EXAM SOLUTIONS

Q1) (a). (i). Any negative multiple of $\underset{\sim}{a}$, ie. $c\underset{\sim}{a}$, $c<0$

i.e. $-\underset{\sim}{a} = [-2, 1, -3]$ (1)

(ii). $\|\underset{\sim}{a}\| = \sqrt{(2)^2+(-1)^2+(3)^2} = \sqrt{4+1+9} = \sqrt{14}$ (1)

(iii). $2\underset{\sim}{a} - 3\underset{\sim}{b} = 2[2,-1,3] - 3[4,1,-2]$

$= [4,-2,6] - [12,3,-6]$ (1) $= [-8,-5,12]$ (1)

(iv). $\underset{\sim}{b}\cdot\hat{\underset{\sim}{a}} = \dfrac{\underset{\sim}{b}\cdot\underset{\sim}{a}}{\|\underset{\sim}{a}\|} = \dfrac{[4,1,-2]\cdot[2,-1,3]}{\sqrt{14}}$ (1) $= \dfrac{8-1-6}{\sqrt{14}} = \dfrac{1}{\sqrt{14}}$ (1)

(v). $\underset{\sim}{b}\times\underset{\sim}{c} = \begin{matrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} & \underset{\sim}{i} & \underset{\sim}{j} \\ 4 & 1 & -2 & 4 & 1 \\ -3 & 0 & 1 & -3 & 0 \end{matrix}$ (1)

$= \underset{\sim}{i}\left((1)(1)-(-2)(0)\right) + \underset{\sim}{j}\left((-2)(-3)-(4)(1)\right) + \underset{\sim}{k}\left((4)(0)-(1)(-3)\right)$ (1)

$= \underset{\sim}{i}(1-0) + \underset{\sim}{j}(6-4) + \underset{\sim}{k}(0+3) = [1,2,3]$ (1)

$\dfrac{\underset{\sim}{b}\times\underset{\sim}{c}}{\|\underset{\sim}{b}\times\underset{\sim}{c}\|} = \dfrac{[1,2,3]}{\sqrt{(1)^2+(2)^2+(3)^2}} = \dfrac{[1,2,3]}{\sqrt{14}} = \left[\dfrac{1}{\sqrt{14}}, \dfrac{2}{\sqrt{14}}, \dfrac{3}{\sqrt{14}}\right]$ (1)

Also $-\dfrac{\underset{\sim}{b}\times\underset{\sim}{c}}{\|\underset{\sim}{b}\times\underset{\sim}{c}\|} = \dfrac{\underset{\sim}{c}\times\underset{\sim}{b}}{\|\underset{\sim}{c}\times\underset{\sim}{b}\|} = \left[\dfrac{-1}{\sqrt{14}}, \dfrac{-2}{\sqrt{14}}, \dfrac{-3}{\sqrt{14}}\right]$ (1)

(b). (i). $A+2B = \begin{bmatrix} 6 & -3 \\ 12 & -6 \end{bmatrix} + 2\begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ 12 & -6 \end{bmatrix} + \begin{bmatrix} 6 & 4 \\ 0 & 8 \end{bmatrix}$ (1) $= \begin{bmatrix} 12 & 1 \\ 12 & 2 \end{bmatrix}$ (1)

(ii). $(BC^T)D = \left(\begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 0 & -1 \end{bmatrix}^T\right)\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

$= \left(\begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 3 & 0 \\ 2 & 2 & -1 \end{bmatrix}\right)$ (1) $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 & 13 & -2 \\ 8 & 8 & -4 \end{bmatrix}$ (1) $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -8 \end{bmatrix}$ (1)

(iii). $A^{-1} = \dfrac{1}{(6)(-6)-(-3)(12)}\begin{bmatrix} -6 & 3 \\ -12 & 6 \end{bmatrix} = \dfrac{1}{0}\begin{bmatrix} -6 & 3 \\ -12 & 6 \end{bmatrix}$ d.n.e (1) since $\det(A)=0$ (1)

(iv). $B^{-1} = \dfrac{1}{(3)(4)-(2)(0)}\begin{bmatrix} 4 & -2 \\ 0 & 3 \end{bmatrix}$ (1) $= \dfrac{1}{12}\begin{bmatrix} 4 & -2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/6 \\ 0 & 1/4 \end{bmatrix}$ (1)

Q2). (a). (i). $\hat{\overrightarrow{OP}} = \dfrac{\overrightarrow{OP}}{\|\overrightarrow{OP}\|} = \dfrac{[1, 3, -4]}{\sqrt{(1)^2+(3)^2+(-4)^2}}$ (1) $= \dfrac{[1, 3, -4]}{\sqrt{1+9+16}}$

$= \left[\dfrac{1}{\sqrt{26}}, \dfrac{3}{\sqrt{26}}, \dfrac{-4}{\sqrt{26}}\right]$ (1)

(ii). $\overrightarrow{OP} . \overrightarrow{OQ} = 0$ (1)

$[1, 3, \alpha] . [-7, 1-\alpha, \alpha] = 0$

$-7 + 3(1-\alpha) + \alpha(\alpha) = 0$ (1) $\Rightarrow -7+3-3\alpha+\alpha^2 = 0$

$\alpha^2 - 3\alpha - 4 = 0 \Rightarrow (\alpha+1)(\alpha-4) = 0$ (1)

$\therefore \alpha = -1$ (1/2) and $\alpha = 4$ (1/2)

(iii). $\left(\sqrt{(-7)^2+(1-\alpha)^2+\alpha^2}\right)^2 = 2\left(\sqrt{1^2+3^2\alpha^2}\right)^2$ (1)

$49 + (1-\alpha)^2 + \alpha^2 = 2(1+9+\alpha^2)$ (1)

$49 + 1 - 2\alpha + \alpha^2 + \alpha^2 = 20 + 2\alpha^2$ (1) $\Rightarrow 20 + 2\alpha^2 = 50 - 2\alpha + 2\alpha^2$

$2\alpha = 30 \Rightarrow \alpha = 15$ (1)

(b). (i). $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = [2, 4, 1] - [1, 2, 3] = [1, 2, -2]$ (1)

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = [3, 5, -3] - [1, 2, 3] = [2, 3, -6]$ (1)

$\cos\theta = \dfrac{\overrightarrow{AB} . \overrightarrow{AC}}{\|\overrightarrow{AB}\| \|\overrightarrow{AC}\|} = \dfrac{[1, 2, -2] . [2, 3, -6]}{\sqrt{(1)^2+(2)^2+(-2)^2}\sqrt{(2)^2+(3)^2+(-6)^2}}$ (1) $= \dfrac{2+6+12}{\sqrt{1+4+4}\sqrt{4+9+16}}$ (1)

$= \dfrac{20}{\sqrt{9}\sqrt{49}} = \dfrac{20}{21}$ (1)

(ii). $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{array}{ccccc} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} & \underset{\sim}{i} & \underset{\sim}{j} \\ 1 & 2 & -2 & 1 & 2 \\ 2 & 3 & -6 & 2 & 3 \end{array}$ (1)

$= \underset{\sim}{i}((2)(-6)-(-2)(3)) + \underset{\sim}{j}((-2)(2)-(1)(-6)) + \underset{\sim}{k}((1)(3)-(2)(2))$ (1)

$= \underset{\sim}{i}(-12+6) + \underset{\sim}{j}(-4+6) + \underset{\sim}{k}(3-4) = [-6, 2, -1]$ (1)

Area of $ABC = \dfrac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\|$ (1)

$= \dfrac{1}{2}\sqrt{(-6)^2+(2)^2+(-1)^2} = \dfrac{1}{2}\sqrt{36+4+1} = \dfrac{\sqrt{41}}{2}$ (1)

Q3).(a). Direction of L_1 $\underset{\sim}{d_1} = [4, -8, 3]$ $\frac{1}{2}$

Let direction of perpendicular line be $\underset{\sim}{a} = [a_1, a_2, a_3]$

Know $\underset{\sim}{a} \cdot \underset{\sim}{d} = 0$

$[a_1, a_2, a_3] \cdot [4, -8, 3] = 0$ (1) $\Rightarrow 4a_1 - 8a_2 + 3a_3 = 0$ (1)

Infinite Solutions

Let $a_3 = 0$ $(\frac{1}{2})$ & $a_2 = 1$ $(\frac{1}{2})$ $\Rightarrow 4a_1 - 8(1) + 3(0) = 0 \Rightarrow 4a_1 - 8 = 0$

$4a_1 = 8 \Rightarrow a_1 = 2$ (1)

$\therefore$ Direction of perpendicular line $\underset{\sim}{a} = [2, 1, 0]$

Any perpendicular vector is fine. 4 marks for determining this vector.

Point $(3, -1, 2)$

$\therefore$ Parametric equations

$x = 3 + 2t$ $(\frac{1}{2})$, $y = -1 + 1t$ $(\frac{1}{2})$, $z = 2 + 0t$ $(\frac{1}{2})$

(b). Direction of L_2 $\underset{\sim}{d_2} = [3, 3, -1]$ $(\frac{1}{2})$

Normal vector of P_1 $\underset{\sim}{n_1} = [1, 3, -1]$ $(\frac{1}{2})$

$$\theta = \cos^{-1}\left(\frac{d_2 \cdot n_2}{\|d_2\| \|n_2\|}\right) = \cos^{-1}\left(\frac{[3, 3, -1] \cdot [1, 3, -1]}{\sqrt{(3)^2 + (3)^2 + (-1)^2}\sqrt{(1)^2 + (3)^2 + (-1)^2}}\right) \text{ (1)}$$

$$= \cos^{-1}\left(\frac{3 + 9 + 1}{\sqrt{9+9+1}\sqrt{1+9+1}}\right) \text{ (1)} = \cos^{-1}\left(\frac{13}{\sqrt{19}\sqrt{11}}\right) = 25.94° \text{ (1)}$$

(c). Sub L_2 into P_2, solve for t

$2(1 + 3t) - (3t) + (2 - t) = -2$ (1)

$2 + 6t - 3t + 2 - t = -2 \Rightarrow 2t + 4 = -2 \Rightarrow 2t = -6 \Rightarrow t = -3$ (1)

Sub back into L_2 to get point

$x = 1 + 3(-3) = -8$, $y = 3(-3) = -9$, $z = 2 - (-3) = 5$ (2)

(d). Solve for x, y, z using G.E.

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 7 \\ 2 & -1 & 1 & -2 \\ 3 & 1 & 2 & -2 \end{array}\right] \text{ (1) } \begin{array}{l} \\ R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 3R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & -1 & 7 \\ 0 & -7 & 3 & -16 \\ 0 & -8 & 5 & -23 \end{array}\right] \text{ (1) } R_3 = 7R_3 - 8R_2$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 7 \\ 0 & -7 & 3 & -16 \\ 0 & 0 & 11 & -33 \end{array}\right] \text{ (1)}$$

Row 3: $11z = -33 \Rightarrow z = -3$ (1)

Row 2: $-7y + 3z = -16 \Rightarrow -7y - 9 = -16 \Rightarrow y = 1$ (1)

Row 1: $x + 3y - z = 7 \Rightarrow x + 3 + 3 = 7 \Rightarrow x = 1$ (1)

$\therefore$ Point of intersection $(1, 1, -3)$

Q4). (a). $\det(A) = \begin{vmatrix} 1 & 0 & 0 \\ 6 & 2 & 1 \\ 4 & 3 & 2 \end{vmatrix}$

Cofactor expansion along 1st row

$\det(A) = 1\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} - 0\begin{vmatrix} 6 & 1 \\ 4 & 2 \end{vmatrix} + 0\begin{vmatrix} 6 & 2 \\ 4 & 3 \end{vmatrix}$ 1

$= 1((2)(2) - (1)(3))$ 1 $= 1(4-3) = 1(1) = 1$ 1 $\neq 0$

Since $\det(A) \neq 0$ 1 $\therefore$ Columns are l.i. 1

(b). $\left[\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 1 & 1 & 3 & 0 \\ 4 & 3 & b & 0 \end{array}\right]$ 1 $\begin{array}{l} R_2 = 2R_2 - R_1 \\ R_3 = R_3 - 2R_1 \end{array}$ $\sim \left[\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 1 & b-4 & 0 \end{array}\right]$ 2 $R_3 = R_3 - R_2$

$\sim \left[\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & b-8 & 0 \end{array}\right]$ 1

(i). $r(A) = n = 3$ 1

$b - 8 \neq 0 \Rightarrow b \neq 8$ 1

(ii) $r(A) = 2 < n = 3$ 1

$b - 8 = 0 \Rightarrow b = 8$ 1

(c). $\left[\begin{array}{ccc|c} 5 & 7 & 3 & 4 \\ 3 & 2 & 26 & 9 \\ 7 & 10 & 2 & 5 \end{array}\right]$ 1 $\begin{array}{l} R_2 = 5R_2 - 3R_1 \\ R_3 = 5R_3 - 7R_1 \end{array}$ $\sim \left[\begin{array}{ccc|c} 5 & 7 & 3 & 4 \\ 0 & -11 & 121 & 33 \\ 0 & 1 & -11 & -3 \end{array}\right]$ 2 $R_3 = 11R_3 + R_2$

$\sim \left[\begin{array}{ccc|c} 5 & 7 & 3 & 4 \\ 0 & -11 & 121 & 33 \\ 0 & 0 & 0 & 0 \end{array}\right]$ 1 $r(A) = 2,\ n = 3$

Need $n - r = 3 - 2 = 1$ parameter

Let $x_3 = t$ 1

Row 2: $-11x_2 + 121x_3 = 33 \Rightarrow -x_2 + 11t = 3 \Rightarrow x_2 = 11t - 3$ 1

Row 1: $5x_1 + 7x_2 + 3x_3 = 4 \Rightarrow 5x_1 + 7(11t - 3) + 3t = 4$

$5x_1 + 77t - 21 + 3t = 4 \Rightarrow 5x_1 = -80t + 25$

$\therefore x_1 = -16t + 5$ 1

Q5) (a). $[A|I] = \left[\begin{array}{ccc|ccc} 0 & 1 & 4 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{array}\right]$ Swap R_1 & R_2 (2)

$R_3 = R_3 - 5R_2$

$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & -4 & -15 & 0 & -5 & 1 \end{array}\right]$ (1) $R_3 = R_3 + 4R_2$ $\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 & -5 & 1 \end{array}\right]$ $R_1 = R_1 - 3R_3$, $R_2 = R_2 - 4R_3$ (1)

$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -12 & 16 & -3 \\ 0 & 1 & 0 & -15 & 20 & -4 \\ 0 & 0 & 1 & 4 & -5 & 1 \end{array}\right]$ $R_1 = R_1 - 2R_2$ (1) $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 18 & -24 & 5 \\ 0 & 1 & 0 & -15 & 20 & -4 \\ 0 & 0 & 1 & 4 & -5 & 1 \end{array}\right]$ (1)

$\therefore A^{-1} = \begin{bmatrix} 18 & -24 & 5 \\ -15 & 20 & -4 \\ 4 & -5 & 1 \end{bmatrix}$ (1)

(b). (i). Let $\underset{\sim}{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ & $\underset{\sim}{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in V$ $\therefore u_1 + u_2 = u_3$

$v_1 + v_2 = v_3$

$\underset{\sim}{u} + \underset{\sim}{v} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}$ (1/2) Notice that $(u_1+v_1)+(u_2+v_2) = u_1+v_1+u_2+v_2$

$= (u_1+u_2)+(v_1+v_2) = u_3+v_3$

$\therefore \underset{\sim}{u} + \underset{\sim}{v} \in U$ (1/2) ie closed under addition

Let $s \in \mathbb{R}$

$s\underset{\sim}{u} = \begin{bmatrix} su_1 \\ su_2 \\ su_3 \end{bmatrix}$ (1/2) Notice that $su_1 + su_2 = s(u_1+u_2) = su_3$ (1/2)

$\therefore s\underset{\sim}{u} \in U$ ie closed under scalar multiplication

$\therefore$ Subspace of $\mathbb{R}^3$ (1)

(ii). Note that $\underset{\sim}{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ does not have its second component equal to -2

ie $\underset{\sim}{0} \notin V$

$\therefore$ Since $\underset{\sim}{0} \in V$ (1) $\therefore$ Not a subspace of $\mathbb{R}^3$ (1)

(c). $A = \begin{bmatrix} 1 & -4 \\ 1 & -2 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}$ & $\underset{\sim}{b} = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 5 \end{bmatrix}$

$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -4 & -2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 1 & -2 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 40 \end{bmatrix}$ 1

$(A^TA)^{-1} = \frac{1}{(4)(40)-(0)(0)} \begin{bmatrix} 40 & 0 \\ 0 & 4 \end{bmatrix} = \frac{1}{160} \begin{bmatrix} 40 & 0 \\ 0 & 4 \end{bmatrix}$ 1

$\text{pinv}(A) = (A^TA)^{-1}A^T = \frac{1}{160} \begin{bmatrix} 40 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -4 & -2 & 2 & 4 \end{bmatrix}$ 1

$= \frac{1}{160} \begin{bmatrix} 40 & 40 & 40 & 40 \\ -16 & -8 & 8 & 16 \end{bmatrix}$ 1

$\therefore \underset{\sim}{\hat{x}} = \text{pinv}(A)\underset{\sim}{b} = \frac{1}{160} \begin{bmatrix} 40 & 40 & 40 & 40 \\ -16 & -8 & 8 & 16 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 2 \\ 5 \end{bmatrix}$ 1

$= \frac{1}{160} \begin{bmatrix} 240 \\ 112 \end{bmatrix}$ 1 $= \begin{bmatrix} 1.5 \\ 0.7 \end{bmatrix}$ 1 $= \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$

$\therefore y = 1.5 + 0.7x$ 1