MATH1019 Mid-Semester Test

Semester 1, 2020

SOLUTIONS

Question 1

The time taken by students to complete an online assignment is Normally distributed. According to past records the completion time, in minutes, had a mean of 98.5 and a standard deviation of 6.7. A new similar online assignment is released. It is found that the total time taken to complete the new assignment by 10 randomly selected students is 1008.8 minutes. You plan to conduct a hypothesis test to study whether the mean completion time for the new assignment is different from the mean completion time of the old assignment, at the 10% significance level.

(You may assume that the times taken for the new assignment are Normally distributed, with standard deviation given as above.)

(a) Explain what is meant by the 10% significance level. (2 marks)

The probability of rejecting the null hypothesis when it is true is 0.1 **2 marks**

(b) State the null and alternative hypotheses. (2 marks)

H_0: $\mu = 98.5$ **1 mark**

H_1: $\mu \neq 98.5$ **1 mark**

(c) Carry out the hypothesis test (be sure to state the critical region, test statistic and conclusion with reasoning) (6 marks)

Critical region: $\{z < -z_{0.05} \text{ or } z > z_{0.05}\}$ i.e. $\{z < -1.645 \text{ or } z > 1.645\}$ **2 marks**

Test statistic:

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{100.88 - 98.5}{6.7/\sqrt{10}} = 1.1233 \ \ (to\ 4dps)$$

1 mark **1 mark**

1 mark

Conclusion: Since $-1.645 < z < 1.645$, the mean completion time for the new assignment is not statistically significantly different from 98.5 minutes, at the 10% level of significance.

1 mark

(d) State the p-value for the test. Does it support your conclusion in part (c)? (3 marks)

1 mark   1 mark

p-value = $2P(z > 1.1233) \approx 2(1 - 0.8686) = 0.2628$ (Note any value within [0.2613,0.2628] is acceptable. Students could have looked up 1.12 in tables or used software)

Since the p-value is greater than 0.1, we reach the same conclusion as above. 1 mark

Question 2

The random variable X takes the values $-1, 1, 2$ and 3 only. X has probability function given by $\frac{c}{x^2}$, where c is a constant.

(a) Find the exact value of c. Hence, construct the probability distribution table for X. (4 marks)

$$\sum P(X = x) = 1 \Rightarrow c + c + \frac{c}{4} + \frac{c}{9} = 1$$ 1 mark

$$\Rightarrow \frac{85}{36}c = 1, \quad \therefore c = \frac{36}{85}$$ 1 mark

Probability distribution for X:

X	-1	1	2	3
P(X=x)	36/85	36/85	9/85	4/85

2 marks

(b) Compute the expected value and variance of X. (5 marks)

1 mark

$$E(X) = -1\left(\frac{36}{85}\right) + \left(\frac{36}{85}\right) + 2\left(\frac{9}{85}\right) + 3\left(\frac{4}{85}\right) = \frac{30}{85} = \frac{6}{17}$$ 1 mark

$$E(X^2) = \left(\frac{36}{85}\right) + \left(\frac{36}{85}\right) + 4\left(\frac{9}{85}\right) + 9\left(\frac{4}{85}\right) = \frac{144}{85}$$ 1 mark

$$Var(X) = E(X^2) - \left(E(X)\right)^2 = \frac{144}{85} - \left(\frac{6}{17}\right)^2 \approx 1.5696$$

1 mark   1 mark

Question 3

Suppose there are two boxes, A and B, with numbered cards. Box A consists of ten cards numbered $1, 2, 2, 3, 3, 3, 4, 4, 4, 4$. Box B consists of five cards $0, 0, 5, 5, 5$. One card is chosen randomly from each box. The random variable X is defined as the sum of the two numbers on the cards. Construct the probability distribution table for X and hence find $E(X)$. (6 marks)

X	P(X=x)
1	1/10×2/5=1/25
2	2/10×2/5=2/25
3	3/10×2/5=3/25
4	4/10×2/5=4/25
6	1/10×3/5=3/50
7	2/10×3/5=3/25
8	3/10×3/5=9/50
9	4/10×3/5=6/25

5 marks

$$E(X) = \frac{1}{25} + \frac{4}{25} + \frac{9}{25} + \frac{16}{25} + \frac{18}{50} + \frac{21}{25} + \frac{72}{50} + \frac{54}{25} = 6$$ 1 mark

Question 4

The Environmental Protection Agency has been studying Miller Creek regarding ammonia nitrogen concentration. For many years, the concentration has been 2.3 mg/L. However, a new golf course and housing developments are raising concern that the concentration may have changed because of lawn fertiliser. Any change (either an increase or a decrease) in the ammonia nitrogen concentration can affect plant and animal life in and around the creek. A sample of eight water tests from the creek gave a sample mean of 2.51 mg/L and a sample standard deviation of 0.35 mg/L.

(a) Construct a 95% confidence interval for μ, the mean ammonia nitrogen concentration. (4 marks)

Degrees of freedom = 8-1=7, α = 0.05 1 mark

95% CI: $\bar{x} \pm t^* \frac{s}{\sqrt{n}} = 2.51 \pm 2.365 \frac{0.35}{\sqrt{8}} = 2.51 \pm 0.2927 = (2.217, 2.803)$

1 mark 1 mark 1 mark

(b) State any assumptions you made in constructing the confidence interval above. (2 marks)

1 mark

We are assuming that we have a simple random sample and that the distribution of ammonia nitrogen concentration under study is normal. 1 mark

(c) Based on your interval in part (a), do you think that there is enough evidence in the sample to conclude that the ammonia nitrogen concentration has changed? Explain. (2 marks)

1 mark

No, at 5% level of significance. This is because the null hypothesis value of 2.3 is in the 95% confidence interval for the mean. 1 mark

Question 5

The amount of coffee dispensed by a coffee machine is normally distributed with mean 205 mL and standard deviation 20 mL.

(a) What is the probability that a cup contains between 191 mL and 209 mL of coffee? (3 marks)

Let X be the amount of coffee dispensed by the coffee machine.

$$P(191 < X < 219) = P\left(\frac{191-205}{20} < Z < \frac{209-205}{20}\right) = P(-0.7 < Z < 0.2)$$ 2 marks

$$= P(Z < 0.2) - P(Z < -0.7) = 0.5793 - 0.2420 = 0.3373$$ 1 mark

(b) Above what value do we get the biggest 25% of the drinks? (3 marks)

$$P(X > x) = 0.25$$

i.e.

$$P\left(Z < \frac{x-205}{20}\right) = 0.75$$ 1 mark

Probability of 0.75 corresponds to a z-value of 0.675. Therefore,

$$\frac{x-205}{20} = 0.675, i.e. x = (0.675)(20) + 205 = 218.5$$ 2 marks

For another coffee machine, the amount coffee is also normally distributed but with mean 210 mL.

(c) If 20% of 220 mL cups will overflow using this particular coffee machine, what is the population standard deviation? (4 marks)

$$P(X > 220) = 0.2$$

i.e.

$$P\left(Z > \frac{220-210}{\sigma}\right) = 0.2$$ 1 mark

i.e.

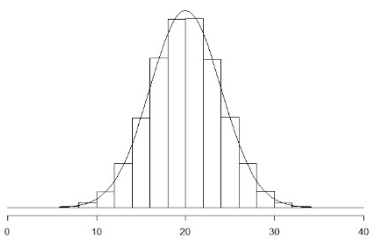
$$P\left(Z < \frac{220-210}{\sigma}\right) = 1 - 0.2 = 0.8$$ 1 mark

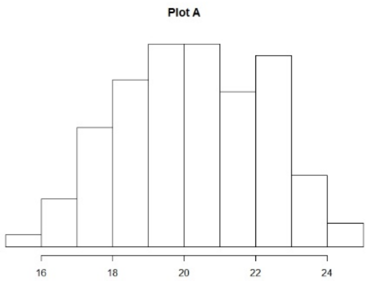
Probability of 0.8 corresponds to a z-value of 0.84. Therefore,

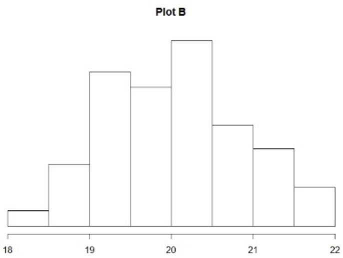
$$\frac{10}{\sigma} \approx 0.84, i.e. \sigma \approx \frac{10}{0.84} = 11.9$$ **2 marks**

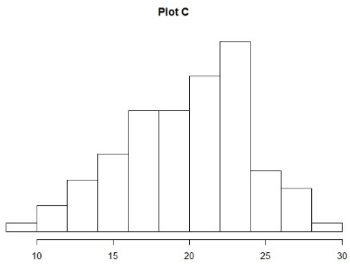
Question 6

Below are four plots. The top plot represents the distribution of a population with μ=20 and σ=4. The remaining three plots show a distribution of : (**1**) a single random sample of 100 values from this population; (**2**) a distribution of 100 sample means from random samples of size 5 from this population; and (**3**) a distribution of 100 sample means from random samples of size 25 from this population. Match each plot **A**, **B**, and **C** with the corresponding description **1**, **2**, and **3**. Justify your answer. (4 marks)









Plot A: 2 **1 mark**

Plot B: 3 **1 mark**

Plot C: 1 **1 mark**

Justification: look at ranges not shapes, since parent population is Normal. **1 mark**

Question 7

Given the points,

$$A(1,-1,2),\ B(0,4,3),\ C(2,1,0),\ D(3,-3,-1),\ E(1,2,-2),\ F(3,0,2)$$

and also given the vectors,

$$\boldsymbol{a} = [-1,4,2],\ \boldsymbol{b} = [0,3,4],\ \boldsymbol{c} = [2,-2,\sqrt{8}],\ \boldsymbol{d} = [2,8,-4],$$

$$\boldsymbol{e} = [4,8,x],\ \boldsymbol{f} = [5,2,-1],\ \boldsymbol{g} = [1,y,-1]$$

Solve the following questions.

(a) Find the position vector of point A. (1 mark)

$\overrightarrow{0A} = [1,-1,2]$ **1 mark**

(b) Find a vector with the same length as $\boldsymbol{a}$ but in the opposite direction. (1 mark)

$-\boldsymbol{a} = -[-1,4,2] = [1,-4,-2]$ **1 mark**

(c) Find the unit vector $\widehat{\overrightarrow{AB}}$. (2 marks)

$\overrightarrow{AB} = [0,4,3] - [1,-1,2] = [-1,5,1]$ **0.5 mark**

$$\widehat{\overrightarrow{AB}} = \frac{\overrightarrow{AB}}{\|\overrightarrow{AB}\|} = \frac{[-1,5,1]}{\sqrt{(-1)^2+(5)^2+(1)^2}} = \frac{[-1,5,1]}{\sqrt{27}} = \left[\frac{-1}{\sqrt{27}}, \frac{5}{\sqrt{27}}, \frac{1}{\sqrt{27}}\right]$$ **1 mark**

0.5 mark

(d) Find a vector with the same length as $\boldsymbol{b}$ but in the direction of $\boldsymbol{c}$. (3 marks)

$\|\boldsymbol{b}\| = \sqrt{(0)^2 + (3)^2 + (4)^2} = \sqrt{25} = 5$ **0.5 mark**

$$\|\boldsymbol{b}\|\hat{\boldsymbol{c}} = \|\boldsymbol{b}\|\frac{\boldsymbol{c}}{\|\boldsymbol{c}\|} = 5\frac{[2,-2,\sqrt{8}]}{\sqrt{(2)^2 + (-2)^2 + \left(\sqrt{8}\right)^2}} = 5\frac{[2,-2,\sqrt{8}]}{\sqrt{16}} = \frac{5}{4}[2,-2,\sqrt{8}] = \left[\frac{5}{2}, \frac{-5}{2}, \frac{5\sqrt{8}}{4}\right]$$ **1 mark**

1 mark **0.5 mark**

(e) Determine whether the vectors $\boldsymbol{a}$ and $\boldsymbol{d}$ are parallel or not. (2 marks)

Not parallel, **1 mark**

since $\boldsymbol{a} \neq m\boldsymbol{d}$ (i.e. $\boldsymbol{a}$ and $\boldsymbol{d}$ are not scalar multiples of each other) **1 mark**

(f) Find the value of x that makes the vectors $\boldsymbol{b}$ and $\boldsymbol{e}$ perpendicular to each other. (2 marks)

Perpendicular if $\boldsymbol{b}.\boldsymbol{e} = 0$ **0.5 mark**

Hence, $\boldsymbol{b}.\boldsymbol{e} = [0,3,4].[4,8,x] = (0)(4) + (3)(8) + (4)(x) = 24 + 4x$ **1 mark**

$\therefore 24 + 4x = 0 \Rightarrow 4x = -24 \Rightarrow x = \frac{-24}{4} = -6$ **0.5 mark**

(g) Find a non-zero vector perpendicular to both $\boldsymbol{d}$ and $\boldsymbol{f}$. (2 marks)

0.5 mark **1 mark**

$\boldsymbol{d} \times \boldsymbol{f} = [(8)(-1) - (-4)(2), (-4)(5) - (2)(-1), (2)(2) - (8)(5)] = [0, -18, -36]$ **0.5 mark**

Any non-zero multiple of $\boldsymbol{d} \times \boldsymbol{f}$ is acceptable.

(i) If the scalar projection of the vector $\boldsymbol{g}$ on $\boldsymbol{f}$ is $\frac{1}{\sqrt{30}}$, find the value of y. (3 marks)

Given the scalar proj $\boldsymbol{g}$ on $\boldsymbol{f}$ is, $p = \boldsymbol{g}.\hat{\boldsymbol{f}} = \frac{\boldsymbol{g}.\boldsymbol{f}}{\|\boldsymbol{f}\|} = \frac{1}{\sqrt{30}}$ **0.5 mark**

0.5 mark

$$p = \boldsymbol{g}.\hat{\boldsymbol{f}} = \frac{\boldsymbol{g}.\boldsymbol{f}}{\|\boldsymbol{f}\|} = \frac{[1, y, -1].[5,2,-1]}{\sqrt{(5)^2 + (2)^2 + (-1)^2}} = \frac{5 + 2y + 1}{\sqrt{30}} = \frac{2y + 6}{\sqrt{30}}$$

0.5 mark

$$\therefore \frac{2y + 6}{\sqrt{30}} = \frac{1}{\sqrt{30}} \Rightarrow 2y + 6 = 1 \Rightarrow 2y = -5 \Rightarrow y = -\frac{5}{2}$$ **1 mark**

0.5 mark

(j) Given a quadrilateral $ABCD$ is formed by the points, A, B, C and D, determine the angle at the vertex C. (4 marks)

$\overrightarrow{CB} = [0,4,3] - [2,1,0] = [-2,3,3]$ **0.5 mark**

$\overrightarrow{CD} = [3,-3,-1] - [2,1,0] = [1,-4,-1]$ **0.5 mark**

$$\theta = \cos^{-1}\left(\frac{\overrightarrow{CB}.\overrightarrow{CD}}{\|\overrightarrow{CB}\|\|\overrightarrow{CD}\|}\right) = \cos^{-1}\left(\frac{[-2,3,3].[1,-4,-1]}{\sqrt{(-2)^2 + (3)^2 + (3)^2}.\sqrt{(1)^2 + (-4)^2 + (-1)^2}}\right)$$ **0.5 mark**

0.5 mark **0.5 mark** **0.5 mark**

$$= \cos^{-1}\left(\frac{(-2)(1) + (3)(-4) + (3)(-1)}{\sqrt{4 + 9 + 9}\sqrt{1 + 16 + 1}}\right) = \cos^{-1}\left(\frac{-17}{\sqrt{22}\sqrt{18}}\right)$$ **0.5 mark**

$\therefore \theta = \cos^{-1}(-0.85428) \approx 148.68^{\circ}$ **0.5 mark**

(k) Determine whether the four points C, D, E and F are coplanar or not. (5 marks)

$\overrightarrow{CD} = [3,-3,-1] - [2,1,0] = [1,-4,-1]$ **0.5 mark**

$\overrightarrow{CE} = [1,2,-2] - [2,1,0] = [-1,1,-2]$ **0.5 mark**

$\overrightarrow{CF} = [3,0,2] - [2,1,0] = [1,-1,2]$ **0.5 mark**

Finding the scalar triple product, $\overrightarrow{CD}.\left(\overrightarrow{CE} \times \overrightarrow{CF}\right)$

1 mark

$\overrightarrow{CE} \times \overrightarrow{CF} = [(1)(2) - (-2)(-1), (-2)(1) - (-1)(2), (-1)(-1) - (1)(1)] = [0,0,0]$ **0.5 mark**

$\therefore \overrightarrow{CD}.\left(\overrightarrow{CE} \times \overrightarrow{CF}\right) = [1,-4,-1].[0,0,0] = 0 + 0 + 0 = 0$ **0.5 mark**

Since $\overrightarrow{CD}.\left(\overrightarrow{CE} \times \overrightarrow{CF}\right) = 0$ they are coplanar. **1 mark**

0.5 mark