WORKSHOP 10 SOLUTIONS

1. Direction of line $\boldsymbol{a} = [2-3, 1-5, -1-(-2)] = [-1, -4, 1] = [a_1, a_2, a_3]$
   Point $P(x_0, y_0, z_0) = (2, 1, -1)$
   Cartesian equations: $\dfrac{x-x_0}{a_1} = \dfrac{y-y_0}{a_2} = \dfrac{z-z_0}{a_3} \;\Rightarrow\; \dfrac{x-2}{-1} = \dfrac{y-1}{-4} = \dfrac{z+1}{1}$

2. This line has the same direction as the given line, $\boldsymbol{a} = [3, -2, 1] = [a_1, a_2, a_3]$
   Given the point $P(x_0, y_0, z_0) = (2, 5, -2)$
   Vector equation: $\boldsymbol{r} = [x_0, y_0, z_0] + t[a_1, a_2, a_3] = [2, 5, -2] + t[3, -2, 1]$
   Parametric equations:
   $x = x + 0 + ta_1, y = y_0 + ta_2, z = z_0 + ta_3 \;\Rightarrow\; x = 2 + 3t, y = 5 - 2t, z = -2 + t$

3. The direction of the line through the points $(7, 2, 2)$ and $(1, 4, -2)$ is,
   $$\boldsymbol{d}_1 = [1-7, 4-2, -2-2] = [-6, 2, -4]$$
   The direction of the given line is, $\boldsymbol{d}_2 = [3, -1, 2]$
   Since $\boldsymbol{d}_1 = -2\boldsymbol{d}_2$ the direction vectors of the two lines are parallel and thus the lines are parallel.

4. The direction of the line is, $\boldsymbol{a} = [4, -2, 2]$.
   Let t_1 be the value of t giving the point M on line that is closest to the point $P(0, 0, 12)$. $\therefore M(4t_1, -2t_1, 2t_1)$
   So $\overrightarrow{PM} = [4t_1, -2t_1, 2t_1 - 12]$, and we know that $\overrightarrow{PM} \cdot \boldsymbol{a} = 0$
   $\Rightarrow\; [4t_1, -2t_1, 2t_1 - 12] \cdot [4, -2, 2] = 0 \;\Rightarrow\; 4(4t_1) - 2(-2t_1) + 2(2t_1 - 12) = 0$
   $\Rightarrow\; 16t_1 + 4t_1 + 4t_1 - 24 = 0 \;\Rightarrow\; 24t_1 = 24 \;\Rightarrow\; t_1 = 1$
   $\therefore \overrightarrow{PM} = [4(1), -2(1), 2(1) - 12] = [4, -2, -10]$
   distance $= \left\|\overrightarrow{PM}\right\| = \sqrt{(4)^2 + (-2)^2 + (-10)^2} = \sqrt{16 + 4 + 100} = \sqrt{120} \;\approx\; 10.95$

5. (i) Direction of L_1, $\boldsymbol{d}_1 = [2, 4, -1]$; Direction of L_2, $\boldsymbol{d}_2 = [4, 2, 4]$
   Since $\boldsymbol{d}_1 \neq m\,\boldsymbol{d}_2$, the lines are not parallel

   Intersect if:
   $$\begin{array}{rrcll} x: & 3 + 2t & = & 1 + 4\tau & (1) \\ y: & -1 + 4t & = & 1 + 2\tau & (2) \\ z: & 2 - t & = & -3 + 4\tau & (3) \end{array}$$

   From equation (3), $t = 2 + 3 - 4\tau \;\Rightarrow\; \boxed{t = 5 - 4\tau}$

   Substitute this into equation (2) to get: $-1 + 4(5 - 4\tau) = 1 + 2\tau$
   $\Rightarrow\; -1 + 20 - 16\tau = 1 + 2\tau \;\Rightarrow\; 18 = 18\tau \;\Rightarrow\; \boxed{\tau = 1}$
   $\therefore\; t = 5 - 4(1) = 1 \;\Rightarrow\; \boxed{t = 1}$

   Substitute $t = 1$, $\tau = 1$ into equation 1:
   LHS $= 3 + 2(1) = 5$, RHS $= 1 + 4(1) = 5 =$ RHS
   Since these are consistent, the lines intersect. The point of intersection is
   $x = 3 + 2(1) = 5,\;\; y = -1 + 4(1) = 3,\;\; z = 2 - 1 = 1$

   So the lines intersect at the point $(5, 3, 1)$.

(ii) Direction of L_1, $\boldsymbol{d}_1 = [2, -1, 3]$; Direction of L_2, $\boldsymbol{d}_2 = [-1, 3, 1]$
Since $\boldsymbol{d}_1 \neq m\,\boldsymbol{d}_2$, the lines are not parallel

Intersect if:

$$\begin{array}{llcll} x: & 1+2t & = & 2-\tau & (1) \\ y: & -1-t & = & 3\tau & (2) \\ z: & 3t & = & 1+\tau & (3) \end{array}$$

From equation (3), $\boxed{\tau = 3t-1}$
Substitute $\tau = 3t - 1$ into equation (2) to get: $-1-t = 3(3t-1)$
$\Rightarrow\ -1-t = 9t-3\ \Rightarrow\ 2 = 10t\ \Rightarrow \boxed{t = \frac{1}{5}}\quad \therefore\ \tau = 3(\frac{1}{5}) - 1\ \Rightarrow \boxed{\tau = -\frac{2}{5}}$
Substitute $t = \frac{1}{5}$, $\tau = -\frac{2}{5}$ into equation (1):

LHS $= 1 + 2(\frac{1}{5}) = \frac{7}{5}$, RHS $= 2 - \frac{2}{5} = \frac{9}{5} \neq$ LHS, so the lines do not intersect and hence they are skew. Find the shortest distance between the two lines:

Point on L_1: $P(1, -1, 0)$, Point on L_2: $Q(2, 0, 1)$

Vector $\overrightarrow{PQ} = [2-1, 0-(-1), 1-0] = [1, 1, 1]$

Perpendicular vector $\boldsymbol{n} = \boldsymbol{d}_1 \times \boldsymbol{d}_2 = \boldsymbol{i}\,(-1-9) + \boldsymbol{j}\,(-3-2) + \boldsymbol{k}\,(6-1)$

$$= [-10, -5, 5]$$

$$\text{Distance} = \left|\overrightarrow{PQ}\cdot\hat{\boldsymbol{n}}\right| = \left|\frac{\overrightarrow{PQ}\cdot\boldsymbol{n}}{||\boldsymbol{n}||}\right| = \left|\frac{[1,1,1]\cdot[-10,-5,5]}{\sqrt{(-10)^2+(-5)^2+(5)^2}}\right| = \left|\frac{-10-5+5}{\sqrt{100+25+25}}\right|$$

$$= \left|\frac{-10}{\sqrt{150}}\right| = \frac{10}{\sqrt{150}} \approx 0.816$$

6. Two vectors in the plane are:

$$\overrightarrow{PQ} = [1-0, 0-1, 1-1] = [1, -1, 0]$$
$$\overrightarrow{PR} = [1-0, 1-1, 0-1] = [1, 0, -1]$$

A normal vector is given by:
$\boldsymbol{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = [1, -1, 0] \times [1, 0, -1]$
$\quad = [(-1)(-1) - (0)(0), (0)(1) - (1)(-1), (1)(0) - (-1)(1)] = [1, 1, 1]$

Hence, equation of the plane takes the form $1x + 1y + 1z = d\ \Rightarrow\ x + y + z = d$
Using point $P(0, 1, 1)\ \Rightarrow\ d = x + y + z = 0 + 1 + 1 = 2$,

$$\therefore\ x + y + z = 2$$

7. Note that a point on the line is $t = 0\ \Rightarrow\ Q(4, 3, 7)$
Since $\overrightarrow{PQ} = [4-6, 3-0, 7-(-2)] = [-2, 3, 9]$ is in the plane, the normal is $\boldsymbol{n} = \overrightarrow{PQ} \times \boldsymbol{a}$ where $\boldsymbol{a} = [-2, 5, 4]$ is the direction of the line, i.e.,
$\boldsymbol{n} = \overrightarrow{PQ} \times \boldsymbol{a} = [-2, 3, 9] \times [-2, 5, 4]$
$\quad = [(3)(4) - (9)(5), (9)(-2) - (-2)(4), (-2)(5) - (3)(-2)] = [-33, -10, -4]$

Hence, equation of the plane takes the form $-33x + -10y + -4z = d$
Using point $P(6, 0, -2)\ \Rightarrow\ d = -33(6) - 10(0) - 4(-2) = -190$,

$$\therefore\ -33x - 10y - 4z = -190$$

8. The normal vectors to the planes are $\boldsymbol{n}_1 = [3, -2, 1]$ and $\boldsymbol{n}_2 = [2, 1, -3]$. The direction of the line is:
$\boldsymbol{a} = \boldsymbol{n}_1 \times \boldsymbol{n}_2 = [3, -2, 1] \times [2, 1, -3]$
$= [(-2)(-3) - (1)(1), (1)(2) - (3)(-3), (3)(1) - (-2)(2)] = [5, 11, 7] = [a_1, a_2, a_3]$
Now all we need is a point on the line. If we set $z = 0$, then
$$\begin{aligned} 3x - 2y &= 1 \\ 2x + y &= 3 \end{aligned}$$
solving these two equations simultaneously gives $x = 1$ and $y = 1$. Hence, a point on the line is $(1, 1, 0) = (x_0, y_0, z_0)$. Finally, the parametric equations of the line are $x = 1 + 5t$, $y = 1 + 11t$, $z = 7t$.

9. (i) The normal vectors to the planes are $\boldsymbol{n}_1 = [1, 0, 1]$ and $\boldsymbol{n}_2 = [0, 1, 1]$.
Since $\boldsymbol{n}_1 \neq m\boldsymbol{n}_2$ the planes are not parallel.
Also as $\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = [1, 0, 1] \cdot [0, 1, 1] = 0 + 0 + 1 = 1 \neq 0$ the planes are not perpendicular. $\therefore$ Neither.

   (ii) The normal vectors to the planes are $\boldsymbol{n}_1 = [-8, -6, 2]$ and $\boldsymbol{n}_2 = [4, 3, -1]$.
Since $\boldsymbol{n}_1 = -2\boldsymbol{n}_2$ the normals are parallel and thus the planes are parallel.

   (ii) The normal vectors to the planes are $\boldsymbol{n}_1 = [1, 4, -3]$ and $\boldsymbol{n}_2 = [-3, 6, 7]$.
Since $\boldsymbol{n}_1 \neq m\boldsymbol{n}_2$ the planes are not parallel.
As $\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = [1, 4, -3] \cdot [-3, 6, 7] = -3 + 24 - 21 = 0$ the normals are perpendicular and thus the planes are perpendicular.

10. The normal vectors to the planes are $\boldsymbol{n}_1 = [1, 1, 1]$ and $\boldsymbol{n}_2 = [1, 2, 3]$
The angle between the normal vectors and hence the planes is,
$$\theta = \cos^{-1}\left(\frac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{||\boldsymbol{n}_1||\,||\boldsymbol{n}_2||}\right) = \cos^{-1}\left(\frac{[1, 1, 1] \cdot [1, 2, 3]}{\sqrt{(1)^2 + (1)^2 + (1)^2}\,\sqrt{(1)^2 + (2)^2 + (3)^2}}\right)$$
$$\therefore\ \theta = \cos^{-1}\left(\frac{6}{\sqrt{3}\,\sqrt{14}}\right) = 22.21^\circ$$

11. Put $x = y = 0$ in the equation of the plane to get $z = 5$, *i.e.* a point on the plane is $A(0, 0, 5)$. We have $\boldsymbol{n} = [4, -6, 1]$, $||\boldsymbol{n}|| = \sqrt{(4)^2 + (-6)^2 + (1)^2} = \sqrt{53}$, and $\overrightarrow{AP} = [3 - 0, -2 - 0, 7 - 5] = [3, -2, 2]$
$$d = \left|\overrightarrow{AP} \cdot \hat{\boldsymbol{n}}\right| = \left|[3, -2, 2] \cdot \frac{[4, -6, 1]}{\sqrt{53}}\right| = \frac{26}{\sqrt{53}} \approx 3.57 \text{ (2 d.p.)}$$