WORKSHOP 9

This workshop will build on material from Lecture 9: Determinants.

During this workshop, students will work towards the following learning outcomes:

- calculate determinants of square matrices of any size.
- evaluate determinants of larger matrices by first applying appropriate elementary row or column operations.
- associate the relationship between the determinant of a matrix and its invertibility.
- solve a system of linear equations by applying Cramer's rule.
- calculate the cross product and scalar triple product using a determinant.

Determinants and Inverses

1. Given the following matrices,

$$A = \begin{bmatrix} 2 & -3 \\ 6 & -9 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} \quad C = \begin{bmatrix} 5 & 0 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 4 \\ 0 & 0 & -2 \end{bmatrix}$$

   (i) Calculate the determinant of the matrix.

   (ii) Given the determinant from (i) is the matrix singular or non-singular?

2. By using elementary row or column operations, calculate the following determinant,

$$\begin{vmatrix} 2 & 1 & 3 & 1 \\ -2 & 3 & -1 & 2 \\ 2 & 1 & 2 & 3 \\ -4 & -2 & -6 & -1 \end{vmatrix}$$

3. Find the inverse of the following matrices, if the inverse exists.

   (i) $\begin{bmatrix} 5 & 3 \\ 7 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} -2 & 4 \\ -3 & 6 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$

Cramer's Rule

4. Use Cramer's rule to solve the following systems of linear equations.

(i) $\begin{aligned} 3x_1 - 2x_2 &= 6 \\ -5x_1 + 4x_2 &= 8 \end{aligned}$ (ii) $\begin{aligned} x_1 + 2x_2 &= 3 \\ 3x_1 + x_2 &= -1 \end{aligned}$

5. Use Cramer's rule to solve the following system for x_3 without solving for the remaining variables.

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ 2x_1 - 5x_2 - 3x_3 &= 10 \\ 4x_1 + 8x_2 + 2x_3 &= 4 \end{aligned}$$

Cross and scalar triple products using determinants

6. For the following pairs of vectors, determine $\boldsymbol{a} \times \boldsymbol{b}$ by taking the determinant of an appropriate matrix.

(i) $\boldsymbol{a} = [3, 2, 1]$, $\boldsymbol{b} = [-1, 1, 4]$ (ii) $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$

7. By using an appropriate determinant, calculate the volume of the parallelepiped formed by the vectors $\boldsymbol{a} = [2, 6, -2]$, $\boldsymbol{b} = [-3, 2, 0]$ and $\boldsymbol{c} = [0, 1, 5]$.