Q 1). (a). $\overrightarrow{AB} = [2-(-1), 0-1, -3-2, 3-4] = [3, -1, -5, -1]$ 1

Dist $= \|\overrightarrow{AB}\| = \sqrt{(3)^2+(-1)^2+(-5)^2+(-1)^2}$ ½ $= \sqrt{36} = 6$ ½

(b). $a.b = [2,-1,4].[3,5,-1] = 6-5-4$ 1 $= -3$ 1

(c). $\theta = \cos^{-1}\left(\dfrac{[3,1,1,-1].[2,0,-4,1]}{\sqrt{3^2+1^2+1^2+(-1)^2}\sqrt{2^2+0^2+(-4)^2+1^2}}\right)$ 2

$= \cos^{-1}\left(\dfrac{6+0-4-1}{\sqrt{9+1+1+1}\sqrt{4+0+16+1}}\right) = \cos^{-1}\left(\dfrac{1}{\sqrt{12}\sqrt{21}}\right)$ 1 1 1

$\approx 86.39°$

(d). $\hat{e} = \dfrac{e}{\|e\|} = \dfrac{[3,2,-1]}{\sqrt{3^2+2^2+(-1)^2}}$ ½ $= \left[\dfrac{3}{\sqrt{14}}, \dfrac{2}{\sqrt{14}}, \dfrac{-1}{\sqrt{14}}\right]$ ½

$\therefore [\cos\alpha, \cos\beta, \cos\gamma] = \left[\dfrac{3}{\sqrt{14}}, \dfrac{2}{\sqrt{14}}, \dfrac{-1}{\sqrt{14}}\right]$ 1

(e). $f \times g = \begin{vmatrix} i & j & k \\ 3 & 2 & 0 \\ -1 & 1 & 1 \end{vmatrix}$ ½ ½ ½

$= i((2)(1)-(0)(1)) - j((3)(1)-(0)(-1)) + k((3)(1)-(2)(-1))$

$= i(2) - j(3) + k(5) = [2, -3, 5]$ 1

$\therefore \dfrac{f \times g}{\|f \times g\|} = \dfrac{[2,-3,5]}{\sqrt{2^2+(-3)^2+5^2}}$ ½ $= \left[\dfrac{2}{\sqrt{38}}, \dfrac{-3}{\sqrt{38}}, \dfrac{5}{\sqrt{38}}\right]$ 1

(f). (i). Vector 1

(ii). Vector 1

(iii). Vector 1

(iv). Scalar 1

(v). Not possible ½ meaningless. Can't add a number $\|c\|$ to a vector $a \times b$ ½

Q2). (a). (i). No $(0,-4,9)$ is not on line. 1

Since no t value that generates point $(0,-4,9)$.

(ii). Yes $(-3,1,0)$ is on line. 1

Since $t=-1$ gives $x=-3,\ y=1,\ z=0$

(b). (i). Yes $(2,1,0)$ is on the plane. 1

Since $(2+2)+4(1-2)-2(0)=0$

(ii) No $(-3,0,-4)$ is not on the plane. 1

Since $(-3+2)+4(0-2)-2(-4)=-1\neq 0$

(c). Direction of line $\underset{\sim}{a}=[-1-(-4),3-0,5-2]=[3,3,3]$ 1

Use point $(-4,0,2)$ or $\underset{\sim}{a}=[-3,-3,-3]$

$\therefore$ Vector equation, $\underset{\sim}{r}=[-4,0,2]+t[3,3,3]$ 2

(d). From, $\dfrac{x+3}{-2}=\dfrac{y+1}{3}$ 1 $\Rightarrow 3(x+3)=-2(y+1)$ 1

$\therefore 3(x+3)+2(y+1)=0$

Also, $\dfrac{x+3}{-2}=\dfrac{z-5}{2}$ 1 $\Rightarrow 2(x+3)=-2(z-5)$ 1

or $\therefore 2(x+3)+2(z-5)=0$

Can also have, $\dfrac{y+1}{3}=\dfrac{z-5}{2}$ 1 $\Rightarrow 2(y+1)=3(z-5)$ or

$\therefore 2(y+1)-3(z-5)=0$ 1

Need any two of the three planes above.

(e). From plane $x+4y-z=6$ we get normal vector,

$\underset{\sim}{n}=[1,4,-1]$ 1

From plane $-2x-y+3z=-6$ we get a point

any (x,y,z) which satisfies plane equation, ie. $(3,0,0)$ 2

$\therefore$ Eqn. of plane,

$(x-3)+4(y-0)-1(z-0)=0$ 1

(f). Show they are parallel

Direction of first line, $\underset{\sim}{a}_1=[2,-1,-3]$ ½

Direction of second line, $\underset{\sim}{a}_2=[-4,2,6]$ ½

Since $\underset{\sim}{a}_1 = m\underset{\sim}{a}_2$ (ie. $m = -\frac{1}{2}$) $\therefore$ parallel. 1

Also need to show they share a common point

Point on second line $(x, y, z) = (-10, 8, 15)$ $\frac{1}{2}$ (ie $t = 0$)

Substituting this point into first line,

$-10 = -4 + 2t \Rightarrow -6 = 2t \Rightarrow t = -3$

$8 = 5 - t \Rightarrow 3 = -t \Rightarrow t = -3$

$15 = 6 - 3t \Rightarrow 9 = -3t \Rightarrow t = -3$ 1

Hence for $t = -3$ we also get the point $(-10, 8, 15)$ $\frac{1}{2}$

$\therefore$ Describe same line, since parallel with common point 1

Q3). (a). Try solving,

$c_1 \underset{\sim}{v_1} + c_2 \underset{\sim}{v_2} + c_3 \underset{\sim}{v_3} = \underset{\sim}{w}$ for c_1, c_2 & c_3.

$$c_1 \begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -3 \end{bmatrix} \quad 1$$

$$\therefore \left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ -4 & 3 & 2 & -1 \\ -2 & 3 & 7 & -3 \end{array}\right] \begin{array}{l} \\ R_2 = R_2 + 2R_1 \\ R_3 = R_3 + R_1 \end{array} \quad 1 \sim \left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & 1 & 4 & -1 \\ 0 & 2 & 8 & -3 \end{array}\right] \begin{array}{l} \\ \\ R_3 = R_3 - 2R_2 \end{array} \quad 1$$

$$\sim \left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & -1 \end{array}\right] \quad 1$$

Since $r(A) = 2 \neq r([A|b])$ 1 $\therefore$ No solution 1

$\therefore$ Cannot be written as a l.c. 1

(b). (i). Since there are more vectors $m=3$ than space $n=1$ (ie. $m>n$) 1 $\therefore$ Set is l.d. 1

(ii). Need to solve $c_1 \underset{\sim}{v_1} + c_2 \underset{\sim}{v_2} + c_3 v_3 = 0$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ -1 & 3 & 2 & 0 \\ 0 & -1 & 0 & 0 \\ 2 & 1 & -3 & 0 \end{array}\right] \begin{array}{l} \\ R_2 = R_2 + R_1 \\ \\ R_4 = R_4 - 2R_1 \end{array} \quad 1 \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -3 & -1 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3 = 5R_3 + R_2 \\ R_4 = 5R_4 + 3R_2 \end{array} \quad 1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 = R_4 + 2R_3 \end{array} \quad 1 \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad 1$$

$r(A) = 3 = m$ 1 $\therefore$ Trivial solution ie $c_1 = c_2 = c_3 = 0$

$\therefore$ Set is l.i. 1

(b)(i). Let $u = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \in U$ ½ since $a = 1 \geq 0$ and $b = -1 \leq 0$

Testing scalar multiplication,

if $s<0$ ie $s = -1$ then $su = -1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \notin U$ ½

As it's not closed under scalar multiplication 1 $\therefore$ Not a subspace (ii). Let $\underset{\sim}{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in V$ ½ and $\underset{\sim}{v} = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} \in V$ ½

(Can choose any two vectors $\underset{\sim}{u}, \underset{\sim}{v} \in V$.)

Testing addition

$$\underset{\sim}{u} + \underset{\sim}{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 9 \end{bmatrix} \notin V \quad (1) \quad \text{since } (3)^2 \neq 5, \ (3)^3 \neq 9$$

As it's not closed under addition (1) $\therefore$ Not a subspace.

4)(a). $[A|\underset{\sim}{0}] = \left[\begin{array}{cccc|c} 1 & 3 & 1 & -2 & 0 \\ -2 & 1 & -2 & 0 & 0 \\ -1 & 4 & -1 & -2 & 0 \end{array}\right] \begin{array}{l} \\ R_2 = R_2 + 2R_1 \\ R_3 = R_3 + R_1 \end{array}$

$\sim \left[\begin{array}{cccc|c} 1 & 3 & 1 & -2 & 0 \\ 0 & 7 & 0 & -4 & 0 \\ 0 & 7 & 0 & -4 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3 = R_3 - R_2 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 3 & 1 & -2 & 0 \\ 0 & 7 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$r(A) = 2 < n = 4$ $\therefore$ Need $n - r = 4 - 2 = 2$ parameters.

Let $x_3 = s$, $x_4 = t$

Row 2: $7x_2 - 4x_4 = 0 \Rightarrow 7x_2 = 4t$ $\therefore x_2 = 4t/7$

Row 1: $x_1 + 3x_2 + x_3 - 2x_4 = 0 \Rightarrow x_1 + 3(4t/7) + s - 2t = 0$

$\therefore x_1 = 2t/7 - s$

(b). $\begin{vmatrix} 0 & 2 & 0 & 0 & 1 \\ 4 & 0 & -2 & 0 & 0 \\ 0 & -3 & 0 & 2 & 0 \\ 6 & 0 & 0 & 0 & -2 \\ 0 & 0 & 3 & 2 & 0 \end{vmatrix}$

Cofactor expansion along 1st row

$= -2 \begin{vmatrix} 4 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 6 & 0 & 0 & -2 \\ 0 & 3 & 2 & 0 \end{vmatrix} + 1 \begin{vmatrix} 4 & 0 & -2 & 0 \\ 0 & -3 & 0 & 2 \\ 6 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 \end{vmatrix}$

Cofactor exp along 2nd row   Cofactor exp. along 3rd row

$= -2\left[-2 \begin{vmatrix} 4 & -2 & 0 \\ 6 & 0 & -2 \\ 0 & 3 & 0 \end{vmatrix}\right] + 1\left[6 \begin{vmatrix} 0 & -2 & 0 \\ -3 & 0 & 2 \\ 0 & 3 & 2 \end{vmatrix}\right]$

Cofactor exp along 3rd row   Cofactor exp along 1st row

$= 4\left[-3 \begin{vmatrix} 4 & 0 \\ 6 & -2 \end{vmatrix}\right] + 6\left[-(-2) \begin{vmatrix} -3 & 2 \\ 0 & 2 \end{vmatrix}\right]$

$= -12(-8 - 0) + 12(-6 - 0)$

$= 96 - 72$

$= 24$

(c). $A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 2 \\ -1 & 3 & 1 \end{bmatrix}$ ½

Cofactor expansion along 1st row

$\det(A) = 1\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 2 \\ -1 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix}$ 1

$= 1(1-6) + 1(2-(-2)) + 3(6-(-1))$ ½

$= 1(-5) + 1(4) + 3(7)$

$= 20$ ½

$A_1 = \begin{bmatrix} -6 & -1 & 3 \\ 2 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$ ½

Cofactor expansion along 1st row

$\det(A_1) = -6\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix}$ 1

$= -6(1-6) + 1(2-4) + 3(6-2)$ ½

$= -6(-5) + 1(-2) + 3(4)$

$= 40$ ½

$\therefore\ x_1 = \dfrac{\det(A_1)}{\det(A)} = \dfrac{40}{20} = 2$ 1

Q 5) (a).

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 3 & 8 & 2 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 3R_1 \end{array}$$ 1

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 5 & -1 & -3 & 0 & 1 \end{array}\right] \; R_3 = R_3 - 5R_2$$ 1

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 7 & -5 & 1 \end{array}\right] \quad \begin{array}{l} R_1 = R_1 + R_3 \\ R_3 \times -1 \end{array}$$ 1

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 8 & -5 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array}\right] \; R_1 = R_1 - R_2$$ 1

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -6 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array}\right]$$ 1

$$\therefore \underset{\sim}{x} = A^{-1}\underset{\sim}{b} = \begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 13 \end{bmatrix} = \begin{bmatrix} 20-30+13 \\ -4+5+0 \\ -14+25-13 \end{bmatrix}$$ 1

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$ 1

(b).

$$A = \begin{bmatrix} 1 & -2 & 4 & -8 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{bmatrix}$$ 1

$$A^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \\ -8 & -1 & 0 & 1 & 8 \end{bmatrix}$$ 1

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \\ -8 & -1 & 0 & 1 & 8 \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 & -8 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{bmatrix} \; \tfrac{1}{2} \; = \begin{bmatrix} 5 & 0 & 10 & 0 \\ 0 & 10 & 0 & 34 \\ 10 & 0 & 34 & 0 \\ 0 & 34 & 0 & 130 \end{bmatrix}$$ 1

$$A^T\underset{\sim}{b} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \\ -8 & -1 & 0 & 1 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \\ 2 \\ 0 \end{bmatrix} \; \tfrac{1}{2} \; = \begin{bmatrix} 1 \\ 2 \\ 4 \\ -4 \end{bmatrix}$$ 1

$$[A^TA \mid A^T\underline{b}] = \left[\begin{array}{cccc|c} 5 & 0 & 10 & 0 & 1 \\ 0 & 10 & 0 & 34 & 2 \\ 10 & 0 & 34 & 0 & 4 \\ 0 & 34 & 0 & 130 & -4 \end{array}\right] \quad 1$$

$R_3 = R_3 - 2R_1$

$R_4 = 10R_4 - 34R_2$

$$\sim \left[\begin{array}{cccc|c} 5 & 0 & 10 & 0 & 1 \\ 0 & 10 & 0 & 34 & 2 \\ 0 & 0 & 14 & 0 & 2 \\ 0 & 0 & 0 & 144 & -108 \end{array}\right] \quad 2$$

Row 4: $144a_3 = -108 \Rightarrow a_3 = -108/144 = -3/4$ 1

Row 3: $14a_2 = 2 \Rightarrow a_2 = 2/14 = 1/7$ 1

Row 2: $10a_1 + 34a_3 = 2 \Rightarrow 10a_1 + 34(-3/4) = 2$

$\Rightarrow 10a_1 = 55/2 \Rightarrow a_1 = 11/4$ 1

Row 1: $5a_0 + 10a_2 = 1 \Rightarrow 5a_0 + 10(1/7) = 1$

$\Rightarrow 5a_0 = -3/7 \Rightarrow a_0 = -3/35$ 1