true**
  - ***Inductive Step:***

Prove $\forall k\ ([P(1) \wedge P(2) \wedge \ldots \wedge P(k)] \rightarrow P(k+1))$