MATH1019 Linear Algebra and Statistics for Engineers

Lecture 2: Probability Distributions

Overview: In this lecture, we look at an important concept, namely, that of a *distribution*. A distribution describes the variability among measurements and is a fundamental concept in statistics.

Motivation: Some practical uses of probability distributions are in estimation and hypothesis testing, and simulation studies. The design of various engineering structures, such as buildings, infrastructure, aircraft, ships, as well as microelectronic components and medical implants must ensure a low probability of failure during their service lifetime. This probability is determined using probability distributions derived from physically-based probabilistic models.

Learning outcomes

In today's lecture we will learn how to:

- Distinguish between discrete and continuous random variables
- Determine the mean and variance of random variables
- Define probability functions and probability density functions
- Recognise and apply some basic probability distributions

Key concepts in this lecture:

- Define discrete & continuous random variable;
- Mean and variance of discrete and continuous random variable;
- Probability density function
- Cumulative distribution function
- Examples of specific discrete and continuous distributions
- Normal Distribution

WHAT IS RANDOM ?

- In statistics we basically want to make inferences about populations and their characteristics. The study of these populations results in chance outcomes.

- We can’t be certain of the outcomes

TERMS ASSOCIATED WITH RANDOMNESS:

A ***Random experiment*** is a process that, when performed, results in one and only one of many observations. These observations are called ***outcomes*** of the experiment. The collection of all outcomes for an experiment is called a ***sample space***. Each outcome in a sample space is called an ***element***. A subset of a sample space is called an ***event***.

Examples

Experiment	Outcomes	Sample Space
Toss a coin once	Head, Tail	S = {Head, Tail}
Roll a die once	1, 2, 3, 4, 5, 6	S = {1, 2, 3, 4, 5, 6}
Toss a coin twice	*HH, HT, TH, TT*	S = {*HH, HT, TH, TT*}
Play lottery	Win, Lose	S = {Win, Lose}
Take a test	Pass, Fail	S = {Pass, Fail}
Select a student	Male, Female	S = {Male, Female}

Random Variable

Definition: If S is a sample space and X is a real-valued function defined over the elements of S, then X is called a random variable.

We usually denote random variables by capital letters and their values by the corresponding lowercase letters,

i.e. x denotes a value of the random variable X.

X = x is interpreted as the set of elements of the sample space for which the random variables takes on the value x.

Example

A section of an electrical circuit has two relays, numbered 1 and 2, operating in parallel. The current will flow when a switch is thrown if either one or both of the relays close. The probability of a relay closing properly is 0.8, which is the same for each relay. The relays operate independently. When the switch is thrown, a numerical event of some interest to the operator of this system is *X*, the number of relays that close properly.

Let R_i denote the outcome that a relay closes properly and R'_i that it does not ($i = 1, 2$).

List all the possibilities in terms R_i and R'_i, the corresponding probabilities and the values x.

Solution: Begin by listing the elements of the sample space,

$$S = \{R_1 R_2, R_1 R'_2, R'_1 R_2, R'_1 R'_2\}$$

Now construct a table as follows,

Sample Space	Probability	Number of relays that close, x
$R_1 R_2$	(0.8)(0.8) = 0.64	2
$R_1 R'_2$	(0.8)(0.2) = 0.16	1
$R'_1 R_2$	(0.2)(0.8) = 0.16	1
$R'_1 R'_2$	(0.2)(0.2) = 0.04	0

Two types of random variables, discrete and continuous.

A **discrete random variable** is a random variable whose range is finite or countably infinite, that is its set of possible outcomes is countable.

The random variable X in the previous example is a discrete random variable.

A **continuous random variable** is a random variable that can take on values from a continuous scale.

Examples of continuous random variables include:
- Time taken for an email to go from point A to point B
- Service time
- Waiting time

Discrete Probability Distributions

Usually it is preferable to express the probabilities associated with the values of a random variable by means of a function such that its values, f(x), equal P(X = x) for each x within the range of the random variable X,

i.e. f(x) = Probability that X takes on the value x
= P(X = x)

Definition: A random variable X is discrete if it takes **finite** or **countably infinite** values. If X is a discrete random variable then

$$f(x) = P(X = x)$$

is called the **probability function (pf)** (OR **probability mass function (pmf)**) of X.

Properties of f(x)

(i) $f(x) \geq 0$ for all x (ii) $\sum_{x} f(x) = 1$

Example

For the random variable X in the previous example find the probability function.

Solution: Substituting the values of x = 0, 1, 2 into f(x) we get,

$f(0) = P(X = 0) = 0.04$
$f(1) = P(X = 1) = 0.16+0.16 = 0.32$
$f(2) = P(X = 2) = 0.64$

The probabilities are tabulated below

x	**0**	**1**	**2**
f(x)	**0.04**	**0.32**	**0.64**

Mean and Variance

Definition: If X is a discrete random variable, the mean or EXPECTED VALUE OF X is defined by

$$\mu = E(X) = \sum_x xP(X = x) = \sum_x xf(x)$$

Definition: For a random variable X with mean μ, the variance of X is defined by

$$\sigma^2 = Var(X) = E\left(\left(X - E(X)\right)^2\right) = E\left(\left(X - \mu\right)^2\right)$$

i.e

$$\sigma^2 = Var(X) = \sum_x (x - \mu)^2 f(x)$$

Example

Find the mean and variance of the random variable in the previous example.

Solution: The probability function, $f(x)$, is given by:

x	0	1	2
f(x)	**0.04**	**0.32**	**0.64**

$$\mu = E(X) = \sum_{x=0}^{2} xf(x) = 0(0.04) + 1(0.32) + 2(0.64) = 1.6$$

$$\sigma^2 = Var(X) = \sum_{x=0}^{2} (x-\mu)^2 f(x) = (0-1.6)^2(0.04) + (1-1.6)^2(0.32)$$

$$+ (2-1.6)^2(0.64) = 0.32$$

Cumulative Distribution Function (cdf) of *X*

There are many problems in which it is of interest to know the probability that the value of a variable is less than or equal to some real number x, i.e. $P(X \leq x)$

The cumulative distribution function of a random variable *X* is defined by, for any real number x,

$$
\begin{aligned}
F(x) &= P(X \leq x) \\
&= \sum_{y \leq x} p(y) \\
&= \text{Add all the probabilities up to (including at) x.}
\end{aligned}
$$

Example

Consider the random variable X in the previous example. Obtain the cdf of X.

Solution: Substituting the values of x = 0, 1, 2 into F(x) we get,

F(0) = P(X ≤ 0) = f(0) = 0.04
F(1) = P(X ≤ 1) = f(0) + f(1) = 0.04 + 0.32 = 0.36
F(2) = P(X ≤ 2) = f(0) + f(1) + f(2) = 0.04 + 0.32 + 0.64 = 1

$$\mathrm{F(x)} = \begin{cases} 0 & x < 0 \\ 0.04 & 0 \leq \mathrm{x} < 1 \\ 0.36 & 1 \leq \mathrm{x} < 2 \\ 1 & x \geq 2 \end{cases}$$

Bernoulli and Binomial Distributions

Example Suppose a certain program has 12 subroutines. When the program is run there is a 10% chance that each subroutine has a “bug”, which is independent of occurrence of bugs in other subroutines. Let X_i denote the number of bugs in the i^{th} subroutine, and Y denote the total number of bugs in the program when it is run once.

(a) Write down the probability function of X_i.

(b) What is the mean and the variance of the number of bugs in a subroutine?

(c) Write down the probability function of Y.

(d) Find the probability that there are at most 6 bugs when the program is run.

(a) What is the mean number of bugs in the program when it is run once?

In the above example we notice that the random variable *X* either has a bug or doesn't have a bug, i.e. There are two possible outcomes.

Definition

Bernoulli trial: An experiment in which there are only two possible outcomes, classified as "success" or "failure", with *P*("success") = *p*, $0 \leq p \leq 1.$ the probability of "failure" is *P*("failure") = 1 – *p*.

Bernoulli Random Variable: Measures the outcome of a Bernoulli trial as 1 or 0;

X = 1, if "success",
= 0, if "failure"

The probability function of *X* is given by,

$f(x) = p^x(1-p)^{1-x}$,   In tabular form:

x	***0***	***1***
f(x)	***1-p***	***p***

In this case we write ***X ~ Bernoulli(p)***

Mean and Variance of the Bernoulli random variable X:

$$\begin{aligned}
\therefore E(X) &= 0(1-p)+1p = p \\
E(X^2) &= 0^2(1-p)+1^2 p = p \\
Var(X) &= p-p^2 = p(1-p)
\end{aligned}$$

Definition Bernoulli Process of *n* trials: An experiment of *n* repeated Bernoulli trials, such that

- the probability of success *p**,*** remains the same from trial to trial;
- each trial is independent of the other trials.

Binomial Random Variable

Binomial Random Variable:

X : number of "successes" in a Bernoulli process of *n* trials

Probability function of the Binomial random variable *X*:

$$f(x)=\binom{n}{x}p^{x}(1-p)^{n-x} \qquad \text{for } x = 0, 1, 2, ..., n$$

In this case we write ***X ~ Binomial(n, p)***

Note: The values of **pf** and **cdf** of binomial random variables are listed in the statistical tables for different values of *p.*

Theorem If ***X ~ Binomial(n, p)*** then *E(X) = np* and *Var(X) = np(1 - p)*

Solution to Example:

(a). X_i has two possible outcomes, either it has a bug or doesn't have a bug, referred to as a "success" & "failure" respectively, then

$$X_i \sim Bernoulli(p = 0.1)$$

(b). $E(X_i) = p = 0.1$, $Var(X) = p(1 - p) = 0.1(1 - 0.1) = 0.1(0.9) = 0.09$

(c). Y is a binomial random variable. The program is a bernoulli process of $n = 12$ subroutines, bernoulli trials,

$$Y \sim Binomial(n = 12, p = 0.1)$$

$$f(y) = P(Y = y) = \binom{12}{y} 0.1^y \, 0.9^{12-y}, \qquad y = 0, 1, 2, \ldots, 12$$

(d). $P(\text{At most 6 bugs}) = P(Y \le 6) = \mathrm{P}(Y = 0) + \mathrm{P}(Y = 1) + \ldots + \mathrm{P}(Y = 6)$

$= 0.9999$ (using binomial tables – see next slide)

(e). Mean number of bugs $= E(Y) = np = 12(0.1) = 1.2$

Table A.1 (continued) Binomial Probability Sums $\sum_{x=0}^{r} b(x; n, p)$

		p									
n	r	**0.10**	**0.20**	**0.25**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.80**	**0.90**
12	**0**	0.2824	0.0687	0.0317	0.0138	0.0022	0.0002	0.0000			
	1	0.6590	0.2749	0.1584	0.0850	0.0196	0.0032	0.0003	0.0000		
	2	0.8891	0.5583	0.3907	0.2528	0.0834	0.0193	0.0028	0.0002	0.0000	
	3	0.9744	0.7946	0.6488	0.4925	0.2253	0.0730	0.0153	0.0017	0.0001	
	4	0.9957	0.9274	0.8424	0.7237	0.4382	0.1938	0.0573	0.0095	0.0006	0.0000
	5	0.9995	0.9806	0.9456	0.8822	0.6652	0.3872	0.1582	0.0386	0.0039	0.0001
	6	0.9999	0.9961	0.9857	0.9614	0.8418	0.6128	0.3348	0.1178	0.0194	0.0005
	7	1.0000	0.9994	0.9972	0.9905	0.9427	0.8062	0.5618	0.2763	0.0726	0.0043
	8		0.9999	0.9996	0.9983	0.9847	0.9270	0.7747	0.5075	0.2054	0.0256
	9		1.0000	1.0000	0.9998	0.9972	0.9807	0.9166	0.7472	0.4417	0.1109
	10				1.0000	0.9997	0.9968	0.9804	0.9150	0.7251	0.3410
	11					1.0000	0.9998	0.9978	0.9862	0.9313	0.7176
	12						1.0000	1.0000	1.0000	1.0000	1.0000
13	**0**	0.2542	0.0550	0.0238	0.0097	0.0013	0.0001	0.0000			
	1	0.6213	0.2336	0.1267	0.0637	0.0126	0.0017	0.0001	0.0000		
	2	0.8661	0.5017	0.3326	0.2025	0.0579	0.0112	0.0013	0.0001		
	3	0.9658	0.7473	0.5843	0.4206	0.1686	0.0461	0.0078	0.0007	0.0000	
	4	0.9935	0.9009	0.7940	0.6543	0.3530	0.1334	0.0321	0.0040	0.0002	
	5	0.9991	0.9700	0.9198	0.8346	0.5744	0.2905	0.0977	0.0182	0.0012	0.0000
	6	0.9999	0.9930	0.9757	0.9376	0.7712	0.5000	0.2288	0.0624	0.0070	0.0001
	7	1.0000	0.9988	0.9944	0.9818	0.9023	0.7095	0.4256	0.1654	0.0300	0.0009
	8		0.9998	0.9990	0.9960	0.9679	0.8666	0.6470	0.3457	0.0991	0.0065
	9		1.0000	0.9999	0.9993	0.9922	0.9539	0.8314	0.5794	0.2527	0.0342
	10			1.0000	0.9999	0.9987	0.9888	0.9421	0.7975	0.4983	0.1339
	11				1.0000	0.9999	0.9983	0.9874	0.9363	0.7664	0.3787
	12					1.0000	0.9999	0.9987	0.9903	0.9450	0.7458
	13						1.0000	1.0000	1.0000	1.0000	1.0000

Example Suppose a fair coin is tossed 10 times. Let X denote the number of heads obtained. Find the following probabilities.

(a) $P(X < 3)$

(b) $P(X > 7)$

(c) $P(3 < X < 8)$

(d) $P(X < 3 \text{ or } X > 8)$

Solution: Think of getting a head as a "success". Probability of getting a head $p = ½$. The coin is tossed $n = 10$ times. Each toss is independent.

$$X \sim Binomial(n = 10, p = 0.5)$$

$$f(x) = P(X = x) = \binom{10}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{10-x} = \binom{10}{x}\left(\frac{1}{2}\right)^{10}, \qquad x = 0, 1, 2, \dots, 10$$

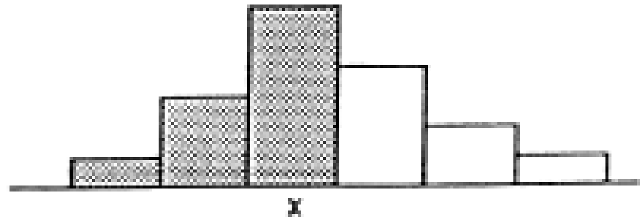
(a). $P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2) = P(X \leq 2) = 0.0547$ (from binomial tables).

(b). $P(X > 7) = 1 – P(X \leq 7) = 1 – 0.9453 = 0.0547$

(c). $P(3 < X < 8) = P(X = 4) + P(X = 5) + P(X = 6) + P(X = 7)$

$= P(X \leq 7) – P(X \leq 3) = 0.9453 – 0.1719 = 0.7734$

CUMULATIVE PROBABILITIES FOR THE BINOMIAL DISTRIBUTION



$P(X \le x)$ where $X \sim B(n,p)$

Read blank entries as 0.0000 or 1.0000 as appropriate

n	x \ p	0.10	0.20	0.25	0.30	0.40	0.50	0.60	0.70	0.75	0.80	0.90
2	0	0.8100	0.6400	0.5625	0.4900	0.3600	0.2500	0.1600	0.0900	0.0625	0.0400	0.0100
	1	0.9900	0.9600	0.9375	0.9100	0.8400	0.7500	0.6400	0.5100	0.4375	0.3600	0.1900
	2	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
3	0	0.7290	0.5120	0.4219	0.3430	0.2160	0.1250	0.0640	0.0270	0.0156	0.0080	0.0010
	1	0.9720	0.8960	0.8438	0.7840	0.6480	0.5000	0.3520	0.2160	0.1563	0.1040	0.0280
	2	0.9990	0.9920	0.9844	0.9730	0.9360	0.8750	0.7840	0.6570	0.5781	0.4880	0.2710
	3	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
10	0	0.3487	0.1074	0.0563	0.0283	0.0061	0.0010	0.0001	0.0000			
	1	0.7361	0.3758	0.2440	0.1493	0.0464	0.0107	0.0017	0.0001	0.0000	0.0000	
	2	0.9298	0.6778	0.5256	0.3828	0.1673	0.0547	0.0123	0.0016	0.0004	0.0001	
	3	0.9872	0.8791	0.7759	0.6496	0.3823	0.1719	0.0548	0.0106	0.0035	0.0009	0.0000
	4	0.9984	0.9672	0.9219	0.8497	0.6331	0.3770	0.1662	0.0474	0.0197	0.0064	0.0002
	5	0.9999	0.9936	0.9803	0.9527	0.8338	0.6231	0.3669	0.1503	0.0781	0.0328	0.0016
	6	1.0000	0.9991	0.9965	0.9894	0.9452	0.8281	0.6177	0.3504	0.2241	0.1209	0.0128
	7		0.9999	0.9996	0.9984	0.9877	0.9453	0.8327	0.6172	0.4744	0.3222	0.0702
	8		1.0000	1.0000	0.9999	0.9983	0.9893	0.9536	0.8507	0.7560	0.6242	0.2639
	9				1.0000	0.9999	0.9990	0.9940	0.9718	0.9437	0.8926	0.6513
	10					1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

(d). $P(X < 3 \text{ or } X > 8) = P(X < 3) + P(X > 8)$

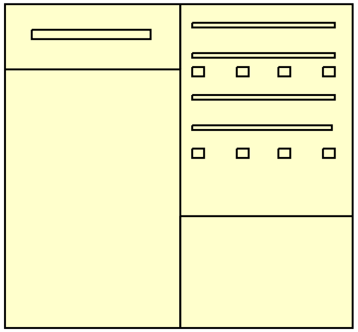
$$= 0.0547 + (1 - P(X \leq 8))$$

$$= 0.0547 + (1 - 0.9893)$$

$$= 0.0547 + 0.0107$$

$$= 0.0654$$

Poisson Distribution



<u>Example:</u> Suppose a web server receives on an average 20 requests per second.

(a) What is the probability that the sever will receive no requests in a second?

(b) Suppose the server will be overloaded if it receives more than 30 requests. Find the probability that the severer will be overloaded in a particular second.

<u>Definition:</u> **<u>Poisson random variable</u>**:

X = number of certain events occurring in a time interval or a region

<u>Probability function of a Poisson Distribution</u>:

$$f(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \qquad x = 0,1,2,\ldots$$

λ = mean number of occurrence per unit time or region

Denoted by ***X ~ Poisson(λ)***

Theorem: If $\boldsymbol{X \sim Poisson(\lambda)}$ then $\boldsymbol{E(X) = \lambda}$ and $\boldsymbol{Var(X) = \lambda}$

Note: If the number of events occurring in a time interval (or region) has Poisson distribution with mean occurrence λ per unit time, then,

Y = number of events occurring in t units of time

has ***Poisson*(*α*)** distribution, where $\alpha = \lambda t$ (Assuming Poisson process properties in the interval).

For the Poisson(λt) distribution, $E(X) = \lambda t$ and $Var(X) = \lambda t$

Situations where Poisson model is used:

- Number particles emitted by a radioactive substance
- Number of calls received in a telephone exchange
- Number of bulbs replaced

Solution to Example: X = number of requests in a second.
We assume,

$$X \sim Poisson(\lambda = 20)$$

$$f(x) = P(X = x) = \frac{e^{-20} 20^x}{x!}$$

(a). P(Server will receive no requests in a second) = $P(X = 0)$

$$= \frac{e^{-20} 20^0}{0!} = e^{-20} = (2.06)10^{-9}$$

(b). P(Server will be overloaded) = $P(X \geq 31) = 1 - P(X \leq 30)$

$= 1 - 0.9865$

$= 0.0135$ (using the Poisson tables)

Example: The average number of trucks arriving during any one hour at a truck depot in a certain city is known to be 2. What is the probability that during an eight hour period fewer than twelve trucks will arrive at the depot?

Solution: Let X = the number of trucks arriving in an eight hour period. Then, since the unit of time is 1 hour, we have

$$\alpha = \lambda t = 2(8) = 16$$

So, $$f(y) = P(y = y) = \frac{e^{-16}16^y}{y!}$$

Therefore, P(Fewer than 12 trucks will arrive) =

$$P(Y < 12) = \sum_{y=0}^{11} P(Y = y) = 0.1270$$ (using the Poisson tables, see next slide)

Table A.2 (continued) Poisson Probability Sums $\sum_{x=0}^{r} p(x;\mu)$

	μ								
r	**10.0**	**11.0**	**12.0**	**13.0**	**14.0**	**15.0**	**16.0**	**17.0**	**18.8**
0	0.0000	0.0000	0.0000						
1	0.0005	0.0002	0.0001	0.0000	0.0000				
2	0.0028	0.0012	0.0005	0.0002	0.0001	0.0000	0.0000		
3	0.0103	0.0049	0.0023	0.0011	0.0005	0.0002	0.0001	0.0000	0.0000
4	0.0293	0.0151	0.0076	0.0037	0.0018	0.0009	0.0004	0.0002	0.0001
5	0.0671	0.0375	0.0203	0.0107	0.0055	0.0028	0.0014	0.0007	0.0003
6	0.1301	0.0786	0.0458	0.0259	0.0142	0.0076	0.0040	0.0021	0.0010
7	0.2202	0.1432	0.0895	0.0540	0.0316	0.0180	0.0100	0.0054	0.0029
8	0.3328	0.2320	0.1550	0.0998	0.0621	0.0374	0.0220	0.0126	0.0071
9	0.4579	0.3405	0.2424	0.1658	0.1094	0.0699	0.0433	0.0261	0.0154
10	0.5830	0.4599	0.3472	0.2517	0.1757	0.1185	0.0774	0.0491	0.0304
11	0.6968	0.5793	0.4616	0.3532	0.2600	0.1848	0.1270	0.0847	0.0549
12	0.7916	0.6887	0.5760	0.4631	0.3585	0.2676	0.1931	0.1350	0.0917
13	0.8645	0.7813	0.6815	0.5730	0.4644	0.3632	0.2745	0.2009	0.1426
14	0.9165	0.8540	0.7720	0.6751	0.5704	0.4657	0.3675	0.2808	0.2081
15	0.9513	0.9074	0.8444	0.7636	0.6694	0.5681	0.4667	0.3715	0.2867
16	0.9730	0.9441	0.8987	0.8355	0.7559	0.6641	0.5660	0.4677	0.3751
17	0.9857	0.9678	0.9370	0.8905	0.8272	0.7489	0.6593	0.5640	0.4686
18	0.9928	0.9823	0.9626	0.9302	0.8826	0.8195	0.7423	0.6550	0.5622
19	0.9965	0.9907	0.9787	0.9573	0.9235	0.8752	0.8122	0.7363	0.6509
20	0.9984	0.9953	0.9884	0.9750	0.9521	0.9170	0.8682	0.8055	0.7307

Continuous Probability Distributions

Definition A random variable X is continuous if

$$P(X = x) = 0, \text{ for real numbers } x,$$

That is the probability that the random variable X assumes the exact value x is zero.

For example, suppose that diameters of machined rods from a certain industrial process may be anywhere from 1.2 to 1.5 centimetres. What is the probability that the diameter of a rod selected at random is 1.3 cm? There are an infinite amount of points for the value of the diameter. Therefore, the probability that the value would be exactly 1.3 cm is zero.

For continuous random variables we typically deal with intervals and its relevant probability.

To determine the probability associated with an interval a to b we require the existence of a probability density function, f(x), such that the area under the curve between a and b corresponds to the probability.

Definition The function f(x) is called the probability density function (pdf) of a continuous random variable X, if

(i). $f(x) \geq 0$ for all x

(ii). $$\int_{-\infty}^{\infty} f(x)dx = 1$$

(iii). $$P(a < X < b) = \int_{a}^{b} f(x)dx.$$ for a < b.

Definition The cumulative distribution function F(x) of a continuous random variable X with pdf f(x) is defined by, for all real number x,

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(y)dy$$

Example

The life-length (measured in hundreds of hours), *X*, of a particular type of battery can be modelled by the following probability density function:

$$f(x) = \begin{cases} \frac{1}{2} e^{-x/2}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

Find the probability that the life-length of a particular battery of this type is greater than 400 hours.

Solution:

$$P(X > 4) = \int_4^{\infty} \frac{1}{2} e^{-x/2} dx = \lim_{b \to \infty} \left[-e^{-x/2}\right]_4^b = \lim_{b \to \infty} \left(-e^{-b/2} + e^{-2}\right) = e^{-2} \approx 0.1353$$

Mean and Variance

Definition: If X is a continuous random variable, with pdf f(x), the mean or EXPECTED VALUE OF X is defined by

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

Definition For a random variable X with mean **μ**, the variance of X is defined by

$$\sigma^2 = Var(X) = E\left(\left(X - E(X)\right)^2\right) = E\left(\left(X - \mu\right)^2\right) = E\left(X^2\right) - \mu^2$$

i.e

$$\sigma^2 = Var(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$

Example

Find the mean and variance of life-lengths of batteries in the previous example.

Solution: (using integration by parts)

$$\mu = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{0}^{\infty} x \frac{1}{2} e^{-x/2} dx$$

$$= \frac{1}{2} \lim_{b\to\infty} \left\{ \left[-2xe^{-x/2} \right]_0^b + 2\int_0^b e^{-x/2} dx \right\}$$

$$= \frac{1}{2} \lim_{b\to\infty} \left\{ -2be^{-b/2} - 0 - 4\left[e^{-b/2} - 1 \right] \right\} = 2$$

Example cont.

$$\sigma^2 = E\left(X^2\right) - \mu^2 = \int_{-\infty}^{\infty} x^2 f(x)dx - 2^2 = \int_0^{\infty} x^2 \frac{1}{2} e^{-x/2} dx - 4$$

$$= \frac{1}{2} \lim_{b\to\infty} \left[-2x^2 e^{-x/2} - 8xe^{-x/2} - 16e^{-x/2}\right]_0^b - 4$$

$$= \frac{1}{2} \lim_{b\to\infty} \left[-2b^2 e^{-b/2} - 8be^{-b/2} - 16e^{-b/2} + 16\right] - 4$$

$$= 8 - 4 = 4$$

Normal Distribution

- Most important and often used distribution
- First discovered by the English mathematician Abraham De Moivre (1667-1754)
- Early applications included astronomy (Laplace (1749-1827)) and physics (Gauss (1777-1855))
- The normal distribution works well as a model for many different types of measurements generated in real experiments

Normal Distribution

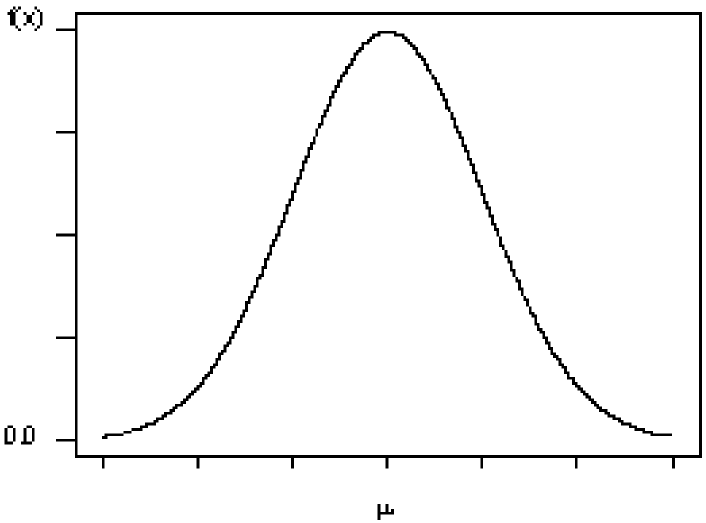
<u>Definition</u>: A random variable *X* is said to have **Normal Distribution** if the pdf of *X* has the form

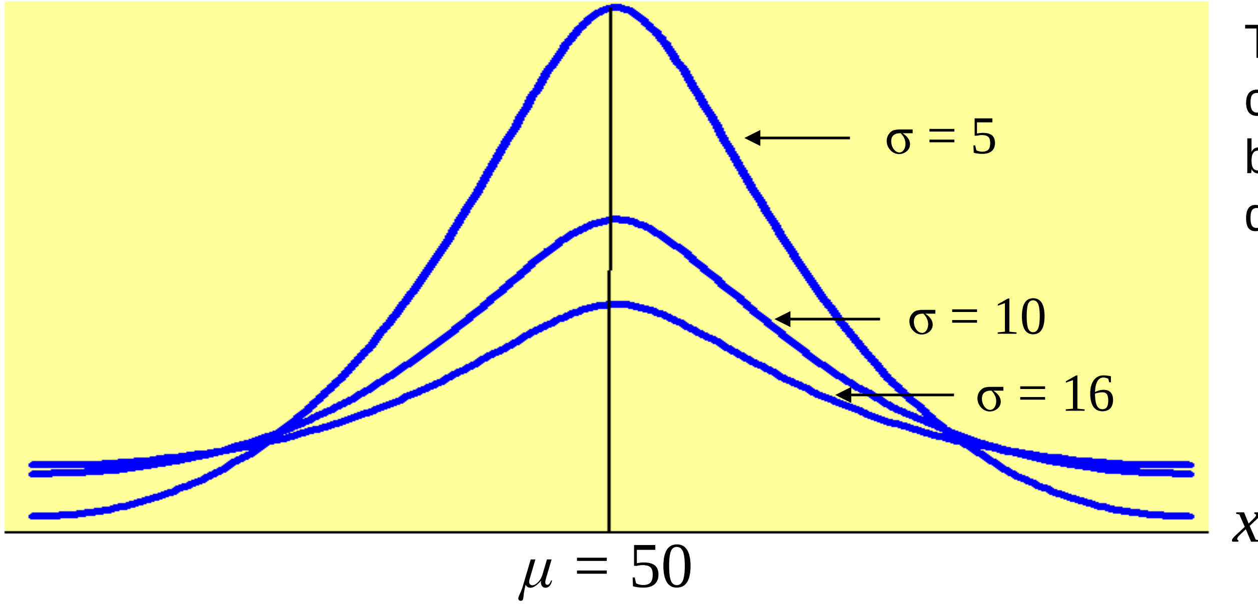
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \text{ if} -\infty < x < \infty$$

μ is the mean of *X* and σ is the standard deviation of *X*, assume these are specified or known

Denoted by $\boldsymbol{X \sim N(\mu, \sigma^2)}$

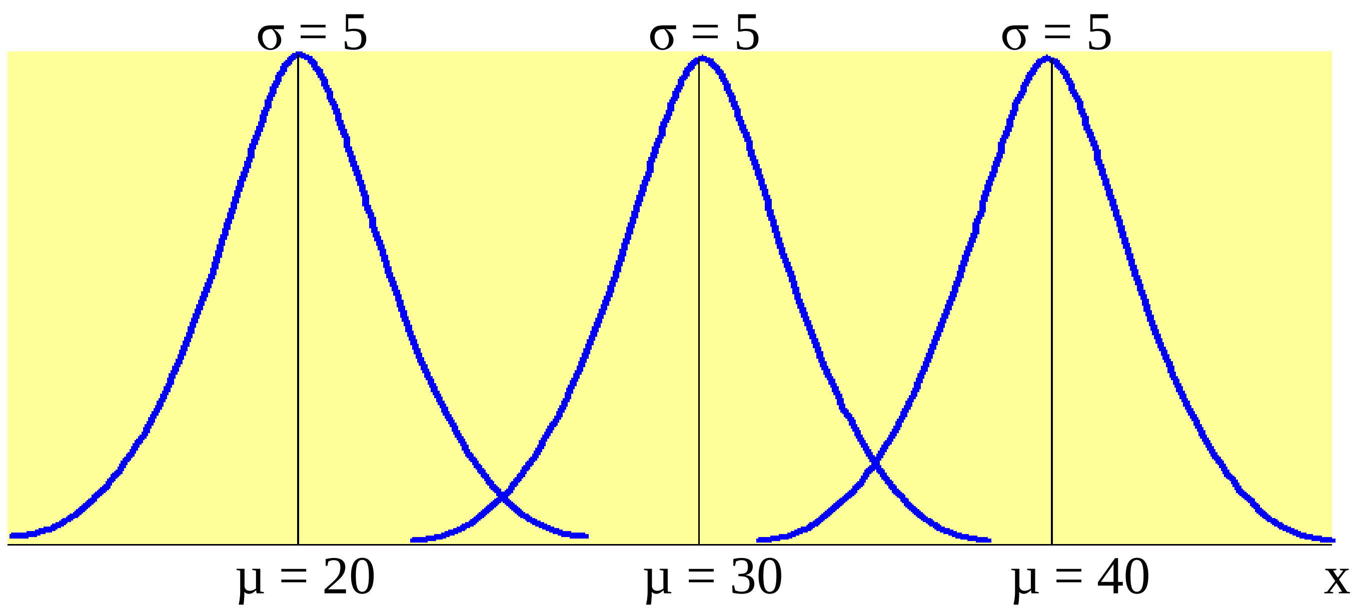
Graph of the normal distribution is known as a bell or normal curve





Three Normal Distribution curves with the same mean but different standard deviation

Three Normal Distribution curves with different means but same standard deviation



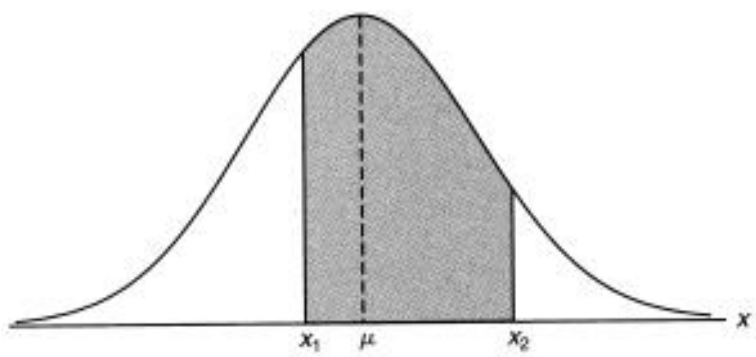
<u>Definition</u>: The normal distribution with $\mu = 0$ and $\sigma^2 = 1$ is referred to as the **Standard Normal Distribution**. It's pdf has the form

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \text{ if } -\infty < x < \infty$$

<u>Theorem</u>: If $\boldsymbol{X \sim N(\mu, \sigma^2)}$ then $\boldsymbol{E(X) = \mu}$ and $\boldsymbol{Var(X) = \sigma^2}$

We know the area under the normal curve between x_1 and x_2 equals

$$P(x_1 < X < x_2) = \int_{x_1}^{x_2} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$



- Unfortunately it's density cannot be integrated directly

- Need a tabulation of normal curve areas, like Binomial and Poisson

- The areas for the Standard Normal Distribution are tabulated

- Can transform random variables X having normal distributions into standard normal variables Z using, $Z = (X - \mu)/\sigma$

- That is, the normal distribution with mean μ and variance σ is transformed into a standard normal distribution with mean 0 and variance 1

Theorem: If $\boldsymbol{X \sim N(\mu, \sigma^2)}$ and $\boldsymbol{Z = (X - \mu)/\sigma}$ then $\boldsymbol{Z \sim N(0, 1)}$

Therefore when X assumes the value x, the corresponding value of Z is $z = (x - \mu)/\sigma$

So to calculate $P(x_1 < X < x_2)$, we need to first determine the corresponding $z_1 = (x_1 - \mu)/\sigma$ & $z_2 = (x_2 - \mu)/\sigma$, then

$$P(x_1 < X < x_2) = \int_{z_1}^{z_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = P(z_1 < Z < z_2)$$

Now it's possible to use the standard normal distribution tables

Example Suppose Z has standard normal distribution. Evaluate the following

(a) $P(Z < 1.9)$

(b) $P(Z > -1.9)$

(c) $P(-1 < Z < 1)$

(d) $P(-2.1 < Z < 4.2)$

Solution: $Z \sim N(0, 1)$

(a). $P(Z < 1.9) = 0.9713$

(b). $P(Z > -1.9) = 1 - P(Z \leq -1.9) = 1 - 0.0287 = 0.9713$

or use the fact that the normal curve is symmetric $P(Z < -k) = P(Z > k)$

(c). $P(-1 < Z < 1) = 1 - 2P(Z < -1)$ (by symmetry)

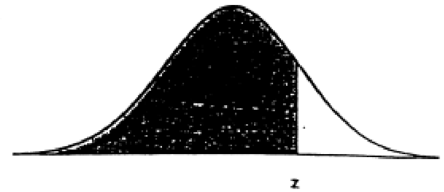
$= 1 - 2(0.1587) = 1 - 0.3174 = 0.6826$

or

$= P(Z < 1) - P(Z < -1) = 0.8413 - 0.1587 = 0.6826$

(d). $P(-2.1 < Z < 4.2) = P(Z < 4.2) - P(Z < -2.1) = 1 - 0.0179 = 0.9821$

CUMULATIVE PROBABILITIES FOR THE STANDARD NORMAL DISTRIBUTION



$P(Z \leq z)$ where $Z \sim N(0,1)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
-0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879

$P(Z \le z)$ where $Z \sim N(0,1)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997

Example Suppose X has normal distribution with mean 10 and variance 16. Evaluate the following

(a) $P(X < 8)$

(b) $P(X > 6)$

(c) $P(4 < X < 12)$

Solution: $X \sim N(10, 16) \Rightarrow Z = (X - 10)/4 \sim N(0, 1)$

(a). $P(X < 8) = P(Z < (8 - 10)/4) = P(Z < -0.5) = 0.3085$

(b). $P(X > 6) = P(Z > (6 - 10)/4) = P(Z > -1) = 1 - P(Z \leq -1)$

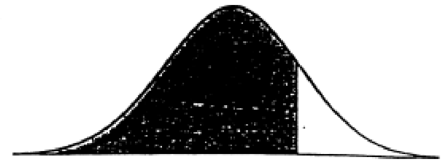
$= 1 - 0.1587 = 0.8413$

(c). $P(4 < X < 12) = P((4 - 10)/4 < Z < (12 - 10)4) = P(-1.5 < Z < 0.5)$

$= P(Z < 0.5) - P(Z < -1.5) = 0.6915 - 0.0668$

$= 0.6247$

CUMULATIVE PROBABILITIES FOR THE STANDARD NORMAL DISTRIBUTION



z

$P(Z \leq z)$ where $Z \sim N(0,1)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
-0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879

$P(Z \le z)$ where $Z \sim N(0,1)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997

Example

With an eye toward improving performance, industrial engineers studied the ability of scanners to read the bar codes of various food and household products. The maximum reduction in power, just before the scanner cannot read the bar code at a fixed distance, is called the maximum attenuation. This quantity, measured in decibels, varies from product to product. After collecting considerable data, the engineers decided to model the variation in maximum attenuation as a normal distribution with mean 10.1 dB and standard deviation 2.7 dB

a) For the next food or product, what is the probability that its maximum attenuation is between 8.5 dB and 13.0 dB?

b) What proportion of the products have maximum attenuation greater than 15.1 dB?

Solution

a)

- For x = 8.5: $$z = \frac{8.5 - 10.1}{2.7} = -0.59$$

- For x = 13.0: $$z = \frac{13.0 - 10.1}{2.7} = 1.07$$

- P (8.5 ≤ X ≤ 13.0)

  = P (-0.59 ≤ Z ≤ 1.07) = 0.8577-0.2776

  = 0.5801

Solution cont.

b)

- For x = 15.1: $z = \dfrac{15.1 - 10.1}{2.7} = 1.85$

- P (X >15.1) = P(Z > 1.85)

  = 1 – P(Z < 1.85)

  = 1 – 0.9678

  = 0.0322

Lecture Summary

- A **discrete random variable** is a random variable whose range is finite or countably infinite, that is its set of possible outcomes is countable.
- A **continuous random variable** is a random variable that can take on values from a continuous scale.
- Discrete probability distribution

Properties: (i) f(x) $\geq$ 0 for all x (ii) $\sum_x f(x) = 1$

Mean: $\mu = E(X) = \sum_x xP(X = x) = \sum_x xf(x)$ Variance: $\sigma^2 = Var(X) = \sum_x (x-\mu)^2 f(x)$

- Continuous probability Distribution

Properties: (i) f(x) $\geq$ 0 for all x; (ii) $\int_{-\infty}^{\infty} f(x)dx = 1$ (iii) $\mathrm{P(a < X < b) = \int_a^b f(x)dx.}$

Mean & variance:

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx \qquad \sigma^2 = Var(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx$$

- Normal probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \text{ if } -\infty < x < \infty$$

- If ***X* ~ *N*(*μ*, *σ*^{2})** then ***E*(*X*) = *μ*** and ***Var*(*X*) = *σ*^{2}**
- If ***X* ~ *N*(*μ*, *σ*^{2})** and ***Z* = (*X* - *μ*)/*σ*** then ***Z* ~ *N*(*0, 1*)**