Systems of Linear Equations & Gaussian Elimination

Motivation: Virtually all engineering problems involving a reasonable degree of complexity will require the solution of systems of linear equations, usually at more than one level. Practical problems can easily give rise to systems involving hundreds or thousands of variables and equations. Although these systems and methods for their solution are often buried inside software tools and certainly not solved by hand, it is important to understand the various solution techniques that exist and in which instance to apply them. This knowledge is best gained by working through a variety of elementary examples by hand. You're probably not going to solve more than 20-30 systems of linear equations in your lifetime, so put the calculator (or Maple, for that matter) aside for these exercises and focus on the solution techniques rather than on the arithmetic or the solutions themselves.

Most linear systems arising in practice do not have a unique solution and hence we need to know how to identify and deal with the other two cases (i.e. no solution or infinite solutions).

Outcomes In today's lecture we will learn how to:

- Recognize the three basic scenarios that can arise when solving systems of linear equations.
- Recognize a row echelon form.
- Solve for a unique solution by back substitution.
- Understand the basis of Gaussian elimination.
- Recognize the type of solution of a linear system by identifying the ranks of the coefficient and augmented matrices.
- Describe infinitely many solutions in terms of parameters.

Contents

- Systems of equations in 2 or 3 variables and geometric interpretation.
- The augmented matrix.
- Row echelon form.
- Back substitution.
- Gaussian Elimination and notation.

- Linear dependence and independence of vectors.
- The rank of a matrix and how to compute it.
- Relation to solving linear systems by Gaussian elimination.
- Parametrizing infinitely many solutions.
- Some hints on Gaussian elimination.

Exercises

1. Which of the following are linear equations in x_1, x_2 and x_3?

   (a) $x_1 + 5x_2 - \sqrt{2}x_3 = 1$ (b) $x_1 + 3x_2 + x_1x_2 = 2$ (c) $x_1 = -7x_2 + 3x_3$

   (d) $\frac{1}{x_1} + x_2 + 8x_3 = 5$ (e) $x_1^{\frac{3}{5}} - 2x_2 + x_3 = 4$ (f) $\pi x_1 - \sqrt{3}x_2 + \frac{1}{3}x_3 = \sqrt[3]{7}$

2. Find the augmented matrix for each of the following system:

   (a) $$\begin{array}{rcl} 3x_1 + x_3 & = & 11 \\ 4x_2 - 3x_3 & = & 9 \\ x_1 - x_3 & = & -3 \end{array}$$

   (b) $$\begin{array}{rcl} x_1 + 2x_2 - x_4 + x_5 & = & 3 \\ 2x_2 - 4x_3 + 2x_5 & = & 13 \\ x_3 - 7x_4 & = & 21 \end{array}$$

3. Which of the following matrices are in row echelon form?

   (a) $\begin{bmatrix} 3 & 1 & 0 & 2 & 5 \\ 0 & 0 & 2 & 1 & -2 \\ 0 & 0 & 0 & 0 & -17 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 5 \\ 3 & 2 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 7 & 3 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} -8 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

4. For the following augmented matrix in row echelon form, solve the corresponding linear system.

   $$\left[\begin{array}{ccc|c} 2 & -3 & 4 & 7 \\ 0 & 4 & 2 & 2 \\ 0 & 0 & 5 & 15 \end{array}\right]$$

5. Use Gaussian Elimination to find the unique solution to the following system of linear equations.

   (a) $$\begin{array}{rcl} x_1 + x_2 + 6x_3 & = & 3 \\ x_1 + x_2 + 3x_3 & = & 3 \\ x_1 + 2x_2 + 4x_3 & = & 7 \end{array}$$

   (b) $$\begin{array}{rcl} x_1 + x_2 + x_3 & = & 4 \\ -x_1 + 2x_2 + 3x_3 & = & 17 \\ 2x_1 - x_2 & = & -7 \end{array}$$

   (c) $$\begin{array}{rcl} 2x_1 + x_2 & = & 7 \\ 2x_1 - x_2 + x_3 & = & 6 \\ 3x_1 - 2x_2 + 4x_3 & = & 11 \end{array}$$

6. Exercise 11(d) (p. 606): 1, 5.

7. In each of the cases below, an augmented matrix for a given system, reduced to row echelon form, is given. Determine r(A) and r([A| $\boldsymbol{b}$]) and find the solution of the system in each case.

(a) $\left[\begin{array}{ccccc|c} 1 & 7 & -2 & 0 & -8 & -3 \\ 0 & 0 & 1 & 1 & 6 & 5 \\ 0 & 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$ (b) $\left[\begin{array}{ccc|c} 1 & -3 & 7 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

8. Solve $\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$

9. Solve $\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 0 \\ -2x_1 + 5x_2 + 2x_3 &= 1 \\ 8x_1 + x_2 + 4x_3 &= -1 \end{aligned}$

10. Solve $\begin{aligned} -2x_2 + 3x_3 &= 1 \\ 3x_1 + 6x_2 - 3x_3 &= -2 \\ 6x_1 + 6x_2 + 3x_3 &= 5 \end{aligned}$

11. Exercise 11(e) (p. 614): 3.

12. For each of the following sets of vectors, determine whether they are linearly dependent or independent.

(a) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -7 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} -2 \\ 6 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} \right\}$

These exercises should take around 70 minutes to complete.

(Answers: 1. (a), (c) & (f) only; 2.(a) $\left[\begin{array}{ccc|c} 3 & 0 & 1 & 11 \\ 0 & 4 & -3 & 9 \\ 1 & 0 & -1 & -3 \end{array}\right]$ (b) $\left[\begin{array}{ccccc|c} 1 & 2 & 0 & -1 & 1 & 3 \\ 0 & 2 & -4 & 0 & 2 & 13 \\ 0 & 0 & 1 & -7 & 0 & 21 \end{array}\right]$; 3. (a) only; 4. $x_1 = -4$, $x_2 = -1$, $x_3 = 3$; 5.(a) $x_1 = -1$, $x_2 = 4$, $x_3 = 0$ (b) $x_1 = -2$, $x_2 = 3$, $x_3 = 3$ (c) $x_1 = 3$, $x_2 = 1$, $x_3 = 1$; 7.(a) r(A) = 3, r([A| $\boldsymbol{b}$]) = 3, $x_1 = -7s+2t-11$, $x_2 = s$, $x_3 = -3t-4$, $x_4 = -3t+9$, $x_5 = t$ (b) r(A) = 2, r([A| $\boldsymbol{b}$]) = 3, No solutions; 8. $x_1 = 3$, $x_2 = 1$, $x_3 = 2$; 9. $x_1 = -\frac{1}{7} - \frac{3}{7}t$, $x_2 = \frac{1}{7} - \frac{4}{7}t$, $x_3 = t$; 10. No solutions; 12.(a) independent (b) dependent)

Linear Systems of Equations

A linear system of algebraic equations is simply a set of equations in which all of the unknown variables occur linearly (*i.e.* no polynomial terms, products of variables, sines, cosines *etc.* are allowed). A general linear system has m equations in n variables and is usually written in the form

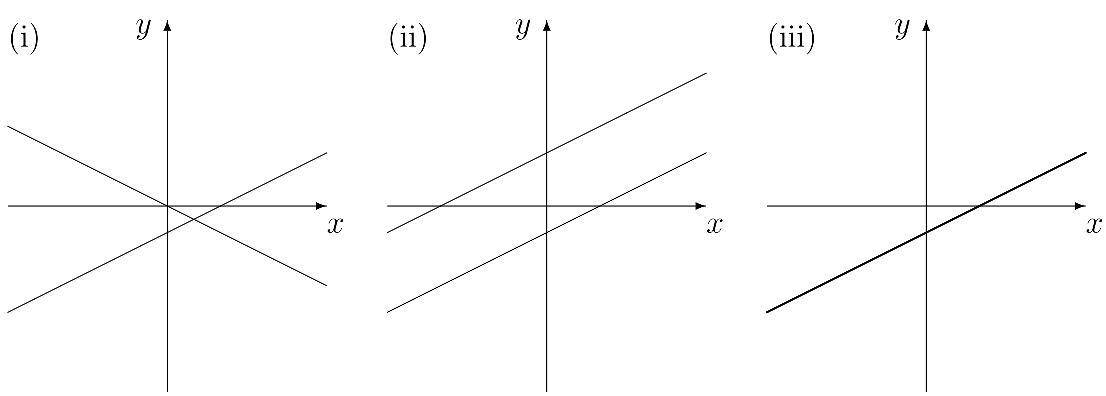
$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n &= b_3 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn} &= b_m
\end{aligned}$$

where the variables are x_1, x_2, $\ldots$, x_n and the coefficients a_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n$, as well as the right hand side values b_i, $i = 1, m$, are given.

The simplest example is a pair of linear equations in two variables, *i.e.*

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 &= b_1 \\
a_{21}x_1 + a_{22}x_2 &= b_2
\end{aligned}$$

Note that this pair of equations simply represents two lines in 2 space. The solution to the equations is given by their intersection. Clearly, one of three possible things can happen:



(i) The lines intersect at a single point, *i.e.* the system has a unique solution.

(ii) The lines are parallel and don't intersect, *i.e.* the system does not have a solution.

(iii) The two equations actually describe the same line. In other words the two lines coincide completely and any point along them is a solution, *i.e.* the system has infinitely many solutions.

Although the example of just two equations in two unknowns appears elementary, it turns out that all systems of linear equations lead to one of these three scenarios, *i.e.* they may have a unique solution, they may have infinitely many solutions or they may have no solution at all. In particular, note how it is not possible for a linear system of equations to have a finite (*e.g.* 2 or 3) number of solutions. If solutions exists, there's either only one of them or infinitely many of them. See the exercises for the case of equations in 3 unknowns.

Now to the issue of how to determine solutions for systems of linear equations. Let us start with an example of 3 equations in three unknowns.

$$\begin{aligned} 2x_1 + 4x_2 - 3x_3 &= 1 \\ x_1 + x_2 + 2x_3 &= 9 \\ 3x_1 + 6x_2 - 5x_3 &= 0 \end{aligned}$$

Swapping the first and second equations, we have

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 9 \\ 2x_1 + 4x_2 - 3x_3 &= 1 \\ 3x_1 + 6x_2 - 5x_3 &= 0 \end{aligned}$$

Next, we subtract 2 lots of the first equation from the second and 3 lots of the first equation from the third to obtain

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 9 \\ 2x_2 - 7x_3 &= -17 \\ 3x_2 - 11x_3 &= -27 \end{aligned}$$

Notice how this has eliminated x_1 from the last two equations. Next, we multiply the second equation by $\frac{1}{2}$ to obtain

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 9 \\ x_2 - \frac{7}{2}x_3 &= -\frac{17}{2} \\ 3x_2 - 11x_3 &= -27 \end{aligned}$$

We subtract 3 times the second equation from the third.

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 9 \\ x_2 - \frac{7}{2}x_3 &= -\frac{17}{2} \\ -\frac{1}{2}x_3 &= -\frac{3}{2} \end{aligned}$$

Then, we finish by multiplying the third equation by -2,*i.e.*

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 9 \\ x_2 - \frac{7}{2}x_3 &= -\frac{17}{2} \\ x_3 &= 3 \end{aligned}$$

This gives the solution for x_3. Substituting into the second equation, we have $x_2 = -\frac{17}{2} + \frac{7}{2}x_3 = -\frac{17}{2} + \frac{7}{2}(3) = 2$. Substituting the x_2 and x_3 values into the first equation, we get $x_1 = 9 - x_2 - 2x_3 = 9 - (2) - 2(3) = 1$.

The process we used in solving these equations consisted of three basic operations:

(i) We could multiply an equation by a nonzero constant.

(ii) We could add a multiple of one equation to another.

(iii) We could interchange two equations.

Note that none of these operations actually changed the solution set of the system of equations. (Be careful with multiplying an equation by a constant. This constant must be nonzero, otherwise we would wipe out the equation altogether and change the solution set.)

However, it is rather tedious to carry along the variables when we are writing down the equations after each step. Instead, it pays to work with a short form of these equations known as an *augmented matrix.*

Firstly, notice that the general system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n &= b_3 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn} &= b_m \end{aligned}$$

can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix},$$

i.e. $A\boldsymbol{x} = \boldsymbol{b}$, where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$.

The augmented matrix of the system then is

$$[A|\,\boldsymbol{b}] = \left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right],$$

i.e. it's just the A matrix with the $\boldsymbol{b}$ vector appended to the right hand side. Note how the augmented matrix contains all of the essential information to allow us to write down the original set of equations.

If the augmented matrix has a certain *row echelon form*, it will be easy to determine the solution(s) of the systems of equations, as we will see below.

For any matrix, the first non zero entry in a particular row is called the leading entry of that row. *e.g.* the leading entries in the matrix below are indicated in green:

$$\begin{bmatrix} 2 & 1 & 4 & 7 & 1 \\ 0 & 0 & 5 & 3 & 2 \\ 0 & 7 & 4 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that a row consisting entirely of zeros does not have a leading entry. A matrix is said to be in row echelon form if the leading entry in each row lies to the right of that in the preceding row. Any row of zeros must appear below the non zero rows.

$$e.g. \qquad \begin{bmatrix} 4 & 2 & 3 & 4 \\ 0 & 6 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

are in row echelon form, while

$$\begin{bmatrix} 2 & 1 & 4 & 7 & 1 \\ 0 & 0 & 5 & 3 & 2 \\ 0 & 7 & 4 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 4 & 3 & 6 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 5 & 0 & 3 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & -3 & 2 & 0 & 1 \\ 0 & 0 & 3 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

are not in row echelon form.

What is so important about a row echelon form? It turns out that if the augmented matrix $[A|\boldsymbol{b}]$ for a system $A\boldsymbol{x} = \boldsymbol{b}$ is in row echelon form, we can easily determine the solutions of the system (or detect the case of no solutions). For example, suppose

$$[A|\boldsymbol{b}] = \left[\begin{array}{ccc|c} 2 & -1 & 0 & 3 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

This matrix clearly belongs to a system of 3 equations in 3 unknowns (let's call them x_1, x_2 and x_3 to be consistent with our earlier notation). The last line in the matrix corresponds to the equation $x_3 = 3$, immediately giving us part of the solution. The second last line corresponds to $-x_2 + x_3 = 2$. Hence, $x_2 = x_3 - 2 = 3 - 2 = 1$. Finally, the first line corresponds to $2x_1 - x_2 = 3$, *i.e.* $2x_1 = 3 + x_2 = 3 + 1 = 4$ and $x_1 = 2$. This process of determining the variables is known as back substitution, but it is only possible if $[A|\boldsymbol{b}]$ is in row echelon form.

Clearly, for a given linear system, $[A|\boldsymbol{b}]$ will not always be in row echelon form. Instead, we usually need to first apply the process of *Gaussian elimination* to $[A|\boldsymbol{b}]$.

Gaussian Elimination: Reduce $[A|\,\boldsymbol{b}]$ to row echelon form.

How is this reduction to be carried out? We are allowed to use three of types of elementary row operations (e.r.o.'s):

(i) We can multiply a row by a non zero constant.

(ii) We can add a multiple of one row to another.

(iii) We can interchange two rows.

Note that these operations on the rows of the augmented matrix are equivalent to the three types of equation manipulations we used in our first example today. Hence, **they do not change the solutions (or non-solvability) of the underlying system!**

Before we go into the details of an example of Gaussian elimination, some explanation of notation is in order. We are going to manipulate $[A|\boldsymbol{b}]$ in a number of steps, each involving one or more elementary row operations. The matrices generated at each step are said to be *row equivalent*. In general, two matrices A and B are row equivalent if one can be obtained from the other by using one or more e.r.o.'s. We write $A \sim B$ in this case. In addition, it is good practice to indicate the details of a particular row operation at the side of a matrix to allow readers to check your working.

<u>Ex:</u> Solve the system

$$\begin{array}{rcl} 2x_1 + 4x_2 - 3x_3 & = & 1 \\ x_1 + x_2 + 2x_3 & = & 9 \\ 3x_1 + 6x_2 - 5x_3 & = & 0 \end{array}$$

Soln: The augmented matrix is $\left[\begin{array}{ccc|c} 2 & 4 & -3 & 1 \\ 1 & 1 & 2 & 9 \\ 3 & 6 & -5 & 0 \end{array}\right] \begin{array}{l} \mathrm{R}_1 \to \mathrm{R}_2 \\ \mathrm{R}_2 \to \mathrm{R}_1 \\ \end{array}$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right] \begin{array}{l} \\ \mathrm{R}_2 \to \mathrm{R}_2 - 2\mathrm{R}_1 \\ \mathrm{R}_3 \to \mathrm{R}_3 - 3\mathrm{R}_1 \end{array} \quad \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array}\right] \mathrm{R}_2 \to \tfrac{1}{2}\mathrm{R}_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{array}\right] \mathrm{R}_3 \to \mathrm{R}_3 - 3\mathrm{R}_2 \quad \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{array}\right] \mathrm{R}_3 \to -2\mathrm{R}_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{array}\right] \quad \textit{i.e.} \quad x_3 = 3, \quad x_2 - \frac{7}{2}x_3 = -\frac{17}{2}, \quad x_2 = \frac{7}{2}x_3 - \frac{17}{2} = \frac{7}{2}(3) - \frac{17}{2} = 2,$$

and $x_1 + x_2 + 2x_3 = 9$, *i.e.* $x_1 = -x_2 - 2x_3 + 9 = -(2) - 2(3) + 9 = 1$ completes the solution.

Linear Dependence/Independence

Before looking into the details of linear systems again, we first introduce the ideas of *linear dependence/independence* and the *rank* of a matrix. We will also use the idea of linear independence in the case of describing infinitely many solutions to a system of linear equations.

Consider a set of vectors $\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_n\}$ and the equation

$$c_1\boldsymbol{r}_1 + c_2\boldsymbol{r}_2 + \ldots + c_n\boldsymbol{r}_n = \mathbf{0}$$

where c_1, c_2, …, c_n are scalars. If the only solution to the equation is $c_1 = c_2 = \ldots = c_n = 0$, then we say the set of vectors is linearly independent. If there is a solution to the equation with at least one $c_i \neq 0$, then the set is linearly dependent.

The practical meaning of these phrases is quite simple. In a linearly dependent set of vectors, it will always be possible to express one of the vectors as a combination of the others. For a linearly independent set, this cannot be done.

Ex: Check that $\{\boldsymbol{r}_1, \boldsymbol{r}_2\}$ is linearly independent, where $\boldsymbol{r}_1 = [3, 2]$ and $\boldsymbol{r}_2 = [0, 4]$.

Soln: $c_1\boldsymbol{r}_1 + c_2\boldsymbol{r}_2 = \mathbf{0}$ gives $c_1[3, 2] + c_2[0, 4] = [0, 0]$. Writing down the equations for the individual components, we have $3c_1 = 0$, *i.e.* $c_1 = 0$ and $2c_1 + 4c_2 = 0$, *i.e.* $4c_2 = 0$, so $c_2 = 0$. As the only solution to the vector equation was $c_1 = c_2 = 0$, the vectors are linearly independent.

Note that a set of just two vectors is easy to analyze. If the vectors are parallel (*i.e.* a scalar multiple of one another), they are linearly dependent. Otherwise, they are linearly independent.

Ex: Show that $\{\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3\}$ is linearly dependent, where $\boldsymbol{r}_1 = [1, 1, 2]$, $\boldsymbol{r}_2 = [1, 0, 0]$ and $\boldsymbol{r}_3 = [0, 1, 2]$.

Soln: By inspection, $\boldsymbol{r}_1 = \boldsymbol{r}_2 + \boldsymbol{r}_3$, *i.e.* $(1)\boldsymbol{r}_1 + (-1)\boldsymbol{r}_2 + (-1)\boldsymbol{r}_3 = \mathbf{0}$, *i.e.* the equation $c_1\boldsymbol{r}_1 + c_2\boldsymbol{r}_2 + c_3\boldsymbol{r}_3 = \mathbf{0}$ is satisfied for at least one nonzero set of c_1, c_2 and c_3 values, so the vectors must be linearly dependent.

Exercise: Show that $\boldsymbol{i} = [1, 0, 0]$, $\boldsymbol{j} = [0, 1, 0]$ and $\boldsymbol{k} = [0, 0, 1]$ form a linearly independent set. (This is part of the reason we refer to these as *basis* vectors.)

The rank of a matrix A, denoted $\mathrm{r}(A)$, is simply the number of linearly independent rows in A.

Unless A has just two rows (in which case we just check whether they are scalar multiples of each other or not), it is usually difficult to determine $\mathrm{r}(A)$ by inspection. *e.g.* let

$$A = \begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 3 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}$$

Given our earlier examples, all we can say at this stage is that $\mathrm{r}(A) = 2$, $\mathrm{r}(B) < 3$ and $\mathrm{r}(C) \leq 3$. Clearly, we need a systematic way to determine ranks of matrices.

Although we won't concern ourselves with the details here, it can be shown that

- The rank of a matrix in row echelon form is equal to the number of nonzero rows.
- Elementary row operations <u>do not change the rank of a matrix</u>.

Ex: Find the rank of

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 & 2 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Soln: As A is in row echelon form, $\mathrm{r}(A) = \#$ of nonzero rows $= 3$.

For a matrix which is not in row echelon form, we first use e.r.o.'s to reduce it to row echelon form and then simply count the number of nonzero rows.

Ex: Find the rank of

$$B = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 5 & 3 \\ 3 & 8 & 4 \end{bmatrix}$$

Soln: $B = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 5 & 3 \\ 3 & 8 & 4 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{bmatrix} R_3 \to R_3 - 2R_2 \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$,
i.e. $\mathrm{r}(B) = 2$.

Recognizing the Nature of Solutions

Consider a general system of m linear equations in n unknowns:

$$A\boldsymbol{x} = \boldsymbol{b}$$

i.e. A is $m \times n$, $\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$.

Let $[A|\,\boldsymbol{b}]$ be the augmented matrix for the system.

Then, one of the following will always occur:

(i) If $\mathrm{r}(A) \neq \mathrm{r}([A|\,\boldsymbol{b}])$, then the equations are inconsistent and there are no solutions.

(ii) If $\mathrm{r}(A) = \mathrm{r}([A|\,\boldsymbol{b}]) = n$, then the equations are consistent and have a unique solution.

(iii) If $\mathrm{r}(A) = \mathrm{r}([A|\,\boldsymbol{b}]) = r < n$, then the equations are consistent and have infinitely many solutions, which we can express in terms of $n - r$ parameters ...see below.

Ex: Solve, if possible:

$$\begin{array}{rrrcr} 2x_1 & +3x_2 & -x_3 & = & 1 \\ 4x_1 & +6x_2 & +x_3 & = & 0 \\ 2x_1 & +3x_2 & +2x_3 & = & 2 \end{array}$$

Soln: $[A|\,\boldsymbol{b}] = \left[\begin{array}{ccc|c} 2 & 3 & -1 & 1 \\ 4 & 6 & 1 & 0 \\ 2 & 3 & 2 & 2 \end{array}\right] \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \sim \left[\begin{array}{ccc|c} 2 & 3 & -1 & 1 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 3 & 1 \end{array}\right] R_3 \to R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 2 & 3 & -1 & 1 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 3 \end{array}\right].$$

Clearly, $\mathrm{r}(A) = 2$ and $\mathrm{r}([A|\,\boldsymbol{b}]) = 3 \neq \mathrm{r}(A)$. Hence, the equations are inconsistent and there is no solution. This can also be seen when we rewrite the last line of the augmented matrix as $(0)x_1 + (0)x_2 + (0)x_3 = 3$, *i.e.* $0 = 3$ which indicates the inconsistency more clearly.

Ex: Solve, if possible:

$$\begin{array}{rrrrr} x_1 & +3x_2 & +2x_3 & = & 3 \\ 2x_1 & +x_2 & -x_3 & = & 6 \\ 3x_1 & -x_2 & +x_3 & = & 4 \end{array}$$

Soln: $[A|\,\boldsymbol{b}] = \left[\begin{array}{ccc|c} 1 & 3 & 2 & 3 \\ 2 & 1 & -1 & 6 \\ 3 & -1 & 1 & 4 \end{array}\right] \begin{array}{l} \\ \mathrm{R}_2 \to \mathrm{R}_2 - 2\mathrm{R}_1 \\ \mathrm{R}_3 \to \mathrm{R}_3 - 3\mathrm{R}_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 3 & 2 & 3 \\ 0 & -5 & -5 & 0 \\ 0 & -10 & -5 & -5 \end{array}\right] \mathrm{R}_3 \to \mathrm{R}_3 - 2\mathrm{R}_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 2 & 3 \\ 0 & -5 & -5 & 0 \\ 0 & 0 & 5 & -5 \end{array}\right].$$

Here, $\mathrm{r}(A) = \mathrm{r}([A|\,\boldsymbol{b}]) = 3$ and we get a unique solution. Solving by back substitution, $5x_3 = -5$ gives $x_3 = -1$. $-5x_2 - 5(-1) = 0$, *i.e.* $x_2 = 1$. $x_1 + 3(1) + 2(-1) = 3$, *i.e.* $x_1 = 2$.

The case of infinitely many solutions requires more care, as we can see in the example below.

Ex: Solve, if possible:

$$\begin{array}{rrrrrr} x_1 & +2x_2 & +x_3 & +4x_4 & = & 1 \\ x_1 & +2x_2 & +2x_3 & +6x_4 & = & 2 \\ 2x_1 & +4x_2 & +5x_3 & +14x_4 & = & 5 \\ 3x_1 & +6x_2 & +7x_3 & +20x_4 & = & 7 \end{array}$$

Soln: $[A|\,\boldsymbol{b}] = \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 1 \\ 1 & 2 & 2 & 6 & 2 \\ 2 & 4 & 5 & 14 & 5 \\ 3 & 6 & 7 & 20 & 7 \end{array}\right] \begin{array}{l} \\ \mathrm{R}_2 \to \mathrm{R}_2 - \mathrm{R}_1 \\ \mathrm{R}_3 \to \mathrm{R}_3 - 2\mathrm{R}_1 \\ \mathrm{R}_4 \to \mathrm{R}_4 - 3\mathrm{R}_1 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 3 & 6 & 3 \\ 0 & 0 & 4 & 8 & 4 \end{array}\right] \begin{array}{l} \\ \\ \mathrm{R}_3 \to \mathrm{R}_3 - 3\mathrm{R}_2 \\ \mathrm{R}_4 \to \mathrm{R}_4 - 4\mathrm{R}_2 \end{array}$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Here, we have 4 variables, *i.e.* $n = 4$. Also, from the final augmented matrix, we can see that $\mathrm{r}(A) = \mathrm{r}([A|\,\boldsymbol{b}]) = r = 2 < n$, *i.e.* the equations are consistent, but the system has infinitely many solutions.

Before describing the solution set, we separate the variables as follows. Notice that each of the first 4 columns in the augmented matrix is associated with a corresponding variable (*e.g.* column 1 with x_1, column 2 with x_2 *etc.*). In the final augmented matrix, the leading entries (recall: leading entry = first nonzero entry in a row) clearly occur in columns 1 and 3. Hence, we refer to x_1 and x_3 as the leading variables of the system.

Consequently, x_2 and x_4 are called the non-leading variables, since their corresponding columns do not contain a leading entry.

Since we have infinitely many solutions, we describe these in terms of a set of *parameters*. The number of parameters required is equal to $n - r = 4 - 2 = 2$ in this case. Now, *the convention is to assign these parameters to the non-leading variables.* The names chosen for the parameters are up to you, but we typically use s, t, *etc.* . In this example, since x_2 and x_4 are the non-leading variables, we put $x_2 = s$ and $x_4 = t$. It then remains to express the other variables in terms of s and t.

From the second line in the final augmented matrix, we obtain $x_3 + 2x_4 = 1$, *i.e.* $x_3 = 1 - 2x_4$, *i.e.* $x_3 = 1 - 2t$. Similarly, from the first line, $x_1 + 2x_2 + x_3 + 4x_4 = 1$, *i.e.* $x_1 = 1 - 2x_2 - x_3 - 4x_4 = 1 - 2s - (1 - 2t) - 4t = -2s - 2t$.

Finally, in the case of infinitely many solutions, it is useful to write them in vector form as follows:

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s - 2t \\ s \\ 1 - 2t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

Any combination of values for s and t will generate a solution $\boldsymbol{x}$ for this system. Since there are infinitely many possible values for s and t, we get infinitely many solutions.

Ex: Determine the solution set for the system $A\boldsymbol{x} = \boldsymbol{b}$ with the augmented matrix

$$[A|\boldsymbol{b}] = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Soln: A lot of students get mixed up with the column of zeros (associated with x_2) and simply put $x_2 = 0$. This is not correct. The zeros in the second column simply mean that x_2 does not actually appear in any of the equations, *i.e.* it is an unrestricted variable and can take on any value! In the context of the terminology we used in the previous example, note that column 2 does not have a leading entry. Hence, x_2 is a non-leading variable and should be assigned a parameter (which reflects the fact that it can take on any real value), *i.e.* set $x_2 = s$. As leading entries do appear in columns 1 and 3, x_1 and x_3 are leading variables and we need to solve for them. In this case, we have $x_3 = 2$ and $x_1 = 3 - 2x_3 = 3 - 4 = -1$. In vector form, the solution may be written as

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ s \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Some Comments on Gaussian Elimination

We've used Gaussian elimination in a number of examples now. As you look back through these, take note of the order in which we applied e.r.o.'s:

- Interchanging of rows is usually only needed when we want a convenient leading entry (typically a 1) to appear in the row which we use to manipulate the remaining rows. This leading entry is often called a pivot.
- We can apply more than one e.r.o. at a time, as long as two different rows are being manipulated. Beware of performing two e.r.o.'s which change the same row (*e.g.* adding a multiple of one row to another whilst swapping that row with a third row).
- Work on one column at a time, reducing all entries below a leading entry in that column to zero. Once done, don't come back to that column later, but move to the next column on the right. This way, you won't end up applying e.r.o.'s which get you away from the row echelon form.
- **Never use elementary column operations!** Unlike e.r.o.'s, these will change the solution set of your system of equations. Computer software solving linear systems of equations does actually use column swaps to improve the numerical stability of the solution process, but it also keeps track of the corresponding swaps in variables (which is too tedious for hand calculations).