WORKSHOP 9 SOLUTIONS

1. $A = \begin{bmatrix} 2 & -3 \\ 6 & -9 \end{bmatrix}$

   (i) $\det(A) = (2)(-9) - (-3)(6) = -18 + 18 = 0$

   (ii) Singular as $\det(A) = 0$, no inverse exists.

   $B = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$

   (i) $\det(B) = (2)(-7) - (5)(-3) = -14 + 15 = 1$

   (ii) Non-singular as $\det(B) \neq 0$, inverse exists.

   $C = \begin{bmatrix} 5 & 0 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{bmatrix}$

   (i) $\det(C) = \begin{vmatrix} 5^{+} & 0^{-} & -1^{+} \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{vmatrix}$ Cofactor expansion along 1st row:

   $\det(C) = (5)\begin{vmatrix} -3 & -2 \\ 5 & 3 \end{vmatrix} - (0) + (1)\begin{vmatrix} 1 & -3 \\ 0 & 5 \end{vmatrix}$

   $= 5(-9 - (-10)) - 1(5 - 0) = 5(1) - 1(5) = 5 - 5 = 0.$

   (ii) Singular as $\det(C) = 0$, no inverse exists.

   $D = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$

   (i) $\det(D) = \begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0^{+} & -2^{-} & 0^{+} \end{vmatrix}$ Cofactor expansion along 3rd row:

   $\det(D) = (0) - (-2)\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} + (0)$

   $= 2(-1 - 0) = 2(-1) = -2.$

   (ii) Non-singular as $\det(D) \neq 0$, inverse exists.

   $E = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 4 \\ 0 & 0 & -2 \end{bmatrix}$

   (i) $\det(E) = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 4 & 4 \\ 0 & 0 & -2 \end{vmatrix}$ Since it's an upper triangular matrix:

   $\det(E) = (1)(4)(-2) = -8$

(ii) Non-singular as $\det(E) \neq 0$, inverse exists.

2. $$\begin{vmatrix} 2 & 1 & 3 & 1 \\ -2 & 3 & -1 & 2 \\ 2 & 1 & 2 & 3 \\ -4 & -2 & -6 & -1 \end{vmatrix} \begin{matrix} \\ R_2 \to R_2 + R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 + 2R_1 \end{matrix} = \begin{vmatrix} 2 & 1 & 3 & 1 \\ 0 & 4 & 2 & 3 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Since it's an upper triangular matrix:
$= (2)(4)(-1)(1) = -8$

3. (i) $$\begin{bmatrix} 5 & 3 \\ 7 & 4 \end{bmatrix}^{-1} = \frac{1}{(5)(4)-(3)(7)} \begin{bmatrix} 4 & -3 \\ -7 & 5 \end{bmatrix} = \frac{1}{-1} \begin{bmatrix} 4 & -3 \\ -7 & 5 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 7 & -5 \end{bmatrix}$$

(ii) $$\begin{bmatrix} -2 & 4 \\ -3 & 6 \end{bmatrix}^{-1} = \frac{1}{(-2)(6)-(4)(-3)} \begin{bmatrix} 6 & -4 \\ 3 & -2 \end{bmatrix} = \frac{1}{0} \begin{bmatrix} 6 & -4 \\ 3 & -2 \end{bmatrix}$$

Singular matrix, no inverse exists.

(iii) $$\begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}^{-1} = \frac{1}{(3)(4)-(5)(2)} \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{5}{2} \\ -1 & \frac{3}{2} \end{bmatrix}$$

4. (i) $A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}$, $\det(A) = (3)(4) - (-2)(-5) = 12 - 10 = 2.$

$A_1 = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix}$, $\det(A_1) = (6)(4) - (-2)(8) = 24 + 16 = 40.$

$A_2 = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix}$, $\det(A_2) = (3)(8) - (6)(-5) = 24 + 30 = 54.$

Hence, $x_1 = \dfrac{\det(A_1)}{\det(A)} = \dfrac{40}{2} = 20$ and $x_2 = \dfrac{\det(A_2)}{\det(A)} = \dfrac{54}{2} = 27$

(ii) $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$, $\det(A) = (1)(1) - (2)(3) = 1 - 6 = -5.$

$A_1 = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$, $\det(A_1) = (3)(1) - (2)(-1) = 3 + 2 = 5.$

$A_2 = \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$, $\det(A_2) = (1)(-1) - (3)(3) = -1 - 9 = -10.$

Hence, $x_1 = \dfrac{\det(A_1)}{\det(A)} = \dfrac{5}{-5} = -1$ and $x_2 = \dfrac{\det(A_2)}{\det(A)} = \dfrac{-10}{-5} = 2$

5. $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -5 & -3 \\ 4 & 8 & 2 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 0 \\ 10 \\ 4 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -5 & 10 \\ 4 & 8 & 4 \end{bmatrix}$

$\det(A) = \begin{vmatrix} 1^{+} & 1^{-} & 1^{+} \\ 2 & -5 & -3 \\ 4 & 8 & 2 \end{vmatrix}$ Cofactor expansion along 1st row:

$\det(A) = (1)\begin{vmatrix} -5 & -3 \\ 8 & 2 \end{vmatrix} - (1)\begin{vmatrix} 2 & -3 \\ 4 & 2 \end{vmatrix} + (1)\begin{vmatrix} 2 & -5 \\ 4 & 8 \end{vmatrix}$
$= 1(-10-(-24)) - 1(4-(-12)) + 1(1-(-20)) = 1(14) - 1(16) + 1(36)$
$= 14 - 16 + 36 = 34$

$\det(A_3) = \begin{vmatrix} 1^{+} & 1^{-} & 0^{+} \\ 2 & -5 & 10 \\ 4 & 8 & 4 \end{vmatrix}$ Cofactor expansion along 1st row:

$\det(A_3) = (1)\begin{vmatrix} -5 & 10 \\ 8 & 4 \end{vmatrix} - (1)\begin{vmatrix} 2 & 10 \\ 4 & 4 \end{vmatrix} + (0)$

$= 1(-20-80) - 1(8-40) + 0 = 1(-100) - 1(-32) = -100 + 32 = -68$

Hence, $x_3 = \dfrac{\det(A_3)}{\det(A)} = \dfrac{-68}{34} = -2$

6. (i) $\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 2 & 1 \\ -1 & 1 & 4 \end{vmatrix}$, doing a cofactor expansion along the first row we get:

$$\begin{aligned} \boldsymbol{a} \times \boldsymbol{b} &= \boldsymbol{i}\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix} \\ &= (8-1)\boldsymbol{i} - (12-(-1))\boldsymbol{j} + (3-(-2))\boldsymbol{k} \\ &= (7)\boldsymbol{i} - (13)\boldsymbol{j} + (5)\boldsymbol{k} \\ &= [7, -13, 5] \end{aligned}$$

(ii) $\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix}$, doing a cofactor expansion along the first row we get:

$$\begin{aligned} \boldsymbol{a} \times \boldsymbol{b} &= \boldsymbol{i}\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} \\ &= (0-1)\boldsymbol{i} - (-2-1)\boldsymbol{j} + (2-0)\boldsymbol{k} \\ &= (-1)\boldsymbol{i} - (-3)\boldsymbol{j} + (2)\boldsymbol{k} \\ &= [-1, 3, 2] \end{aligned}$$

7. $\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} 2 & 6 & -2 \\ -3 & 2 & 0 \\ 0 & 1 & 5 \end{vmatrix}$, doing a cofactor expansion along the first row we get,

$$\begin{aligned} \boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c}) &= 2\begin{vmatrix} 2 & 0 \\ 1 & 5 \end{vmatrix} - 6\begin{vmatrix} -3 & 0 \\ 0 & 5 \end{vmatrix} + (-2)\begin{vmatrix} -3 & 2 \\ 0 & 1 \end{vmatrix} \\ &= 2(10-0) - 6(-15-0) - 2(-3-0) \\ &= 20 + 90 + 6 \\ &= 116 \end{aligned}$$

Since the volume of the parallelepiped is given by the absolute value of the scalar triple product, we have $|\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c})| = |116| = 116$.