WORKSHOP 8 SOLUTIONS

1. (i) $\left[\begin{array}{cc|c} 4 & 3 & 0 \\ -2 & 1 & 0 \end{array}\right] \begin{array}{l} R_2 = 2R_2 + R_1 \end{array} \sim \left[\begin{array}{cc|c} 4 & 3 & 0 \\ 0 & 5 & 0 \end{array}\right]$

$r(A) = 2 = n \;\Rightarrow$ Unique/Trivial solution:

$x_1 = 0, \quad x_2 = 0$

(ii) $\left[\begin{array}{cc|c} 2 & 3 & 0 \\ 6 & 9 & 0 \end{array}\right] \begin{array}{l} R_2 = R_2 - 3R_1 \end{array} \sim \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$r(A) = 1 < n = 2 \;\Rightarrow$ Infinitely many solutions

Need $n - r = 2 - 1 = 1$ parameter

Let $x_2 = t, \; t \in \mathbb{R}$

Row 1: $2x_1 + 3x_2 = 0 \;\Rightarrow\; 2x_1 + 3t = 0 \;\Rightarrow\; x_1 = -\frac{3t}{2}$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

(iii) $\left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right] \begin{array}{l} R_2 = R_2 + R_1 \\ R_3 = R_3 - 2R_1 \end{array} \sim \left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{array}\right] R_3 = R_3 + 3R_2$

$\sim \left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $r(A) = 2 < n = 3 \;\Rightarrow$ Infinitely many solutions
Need $n - r = 3 - 2 = 1$ parameter

Let $x_3 = t, \; t \in \mathbb{R}$

Row 2: $3x_2 = 0 \;\Rightarrow\; x_2 = 0$

Row 1: $3x_1 + 5x_2 - 4x_3 = 0 \;\Rightarrow\; 3x_1 - 4t = 0 \;\Rightarrow\; x_1 = \frac{4t}{3}$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

2. (i) $\left[\begin{array}{cc|c} -2 & 3 & 13 \\ 4 & 2 & -2 \end{array}\right] R_2 = R_2 + 2R_1 \sim \left[\begin{array}{cc|c} -2 & 3 & 13 \\ 0 & 8 & 24 \end{array}\right] R_2 = R_2 \div 8$

$\sim \left[\begin{array}{cc|c} -2 & 3 & 13 \\ 0 & 1 & 3 \end{array}\right] R_1 = R_1 - 3R_2 \sim \left[\begin{array}{cc|c} -2 & 0 & 4 \\ 0 & 1 & 3 \end{array}\right] R_1 = R_2 \div (-2) \sim \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 3 \end{array}\right]$

$\therefore\; x_1 = -2, \quad x_2 = 3$

(ii) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right] \begin{array}{l} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 3R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right] R_3 = 7R_3 - 5R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & -50 & -150 \end{array}\right] R_3 = R_3 \div (-50) \quad \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} R_1 = R_1 - 3R_3 \\ R_2 = R_2 + 4R_3 \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & -3 \\ 0 & -7 & 0 & 14 \\ 0 & 0 & 1 & 3 \end{array}\right] R_2 = R_2 \div (-7) \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right] R_1 = R_1 - 2R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right] \qquad \therefore\ x_1 = 1,\ x_2 = -2,\ x_3 = 3$$

3. (i) $[A|I] = \left[\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 6 & -9 & 0 & 1 \end{array}\right]$ $\mathrm{R}_2 \to \mathrm{R}_2 - 3\mathrm{R}_1$

$\sim \left[\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{array}\right]$

i.e. Due to the row of zeros, the matrix A is not invertible.

(ii) $[B|I] = \left[\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ -3 & -7 & 0 & 1 \end{array}\right]$ $\mathrm{R}_2 \to 2\mathrm{R}_2 + 3\mathrm{R}_1$

$$\sim \left[\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 0 & 1 & 3 & 2 \end{array}\right] \mathrm{R}_1 \to \mathrm{R}_1 - 5\mathrm{R}_2$$

$$\sim \left[\begin{array}{cc|cc} 2 & 0 & -14 & -10 \\ 0 & 1 & 3 & 2 \end{array}\right] \mathrm{R}_1 \to \mathrm{R}_1 \div (2)$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & -7 & -5 \\ 0 & 1 & 3 & 2 \end{array}\right] = [I|B^{-1}]$$

i.e. $B^{-1} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$

(iii) $[C|I] = \left[\begin{array}{cc|cc} -4 & -8 & 1 & 0 \\ -2 & -3 & 0 & 1 \end{array}\right]$ $\mathrm{R}_2 \to -2\mathrm{R}_2 + \mathrm{R}_1$

$$\sim \left[\begin{array}{cc|cc} -4 & -8 & 1 & 0 \\ 0 & -2 & 1 & -2 \end{array}\right] \mathrm{R}_1 \to \mathrm{R}_1 - 4\mathrm{R}_2$$

$$\sim \left[\begin{array}{cc|cc} -4 & 0 & -3 & 8 \\ 0 & -2 & 1 & -2 \end{array}\right] \begin{array}{l} \mathrm{R}_1 \to \mathrm{R}_1 \div (-4) \\ \mathrm{R}_2 \to \mathrm{R}_1 \div (-2) \end{array}$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{4} & -2 \\ 0 & 1 & -\frac{1}{2} & 1 \end{array}\right] = [I|C^{-1}]$$

i.e. $C^{-1} = \begin{bmatrix} \frac{3}{4} & -2 \\ -\frac{1}{2} & 1 \end{bmatrix}$

(iv) $[D|I] = \left[\begin{array}{ccc|ccc} 5 & 0 & -1 & 1 & 0 & 0 \\ 1 & -3 & -2 & 0 & 1 & 0 \\ 0 & 5 & 3 & 0 & 0 & 1 \end{array}\right]$ $\mathrm{R}_2 \to 5\mathrm{R}_2 - \mathrm{R}_1$

$$\sim \left[\begin{array}{ccc|ccc} 5 & 0 & -1 & 1 & 0 & 0 \\ 0 & -15 & -9 & -1 & 5 & 0 \\ 0 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \mathrm{R}_3 \to 3\mathrm{R}_3 + \mathrm{R}_2$$

$$\sim \left[\begin{array}{ccc|ccc} 5 & 0 & -1 & 1 & 0 & 0 \\ 0 & -15 & -9 & -1 & 5 & 0 \\ 0 & 0 & 0 & -1 & 5 & 3 \end{array}\right]$$

i.e. Due to the row of zeros, the matrix D is not invertible.

(v) $[E|I] = \left[\begin{array}{ccc|ccc} 1 & 5 & 0 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 \end{array}\right] \quad \mathrm{R}_2 \to \mathrm{R}_2 - 2\mathrm{R}_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 5 & 0 & 1 & 0 & 0 \\ 0 & -6 & -1 & -2 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 \end{array}\right] \quad \mathrm{R}_3 \to 3\mathrm{R}_3 - \mathrm{R}_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 5 & 0 & 1 & 0 & 0 \\ 0 & -6 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 2 & -1 & 3 \end{array}\right] \quad \mathrm{R}_2 \to \mathrm{R}_2 + \mathrm{R}_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 5 & 0 & 1 & 0 & 0 \\ 0 & -6 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 & -1 & 3 \end{array}\right] \quad \mathrm{R}_2 \to \mathrm{R}_2 \div (-6)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 5 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 2 & -1 & 3 \end{array}\right] \quad \mathrm{R}_1 \to \mathrm{R}_1 - 5\mathrm{R}_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -0 & \frac{5}{2} \\ 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 2 & -1 & 3 \end{array}\right] = [I|E^{-1}]$$

i.e. $E^{-1} = \begin{bmatrix} 1 & 0 & \frac{5}{2} \\ 0 & 0 & -\frac{1}{2} \\ 2 & -1 & 3 \end{bmatrix}$

4. (i) $[A|I] = \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 4 & 3 & 0 & 1 \end{array}\right] \quad \mathrm{R}_2 \to \mathrm{R}_2 - 2\mathrm{R}_1$

$$\sim \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \quad \mathrm{R}_1 \to \mathrm{R}_1 - \mathrm{R}_2$$

$$\sim \left[\begin{array}{cc|cc} 2 & 0 & 3 & -1 \\ 0 & 1 & -2 & 1 \end{array}\right] \quad \mathrm{R}_1 \to \mathrm{R}_1 \div (2)$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 1 & -2 & 1 \end{array}\right] = [I|A^{-1}]$$

i.e. $A^{-1} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{bmatrix}$

Thus,

$$\boldsymbol{x} = A^{-1}\boldsymbol{b} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} \frac{15}{2} - \frac{9}{2} \\ -10 + 9 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(ii) $[A|I] = \left[\begin{array}{cc|cc} 3 & 5 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right]$ $\mathrm{R}_2 \to 3\mathrm{R}_2 - \mathrm{R}_1$

$$\sim \left[\begin{array}{cc|cc} 3 & 5 & 1 & 0 \\ 0 & 1 & -1 & 3 \end{array}\right] \quad \mathrm{R}_1 \to \mathrm{R}_1 - 5\mathrm{R}_2$$

$$\sim \left[\begin{array}{cc|cc} 3 & 0 & 6 & -15 \\ 0 & 1 & -1 & 3 \end{array}\right] \quad \mathrm{R}_1 \to \mathrm{R}_1 \div (3)$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 2 & -5 \\ 0 & 1 & -1 & 3 \end{array}\right] = [I|A^{-1}]$$

i.e. $A^{-1} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$

Thus,

$$\boldsymbol{x} = A^{-1}\boldsymbol{b} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0+0 \\ 0+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(iii) $[A|I] = \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 2 & -1 & 4 & 0 & 1 & 0 \\ -1 & 2 & -4 & 0 & 0 & 1 \end{array}\right]$ $\begin{array}{l} \mathrm{R}_2 \to \mathrm{R}_2 - 2\mathrm{R}_1 \\ \mathrm{R}_3 \to \mathrm{R}_3 + \mathrm{R}_1 \end{array}$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 \end{array}\right] \quad \mathrm{R}_3 \to \mathrm{R}_3 - \mathrm{R}_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3 & -1 & 1 \end{array}\right] \quad \begin{array}{l} \mathrm{R}_1 \to \mathrm{R}_1 - 3\mathrm{R}_3 \\ \mathrm{R}_2 \to \mathrm{R}_2 + 2\mathrm{R}_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -8 & 3 & -3 \\ 0 & 1 & 0 & 4 & -1 & 2 \\ 0 & 0 & 1 & 3 & -1 & 1 \end{array}\right] \quad \mathrm{R}_1 \to \mathrm{R}_1 + \mathrm{R}_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 2 & -1 \\ 0 & 1 & 0 & 4 & -1 & 2 \\ 0 & 0 & 1 & 3 & -1 & 1 \end{array}\right] = [I|A^{-1}]$$

i.e. $A^{-1} = \begin{bmatrix} -4 & 2 & -1 \\ 4 & -1 & 2 \\ 3 & -1 & 1 \end{bmatrix}$

$$\boldsymbol{x} = A^{-1}\boldsymbol{b} = \begin{bmatrix} -4 & 2 & -1 \\ 4 & -1 & 2 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 11 \\ -11 \end{bmatrix} = \begin{bmatrix} -32+22+11 \\ 32-11-22 \\ 24-11-11 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$