Design and Analysis of Algorithms (COMP3001)

Tutorial 3

Divide and Conquer

Question 1.

a) Using Figure 2.4 (textbook 3rd edition) as a model, illustrate the operation of MERGESORT on the array *A* = <13, 19, 9, 5, 12, 8, 7, 4, 11, 2, 6, 21>.

b) Using Figure 7.1 (textbook 3rd edition) as a model, illustrate the operation of PARTITION on the array

- *A* = <13, 19, 9, 5, 12, 8, 7, 4, 11, 2, 6, 21>
- *A* = <13, 19, 9, 5, 12, 8, 7, 4, 11, 2, 6>

c) Using Figure 2.2 (textbook new edition) as a model, illustrate the operation of INSERTION-SORT on the array *A* = <13, 19, 9, 5, 12, 8, 7, 4, 11, 2, 6, 21>. Read the analysis of Insertion Sort.

Question 2.

Exercise 7.2-2 (textbook, Cormen, et al). What is the running time of QUICKSORT when all elements of array *A* have the same value? Is this the best case for QUICKSORT? Why?

Question 3.

a) Show that Quicksort's best case running time is $\Omega(n\log n)$.

b) Show that the running time of QUICKSORT is $\Theta(n^2)$ when the array A contains distinct elements and is sorted in decreasing order.

Question 4.

a) Design an algorithm that will take as input a set of real numbers *S* and a target real number *x*, and output TRUE if there are two numbers in *S* that sum to *x*, and FALSE otherwise.

b) How does the running time of your algorithm change if *S* is sorted or unsorted?

Question 5.

Chapter 8 (Textbook) provides the following Counting Sort algorithm. The algorithm considers an input array A [1 .. n], and assumes each element in array A is an integer in the range 0 to k. Note that $A.length$ is size of array A, and thus $A.length = n$. Further, the algorithm uses array B [1 .. n] to keep the sorted output, and arry C [0 .. k] for temporary storage.

Counting-Sort (A, B, k)

```
1    let C [0 .. k] be a new array
2    for i = 0 to k
3        C [i] = 0
4    for j = 1 to A.length
5        C [ A [j] ] = C [ A [j] ] + 1
6    // C [i] now contains the number of elements equal to i
7    for i = 1 to k
8        C [i ] = C [i] + C [i - 1]
9    // C [i] now contains the number of elements less than or equal to i
10   for j = A.length  downto 1
11       B [ C [ A [j] ]] = A [j]
12       C [ A [j] ] = C [ A [j] ] - 1
```

a) Show that Counting-Sort has a running time complexity of $O(n)$.

b) (Exercise 8.2.1 – textbook). Using Figure 8.2 as a model, illustrate the operation of Counting-Sort on the array A = [6, 0, 2, 0, 1, 3, 4, 6, 1, 3, 2], i.e.,

- Show the contents of array A and C after line 5.
- Show the content of array C after line 8.
- Show the content of array B and C after one, two, and three iterations of the loop in lines 10-12
- Show the final sorted output array B.

Question 6.

Consider an array A[1..n] of integers. Design a divide and conquer algorithm to find the minimum element in A. For example, if A=[3, 6, 1, 5, 7, 2, 1], your algorithm should obtain 1. **Your algorithm must be computationally as efficient as possible.**

(i) Write the pseudocode of your algorithm. **Hint.** Similar to MERGE_SORT.

(ii) Show how your algorithm works on the input A = [3, 6, 1, 5, 7, 2, 1].

(iii) Write the recurrence of the time complexity of your algorithm and solve the recurrence to find its time complexity in Big Oh.

Question 7.

a) Strassen's algorithm (discussed in the lecture) generates the following equations to produce:

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \times \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$S_1 = B_{12} - B_{22}$ $\quad S_6 = B_{11} + B_{22}$
$S_2 = A_{11} + A_{12}$ $\quad S_7 = A_{12} - A_{22}$
$S_3 = A_{21} + A_{22}$ $\quad S_8 = B_{21} + B_{22}$
$S_4 = B_{21} - B_{11}$ $\quad S_9 = A_{11} - A_{21}$
$S_5 = A_{11} + A_{22}$ $\quad S_{10} = B_{11} + B_{12}$

$P_1 = A_{11} \times S_1$
$P_2 = S_2 \times B_{22}$
$P_3 = S_3 \times B_{11}$
$P_4 = A_{22} \times S_4$
$P_5 = S_5 \times S_6$
$P_6 = S_7 \times S_8$
$P_7 = S_9 \times S_{10}$

$C_{11} = P_5 + P_4 - P_2 + P_6$
$C_{12} = P_1 + P_2$
$C_{21} = P_3 + P_4$
$C_{22} = P_5 + P_1 - P_3 - P_7$

Verify that the following equations are correct, and hence Strassen's algorithm is correct.

$C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21}$
$C_{12} = A_{11} \times B_{12} + A_{12} \times B_{22}$
$C_{21} = A_{21} \times B_{11} + A_{22} \times B_{21}$
$C_{22} = A_{21} \times B_{12} + A_{22} \times B_{22}$

b) Exercise 4.2-1. Use Strassen's algorithm to compute the matrix product. Show your work.

$$\begin{pmatrix} 1 & 3 \\ 7 & 5 \end{pmatrix} \begin{pmatrix} 6 & 8 \\ 4 & 2 \end{pmatrix}$$