WORKSHOP 5 SOLUTIONS

1. (i) $\boldsymbol{a}+\boldsymbol{b} \;=\; [2,-1,3]+[4,0,-3] \;=\; [2+4,-1+0,3-3] \;=\; [6,-1,0]$

   (ii) $3\boldsymbol{a}-4\boldsymbol{c} \;=\; 3[2,-1,3]-4[1,-2,2] \;=\; [6,-3,9]-[4,-8,8]$
   $\;=\; [6-4,-3-(-8),9-8] \;=\; [2,5,1]$

   (iii) $||\boldsymbol{b}|| = \sqrt{(4)^2+(0)^2+(-3)^2} = \sqrt{16+0+9} = \sqrt{25} = 5$

   (iv) $\hat{\boldsymbol{b}} = \dfrac{\boldsymbol{b}}{||\boldsymbol{b}||} = \dfrac{[4,0,-3]}{5} = \left[\dfrac{4}{5},\ 0,\ -\dfrac{3}{5}\right]$

   (v) $||\boldsymbol{c}||\,\hat{\boldsymbol{b}} = \sqrt{(1)^2+(-2)^2+(2)^2}\left[\dfrac{4}{5},\ 0,\ -\dfrac{3}{5}\right] = \sqrt{9}\left[\dfrac{4}{5},\ 0,\ -\dfrac{3}{5}\right] = \left[\dfrac{12}{5},\ 0,\ -\dfrac{9}{5}\right]$

2. $\boldsymbol{a} = \overrightarrow{OA} = [2,-3]$
   $\boldsymbol{b} = \overrightarrow{AB} = [4-2,\ 1-(-3)] = [2,4]$

3. (a) (i) $\boldsymbol{a}\cdot\boldsymbol{b} = [2,-4,\sqrt{5}]\cdot[-2,4,-\sqrt{5}]$
   $= (2)(-2)+(-4)(4)+(\sqrt{5})(-\sqrt{5}) \;=\; -4-16-5 \;=\; -25$

   (ii) $\theta = \cos^{-1}\left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{||\boldsymbol{a}||\,||\boldsymbol{b}||}\right) = \cos^{-1}\left(\frac{-25}{\sqrt{(2)^2+(-4)^2+(\sqrt{5})^2}\sqrt{(-2)^2+(4)^2+(-\sqrt{5})^2}}\right)$
   $= \cos^{-1}\left(\dfrac{-25}{\sqrt{25}\sqrt{25}}\right) = \cos^{-1}(-1) = 180^\circ$

   (iii) scalar projection, $p = \boldsymbol{a}\cdot\hat{\boldsymbol{b}} = \dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{||\boldsymbol{b}||} = \dfrac{-25}{5} = -5$

   (iv) vector projection, $\boldsymbol{p} = p\,\hat{\boldsymbol{b}} = p\,\dfrac{\boldsymbol{b}}{||\boldsymbol{b}||} = \dfrac{-5\,[-2,4,-\sqrt{5}]}{5} = [2,-4,\sqrt{5}]$

   (b) (i) $\boldsymbol{a}\cdot\boldsymbol{b} = (2\boldsymbol{i}+10\boldsymbol{j}-11\boldsymbol{k})\cdot(2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k})$
   $= (2)(2)+(10)(2)+(-11)(1) \;=\; 4+20-11 \;=\; 13$

   (ii) $\theta = \cos^{-1}\left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{||\boldsymbol{a}||\,||\boldsymbol{b}||}\right) = \cos^{-1}\left(\frac{13}{\sqrt{(2)^2+(10)^2+(-11)^2}\sqrt{(2)^2+(2)^2+(1)^2}}\right)$
   $= \cos^{-1}\left(\dfrac{13}{\sqrt{225}\sqrt{9}}\right) = \cos^{-1}\left(\dfrac{13}{(15)(3)}\right) \approx 73.21^\circ$

   (iii) scalar projection, $p = \boldsymbol{a}\cdot\hat{\boldsymbol{b}} = \dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{||\boldsymbol{b}||} = \dfrac{13}{3}$

   (iv) vector projection, $\boldsymbol{p} = p\,\hat{\boldsymbol{b}} = p\,\dfrac{\boldsymbol{b}}{||\boldsymbol{b}||} = \dfrac{13}{3}\dfrac{[2,2,1]}{3} = \left[\dfrac{26}{9},\dfrac{26}{9},\dfrac{13}{9}\right]$

   (c) (i) $\boldsymbol{a}\cdot\boldsymbol{b} = (\boldsymbol{i}+\boldsymbol{k})\cdot(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})$
   $= (1)(1)+(0)(1)+(1)(1) \;=\; 1+0+1 \;=\; 2$

(ii) $\theta = \cos^{-1}\left(\frac{\boldsymbol{a}.\boldsymbol{b}}{||\boldsymbol{a}||\,||\boldsymbol{b}||}\right) = \cos^{-1}\left(\frac{2}{\sqrt{(1)^2+(0)^2+(1)^2}\sqrt{(1)^2+(1)^2+(1)^2}}\right)$

$= \cos^{-1}\left(\frac{2}{\sqrt{2}\sqrt{3}}\right) = \cos^{-1}\left(\frac{2}{\sqrt{6}}\right) \approx 35.26°$

(iii) scalar projection, $p = \boldsymbol{a}\cdot\hat{\boldsymbol{b}} = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{||\boldsymbol{b}||} = \frac{2}{\sqrt{3}}$

(iv) vector projection, $\boldsymbol{p} = p\,\hat{\boldsymbol{b}} = p\,\frac{\boldsymbol{b}}{||\boldsymbol{b}||} = \frac{2}{\sqrt{3}}\,\frac{[1,1,1]}{\sqrt{3}} = \left[\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right]$

4. Consider a cube of side length 1 placed in 3 dimensional space so that one of the vertices of the cube is at the origin $(0,0,0)$ and the opposite vertex is at the point $(1,1,1)$. Then the vector $[1,0,0]$ represents an edge member (i.e. the edge falling along the x-axis) and $[1,1,1]$ is the diagonal member (i.e. the edge going from $(0,0,0)$ to $(1,1,1)$). The angle between these two members is,

$$\theta = \cos^{-1}\left(\frac{[1,0,0]\cdot[1,1,1]}{\sqrt{(1)^2+(0)^2+(0)^2}\sqrt{(1)^2+(1)^2+(1)^2}}\right) \quad = \cos^{-1}\left(\frac{1+0+0}{\sqrt{1}\sqrt{3}}\right)$$

$$= \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = 54.74°$$

5. $\boldsymbol{u}\cdot\boldsymbol{v} = [2,-2,-1]\cdot[3,5,-4] = (2)(3)+(-2)(5)+(-1)(-4) = 6-10+4 = 0$

   Since $\boldsymbol{u}\cdot\boldsymbol{v} = 0$, $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular.

6. Displacement $\boldsymbol{s} = [1,1,1]-[0,0,0] = [1,1,1]$
   Force $\boldsymbol{F} = 5\boldsymbol{j} = [0,5,0]$
   Work $= \boldsymbol{F}\cdot\boldsymbol{s} = [0,5,0]\cdot[1,1,1] = 0+5+0 = 5\,J$

7. $||\boldsymbol{s}|| = 15$, $||\boldsymbol{F}|| = 150$, $\theta = 45°$
   Work $= \boldsymbol{F}\cdot\boldsymbol{s} = ||\boldsymbol{F}||\,||\boldsymbol{s}||\cos\theta = (150)(15)\cos(45°) = \frac{2,250}{\sqrt{2}}\,J$

8. Direction cosines of $\boldsymbol{a}$ are given by $\hat{\boldsymbol{a}} = [\cos\alpha,\, \cos\beta,\, \cos\gamma]$

$$\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{||\boldsymbol{a}||} = \frac{[a_1,\, a_2,\, a_3]}{\sqrt{a_1^2+a_2^2+a_3^2}} = \left[\frac{a_1}{\sqrt{a_1^2+a_2^2+a_3^2}}, \frac{a_2}{\sqrt{a_1^2+a_2^2+a_3^2}}, \frac{a_3}{\sqrt{a_1^2+a_2^2+a_3^2}}\right]$$

$$\therefore\ \cos\alpha = \frac{a_1}{\sqrt{a_1^2+a_2^2+a_3^2}},\ \cos\beta = \frac{a_2}{\sqrt{a_1^2+a_2^2+a_3^2}},\ \cos\gamma = \frac{a_3}{\sqrt{a_1^2+a_2^2+a_3^2}}$$

$$\text{Thus}\ \ \cos^2\alpha+\cos^2\beta+\cos^2\gamma = \left(\frac{a_1}{\sqrt{a_1^2+a_2^2+a_3^2}}\right)^2 + \left(\frac{a_2}{\sqrt{a_1^2+a_2^2+a_3^2}}\right)^2 + \left(\frac{a_3}{\sqrt{a_1^2+a_2^2+a_3^2}}\right)^2$$

$$= \frac{a_1^2}{a_1^2+a_2^2+a_3^2} + \frac{a_2^2}{a_1^2+a_2^2+a_3^2} + \frac{a_3^2}{a_1^2+a_2^2+a_3^2}$$

$$= \frac{a_1^2+a_2^2+a_3^2}{a_1^2+a_2^2+a_3^2} \ = \ 1$$