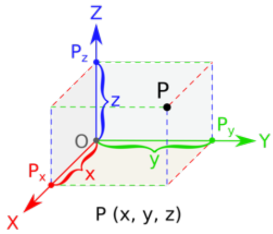
Trigonometric Ratios

	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$\mathrm{n.d.}$

3-Dimensional Cartesian Plane



Lecture 5: Vectors

Vector Operations

- The **negative vector of a vector $\boldsymbol{u}$** is denoted by $-\boldsymbol{u}$ and it has the same length as $\boldsymbol{u}$ but is opposite in direction
- In either case, the magnitude (or length) of $c\boldsymbol{a}$ is the length of c multiplied by the length of $\boldsymbol{a}$. i.e. $\|c\boldsymbol{a}\| = |c|\|\boldsymbol{a}\|$
- For any vector $\boldsymbol{a} \neq \boldsymbol{0}$, the **unit vector** in the direction of $\boldsymbol{a}$ is denoted by $\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|}$
- Finally, notice that $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel if and only if $\boldsymbol{a} = c\boldsymbol{b}$ for some scalar

Analytic Representation

- Consider the Cartesian plane with origin O. For any $P(a_1,\ a_2)$, we can write its **position vector** as $\overrightarrow{OP} = [a_1,\ a_2]$
- Furthermore, consider $P(a_1, a_2)$, for any $Q(b_1, b_2)$, we can write its position vector as $\overrightarrow{PQ} = [b_1 - a_1, b_2 - a_2]$
- For $\boldsymbol{a} = [a_1,\ a_2,\ a_3]$ and $\boldsymbol{b} = [b_1,\ b_2,\ b_3]$ we find:
  $\boldsymbol{a} + \boldsymbol{b} = [a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3]$, $c\boldsymbol{a} = [ca_1,\ ca_2,\ ca_3]$ and $\|\boldsymbol{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Standard Unit Basis Vectors

- $\boldsymbol{i} = [1, 0]$ and $\boldsymbol{j} = [0, 1]$ in 2 space and $\boldsymbol{i} = [1, 0, 0]$ and $\boldsymbol{j} = [0, 1, 0]$ and $\boldsymbol{k} = [0, 0, 1]$ in 3 space
- For any $\boldsymbol{a} = [a_1, a_2, a_3]$,
  $\boldsymbol{a} = [a_1, a_2, a_3] = [a_1, 0, 0] + [0, a_2, 0] + [0, 0, a_3] = a_1[1, 0, 0] + a_2[0, 1, 0] + a_3[0, 0, 1] = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$

Dot Product

- The **dot product** (or **scalar product)** of $\boldsymbol{a} = [a_1,\ a_2,\ a_3]$ and $\boldsymbol{b} = [b_1,\ b_2,\ b_3]$ is $\boldsymbol{a}.\boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$
- Consider $\boldsymbol{a}$ and $\boldsymbol{b}$ with an angle θ between them, $0° \le \theta \le 180°$. Suppose we locate both vectors with their initial points at the origin, $\boldsymbol{a}.\boldsymbol{b} = \|\boldsymbol{a}\|\|\boldsymbol{b}\|\cos\theta$ and if $\boldsymbol{a} \neq 0$ and $\boldsymbol{b} \neq 0$, then $\cos\theta = \frac{\boldsymbol{a}.\boldsymbol{b}}{\|\boldsymbol{a}\|\|\boldsymbol{b}\|}$
- We say that $\boldsymbol{a}$ and $\boldsymbol{b}$ are **orthogonal** (or perpendicular) if the angle between them is $90°$.
- In this case, $\frac{\boldsymbol{a}.\boldsymbol{b}}{\|\boldsymbol{a}\|\|\boldsymbol{b}\|} = \cos 90° = 0 \to \boldsymbol{a}.\boldsymbol{b} = 0$

Projection: The **scalar projection** of $\boldsymbol{a}$ on $\boldsymbol{b}$ is $p_s = \|\overrightarrow{OB}\| = \boldsymbol{a}.\hat{\boldsymbol{b}}$ and the **vector projection** of $\boldsymbol{a}$ on $\boldsymbol{b}$ is $\boldsymbol{p}_v = \|\overrightarrow{OB}\| = p_s\hat{\boldsymbol{b}}$

Work Done by a Force: $\text{Work} = \boldsymbol{F}.\boldsymbol{s} = \|\boldsymbol{F}\|\cos\theta\,\|\boldsymbol{s}\|$

Direction Cosines

- Consider a vector $\boldsymbol{a} = [a_1,\ a_2,\ a_3]$ and let α, β and γ be the angles which $\boldsymbol{a}$ makes with $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$, respectively
- Then $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are the **direction cosines** of $\boldsymbol{a}$
- We have $\cos\alpha = \frac{a_1}{\|\boldsymbol{a}\|}$, $\cos\beta = \frac{a_2}{\|\boldsymbol{a}\|}$, $\cos\gamma = \frac{a_1}{\|\boldsymbol{a}\|}$. Therefore, $[\cos\alpha, \cos\beta, \cos\gamma] = \frac{a}{\|\boldsymbol{a}\|} = \hat{\boldsymbol{a}}$

Lecture 6: Vectors & Introduction to Matrices

Cross Product

- The **cross product** of $\boldsymbol{a} = [a_1,\ a_2,\ a_3]$ and $\boldsymbol{b} = [b_1,\ b_2,\ b_3]$ is $\boldsymbol{a} \times \boldsymbol{b} = [\ a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1]$
- $\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c})$ is called the **scalar triple product of** $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$
- $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ is called the **vector triple product of** $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$
- $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are **coplanar** (lie in the same plane) if and only if $\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c}) = 0$

Area of a Parallelogram	Area of a Triangle	Volume of a Parallelepiped
$\text{Area} = \|\overrightarrow{PQ} \times \overrightarrow{PR}\|$ $\text{Area} = \|\boldsymbol{u} \times \boldsymbol{v}\|$ $= \|\boldsymbol{u}\|\|\boldsymbol{v}\|\sin\theta$	$\text{Area} = \frac{1}{2}\|\overrightarrow{PQ} \times \overrightarrow{PR}\|$ $\text{Area} = \frac{1}{2}\|\boldsymbol{u} \times \boldsymbol{v}\|$ $= \frac{1}{2}\|\boldsymbol{u}\|\|\boldsymbol{v}\|\sin\theta$	$V = \|\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c})\|$ $= \|\boldsymbol{a} \times \boldsymbol{b}\|\|\boldsymbol{c}\|\|\sin\theta\|$

Matrix Addition	Scalar Multiplication	Matrix Multiplication
$C = A + B = [c_{ij}] = [a_{ij} + b_{ij}]$	$kA = k[a_{ij}] = [ka_{ij}]$	$A_{m \times n}B_{n \times p} = C_{m \times p}$ where $c_{ij} = a_{i1} + b_{1j} + a_{i2} + b_{2j} + a_{i3} + b_{3j} \dots + a_{in} + b_{nj}$

- Valid properties of matrix multiplication: (1) $A(B+C) = AB + AC$ (2) $(A+B)C = AC + BC$ (3) $AB(C) = A(BC)$
- The **transpose** of an $m \times n$ matrix $A = [a_{ij}]$ is the matrix $A^T = [a'_{ij}]$ where $a'_{ij} = a_{ji}$
- We say that a square matrix A is **symmetric** if $A^T = A$
- Note: (1) $(AB)^T = A^T B^T$ (2) $(A^T)^T = A$ (3) $(A+B)^T = A^T + B^T$

Identities and Inverses

- An **identity matrix** is a square matrix with all main diagonal entries equal to 1 and all other entries equal to 0
- E.g. $I_1 = [1]$, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- Note: I is the **identity element** of matrix multiplication, i.e. for $A_{m \times n}$, $AI_n = A$ and $I_m A = A$
- If A is square and of order n, and if there exists a B such that $AB = BA = I_n$, then we say that A **is invertible** and we call B the **inverse** of A and we write $B = A^{-1}$ to denote the inverse

Lecture 7: Systems of Linear Equations & Gaussian Elimination

Linear system of m equations in n variables	Matrix of system	Augmented matrix of system
$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$ $a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$ $a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$ $\cdots$ $a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$	$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ i.e. $A\boldsymbol{x} = \boldsymbol{b}$	$\left[\begin{array}{ccc\|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$ i.e. $[A\|\boldsymbol{b}]$

- A system of linear equations can have no solutions, one solution or infinitely many solutions
- If $[A|\boldsymbol{b}]$ is in **row echelon** form, it will be easy to determine the solution(s)
- For any matrix, the first non-zero entry in a particular row is called the **leading entry** of that particular row
- A matrix is said to be in row echelon form if the leading entry in each row lies to the right of that in the preceding row

Gaussian Elimination: Method for reducing $[A|\boldsymbol{b}]$ to row echelon form

- **ERO**s: (1) Multiply a row by a non-zero constant (2) Add or subtract a multiple of one row to or from another, respectively (3) Interchange two rows

Linear Dependence and Independence

- Consider a set of vectors $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \ldots, \boldsymbol{u}_n\}$ and the equation $c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + c_3\boldsymbol{u}_3 + \cdots c_n\boldsymbol{u}_n = 0$,
- Given the above, one of the following will always occur:
    1. $c_1 = c_2 = c_3 = \cdots = c_n = 0$, set of vectors is **linearly independent**
    2. Solution with at least one $c_i \neq 0$, set of vectors is **linearly dependent**

Rank of a Matrix: The **rank** of a matrix A, denoted by $r(A)$, is simply the number of linearly independent rows in A

Recognising the Nature of Solutions

- Consider a general system of m linear equations in n unknowns, $A\boldsymbol{x} = b$
- i.e. A is $m \times n$, $\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$. Let $[A|\boldsymbol{b}]$ be the augmented matrix for the system
    1. $\mathrm{r}(A) \neq \mathrm{r}([A|\boldsymbol{b}])$, equations are **inconsistent** and there are **no solutions**
    2. $\mathrm{r}(A) = \mathrm{r}([A|\boldsymbol{b}]) = n$, equations are **consistent** and have a **unique solution**
    3. $\mathrm{r}(A) = \mathrm{r}([A|\boldsymbol{b}]) = r < n$, equations are **consistent** and have **infinitely many solutions**, which we can express in terms of $n - r$ parameters

Lecture 8: More on Linear Systems & Inverses

Homogenous System of m Equations in n Variables	Matrix of System
$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = 0$ $a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = 0$ $a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = 0$ $\cdots$ $a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = 0$	$A\boldsymbol{x} = 0$

- Note: $\boldsymbol{x} = 0$ is always a solution (the **trivial solution**), i.e. a homogenous system is always consistent
    1. $r(A) = n$, **trivial solution is the only solution**
    2. $r(A) < n$, **infinitely many solutions**. Includes **trivial solution** and **infinite non-trivial solutions**

Calculating Inverses

- To find the inverse of A, form the augmented matrix $[A|I]$, then apply EROs to this augmented matrix until the LHS turns into the identity. The resulting RHS will represent A^{-1}

Invertibility and Solutions of Systems

- Consider a system of n equations in n unknowns, i.e. $Ax = b$, where A is $n \times n, \boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$
- Clearly, if A is invertible with inverse A^{-1}, we have $A^{-1}A\boldsymbol{x} = A^{-1}\boldsymbol{b} \Rightarrow I\boldsymbol{x} = A^{-1}\boldsymbol{b} \Rightarrow \boldsymbol{x} = A^{-1}\boldsymbol{b}$