Vectors & Introduction to Matrices

Motivation: The cross product has many uses in engineering problems. In particular, it is fundamental in the understanding of rotational forces and it is routinely used in the description of fluid mechanics problems.

Matrices are fundamental computational tools in all areas of engineering. They also arise in the numerical solution of differential equations which themselves are used to solve many engineering dynamics problems.

Outcomes In today's lecture we will learn how to:

- Calculate the cross product.
- Determine a cross product geometrically.
- Determine areas of parallelograms and volumes of parallelepipeds.
- Recognize a matrix as a 2 dimensional array.
- Add, multiply and transpose matrices.
- Recognize square and symmetric matrices.
- Identify inverse matrices.

Contents

- Introduction of the cross product and an easy to remember rule.
- Geometric meaning of the cross product - the right hand rule.
- Parallelograms and parallelepipeds.
- Scalar and vector triple products.
- Concept of a matrix and its order.
- Matrix addition.
- Scalar multiplication.
- Matrix multiplication.
- Matrix transposition.
- Square matrices.
- Symmetric matrices.
- Inverse matrices.

<u>Exercises</u>

1. For the following pairs of vectors, determine $\boldsymbol{a} \times \boldsymbol{b}$ then show it's orthogonal to both $\boldsymbol{a}$ and $\boldsymbol{b}$.

   (a) $\boldsymbol{a} = [1, 1, -1]$, $\boldsymbol{b} = [2, 4, 6]$ (b) $\boldsymbol{a} = \boldsymbol{j} + 7\boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$

2. Exercise 12(e) (p. 668): 2, 4(see p. 666 of the text), 6.

3. For each of the following expressions, decide whether it makes sense algebraically.

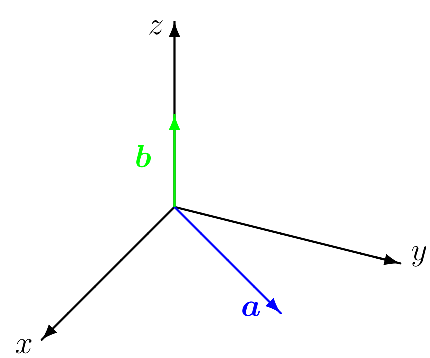
   (a) $\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c})$ (b) $\boldsymbol{a} \times (\boldsymbol{b}.\boldsymbol{c})$ (c) $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$

   (d) $(\boldsymbol{a}.\boldsymbol{b}) \times \boldsymbol{c}$ (e) $(\boldsymbol{a}.\boldsymbol{b}) \times (\boldsymbol{c}.\boldsymbol{d})$ (f) $(\boldsymbol{a} \times \boldsymbol{b}).(\boldsymbol{c} \times \boldsymbol{d})$

4. The figure below shows a vector $\boldsymbol{a}$ in the xy plane and a vector $\boldsymbol{b}$ in the direction of $\boldsymbol{k}$ (*i.e.* in the direction of the z axis). Given that $||\boldsymbol{a}|| = 3$ and $||\boldsymbol{b}|| = 2$,

   (a) Find $||\boldsymbol{a} \times \boldsymbol{b}||$.

   (b) Using the right hand rule, for each component of $\boldsymbol{a} \times \boldsymbol{b}$, decide whether it is positive, negative or zero.



5. Find two unit vectors orthogonal to both $\boldsymbol{a} = [0, 1, -1]$ and $\boldsymbol{b} = [1, 1, 0]$.

6. A parallelogram has the vertices $A(1, 2, 3)$, $B(1, 3, 6)$, $C(3, 8, 6)$ and $D(3, 7, 3)$, determine the area of the parallelogram.

7. A triangle has the vertices $A(0, 0, -3)$, $B(4, 2, 0)$ and $C(3, 3, 1)$, determine the area of the triangle.

8. A parallelepiped is formed by the vectors $\boldsymbol{a} = [1, 1, 0]$, $\boldsymbol{b} = [0, 1, 1]$ and $\boldsymbol{c} = [1, 1, 1]$, determine the volume of the parallelepiped.

9. Show that the four points $A(1, 3, 2)$, $B(3, -1, 6)$, $C(5, 2, 0)$ and $D(3, 6, -4)$ are coplanar.

10. Exercise 11(a) (p. 573): 1.

11. Given the matrices

$$A = \begin{bmatrix} 2 & -5 \\ 0 & 7 \end{bmatrix} \quad B = \begin{bmatrix} 3 & \frac{1}{2} & 5 \\ 1 & -1 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 2 & -\frac{5}{2} & 0 \\ 0 & 2 & -3 \end{bmatrix} \quad D = \begin{bmatrix} 7 & 3 \end{bmatrix}$$

find the following if possible, if it's not possible explain why.

(a) $C - B$ (b) $2A + D$ (c) $D(AB)$

12. Exercise 11(c) (p. 597): 2.(a)-(c).

13. Verify that B is the inverse of A by calculating the products AB and BA.

(a) $A = \begin{bmatrix} 2 & -3 \\ 4 & -7 \end{bmatrix} \quad B = \begin{bmatrix} \frac{7}{2} & -\frac{3}{2} \\ 2 & -1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 1 & -6 \\ 2 & 1 & 12 \end{bmatrix} \quad B = \begin{bmatrix} 9 & -10 & -8 \\ -12 & 14 & 11 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

14. Consider the matrix $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$. Show that $A^2 = A$ (a matrix with this property is called *idempotent* and appears in problems of data analysis). What is A^n for any positive integer n?

15. Diagonal matrices are easy to work with. Let $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Find D^2 and D^3. Make a guess as to what the inverse, D^{-1}, should be. Check your guess by multiplying with D.

16. Show that if A is square, then $(A + A^\top)$ and $AA^\top$ are both symmetric.

These exercises should take around 2 hours to complete.

(Answers: 1.(a) $10\boldsymbol{i} - 8\boldsymbol{j} + 2\boldsymbol{k}$ (b) $11\boldsymbol{i} + 14\boldsymbol{j} - 2\boldsymbol{k}$; 2.(a) Yes (b) No (c) Yes (d) No (e) No (f) Yes; 3.(a) 6 (b) first positive, second negative, third zero; 5. $[\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}]$ & $[-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}]$; 6. $\sqrt{265}$; 7. $\dfrac{\sqrt{86}}{2}$; 8. 1; 11.(a) $\begin{bmatrix} -1 & -3 & -5 \\ -1 & 3 & -6 \end{bmatrix}$ (b) undefined, incompatible dimensions (c) $[28 \;\; 21\; 28]$; 14. $A^n = A$ for any positive integer n; 15. $D^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$, $D^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 27 \end{bmatrix}$, $D^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$)

The Cross Product

The cross product only applies to vectors in 3 space and there is no meaningful equivalent of it in other spaces. It can be easily described as follows.

The cross product of $\boldsymbol{a} = [a_1, a_2, a_3]$ and $\boldsymbol{b} = [b_1, b_2, b_3]$ is

$$\boxed{\boldsymbol{a} \times \boldsymbol{b} = [a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1]}$$

Note the following:

(i) The cross product is itself a vector (note the 3 individual components).

(ii) For this reason, the cross product is also known as the vector product.

(iii) It is only defined for vectors in 3 space.

(iv) $\boldsymbol{a} \times \boldsymbol{b} \neq \boldsymbol{b} \times \boldsymbol{a}$!

Trying to memorize the cross product as it is defined above is difficult, because all three components look similar. Instead, people have developed several simple visual memory aids. One method to find the cross product uses the determinant of a matrix. However since we haven't learnt about matrices and determinants as yet, we instead have the following scheme.

$$\begin{array}{cccccc} + & + & + & - & - & - \\ & \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} & \boldsymbol{i} & \boldsymbol{j} \\ & a_1 & a_2 & a_3 & a_1 & a_2 \\ & b_1 & b_2 & b_3 & b_1 & b_2 \end{array}$$

Briefly put, we construct a 3×3 array, consisting of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ in the first row, the components of $\boldsymbol{a}$ in the second row, and the components of $\boldsymbol{b}$ in the third row. Then extend this to a 5×3 array by repeating the first two columns at the back. Draw the three forward diagonals shown in green as well as three backward diagonals drawn in red. Multiply the entries along each diagonal and add these products according the signs indicated (*i.e.* add the green ones and subtract the red ones). Following this scheme, we get

$$\begin{aligned}
&\boldsymbol{i}(a_2b_3) + \boldsymbol{j}(a_3b_1) + \boldsymbol{k}(a_1b_2) - \boldsymbol{k}(a_2b_1) - \boldsymbol{i}(a_3b_2) - \boldsymbol{j}(a_1b_3) \\
&= \boldsymbol{i}(a_2b_3 - a_3b_2) + \boldsymbol{j}(a_3b_1 - a_1b_3) + \boldsymbol{k}(a_1b_2 - a_2b_1) \\
&= [a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1] \\
&= \boldsymbol{a} \times \boldsymbol{b},
\end{aligned}$$

as required. This approach is actually equivalent to calculating the determinant of a 3×3 matrix, as we will see later.

Ex: Find $\boldsymbol{a} \times \boldsymbol{b}$ if $\boldsymbol{a} = [1, 0, 2]$ and $\boldsymbol{b} = [3, 1, -1]$.

Soln:

$$\begin{array}{cccccc} + & + & + & - & - & - \\ & \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} & \boldsymbol{i} & \boldsymbol{j} \\ & 1 & 0 & 2 & 1 & 0 \\ & 3 & 1 & -1 & 3 & 1 \end{array}$$

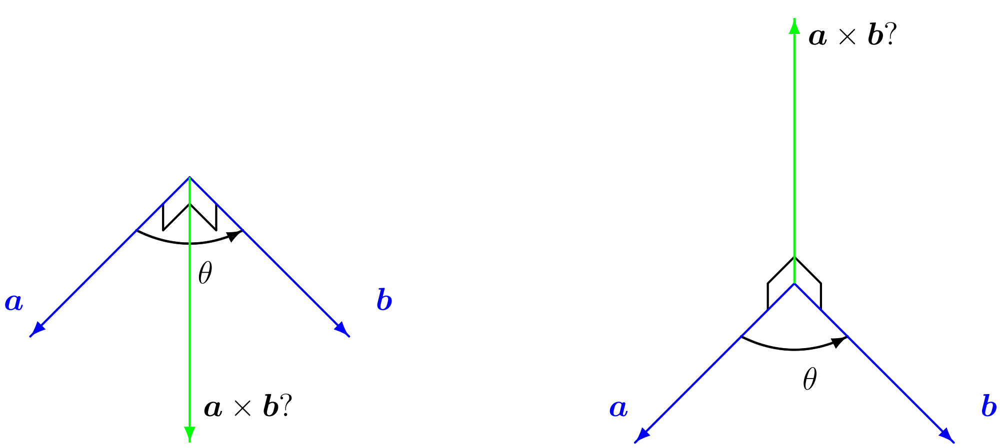
$$\begin{aligned} \boldsymbol{a} \times \boldsymbol{b} &= \boldsymbol{i}((0)(-1) - (1)(2)) + \boldsymbol{j}((2)(3) - (1)(-1)) + \boldsymbol{k}((1)(1) - (0)(3)) \\ &= -2\boldsymbol{i} + 7\boldsymbol{j} + \boldsymbol{k} \\ &= [-2, 7, 1] \end{aligned}$$

To get and idea of the geometric interpretation of the cross product, firstly note that

$$\begin{aligned} (\boldsymbol{a} \times \boldsymbol{b}).\boldsymbol{a} &= [a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1].[a_1, a_2, a_3] \\ &= a_1a_2b_3 - a_1a_3b_2 + a_2a_3b_1 - a_1a_2b_3 + a_1a_3b_2 - a_2a_3b_1 \\ &= 0, \end{aligned}$$

i.e. $\boldsymbol{a} \times \boldsymbol{b}$ is perpendicular to $\boldsymbol{a}$! Similarly, we can show that $(\boldsymbol{a} \times \boldsymbol{b}).\boldsymbol{b} = 0$, *i.e.* $\boldsymbol{a} \times \boldsymbol{b}$ is perpendicular to $\boldsymbol{b}$ as well!

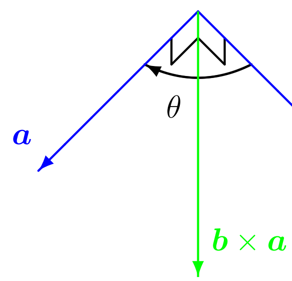
This leaves two possible directions in which $\boldsymbol{a} \times \boldsymbol{b}$ can point:



It turns out the latter is correct. To remember the direction of $\boldsymbol{a} \times \boldsymbol{b}$, we have the following.

RIGHT HAND RULE: If the fingers of your right hand curl in the direction of rotation from $\boldsymbol{a}$ to $\boldsymbol{b}$ (through an angle $0^\circ \leq \theta \leq 180^\circ$), then your extended thumb points in the direction of $\boldsymbol{a} \times \boldsymbol{b}$.

Note that the vector we saw in the first diagram above was actually $\boldsymbol{b} \times \boldsymbol{a}$, *i.e.*



Having worked out the direction of $\boldsymbol{a} \times \boldsymbol{b}$, what is the length of it?

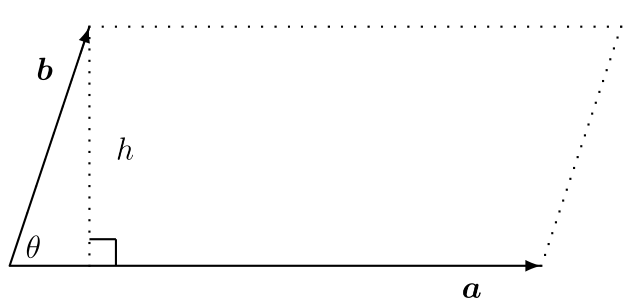
Although a little tedious, it is not difficult to show that

$$\boxed{||\boldsymbol{a} \times \boldsymbol{b}|| = ||\boldsymbol{a}||\,||\boldsymbol{b}||\,\sin\theta}$$

where $0^\circ \leq \theta \leq 180^\circ$ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. Note the symmetry of this formula to the corresponding version for the dot product.

This gives us a way of checking whether two vectors are parallel or not, because parallel means that they have either the same or opposite directions, *i.e.* $\theta = 0^\circ$ or $\theta = 180^\circ$. In either case, $\sin\theta = 0$ and $||\boldsymbol{a} \times \boldsymbol{b}|| = ||\boldsymbol{a}||\,||\boldsymbol{b}||\,\sin\theta = 0$ which, in turn, means that $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}$. In other words, the only way in which the cross product of two vectors can be equal to zero is if they are parallel.

Another interpretation of $||\boldsymbol{a} \times \boldsymbol{b}||$ is the following. Consider the parallelogram formed by $\boldsymbol{a}$ and $\boldsymbol{b}$.

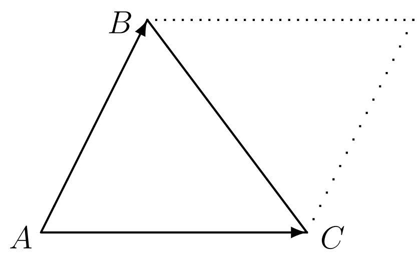


The perpendicular height is given by $h = ||\boldsymbol{b}||\sin\theta$. Now,

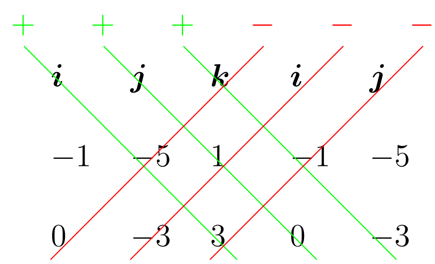
$$||\boldsymbol{a} \times \boldsymbol{b}|| = ||\boldsymbol{a}||\,||\boldsymbol{b}||\,\sin\theta = ||\boldsymbol{a}||\,h = \text{area of parallelogram.}$$

Ex: Find the area of the triangle ABC for points $A(2,4,2)$, $B(1,-1,3)$ and $C(2,1,5)$.

Soln: Note that the area is exactly half the area of the parallelogram formed by $\overrightarrow{AB}$ and $\overrightarrow{AC}$.



We have $\overrightarrow{AB} = [-1,-5,1]$, $\overrightarrow{AC} = [0,-3,3]$,



i.e. $\overrightarrow{AB} \times \overrightarrow{AC} = [-12,3,3]$ and

$$\text{Area } = \frac{1}{2}||\overrightarrow{AB} \times \overrightarrow{AC}|| = \frac{1}{2}\sqrt{(-12)^2+(3)^2+(3)^2} = \frac{9\sqrt{2}}{2}.$$

Here is a summary of other properties of the cross product. Most of these are fairly easy to derive. Some are intuitive, others less so. We won't be making much use of these for the remainder of this unit, but keep these as a reference for work in later units.

(i) $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$

(ii) $(c\boldsymbol{a}) \times \boldsymbol{b} = c(\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{a} \times (c\boldsymbol{b})$

(iii) $\boldsymbol{a} \times (\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}$

(iv) $(\boldsymbol{a}+\boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{c} + \boldsymbol{b} \times \boldsymbol{c}$

(v) $\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}).\boldsymbol{c}$

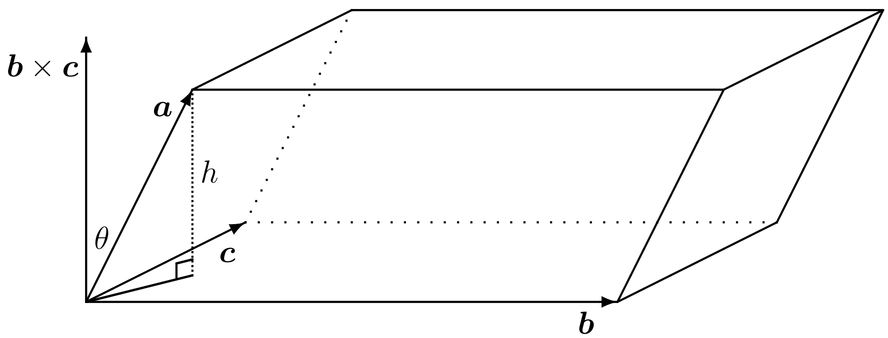
(vi) $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a}.\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}.\boldsymbol{b})\boldsymbol{c}$

Note also that

(i) $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) \neq (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$, in general.

(ii) $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$, $\boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}$ and $\boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$ (check these for yourselves...).

In (v) above, $\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c})$ is called the scalar triple product of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ (since its result is clearly a scalar quantity). It has a nice geometrical interpretation. Consider the *parallelepiped* formed by the triple $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.



Clearly, the area of the base is given by $A = ||\boldsymbol{b} \times \boldsymbol{c}||$, the perpendicular height is $h = ||\boldsymbol{a}|| \cos\theta$ (simply the scalar projection of $\boldsymbol{a}$ on $\boldsymbol{b} \times \boldsymbol{c}$), assuming that $\cos\theta \geq 0$. Finally, the volume of the parallelepiped is given by the area of the base times the perpendicular height, *i.e.*

$$V = A\,h = ||\boldsymbol{b} \times \boldsymbol{c}||\,||\boldsymbol{a}||\,\cos\theta = \boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c}).$$

If $\cos\theta < 0$, this formula gives a negative number (since the body of the parallelepiped appears below the plane). In either case, we can write

$$\boxed{V = |\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c})|}$$

Note the following:

(i) $\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c}) > 0$ for $0^\circ < \theta < 90^\circ$

(ii) $\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c}) < 0$ for $90^\circ < \theta < 180^\circ$

(iii) $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are coplanar (ie. lie in the same plane) if and only if $\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c}) = 0$.

(iv) $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ is called the vector triple product.

Matrices

An $m \times n$ matrix A is a rectangular array of entries (real numbers for our purposes) consisting of m rows and n columns, *i.e.*

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} \end{bmatrix}$$

We say that A is of order (or dimension) $m \times n$. Note that we usually use uppercase letters to denote matrices.

For convenience, we have the following shorthand notation:

$$A = [a_{ij}], \quad 1 \leq i \leq m, \quad 1 \leq j \leq n.$$

In this context, a_{ij} is called the ijth entry of A, *i.e.* the entry which is in row i and column j of the matrix.

Special cases:

$1 \times n$ matrix is a row vector, $[a_1\ a_2\ \ldots\ a_n]$.

$n \times 1$ matrix is a column vector, $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

Ex: Consider the following matrices of various sizes.

(i) $\begin{bmatrix} 1 & 2 \\ 5 & 3 \\ 6 & -5 \end{bmatrix}$ is a 3×2 matrix.

(ii) $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is a 3×1 matrix (ie. a column vector).

(iii) $[4\ \ 0\ \ -1]$ is a 1×3 matrix (ie. a row vector).

(iv) $[3]$ is a 1×1 matrix.

A zero matrix is one where all entries are equal to zero. For example, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a 2×3 zero matrix.
We usually denote a zero matrix simply as 0. Its order will be obvious from the context.

Next, we want to develop the usual algebraic operations for matrices. This first requires the notion of equality.

Two $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal if all their corresponding entries are equal, *i.e.* if

$$a_{ij} = b_{ij}, \qquad 1 \leq i \leq m,\ 1 \leq j \leq n.$$

This implicitly assumes that the matrices have the same order, of course. *e.g.* $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 4 & -1 \end{bmatrix}$ is not equal to $\begin{bmatrix} 2 & 0 \\ 1 & 4 \\ 1 & -1 \end{bmatrix}$, because the matrices are of different orders.

<u>**Ex:**</u> Find the values of a and b, given that $\begin{bmatrix} a+b & 4 \\ 0 & a-b \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix}$.

<u>**Soln:**</u> Clearly, we must have $a + b = 1$ and $a - b = 3$. Adding the two equations, we have $a = 2$ and then $b = -1$.

<u>Operations on Matrices</u>

1. **Matrix Addition:** We can only add matrices of the same order. If $A = [a_{ij}]$ and $B = [b_{ij}]$, then

$$C = A + B = [c_{ij}] = [a_{ij} + b_{ij}]$$

   for all i and j. *e.g.*

$$\begin{bmatrix} 3 & 0 \\ 4 & -2 \\ 1 & 6 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ -4 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & -5 \\ 1 & 6 \end{bmatrix} + \begin{bmatrix} 3 & 4 & 5 \\ 1 & -1 & 2 \end{bmatrix} = \text{ d.n.e.}$$

2. **Scalar Multiplication:** If $A = [a_{ij}]$, then for any scalar k,

$$k\,A = k[a_{ij}] = [k\,a_{ij}]$$

i.e. each entry of the matric gets multiplied by the scalar.

e.g. $2\begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 4 & 2 \\ 10 & -6 \end{bmatrix}$.

3. **Matrix Multiplication:** The matrix product AB is only defined if the number of columns of A is equal to the number of rows of B. In that case

$$A_{m\times n}\, B_{n\times p} = C_{m\times p}$$

If $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = [c_{ij}]$, then

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

i.e. c_{ij} is obtained from the i-th row in A and the j-th column in B. In fact, note how c_{ij} is obtained by effectively taking the dot product of the i-th row in A and the j-th column in B.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \ldots & b_{np} \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & c_{ij} \\ & & \end{bmatrix}$$

<u>**Ex:**</u> Let $A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & -3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 3 & 4 \end{bmatrix}$. Find AB and BA.

<u>**Soln:**</u> $AB = \begin{bmatrix} 3 & 2 \\ 16 & 17 \end{bmatrix}$. $BA = \begin{bmatrix} 3 & -4 & 4 \\ -2 & 1 & 0 \\ 10 & -15 & 16 \end{bmatrix}$.

Clearly, in general $AB \neq BA$. This may be due to a number of different reasons:

- Either product may not exist because the orders don't match.
- AB and BA may be of different order, as in the case above.
- AB and BA may be the same order, but still not necessarily equal. *e.g.* if $A = \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$, then $AB = \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 6 & 6 \end{bmatrix}$ and $BA = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 3 & 3 \end{bmatrix} \neq AB$.

Other valid properties of matrix multiplication are these:

(i) $A(B+C) = AB + AC$

(ii) $(A+B)C = AC + BC$

(iii) (AB)C=A(BC)

But beware of the following:

(a) $AB = 0$ does not necessarily mean $A = 0$ or $B = 0$.
eg. $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -\frac{1}{2} \\ -1 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(b) $AB = AC$ does not necessarily mean $B = C$. *e.g.* $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $AB = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = AC$, but $B \neq C$.

Because matrices are more complicated objects than scalars, there are more manipulations we can perform on them and more interesting ways of classifying them. For example, the transpose of an $m \times n$ matrix $A = [a_{ij}]$ is the matrix $A^\top = [a'_{ij}]$ where $a'_{ij} = a_{ji}$. Taking the transpose effectively means interchanging the rows and columns of the matrix.

<u>Ex:</u> $A = \begin{bmatrix} -1 & 0 & 2 & 3 \\ 1 & 5 & -1 & 4 \\ 4 & 0 & 2 & 1 \end{bmatrix}$, $A^\top = \begin{bmatrix} -1 & 1 & 4 \\ 0 & 5 & 0 \\ 2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$.

A matrix A is said to be square of order n if it is of order $n \times n$. In other words, the number of rows and columns must be the same. In this case, we can identify a main diagonal:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

Notice that the usual exponentiation is possible for square matrices. For example, we may write $A^2 = AA$, or $A^m = \underbrace{AA\ldots A}_{m \text{ times}}$ in general, for a square matrix A.

Finally, we say that a square matrix A is symmetric if $A^\top = A$. Note that A must be square for this to be possible. Symmetric matrices arise frequently in applications and computational algorithms and there are special techniques for manipulating them.

<u>Ex:</u> $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 5 \\ -1 & 5 & 3 \end{bmatrix} = A^\top$ is symmetric.

Note that if $A = [a_{ij}]$ is symmetric, then $a_{ij} = a_{ji}$. In other words, the entries above the main diagonal are a mirror image of the ones below.

In practice, we often have to find the transpose of a product of two or more matrices. Suppose that $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of orders $m \times n$ and $n \times p$ respectively so that AB is defined. Letting $C = [c_{ij}] = AB$, we have $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$ as we saw earlier. Now we want to work out $(AB)^\top = C^\top$. Letting $C^\top = [c'_{ij}]$, we have

$$c'_{ij} = c_{ji} = \sum_{k=1}^{n} a_{jk}b_{ki} = \sum_{k=1}^{n} b_{ki}a_{jk} = \sum_{k=1}^{n} b'_{ik}a'_{k}j,$$

where $[b'_{ik}] = B^\top$ and $[a'_{kj}] = A^\top$, *i.e.*

$$\boxed{(AB)^\top = B^\top A^\top}$$

Satisfy yourself that the following are also true.

$$\boxed{(A^\top)^\top = A} \qquad \boxed{(A+B)^\top = A^\top + B^\top}$$

Identities and Inverses

An identity matrix is a square matrix (*i.e.* of order $n \times n$) with all main diagonal entries equal to 1 and all other entries equal to 0.

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

e.g. $I_1 = [1]$, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *etc.*

Note that I is the *identity element* of matrix multiplication, *i.e.* for $A_{m\times n}$,

$$AI_n = A \qquad \text{and} \qquad I_m A = A.$$

If A is square and of order n, and if there exists a B such that

$$AB = BA = I_n,$$

then we say that A is invertible, we call B the inverse of A and we write $B = A^{-1}$ to denote the inverse.

<u>Ex:</u> Verify that $B = A^{-1}$ if $A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$.

<u>Soln:</u> $AB = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $BA = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
In fact, we really only need to check one of these equations.

Clearly, if B is an inverse of A, then A is an inverse of B. Also, if the inverse of a matrix exists, then it is unique. *i.e.* assume that B and C are inverses of A. Then $AB = I$ and $CA = I$ and $B = IB = (CA)B = C(AB) = CI = C$.

Note that not all square matrices are invertible. We'll look at invertibility and calculating inverses in more detail later.