Lecture 6

Vectors &

Introduction to Matrices

The Cross Product

The cross product of $\boldsymbol{a} = [a_1, a_2, a_3]$ and $\boldsymbol{b} = [b_1, b_2, b_3]$ is

$$\boxed{\boldsymbol{a} \times \boldsymbol{b} = [a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1]}$$

Note the following:

(i) The cross product is itself a vector.

(ii) Also known as the vector product.

(iii) It is only defined for vectors in 3 space.

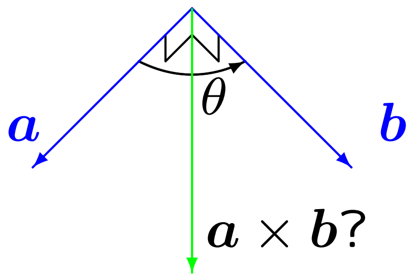
(iv) $\boldsymbol{a} \times \boldsymbol{b} \neq \boldsymbol{b} \times \boldsymbol{a}$!

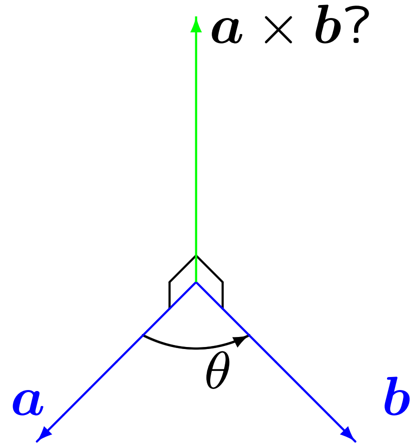
Memory aid:

$$\begin{array}{cccccc} + & + & + & - & - & - \\ & \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} & \boldsymbol{i} & \boldsymbol{j} \\ & a_1 & a_2 & a_3 & a_1 & a_2 \\ & b_1 & b_2 & b_3 & b_1 & b_2 \end{array}$$

<u>**Ex:**</u> Find $\boldsymbol{a} \times \boldsymbol{b}$ if $\boldsymbol{a} = [3, -1, 0]$ and $\boldsymbol{b} = [2, 4, -3]$.

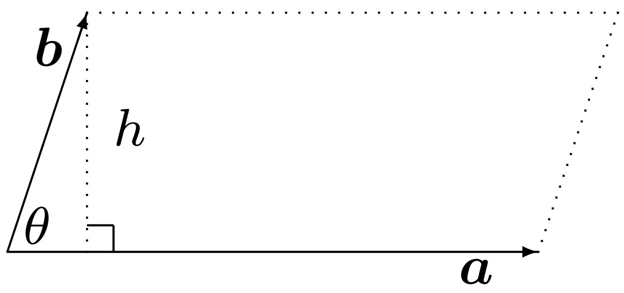
Note that $(\boldsymbol{a} \times \boldsymbol{b}).\boldsymbol{a} = 0$ and $(\boldsymbol{a} \times \boldsymbol{b}).\boldsymbol{b} = 0$, *i.e.* $\boldsymbol{a} \times \boldsymbol{b}$ is perpendicular to $\boldsymbol{a}$ and $\boldsymbol{b}$!





RIGHT HAND RULE: If the fingers of your right hand curl in the direction of rotation from $\boldsymbol{a}$ to $\boldsymbol{b}$ (through an angle $0^\circ \leq \theta \leq 180^\circ$), then your extended thumb points in the direction of $\boldsymbol{a} \times \boldsymbol{b}$.

Consider the parallelogram formed by $\boldsymbol{a}$ and $\boldsymbol{b}$.



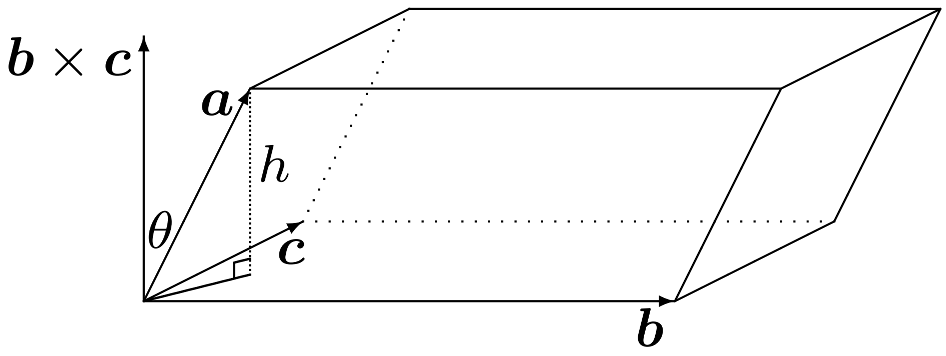
We have

$$||\boldsymbol{a} \times \boldsymbol{b}|| = \text{area of parallelogram.}$$

<u>**Ex:**</u> Find the area of the parallelogram formed by the vectors $\boldsymbol{a} = [3, -1, 0]$ and $\boldsymbol{b} = [2, 4, -3]$.

$\boldsymbol{a}.(\boldsymbol{b}\times\boldsymbol{c})$ is called the scalar triple product of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.

Consider the *parallelepiped* formed by $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.



$$\boxed{V = |\boldsymbol{a}.(\boldsymbol{b}\times\boldsymbol{c})|}$$

$\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are coplanar (ie. lie in the same plane) if and only if $\boldsymbol{a}.(\boldsymbol{b}\times\boldsymbol{c}) = 0$.

Ex: Find the volume of the parallelepiped formed by the vectors $\boldsymbol{a} = [3, 1, 3]$, $\boldsymbol{b} = [0, 1, -4]$ and $\boldsymbol{c} = [2, 2, 0]$.

Matrices

An $m \times n$ matrix A is a rectangular array of entries (real numbers for our purposes) consisting of m rows and n columns, *i.e.*

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

We say that A is of order (or dimension) $m \times n$. Note that we usually use uppercase letters to denote matrices.

For convenience, we have the following shorthand notation:

$$A = \left[a_{ij}\right], \quad 1 \leq i \leq m, \quad 1 \leq j \leq n.$$

In this context, a_{ij} is called the ijth entry of A, *i.e.* the entry which is in row i and column j of the matrix.

$1 \times n$ matrix is a row vector, $[a_1\ a_2\ \ldots\ a_n]$.

$n \times 1$ matrix is a column vector, $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

<u>Ex:</u> Matrices come in various sizes.

(i) $\begin{bmatrix} 1 & 2 \\ 5 & 3 \\ 6 & -5 \end{bmatrix}$ is a 3×2 matrix.

(ii) $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is a 3×1 matrix (ie. a column vector).

(iii) $[4\ \ 0\ \ -1]$ is a 1×3 matrix (ie. a row vector).

(iv) $[3]$ is a 1×1 matrix.

A zero matrix is one where all entries are equal to zero. For example, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a 2×3 zero matrix.
We usually denote a zero matrix simply as 0.

Two $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal if all their corresponding entries are equal, *i.e.* if

$$a_{ij} = b_{ij}, \qquad 1 \leq i \leq m,\ 1 \leq j \leq n.$$

This implicitly assumes that the matrices have the same order, of course. *e.g.* $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 4 & -1 \end{bmatrix}$ is not equal to $\begin{bmatrix} 2 & 0 \\ 1 & 4 \\ 1 & -1 \end{bmatrix}$, because the matrices are of different orders.

<u>Ex:</u> Solve for x and y, given that

$$\begin{bmatrix} x & 3y \\ 3y & x \end{bmatrix} = \begin{bmatrix} 6 & -9 \\ -9 & 6 \end{bmatrix}.$$

Operations on Matrices

1. **Matrix Addition:** We can only add matrices of the same order. If $A = [a_{ij}]$ and $B = [b_{ij}]$, then

$$C = A + B = [c_{ij}] = [a_{ij} + b_{ij}]$$

for all i and j. *e.g.*

$$\begin{bmatrix} 3 & 0 \\ 4 & -2 \\ 1 & 6 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ -4 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & -5 \\ 1 & 6 \end{bmatrix} + \begin{bmatrix} 3 & 4 & 5 \\ 1 & -1 & 2 \end{bmatrix} = \text{ d.n.e.}$$

2. **Scalar Multiplication:** If $A = [a_{ij}]$, then for any scalar k,

$$k\,A = k[a_{ij}] = [k\,a_{ij}]$$

i.e. each entry of the matric gets multiplied by the scalar.

e.g. $\quad 2\begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 4 & 2 \\ 10 & -6 \end{bmatrix}.$

3. **Matrix Multiplication:** The matrix product AB is only defined if the number of columns of A is equal to the number of rows of B. In that case

$$A_{m\times n}\, B_{n\times p} = C_{m\times p}$$

If $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = [c_{ij}]$, then

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

i.e. c_{ij} is obtained from the i-th row in A and the j-th column in B. In fact, note how c_{ij} is obtained by effectively taking the dot product of the i-th row in A and the j-th column in B.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \ldots & b_{np} \end{bmatrix}$$

$$= \begin{bmatrix} & & \\ & & \\ & c_{ij} & \\ & & \end{bmatrix}$$

Ex: Let $A = \begin{bmatrix} 4 & 1 \\ 0 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & -5 \\ 1 & 0 & -2 \end{bmatrix}$ and $C = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}$. Find $A - 3C$, AB and BA.

Clearly, in general $AB \neq BA$. This may be due to a number of different reasons:

- Either product may not exist because the orders don't match, as in the case above.

- AB and BA may be of different order.

- AB and BA may be the same order, but still not necessarily equal. *e.g.* if $A = \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$, then $AB = \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 6 & 6 \end{bmatrix}$ and $BA = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 3 & 3 \end{bmatrix} \neq AB$.

Other valid properties of matrix multiplication are these:

(i) $A(B+C) = AB + AC$

(ii) $(A+B)C = AC + BC$

(iii) (AB)C=A(BC)

But beware of the following:

(a) $AB = 0$ does not necessarily mean $A = 0$ or $B = 0$.
eg. $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -\frac{1}{2} \\ -1 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(b) $AB = AC$ does not necessarily mean $B = C$. *e.g.* $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $AB = AC$, but $B \neq C$.

The transpose of an $m \times n$ matrix $A = [a_{ij}]$ is the matrix $A^\top = [a'_{ij}]$ where $a'_{ij} = a_{ji}$.

Ex: $$A = \begin{bmatrix} -1 & 0 & 2 & 3 \\ 1 & 5 & -1 & 4 \\ 4 & 0 & 2 & 1 \end{bmatrix}, A^\top = \begin{bmatrix} -1 & 1 & 4 \\ 0 & 5 & 0 \\ 2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}.$$

A matrix A is said to be square of order n if it is of order $n \times n$. In this case, we can identify a main diagonal:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

Notice that $A^2 = AA$, or $A^m = \underbrace{AA \ldots A}_{m \text{ times}}$ in general, for a square matrix A.

Ex: Let $A = \begin{bmatrix} 4 & 1 \\ 1 & -3 \end{bmatrix}$, find A^2.

Finally, we say that a square matrix A is symmetric if $A^\top = A$. Note that A must be square for this to be possible.

Ex: $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 5 \\ -1 & 5 & 3 \end{bmatrix} = A^\top$ is symmetric.

Note that if $A = [a_{ij}]$ is symmetric, then $a_{ij} = a_{ji}$.

Identities and Inverses

An identity matrix is a square matrix (*i.e.* of order $n \times n$) with all main diagonal entries equal to 1 and all other entries equal to 0.

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

e.g. $I_1 = [1]$, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Note that I is the *identity element* of matrix multiplication, *i.e.* for $A_{m\times n}$,

$$AI_n = A \qquad \text{and} \qquad I_m A = A.$$

If A is square and of order n, and if there exists a B such that

$$AB = BA = I_n,$$

then we say that A is invertible, we call B the inverse of A and we write $B = A^{-1}$ to denote the inverse.

<u>Ex:</u> Verify that $B = A^{-1}$ if $A = \begin{bmatrix} 2 & 6 \\ 3 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 3 \\ \frac{3}{2} & -1 \end{bmatrix}$.

Clearly, if B is an inverse of A, then A is an inverse of B. Also, if the inverse of a matrix exists, then it is unique.

Note that not all square matrices are invertible. We'll look at invertibility and calculating inverses in more detail later.