MATH1019 Linear Algebra and Statistics for Engineers

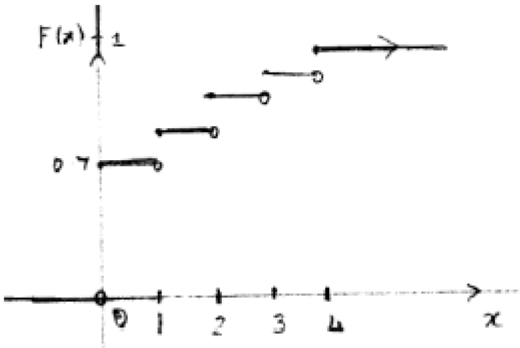
Workshop 2 solutions

Question 1

(a) P $(X \geq 2)$ = = 0.1 + 0.05 + 0.05 = 0.2

(b) F(x)

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 0.7 & \text{if } 0 \leq x < 1 \\ 0.8 & \text{if } 1 \leq x < 2 \\ 0.9 & \text{if } 2 \leq x < 3 \\ 0.95 & \text{if } 3 \leq x < 4 \\ 1 & \text{if } x \geq 4 \end{cases}$$



Question 2

(a)

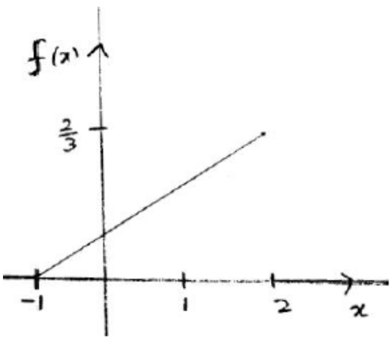
$$\sum_{k=0}^{\infty} p_k = 1 \Leftrightarrow \sum_{k=0}^{\infty} p_0 p^k = 1 \Leftrightarrow p_0 + p_0 p + p_0 p^2 + .. = \frac{p_0}{1-p} = 1 \Leftrightarrow p_0 = 1 - p$$

(b) x = number of customers in the post office.

$P(x \geq 1) = 1 - P(x = 0) = 1 - p_0 = p$

Question 3

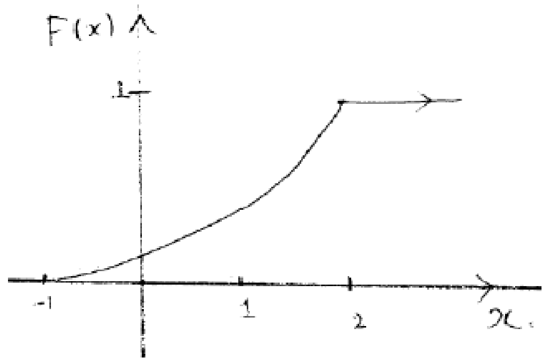
(a)



(b) $F(x) = 0$ if $x < -1$

$$= \frac{2}{9}\left(\frac{x^2}{2} + x\right) + \frac{1}{9} \text{ if } -1 \le x < 2$$

$= 1$ if $x \ge 2$

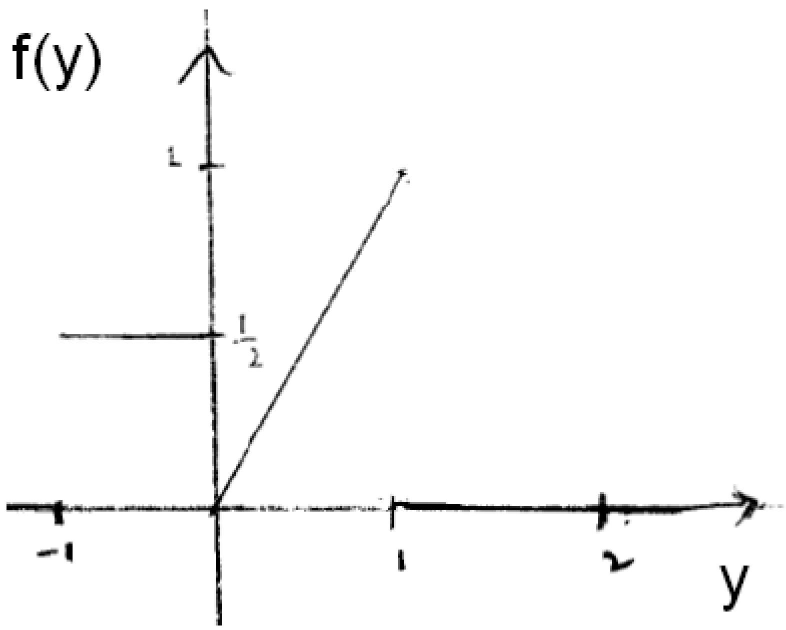


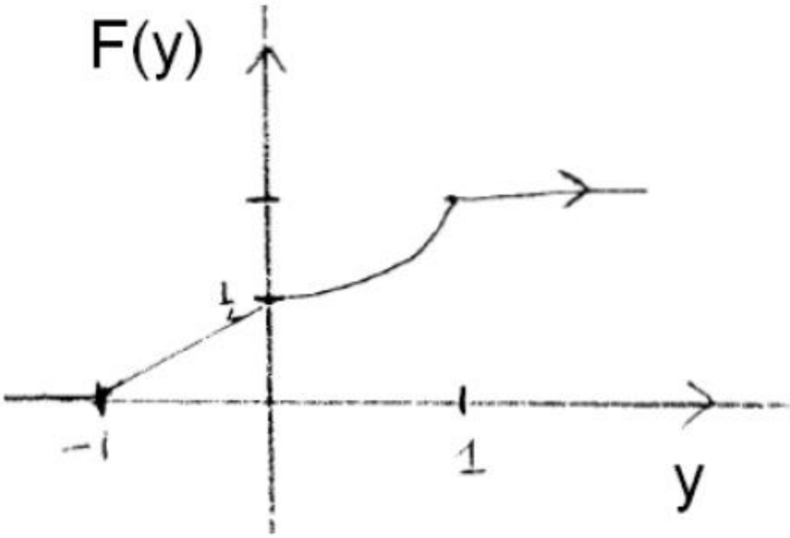
(c) $P(x < 0.25) = F\left(\frac{1}{4}\right) = \frac{2}{9}\left(\frac{1}{32} + \frac{1}{4}\right) + \frac{1}{9} = 0.1736$

Question 4

(a) $f(y) = ½$ if $–1 < y < 0$
$= y$ if $0 < y < 1$
$= 0$, otherwise

(b)





(c) $P(Y \leq 0.8) = \frac{1}{2}(1 + (0.8)^2) = 0.82$

Question 5

$E(X) = (0)(0.7) + (1)(0.1) + (2)(0.1) + 3(0.05) + 4(0.05) = 0.65$

$E(X^2) = (0^2)(0.7) + (1^2)(0.1) + (2^2)(0.1) + (3^2)(0.05) + (4^2)(0.05) = 1.75$

$\therefore \text{Var}(X) = E(X^2) - [E(X)]^2 = 1.75 - 0.65^2 = 1.33$

Question 6

$$E(X) = \int_{-1}^{2} x\left(\tfrac{2}{9}\right)(x+1)dx = \frac{2}{9}\int_{-1}^{2}(x^2+x)dx = \frac{2}{9}\left(\frac{x^3}{3} + \tfrac{x^2}{2}\right)\Bigg]_{-1}^{2}$$
$$= 1.037037 - 0.037037 = 1$$

$$E(X^2) = \frac{2}{9}\int_{-1}^{2}(x^3+x^2)dx = \frac{2}{9}\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_{-1}^{2}$$
$$= 1.481481 - (0.01852) = \frac{3}{2}$$

$$\therefore \text{Var}(X) = E(X^2) - [E(X)]^2 = \tfrac{3}{2} - 1^2 = \tfrac{1}{2}$$

Question 7

Let X = profit. Then

$\mu = E(X) = (250)(0.22) + (150)(0.36) + (0)(0.28) + (-150)(0.14)$
$= \$88.$

Question 8

$E(X) = \int_0^1 x^2 dx + \int_1^2 x(2 - x)dx = 1.$ Therefore, the average number of hours per year is $(1)(100) = 100.$

$$E(X^2) = \int_0^1 x^2 \cdot x\,dx + \int_1^2 x^2(2-x)dx = \int_0^1 x^3\,dx + \int_1^2 2x^2 - x^3\,dx$$

$$= \left.\frac{x^4}{4}\right|_0^1 + \left.\frac{2x^3}{3} - \frac{x^4}{4}\right|_1^2 = 1 + \tfrac{1}{6}$$

$$= 7/6$$

$\therefore$ $Var(X) = E(X^2) - |E(X)|^2 = 7/6 - 1 = 1/6$

Question 9

x	-3	6	9
f(x)	1/6	1/2	1/3
g(x)	25	169	361

$\mu_{g(X)} = E[(2X + 1)^2] = (25)(1/6) + (169)(1/2) + (361)(1/3) = 209.$

Question 10

$\mu = (0)(0.4) + (1)(0.3) + (2)(0.2) + (3)(0.1) = 1.0,$
$E(X^2) = (0)(0.4) + (1)(0.3) + (4)(0.2) + (9)(0.1) = 2.0,$
$\sigma^2 = E(X^2) - \mu^2 = 2.0 - 1.0 = 1.0.$

Question 11

X = number of companies (out of 6) that give employees 4 week of vaccination after 15 years of employment.

X~Binomial(6,0.5)

(a) $P(2 \le X \le 5) = P(X \le 5) - P(X \le 1) = 0.9844 - 0.1094 = 0.8750$
(b) $P(X<3) = P(X \le 2) = 0.3438$

Question 12

X = number of toasters requiring repairs (out of 20)

X~Binomial(20, 0.2)

(a) We need to find x such that $P(X \geq x) < 0.5$, ie $P(X < x) > 0.5$.
From the tables $P(X<5)=0.6296$, $P(X<4)=.4114$. Therefore x = 5
{or use $P(X \geq x) < 0.5$}

(b) Let Y=20 - X = number of toasters that do not require repairs;
Y~Binomial(20,0.8)
Need to find y such that $P(Y \geq y) > 0.8$, ie $P(Y<y) < 0.2$. From the tables y =15.
[or use X~Binomial(20, 0.2) and look for y such that $P(X>20-y)<0.2$]

Question 13

(a) X= number of mice found in an acre. X~Poisson(12)
$P(X<7) = 1-P(X \geq 7) = 0.0458$

(b) Y= number of acres in which fewer than 7 mice found (out of the 3 acres)
Y~Binomial(n=3, p=0.0458)

$$P(Y=2)=\binom{3}{2}(0.0458)^2(1-0.0458)=0.006$$

Question 14

X= number of aircraft arrived in 1 hour. X~Poisson(6)

(a) $P(X=4)= P(X \leq 4) - P(X \leq 3) = 0.1339$

(b) $P(X \geq 4)=1- P(X \leq 3)=0.8488$

(c) Y= number of aircraft arrived in 12 hours. Y~Poisson(72)
$P(Y \geq 75) = 0.3773$ [Not in the Tables – need to use Excel]

Question 15

(a) From Table A.3, k = -1.72.

(b) Since $P(Z > k) = 0.2946$, then $P(Z < k) = 0.7054$. From Table A.3 we find k = 0.54.

(c) The area to the left of z = 0.83 is found from Table A.3 to be 0.1762. Therefore, the total area to the left of k is 0.1762 + 0.7235 = 0.8997, and hence k = 1.28.

Question 16

(a) Area = 0.9236.
(b) Area = 1 - 0.1867 = 0.8133.
(c) Area = 0.2578 - 0.0154 = 0.2424.
(d) Area = 0.0823.
(e) Area = 1 - 0.9750 = 0.0250.
(f) Area = 0.9591 - 0.3156 = 0.6435.

Question 17

(a) z = (224 - 200)/15 = 1.6. Fraction of the cups containing more than 224 milliliters is P(Z > 1.6) = 0.0548.
(b) z_1 = (191 - 200)/15 = -0.6, z_2 = (209 - 200)/15 = 0.6; P(191 < X < 209) = P(-0.6 < Z < 0.6) = 0.7257 - 0.2743 = 0.4514.
(c) z = (230 - 200) /15 = 2.0; P[X>230] = P[Z>2.0] = 0.0228. Therefore, (1000)(0.0228) = 22.8 or approximately 23 cups will overflow.
(d) z = -0.67, x=(15)(-0.67) + 200 = 189.95 milliliters.

Question 18

a. The average strength $\bar{X}$ has approximately a normal distribution with mean $\mu = 14$ and standard deviation $\frac{\sigma}{\sqrt{n}} = \frac{2}{\sqrt{100}} = 0.2$. Thus

$$P(\bar{X} > 14.5) = P(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} > \frac{14.5 - \mu}{\frac{\sigma}{\sqrt{n}}})$$

is approximately equal to

$$P\left(Z > \frac{14.5-14}{0.2}\right) = P\left(Z > \frac{0.5}{0.2}\right) = P(Z > 2.5) = 0.0062.$$

b.

$$P\left(\mu - 1.96\,\frac{\sigma}{\sqrt{n}} \leq \bar{X} \leq \mu + 1.96\frac{\sigma}{\sqrt{n}}\right) = 0.95$$

For a normally distributed $\bar{X}$. In this problem,

$$\mu - 1.96\,\frac{\sigma}{\sqrt{n}} = 14 - 1.96\frac{2}{\sqrt{100}} = 13.6$$

and

$$\mu + 1.96\,\frac{\sigma}{\sqrt{n}} = 14 + 1.96\frac{2}{\sqrt{100}} = 14.4$$

Hence, approximately 95% of sample mean fracture strengths, for samples of size 100, should lie between 13.6 and 14.4.