Lecture 2 Predicates & Quantifiers

Ref.: K. H. Rosen, Section 1.3

Predicate: Definition

Let *H(x,y)* mean that *x* is taller than *y*,

i.e. *H(x,y)* is true iff *x* is taller than *y*.

If we know who *x* and *y* are, we know whether H(x,y) is true or not.

E. g., *H(Tom, John)*= **T**.

H is called a propositional function (predicate)

-- H assigns to every pair x,y a truth value.

Definition

H(Tom, John)=**T** .

Subject:

Is the element of the proposition that determines its value.

A variable is a “place holder” for a subject. It is called a *free variable* unless otherwise specified.

H(x, y)

Propositional Function

"x > 3" can be expressed by function P(x)

> 3 is represented by P

P is the predicate

x is the variable

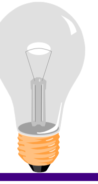
When a value is assigned to each variable of a propositional function, the predicate receives a truth value and thus turns into a proposition.

Examples

Let P(x) denote "x > 3"
Let x = 4, P(4).
4 > 3, therefore P(4) is TRUE

Let Q(x,y,z) denote "x + y = z"
Let x = 2, y = 3, z = 6, Q(2, 3, 6)
2 + 3 = 6 is false, therefore Q(2,3,6) is FALSE

Universal Quantification



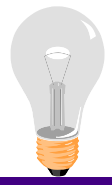
- “There is a person who is taller than 6 feet”.
- “All of us are taller than 4 feet”.

When we make such statements we have in mind a universe (e.g., the students in this room?).

These statements have a truth value and thus are propositions.

These statements are about properties of the universe.

Universal Quantification



Definition:

The *universal quantification of P(x)* is the proposition "P(x) is true for all values of x in the universe of discourse".

Written as: $\forall x\, P(x)$

or For all x, P(x)

or For every x, P(x)

Examples

"Everyone in this room can see me"

can be represented as a universal quantification.

Let P(x) denote: "Person x can see me"

Let R(x) denote: "Person x is in this room"

$$\forall x(R(x) \rightarrow P(x))$$

Universe of discourse: All people

Examples

"$\forall x\, P(x)$ with $P(x)$ being $x^2<10$"

Universe of discourse: {1,2,3,4}

$\forall x\, P(x)$ is the same as $P(1) \wedge P(2) \wedge P(3) \wedge P(4)$

$\forall x\, P(x)$ is false as $P(4)$ is false.

Existential Quantification

Definition:
The *existential quantification of P(x)* is the proposition "There exists an element x in the universe of discourse such that P(x) is true"

Written as: $\exists x\, P(x)$

"There is an x such that P(x)"

or "For some x P(x)"

or "There is at least one x such that P(x)"

Examples

"I can be seen"

Let $P(x)$ denote: "Person x can see me"

$\exists x\, P(x)$

Universe of discourse: All people

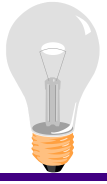
"$\exists x\, P(x)$ with $P(x)$ being $x^2<10$"

Universe of discourse: {1,2,3,4}

$\exists x\, P(x)$ is the same as $P(1) \vee P(2) \vee P(3) \vee P(4)$

$\exists x\, P(x)$ is true as $P(1)$ is true.

∀ and ∃



Statement	When True?	When False?
∀x P(x)	P(x) is true for every x.	There is an x for which P(x) is false.
∃x P(x)	There is an x for which P(x) is true.	P(x) is false for every x.

Example

Take the English sentence :

"All lions are fierce"

Let P(x) denote the statement "x is a lion"
Let Q(x) denote the statement "x is fierce"

$\forall x \in L, Q(x)$ where L is the set of lions

or $\forall x\ (P(x) \rightarrow Q(x))$

Example

"Some programs are object-oriented."

$\exists x \in P$, x is object-oriented where P is the set of programs.

or $\exists x$ (x is a program $\wedge$ x is object-oriented)

Example

"Every human being needs air."

$\forall x \in H$, x needs air where H is the set of human beings.

or $\forall x\ \cancel{(x \text{ is a human being} \wedge x \text{ needs air})}$?

$\forall x\ (x \text{ is a human being} \rightarrow x \text{ needs air})$ ✔

Example

"All even numbers are divisible by 2."

$\forall x \in E$, x is divisible by 2 where E is the set of even numbers.

or $\cancel{\forall x\ (x \text{ is a even number} \wedge x \text{ is divisible by } 2)}$?

or $\forall x\ (x \text{ is a even number} \rightarrow x \text{ is divisible by } 2)$

Example

"Some fish can climb trees."

$\exists x \in F$, x can climb trees where F is the set of fish.

or ~~$\exists x$ (x is a fish $\rightarrow$ x can climb trees)~~ ?

or $\exists x$ (x is a fish $\wedge$ x can climb trees)

Example

"Some CS students are babies."

$\exists x \in S$, x is a baby where S is the set of CS students.

or $\exists x$ ~~(x is a CS students $\Rightarrow$ x is a baby)~~ ?

or $\exists x$ (x is a CS student $\wedge$ x is a baby)

Example

"Every basketball player is tall"

$\forall x$ (x is a basketball player $\rightarrow$ x is tall)

Example

“No dogs have wings.”

$\forall$x (x is a dog $\rightarrow$ x has no wings)

Example

"Some CS students are hardworking."

$\exists x$ (x is a CS students $\wedge$ x is hardwoking)

Example

"If a number is an integer, then it is a rational number."

$$\forall x\ (x \text{ is an integer} \rightarrow x \text{ is a rational number})$$

Negations

“Every student has seen me” $\forall x\, P(x)$

“Not every student has seen me” $\neg\forall x\, P(x)$

$\cancel{\neg\forall x\, P(x) \Leftrightarrow \forall x\, \neg P(x)}$?

$\neg\forall x\, P(x) \Leftrightarrow \exists x\, \neg P(x)$

Negations

“Somebody in this room are hardworking”

$\exists x\ P(x)$

“It’s not the case that somebody in this room are hardworking”

$\neg\exists x\ P(x)$

$$\cancel{\neg\, \exists x\ P(x) \Leftrightarrow \exists x\ \neg P(x)}\quad ?$$

$$\neg\, \exists x\ P(x) \Leftrightarrow \forall x\ \neg P(x)$$

Negations

Negation	Equivalent	When True?	When False?
$\neg\exists x\, P(x)$	$\forall x\, \neg P(x)$	P(x) is false for every x.	There is an x for which P(x) is true.
$\neg\forall x\, P(x)$	$\exists x\, \neg P(x)$	There is an x for which P(x) is false.	P(x) is true for every x.

Example

$\forall x > 0, \exists y > 0$ such that $y < x$

- "For each positive number, there is another positive number smaller than it"
- "Given any positive number, we can find a smaller positive number"
- "There is no smallest positive number"

Example

"Someone is loved by someone"

$\equiv$ $\exists$a person x, $\exists$a person y such that y loves x

"Everyone is loved by everyone"

$\equiv$ $\forall$people x, $\forall$people y, y loves x

"Everyone is loved by someone"

$\equiv$ $\forall$people x, $\exists$a person y such that y loves x

"Someone is loved by everyone"

$\equiv$ $\exists$a person x such that $\forall$people y, y loves x

"Everyone loves someone"

$\equiv$ $\forall$people y, $\exists$a person x such that y loves x

Example

Translate the statement into English:

$\forall x \, (C(x) \vee \exists y \, (C(y) \wedge F(x,y)))$

C(x) is “has a computer”

F(x,y) is “x and y are friends”

Universe of discourse: All students of Curtin.

Every student in Curtin has a computer or has a friend who has a computer.

Example

Translate the statement into English:

$\exists x\ \forall y\ \forall z\ \ (((F(x,y) \wedge F(x,z) \wedge y \neq z)) \rightarrow \neg F(y,z)))$

F(x,y) is “x and y are friends”

Universe of discourse: All students.

There is a student none of whose friends are also friends with each other.

Example

Translate the English sentence :

"Everybody has exactly one best friend"

Let B(x,y) denote the statement "x and y are best friends"

Let the universe of discourse be the set of all people.

$\forall x\ \exists y\ B(x,y)$ is "everybody has a best friend".

$$\forall x\ \exists y \forall z\ (B(x,y) \wedge ((z \neq y) \rightarrow \neg B(x,z)))$$

Summary: ∀ and ∃ for x,y

Statement	When True?	When False?
$\forall x \forall y\ P(x,y)$ $\forall y \forall x\ P(x,y)$	P(x,y) is true for every pair x,y.	There is a pair x,y for which P(x,y) is false.
$\forall x \exists y\ P(x,y)$	For every x there is a y for which P(x,y) is true.	There is an x such that P(x,y) is false for every y.
$\exists x \forall y\ P(x,y)$	There is an x such that P(x,y) is true for every y.	For every x there is a y for which P(x,y) is false.
$\exists x \exists y\ P(x,y)$ $\exists y \exists x\ P(x,y)$	There is a pair x,y for which P(x,y) is true.	P(x,y) is false for every pair x,y.

Order

- **Quantifiers of the <u>same</u> sort can be placed in any order without changing meaning:**

$$\forall x,\ \forall y[x \text{ is mother of } y \rightarrow x \text{ is a parent of } y]$$
$$\equiv \forall y,\ \forall x[x \text{ is mother of } y \rightarrow x \text{ is a parent of } y]$$

Order

- **Quantifiers of <u>different</u> sorts, if placed in different order change the meaning of the sentence**

$\forall x \in$ people $\exists$ a person y s.t. [y is the mother of x]

$\equiv$ "everyone has a mother"

$\exists$ a person y s.t. $\forall x \in$ people [y is the mother of x]

$\equiv$ "someone is everyone's mother"

Truth Value*

- $\forall x\, \exists y\, [P(x) \rightarrow Q(x,y)]$ is true iff

  $\exists y\, [P(x) \rightarrow Q(x,y)]$ is true for each possible value of x

  $\exists y\, [P(x) \rightarrow Q(x,y)]$ is true iff for any ($\geq$ 1) value of y, $P(x) \rightarrow Q(x,y)$ is true.

E.g. $\forall x \in R\ \exists y \in R \text{ s.t. } \left(x \geq 0 \rightarrow y = \sqrt{x}\right)$ is true

$\because$ For any $x \in R$:

when $x \geq 0$: $\exists y \in R \left[x \geq 0 \rightarrow y = \sqrt{x}\right]$ is true

*when $x < 0$: $\exists y \in R \left[x \geq 0 \rightarrow y = \sqrt{x}\right]$ is vacuously true

Truth Value

- $\exists x \forall y\,[P(x) \rightarrow Q(x,y)]$ is true **iff**

  $\forall y\,[P(x) \rightarrow Q(x,y)]$ is true for any value (≥ 1) of x.

  $\forall y\,[P(x) \rightarrow Q(x,y)]$ is true **iff**

  for all value of y, $P(x) \rightarrow Q(x,y)$ is true.

Truth Value: An Example

E.g. Given choices on 4 tables:

salads : { green salad, fruit salad };

main course: { spaghetti, fish };

dessert : { pie, cake };

beverage: { milk, soda, coffee }

Guests' choices:

Joko: green salad, spaghetti, pie, milk

Sidek: fruit salad, fish, pie, cake, milk, coffee

Zhao: spaghetti, fish, pie, soda

Truth Value: An Example (con.)

$\exists$ a guests *g* s.t. $\forall$ table *t*, $\exists$ an item *i* on *t* s.t. *g* chose *i*

$\equiv$ "$\geq$1 guest who chose $\geq$ 1 item from every table":

true; Joko and Sidek

$\forall$ guests *g* and $\forall$ table *t*, $\exists$ an item *i* on *t* s.t. *g* chose i

$\equiv$ "every guest chose $\geq$ 1 item from every table";

false; Zhao did not

Truth Value: A Example (con.)

∃ an item *i* s.t. ∀guests *g*, *g* chose *i*

≡ "there is at least 1 item that's chosen by all guests";

true; e.g. pie

∃ a guests *g* s.t. ∀ item *i*, *g* chose *i*

≡ "there is a guest who chose every available item"

false, none

Negation of multiple quantifiers

Z: Prime Numbers

$\neg(\forall x \in Z, \exists y \in Z\ [(x/y) \in Z \wedge y \neq 1 \wedge y \neq x]) \equiv ?$

$\equiv \exists\ x \in Z$, s.t. $\neg(\exists y \in Z\ [(x/y) \in Z \wedge y \neq 1 \wedge y \neq x])$

$\equiv \exists\ x \in Z$, s.t. $\forall y \in Z\ \{\neg[(x/y) \in Z \wedge y \neq 1 \wedge y \neq x]\}$

$\equiv \exists\ x \in Z$, s.t. $\forall y \in Z\ [(x/y) \notin Z \vee y = 1 \vee y = x]$

Negation of multiple quantifiers

In general, negation is:

$\neg (\forall x \exists y \text{ s.t. } [P(x,y)]) \equiv \exists x \text{ s.t. } \forall y\ [\neg P(x,y)]$

$\neg (\exists x \text{ s.t. } \forall y [P(x,y)]) \equiv \forall x\ \exists y \text{ s.t. } [\neg P(x,y)]$

E.g. $\neg(\forall n \in Z, \exists k \in Z \text{ s.t. } [n=2k])$

Not all integers are even

$\equiv \exists n \in Z, \text{ s.t. } \forall k \in Z\ [n \neq 2k])$

There is at least one non-even integer

Negation of multiple quantifiers

E.g. **$\neg$($\exists$ person *x* s.t.$\forall$ people *y* [*x* loves *y*])**

Negate "someone loves everyone"

$\equiv$ $\forall$ people *x* $\exists$ person *y* s.t. [*x* does not loves *y*]

Nobody loves everybody

Note: negation of $\forall$ and $\exists$: generalized DeMorgans's theorem

Other Extension

- **Consider** $\forall x \in D\ [P(x) \to Q(x)]$
  - **Contrapositive**: $\forall x \in D\ [\neg Q(x) \to \neg P(x)]$

    $\equiv \forall x \in D\ [P(x) \to Q(x)]$
  - **Converse:** $\forall x \in D\ [\ Q(x) \to P(x)]$

    $\cancel{\equiv} \forall x \in D[\ P(x) \to Q(x)\ ]$
  - **Inverse:** $\forall x \in D\ [\ \neg P(x) \to \neg Q(x)]$

    $\cancel{\equiv} \forall x \in D[\ P(x) \to Q(x)\ ]$

Other Extension

- $\forall x$, $R(x)$ is a **sufficient condition** for $S(x)$
  $\equiv \forall x\,[\,R(x) \rightarrow S(x)\,]$
- $\forall x$, $R(x)$ is a **necessary condition** for $S(x)$
  $\equiv \forall x[\neg R(x) \rightarrow \neg S(x)] \equiv \forall x[S(x) \rightarrow R(x)]$
- $\forall x$, $R(x)$ **only if** $S(x)$
  $\equiv \forall x\,[\neg S(x) \rightarrow \neg R(x)\,] \equiv \forall x\,[\,R(x) \rightarrow S(x)\,]$

English — Predicate Logic Translation

With the knowledge you have learnt, you will now be able to translate English sentences to logical expressions.

This will :

- Eliminate ambiguity.
- Enable reasoning.

English — Predicate Logic Translation

- "**<u>All</u>** tourists need visas to come into Australia"

  ≡ $\forall x \in$ tourists of Australia
  [x needs a visa to come into Australia]

  ≡ $\forall x \in$ tourists of Australia$[NeedVisa(x)]$

  ≡ $\forall x \in$ people $[AustraliaTourist(x) \rightarrow NeedVisa(x)]$

- "**<u>Some</u>** tourists need visas to come into Australia"

  ≡ $\exists x \in$ tourists of Australia s.t.
  [x needs a visa to come into Australia]

  ≡ $\exists x \in$ people $[AustraliaTourist(x) \wedge NeedVisa(x)]$

English — Predicate Logic Translation

- **"Mary likes <u>everyone</u> who likes cats"**

  $\equiv \forall x \in$ **people who likes cats[Mary likes** x**]**

  $\equiv \forall x \in$ **people [*Likes*(x, cat) $\rightarrow$ *Likes*(Mary, x)]**

Various predicate symbols must be appropriately pre-defined, Such as
Likes $(x,y) \equiv x$ **likes** y.
The pre-definition is omitted for many examples here.

English — Predicate Logic Translation

- **"A program is correct <u>if</u> it terminates <u>for all</u> suitable inputs <u>and</u> delivers an output which is always in the required relation *S* to the input"**

$\equiv \forall p \in$ programs

$\forall i \in$ suitable input [*p* terminates $\wedge$ (*p* delivers an output *o* s.t. *S*(*i*,*o*))]

$\rightarrow$ *p* is correct]

English — Predicate Logic Translation

$\equiv \forall p \in \textbf{programs}$
$[\forall i \in \textbf{suitable input} \{Terminates(p, i)$
$\wedge (\exists \textbf{output } o \textbf{ s.t.}[Delivers(p,o) \wedge S(i,o)])\}$
$\rightarrow Correct(p)\ ]$

$\equiv \forall p \in \textbf{programs}$
$[\forall i \{SuitableInput(i) \rightarrow$
$Terminates(p,i) \wedge (\ \exists \textbf{output } o \textbf{ s.t.}$
$[Delivers(p,o) \wedge S(i,o)\ ]\ )\ \} \rightarrow Correct(p)\ ]$

English — Predicate Logic Translation

- **"An integer *x* is smaller than an integer *y* if (*x*+1=*y*) or *x* is smaller than a third integer *z* and *z* is smaller than *y*"**

$$\equiv \forall x \in Z\ \forall y \in Z\ [(x+1=y) \vee \exists z \in Z\ [Smaller(x,z) \wedge Smaller(z,y)] \rightarrow Smaller(x,y)]$$

A recursive relationship and the expression itself define the predicate *Smaller*(*x*,*y*)

English — Predicate Logic Translation

- **"A positive integer *x* is a multiple of 5 if *x*=5 or (*x*–5=*y*) and *y* is a multiple of 5"**

$$\equiv \forall x \in Z^{+} \{(x=5) \vee \exists y[(x\text{-}5 = y) \wedge Multiple\ (y,5)] \rightarrow Multiple\ (x,5)\}$$

English — Predicate Logic Translation

- **It is reasonably easy to recognize a correctly formed sentence in predicate logic. It is a little harder to translate such a sentence into English given the informal meaning of ∀ and ∃.**
- **It is even harder to translate English into predicate logic. Some useful rules of thumb are provided here to aid the process of translation.**

Some useful rules

- **Get the structure correct, as regards the use of quantifiers and Boolean connectives; the latter include, "only-if", "sufficient/necessary condition for", 'unless", "if and only if" etc.**

Some useful rules

- **If the sentence has a universal quantifier governing some part(s) of it, it is quite likely that the variable needs qualifying; if so, it is done by an implication.**

E.g.

$$\forall x \in \text{people}\ [\mathit{AustraliaTourist}(x) \rightarrow \mathit{NeedVisa}(x)]$$

$$\forall x \in \text{people}\ [\mathit{Likes}(x, \text{cat}) \rightarrow \mathit{Likes}(\text{Mary}, x)]$$

Some useful rules

- If a sentence is existentially quantified, then there is likely to be a conjunction to link the main part with the qualifying part.

E.g.

$\exists x \in \text{people}\ [\mathit{AustraliaTourist}(x) \wedge \mathit{NeedVisa}(x)]$

Some CS students like FCS152

$\equiv \exists x \in \text{students}\ [\mathit{CS}(x) \wedge \mathit{Likes}(x, \mathit{FCS152})]$

Summary

- Definition of Predicates
- Universal Quantifiers
- Existantial Quantifiers
- Translation to logical forms
- Negation of Quantifiers
- Order of multiple quantifiers
- Truth Value

Summary (cont.)

- Negation of multiple quantifiers
- English-predicate logic translation