LASE EXAM SOLUTIONS

Q1) (a). $\vec{OA} = [2, 1, 1]$ (1 mark)

(b). $\vec{AB} = [0-2, 2-1, -1-1] = [-2, 1, -2]$ (1) $\quad \|\vec{AB}\| = \sqrt{(-2)^2+1^2+(-2)^2} = 3$ (1)

(c). $\|\underset{\sim}{a}\| = \sqrt{(3)^2+(-2)^2+(1)^2} = \sqrt{14}$ (1)

(d). $\underset{\sim}{b} \cdot \underset{\sim}{a} = (-2)(3)+(1)(-2)+(0)(1) = -6-2+0 = -8$ (1)

(e). $\hat{\underset{\sim}{b}} = \dfrac{[-2, 1, 0]}{\sqrt{(-2)^2+(1)^2+0^2}}$ (1) $= \dfrac{[-2, 1, 0]}{\sqrt{5}}$

$\therefore [\cos\alpha, \cos\beta, \cos\gamma]$ (½) $= \hat{\underset{\sim}{b}} = \left[\dfrac{-2}{\sqrt{5}}, \dfrac{1}{\sqrt{5}}, 0\right]$ (1)

$\therefore \alpha = \cos^{-1}(-2/\sqrt{5}) = 153.4^\circ$ (½), $\beta = \cos^{-1}(1/\sqrt{5}) = 63.4^\circ$ (½)

$\gamma = \cos^{-1}(0) = 90^\circ$ (½)

(f). $\vec{DB} = [0-6, 2-(-1), -1-5] = [-6, 3, -6]$ (1)

$$b \times \vec{DB} = \begin{vmatrix} i & j & k \\ -2 & 1 & 0 \\ -6 & 3 & -6 \end{vmatrix}$$

$= (-6-0)\underset{\sim}{i} - (12-0)\underset{\sim}{j} + (-6+6)\underset{\sim}{k}$ (1) $= [-6, -12, 0]$ (1)

(g). $2\hat{\underset{\sim}{b}} = 2\dfrac{\underset{\sim}{b}}{\|\underset{\sim}{b}\|} = 2 \times \dfrac{[-2, 1, 0]}{\sqrt{5}}$ (1) (1) $= \left[\dfrac{-4}{\sqrt{5}}, \dfrac{2}{\sqrt{5}}, 0\right]$

(h). $\vec{AB} = [0-2, 2-1, -1-1] = [-2, 1, -2]$ (½)

$\vec{AC} = [3-2, -1-1, 0-1] = [1, -2, -1]$ (½)

$\vec{AD} = [6-2, -1-1, 5-1] = [4, -2, 4]$ (½)

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ -2 & 1 & -2 \\ 1 & -2 & -1 \end{vmatrix}$$

$= (-1-4)\underset{\sim}{i} - (2+2)\underset{\sim}{j} + (4-1)\underset{\sim}{k}$ (1) $= [-5, -4, 3]$ (1)

$(\vec{AB} \times \vec{AC}) \cdot \vec{AD} = [-5, -4, 3] \cdot [4, -2, 4]$

$= -20 + 8 + 12$ (½)

$= 0$ (1) $\therefore$ Coplanar. (1)

Q2).

(a).(i) Scalar 1

(ii). Not possible $\frac{1}{2}$ Cant take a dotproduct between a scalar ($\|\underset{\sim}{a}\|$) and a vector ($\underset{\sim}{b}$) $\frac{1}{2}$

(iii). Vector. 1

(iv). Vector 1

(v). Scalar 1

(vi). Not possible $\frac{1}{2}$ Cant divide a vector ($\underset{\sim}{a}+\underset{\sim}{b}$) by a vector ($\underset{\sim}{a}\times\underset{\sim}{b}$) $\frac{1}{2}$

(b). Direction of line $\underset{\sim}{a} = [2, -4, 1]$ $\frac{1}{2}$

Normal vector of plane $\underset{\sim}{n} = [-1, 0, 1]$ $\frac{1}{2}$

$$\theta = \cos^{-1}\left(\frac{\underset{\sim}{a}\cdot\underset{\sim}{n}}{\|\underset{\sim}{a}\|\,\|\underset{\sim}{n}\|}\right) = \cos^{-1}\left(\frac{[2,-4,1]\cdot[-1,0,1]}{\sqrt{2^2+(-4)^2+1^2}\,\sqrt{(-1)^2+0^2+1^2}}\right) \quad 1$$

$$= \cos^{-1}\left(\frac{-2+0+1}{\sqrt{4+16+1}\,\sqrt{1+0+1}}\right) \quad 1 \quad = \cos^{-1}\left(\frac{-1}{\sqrt{21}\,\sqrt{2}}\right) \quad 1$$

$$= \cos^{-1}(-0.1543) \approx 98.88^\circ \quad 1$$

(c). Direction of line $\underset{\sim}{a} = [3, -1, -2]$ $\frac{1}{2}$

Normal vector to the plane $\underset{\sim}{n} = [-6, 2, -4]$ $\frac{1}{2}$

$$\underset{\sim}{a}\cdot\underset{\sim}{n} = [3,-1,-2]\cdot[-6,2,-4]$$

$$= -18 - 2 + 8 = -12 \quad 1 \quad \neq 0 \quad 1 \quad \therefore \text{ Not parallel} \quad 1$$

(d). Direction of line $\underset{\sim}{a} = [3, 0, -2]$ $\frac{1}{2}$

Point on line $A(2, 2, -1)$ $\frac{1}{2}$ Given $P(-1, 1, 0)$

$$\overrightarrow{AP} = [-1-2,\ 1-2,\ 0-(-1)] = [-3, -1, 1] \quad 1$$

$$\therefore \text{distance} = \sqrt{\|\overrightarrow{AP}\|^2 - \left(\frac{\overrightarrow{AP}\cdot\underset{\sim}{a}}{\|\underset{\sim}{a}\|}\right)^2} = \sqrt{\left(\sqrt{(-3)^2+(-1)^2+(1)^2}\right)^2 - \left(\frac{[-3,-1,1]\cdot[3,0,-2]}{\sqrt{3^2+0^2+(-2)^2}}\right)^2} \quad 1$$

$$= \sqrt{(\sqrt{11})^2 - \left(\frac{-11}{\sqrt{13}}\right)^2} \quad 1 \quad = \sqrt{11 - \frac{121}{13}} = \sqrt{\frac{22}{13}} \quad 1$$

$$\simeq 1.3$$

Q3).

(a). $B - A$ = d.n.e 1 Not the same dimension. 1

(b). BA = d.n.e 1 # Columns in $B \neq$ # Rows in A 1

(c). $CD = \begin{bmatrix} -1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 & 3 & 0 \\ 4 & 1 & 1 & 0 \end{bmatrix}$ (lose 0.5 each wrong entry)

$= \begin{bmatrix} 1+12 & -2+3 & -3+3 & 0+0 \end{bmatrix} = \begin{bmatrix} 13 & 1 & 0 & 0 \end{bmatrix}$ 2

(d). $AC^T = \begin{bmatrix} -1 & 2 \\ -4 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ 2 $= \begin{bmatrix} 1+6 \\ 4+15 \end{bmatrix} = \begin{bmatrix} 7 \\ 19 \end{bmatrix}$ 1

(e). $\det(C)$ = d.n.e 1 C is not square 1

(f). $A^{-1} = \dfrac{1}{(-1)(5)-(2)(-4)}$ 1 $\begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix}$ 1 $= \dfrac{1}{3} \begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 5/3 & -2/3 \\ 4/3 & -1/3 \end{bmatrix}$ 1

(g). E^{-1} = d.n.e 1 since $\det(E) = (-9)(5) - (-15)(3)$ 1 $= -45 + 45 = 0$ 1

(h). $|B| I_3 = 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 1 1 $= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ 1

Q4).

(a). $A = \begin{bmatrix} 3 & -4 \\ 1 & -2 \end{bmatrix}$ 1/2 $\underset{\sim}{b} = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$

$\det(A) = (3)(-2) - (-4)(1) = -2$ 1/2

$A_1 = \begin{bmatrix} 8 & -4 \\ 5 & -2 \end{bmatrix}$ 1/2

$\det(A_1) = (8)(-2) - (-4)(5) = 4$ 1/2

$A_2 = \begin{bmatrix} 3 & 8 \\ 1 & 5 \end{bmatrix}$ 1/2

$\det(A_2) = (3)(5) - (8)(1) = 7$ 1/2

$\therefore\ p = \dfrac{\det(A_1)}{\det(A)} = \dfrac{4}{-2} = -2$ 1 $\quad q = \dfrac{\det(A_2)}{\det(A)} = \dfrac{7}{-2}$ 1

(b). $\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix}$ 1

$= 3\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix}$ 1

$= 3(-1-1) + 2(1+2) = -6 + 2(3) = 0$ 1

$\therefore$ Since $|A| = 0$ 1 $\therefore$ Set of vectors is l.d. 1

(c). $\left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 2 \\ 0 & 2 & 4 & 2 & 3 \\ 1 & 2 & 1 & 2 & k \end{array}\right]$ 1 $R_3 = R_3 - R_1$ $\sim \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 2 \\ 0 & 2 & 4 & 2 & 3 \\ 0 & 1 & 2 & 1 & k-2 \end{array}\right]$ 1 $R_3 = 2R_3 - R_2$

$\sim \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 2 \\ 0 & 2 & 4 & 2 & 3 \\ 0 & 0 & 0 & 0 & 2k-7 \end{array}\right]$ 1 $\quad r(A) = 2$ 1/2 $\quad r([A|\underset{\sim}{b}]) = 3$ 1/2

For the system to be consistent $\Rightarrow r([A|b]) = r(A)$

Let $2k - 7 = 0$ 1 (for $r([A|\underset{\sim}{b}]) = 2$)

$\therefore k = 7/2$ 1

Need $n - r = 4 - 2 = 2$ parameters

$x_3 = s$ 1

$x_4 = t$ 1

$x_2 = \frac{1}{2}(3-4s-2t) = \frac{3}{2} - 2s - t$ 1

$x_1 = 2 - \frac{3}{2} + 2s + t + s - t = \frac{1}{2} + 3s$ 1

$$\therefore \underset{\sim}{x} = \begin{bmatrix} \frac{1}{2} + 3s \\ \frac{3}{2} - 2s - t \\ s \\ t \end{bmatrix}$$

Q5)

(a). (i). $[A|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array}\right]$ $\begin{array}{l} \\ R_2 = R_2 - R_1 \\ R_3 = R_3 - R_1 \end{array}$ 1

$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 3 & 8 & -1 & 0 & 1 \end{array}\right]$ $R_3 = R_3 - 3R_2$ 2 $\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 2 & 2 & -3 & 1 \end{array}\right]$ $\begin{array}{l} \\ R_2 = R_2 - R_3 \\ R_3 \div 2 \end{array}$ 1

$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 4 & -1 \\ 0 & 0 & 1 & 1 & -\frac{3}{2} & \frac{1}{2} \end{array}\right]$ $R_1 = R_1 - R_3$ 1 $\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & -3 & 4 & -1 \\ 0 & 0 & 1 & 1 & -\frac{3}{2} & \frac{1}{2} \end{array}\right]$ $R_1 = R_1 - R_2$ 1

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -\frac{5}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -3 & 4 & -1 \\ 0 & 0 & 1 & 1 & -\frac{3}{2} & \frac{1}{2} \end{array}\right] = [I|A^{-1}]$ 1

$A^{-1} = \begin{bmatrix} 3 & -\frac{5}{2} & \frac{1}{2} \\ -3 & 4 & -1 \\ 1 & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$ A is not singular, it's non-singular matrix. 1

(ii). Trivial Solution 1

(b). Let $\underset{\sim}{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in U$ $\quad \underset{\sim}{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in U$

$\therefore u_2 = (u_1)^2, \; u_3 = u_1 + u_2$ $\quad \therefore v_2 = (v_1)^2, \; v_3 = v_1 + v_2$

Choose $\underset{\sim}{u} = \begin{bmatrix} 3 \\ 9 \\ 12 \end{bmatrix}$ ½ and $\underset{\sim}{v} = \begin{bmatrix} 5 \\ 25 \\ 30 \end{bmatrix}$ ½

$$\underset{\sim}{u}+\underset{\sim}{v} = \begin{bmatrix} 8 \\ 34 \\ 42 \end{bmatrix}$$

Since $34 \neq (8)^2 = 64$ 1 $\therefore$ Not closed under addition 1

$\therefore$ Not a subspace of $\mathbb{R}^3$ 1

(c). $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ $\underset{\sim}{b} = \begin{bmatrix} 6 \\ 3 \\ 2 \\ -1 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 14 \end{bmatrix}$$ ½ ½

$$(A^TA)^{-1} = \frac{1}{56-16} \begin{bmatrix} 14 & -4 \\ -4 & 4 \end{bmatrix} = \frac{1}{40} \begin{bmatrix} 14 & -4 \\ -4 & 4 \end{bmatrix}$$ 1

$$\text{pinv}(A) = (A^TA)^{-1}A^T = \frac{1}{40} \begin{bmatrix} 14 & -4 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 2 & 3 \end{bmatrix}$$ 1

$$= \frac{1}{40} \begin{bmatrix} 18 & 14 & 6 & 2 \\ -8 & -4 & 4 & 8 \end{bmatrix}$$ 1

$$\hat{\underset{\sim}{x}} = \text{pinv}(A)\underset{\sim}{b} = \frac{1}{40} \begin{bmatrix} 18 & 14 & 6 & 2 \\ -8 & -4 & 4 & 8 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ 2 \\ -1 \end{bmatrix}$$ 1

$$= \frac{1}{40} \begin{bmatrix} 160 \\ -60 \end{bmatrix} = \begin{bmatrix} 4 \\ -3/2 \end{bmatrix}$$ 1

$\therefore y = 4 - 1.5x$ 1