WORKSHOP 6 SOLUTIONS

1. $$\begin{array}{cccccc} + & + & + & - & - & - \\ & \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} & \boldsymbol{i} & \boldsymbol{j} \\ & 2 & -1 & 0 & 2 & -1 \\ & 1 & 3 & -2 & 1 & 3 \end{array}$$

$$\begin{aligned} \boldsymbol{a} \times \boldsymbol{b} &= \boldsymbol{i}\,((-1)(-2) - (0)(3)) + \boldsymbol{j}\,((0)(1) - (2)(-2)) + \boldsymbol{k}\,((2)(3) - (-1)(1)) \\ &= \boldsymbol{i}\,(2-0) + \boldsymbol{j}\,(0+4) + \boldsymbol{k}\,(6+1) \\ &= [2, 4, 7] \end{aligned}$$

$(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{b} = [2, 4, 7] \cdot [1, 3, -2] = (2)(1) + (4)(3) + (7)(-2) = 2 + 12 - 14 = 0$

Since $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{b} = 0$, $\boldsymbol{a} \times \boldsymbol{b}$ and $\boldsymbol{b}$ are perpendicular.

2. Area of triangle $= \frac{1}{2}$ Area of parallelogram

$$= \tfrac{1}{2} \left|\left| \overrightarrow{PQ} \times \overrightarrow{PR} \right|\right|$$

$\overrightarrow{PQ} = [2-1, 0-(-1), -1-2] = [1, 1, -3]$

$\overrightarrow{PR} = [0-1, 2-(-1), 1-2] = [-1, 3, -1]$

$\overrightarrow{PQ} \times \overrightarrow{PR} = \boldsymbol{i}\,(-1-(-9)) + \boldsymbol{j}\,(3-(-1)) + \boldsymbol{k}\,(3-(-1)) \;=\; [8, 4, 4]$

Area $= \dfrac{1}{2}\sqrt{(8)^2 + (4)^2 + (4)^2} = \dfrac{1}{2}\sqrt{64 + 16 + 16} = \dfrac{\sqrt{96}}{2}$

3. Area of parallelogram $= ||\boldsymbol{u} \times \boldsymbol{v}|| = ||\boldsymbol{u}||\ ||\boldsymbol{v}||\ \sin\theta$

Given $||\boldsymbol{u}|| = 16$, $||\boldsymbol{v}|| = 4$, $\cos\theta = \frac{1}{2} \;\Rightarrow\; \theta = \cos^{-1}(\frac{1}{2}) = 60°$

$\therefore$ Area $= (16)(4)\sin(60°) = 64\left(\frac{\sqrt{3}}{2}\right) = 32\sqrt{3}$

4. Coplanar if $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = 0$

$$\begin{array}{cccccc} + & + & + & - & - & - \\ & \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} & \boldsymbol{i} & \boldsymbol{j} \\ & -2 & 0 & 3 & -2 & 0 \\ & 2 & -4 & -4 & 2 & -4 \end{array}$$

$\boldsymbol{b} \times \boldsymbol{c} = \boldsymbol{i}\,(0-(-12)) + \boldsymbol{j}\,(6-8) + \boldsymbol{k}\,(8-0) \;=\; [12, -2, 8]$

$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = [1, 2, -1] \cdot [12, -2, 8] = 12 - 4 - 8 = 0$

So the vectors $\boldsymbol{a} = [1, 2, -1]$, $\boldsymbol{b} = [-2, 0, 3]$ and $\boldsymbol{c} = [2, -4, -4]$ are coplanar.

5. (i) $A+B = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix} + \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix} = \begin{bmatrix} 2+7 & 0+(-5) & -1+1 \\ 4+1 & -5+(-4) & 2+(-3) \end{bmatrix}$

$$= \begin{bmatrix} 9 & -5 & 0 \\ 5 & -9 & -1 \end{bmatrix}$$

(ii) $-4B = -4\begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix} = \begin{bmatrix} -4(7) & -4(-5) & -4(1) \\ -4(1) & -4(-4) & -4(-3) \end{bmatrix} = \begin{bmatrix} -28 & 20 & -4 \\ -4 & 16 & 12 \end{bmatrix}$

(iii) $AC = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$ This is not possible. The number of columns in A is not the same as the number of rows in C.

(iv) $CB = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix} = \begin{bmatrix} 7+2 & -5-8 & 1-6 \\ -14+1 & 10-4 & -2-3 \end{bmatrix} = \begin{bmatrix} 9 & -13 & -5 \\ -13 & 6 & -5 \end{bmatrix}$

(v) $AB^T = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix}\begin{bmatrix} 7 & 1 \\ -5 & -4 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 14+0-1 & 2+0+3 \\ 28+25+2 & 4+20-6 \end{bmatrix} = \begin{bmatrix} 13 & 5 \\ 55 & 18 \end{bmatrix}$

(vi) $C - 3I_2 = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 1-3 & 2 \\ -2 & 1-3 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix}$$

(vii) $C^2 = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1-4 & 2+2 \\ -2-2 & -4+1 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix}$

6. $A_{6\times4}\, B_{n\times m} = (AB)_{6\times8}$
The number of columns in A, 4, equals the number of rows in B, n, $\therefore\ n = 4$
The number of columns in AB, 8, equals the number of columns in B, m, $\therefore\ m = 8$
$\therefore$ order of $B = n \times m = 4 \times 8$

7. $B_{n\times m}\, C_{m\times p} = (BC)_{n\times p}$ Given $n \times p = 4 \times 3$. Hence $n = 4$, i.e., B has 4 rows.

8. $AB = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix} = \begin{bmatrix} 23 & -10+5k \\ -9 & 15+k \end{bmatrix}$

$BA = \begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 23 & 15 \\ 6-3k & 15+k \end{bmatrix}$

Need $-10+5k = 15 \quad\Rightarrow\quad 5k = 25 \quad\Rightarrow\quad k = 5$
and $6-3k = -9 \quad\Rightarrow\quad -3k = -15 \quad\Rightarrow\quad k = 5$

9. $AB = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}\begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -14+15 & -10+10 \\ 21-21 & 15-14 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$BA = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} -14+15 & -35+35 \\ 6-6 & 15-14 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Since $AB = BA = I$, the matrices A and B are the inverse of one another.

10. $AB = 0$ If A is invertible, A^{-1} exists. Multiply both sides by A^{-1}, from the left:
$A^{-1}AB = A^{-1}\,0 \quad\Rightarrow\quad IB = 0 \quad\Rightarrow\quad B = 0$