Lecture 5. Set Theory

Ref.: K H Rosen Section 1.6 & 1.7

What is a set?

- Sets are used to group Objects together.
- Any real world object can be a member of a Set.

  E.g., Students in a Class.
- Sets are not limited to physical objects!

Definition 1: The *objects* in a set are also called *elements* or *members* of the set. A set is said to *contain* its elements.

Examples

- A Set of Vowels:

$$V = \{a, e, i, o, u\}$$

- The Set of Odd Positive numbers less than 10

$$O = \{1,3,5,7,9\}$$

- S={fred, a, 76, New Jersey}

Equality of Sets

Definition 2:
Two sets are equal if and only if they have the same elements.

{1,3,5}

{1,5,3}

{1,1,5,5,3,3,3}

Equal!

Special Cases

Sets are usually denoted with Uppercase Letters. There are 3 Reserved Letters:

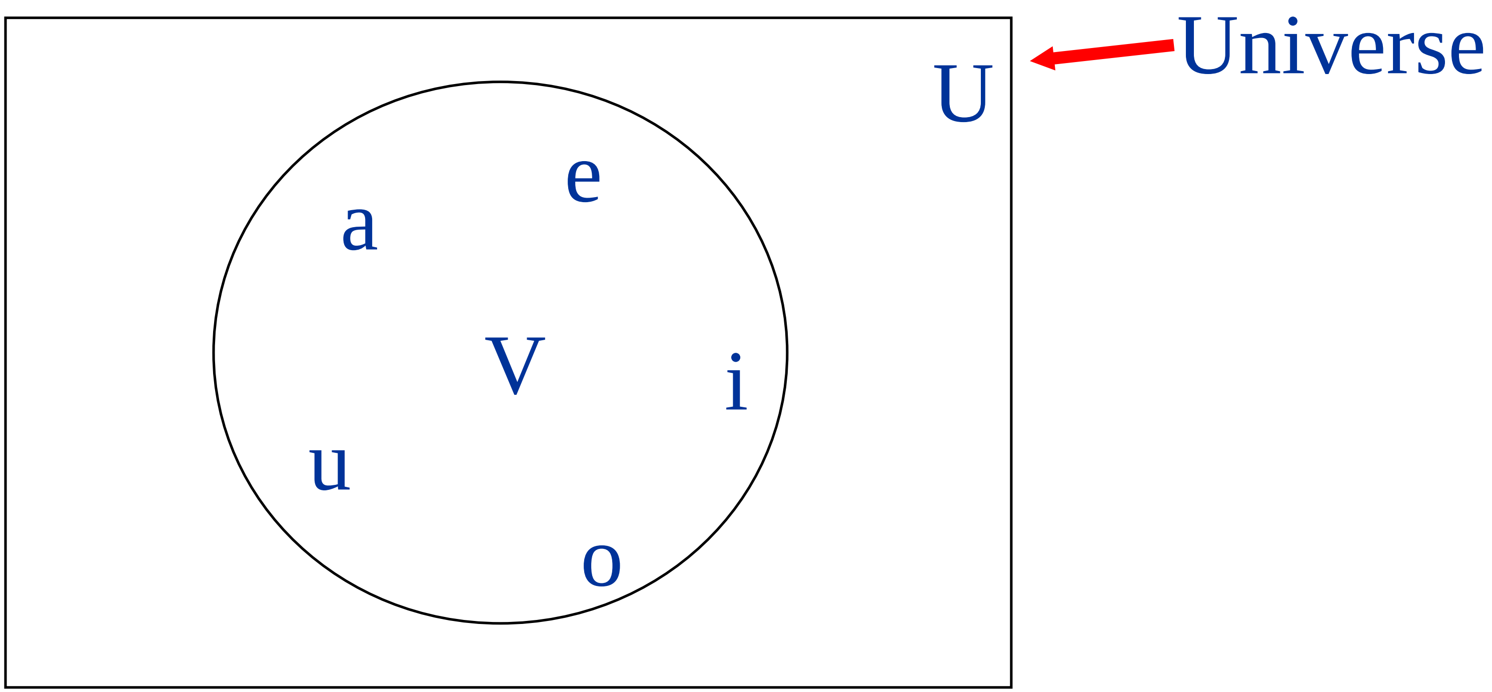
N = {0,1,2,3, …} = {x| x is a natural number}

Z = {….,-2,-1,0,1,2,….} = {x| x is an integer}

R = {….,1.1,….,1.2,….} = {x| x is a real number}

Set builder notation

Graphical Representation



Venn diagram for the set of vowels ‘V’

Membership in Sets

Example set: $A = \{v,w,x,y,z\}$

The letter ‘w’ is a member of this set

The letter ‘p’ is not a member of this set

$\Updownarrow$

$\Updownarrow$

$\mathbf{w \in A}$

$\mathbf{p \notin A}$

The empty set

The set that contains no element is called the *empty set* or *null set.*

The empty set is denoted by $\emptyset$ or by $\{\}$.

Subsets

Definition 3: A set A is said to be a *subset* of B if and only if every element of A is also an element of B.

We use the notation $A \subseteq B$ to indicate that A is a subset of B.

If $A \subseteq B$ and $A \neq B$, A is called a *proper subset* of B, denoted by $A \subset B$.

The empty set is a subset of every set.

Subsets

$S = \{1,2,3,4,5\}$

$T = \{1,2\}$, $U = \{2,3,5\}$

$T \subseteq S$

$U \subseteq S$

$\varnothing \subseteq S$

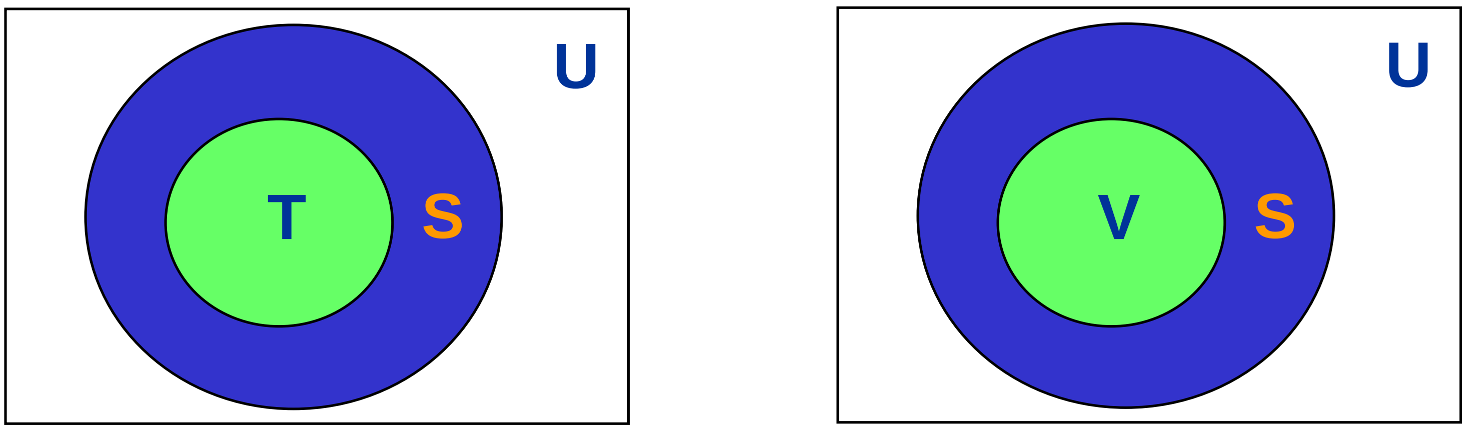
$T \subseteq S$ if and only if the quantification is true:

$$\forall x\ (x \in T \rightarrow x \in S)$$

The Venn Diagram

The Venn diagram is useful for showing subsets.

$$T \subseteq S$$
$$V \subseteq S$$



Venn diagrams showing T and V as subsets of S, in universe of discourse U.

Remember!

Every set is a subset of itself!

Let P be a set then:

$$\varnothing \subseteq \mathbf{P}$$

and

$$\mathbf{P} \subseteq \mathbf{P}$$

Equality and Cardinality

Two sets A and B are equal iff

$A \subseteq B$ and $B \subseteq A$

Definition 4:
Let S be a set. If there are exactly *n distinct elements* in S where n is a nonnegative integer, we say that S is a ***finite*** set. S is called ***infinite*** otherwise.
The cardinality of S is denoted by $|S|$ $(= n)$.

Examples

Finite:

A ={a,b,c,d,e,f,g,h,I,j,k,l,m,n,o,p,q,r,s,t,u, v,w,x,y,z}

The set A has cardinality: $|A| = 26$.

Odd positive integers less then 10:

$P=\{1,3,5,7,9\}$, $|P|=5$.

Empty set: $\mathbf{|\varnothing| = 0}$;

Infinite: positive integers, real numbers, odd integers.

A Set as an Element

$S=\{a,b\}$

$P(S)=\{\varnothing, \{a\}, \{b\}, \{a,b\}\}$

In set builder notation:

$P(S)=\{x \mid x$ is a subset of $S\}$

The Power Set

Definition 5:

Given a set S, the power set of S is the set of all subsets of S.

The power set of S is denoted by P(S).

Examples:

$P(\{1,2,3\})$

$=\{\emptyset,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$

$P(\emptyset) = \{\emptyset\}$; $|P(\emptyset)|=1$

$P(\{\emptyset\}) = \{\emptyset,\{\emptyset\}\}$; $|P(\{\emptyset\})|=2$

Order in Sets

Now consider the order of elements in a set…

Example: First name, last name, street address, city, … in a database.

Definition 6:

The ***ordered n-tuple*** $(a_1, a_2,\ldots,a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, … and a_n as its nth element.

2-tuples are called ***ordered pairs.***

Equality of ordered n-tuples

Equality: $(a_1, a_2\ldots,a_n) = (b_1, b_2\ldots,b_n)$
if and only if $a_i = b_i$ for $i=1, \ldots, n$.

E.g. $(a,b,c,d) \neq (a,c,b,d)$

$(a,b)=(b,a) \leftrightarrow a=b$

Cartesian Products

Definition 7:
Let A and B be sets. The ***Cartesian product*** of A and B, denoted by $A \times B$, is the set of all ordered pairs (a,b) where $a \in A$ and $b \in B$.
Hence $A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$.

E.g. $A = \{1,2\}$, $B = \{x,y,z\}$

$$A \times B = \{(1,x), (1,y), (1,z), (2,x), (2,y), (2,z)\}$$
$$B \times A = \{(x,1), (x,2), (y,1), (y,2), (z,1), (z,2)\}$$

Cartesian Products

Definition 8:

The ***Cartesian product*** of the sets A_1, A_2, …, A_n , denoted by $A_1 \times A_2 \times \ldots \times A_n$, is the set of all ordered n-tuples $(a_1,a_2, \ldots , a_n)$, where $a_i \in A_i$ for $i=1,2, \ldots, n$.

Hence $A_1 \times A_2 \times \ldots \times A_n = \{(a_1,a_2, \ldots , a_n) \mid a_i \in A_i$ for $i=1,2, \ldots, n\}$.

Cartesian Products

E.g.: $A=\{1,2\}$, $B=\{5\}$, $C=\{2,3\}$

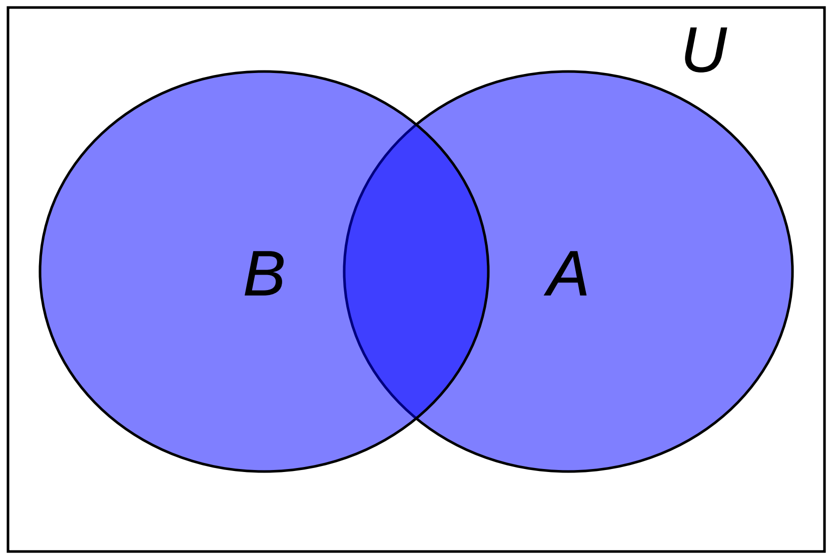
$A \times B \times C = \{(1,5,2),(1,5,3),(2,5,2),(2,5,3)\}$

$$\boxed{|A_1 \times A_2 \times \ldots \times A_n| = |A_1|\,|A_2|\ldots|A_n|}$$

$$\boxed{\text{If } A=\varnothing \text{ then } A \times B = B \times A = \varnothing}$$

Union

Venn diagram representing the Union of *A* and *B*



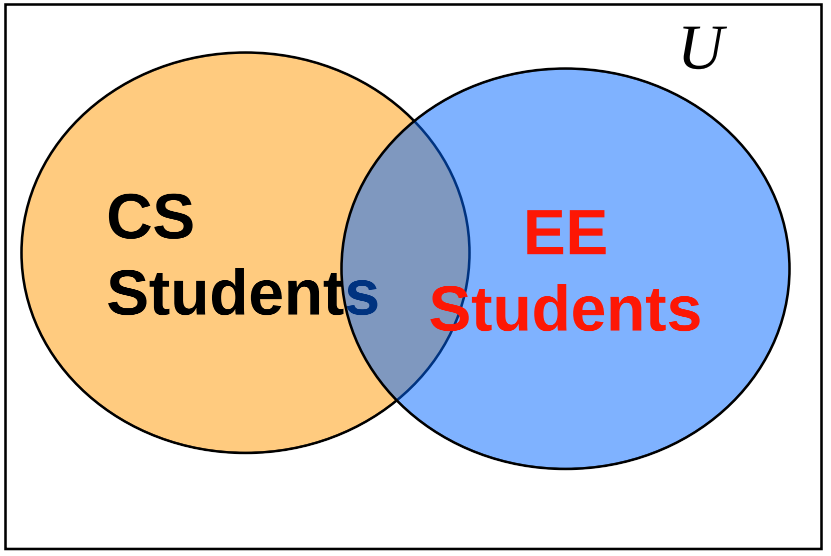
***A* ∪ *B* is shown by the blue area**

Union

Definition 1: The union of the sets A and B is the set of those elements that are either in A or in B or in both.

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

Example 1

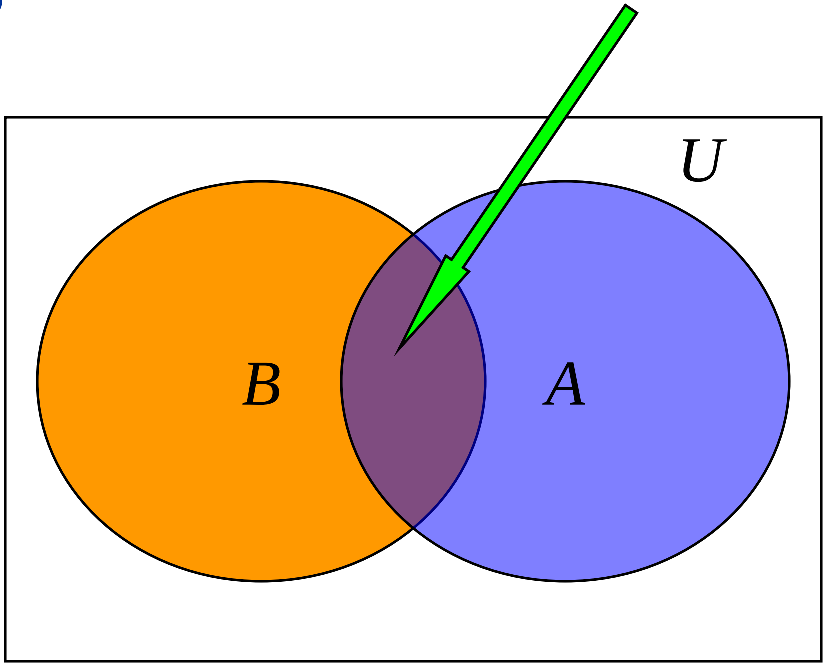


CS Students ∪ EE Students

= CS and/or EE Students

Intersection

Venn diagram representing the Intersection of *A* and *B*

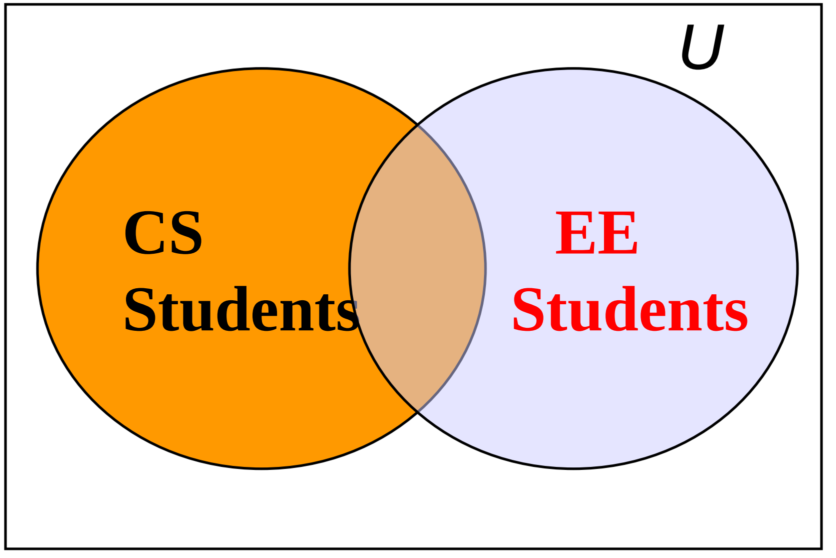


Intersection

Definition 2: The intersection of the sets A and B is the set of all elements that are in both A and B.

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

Example 2

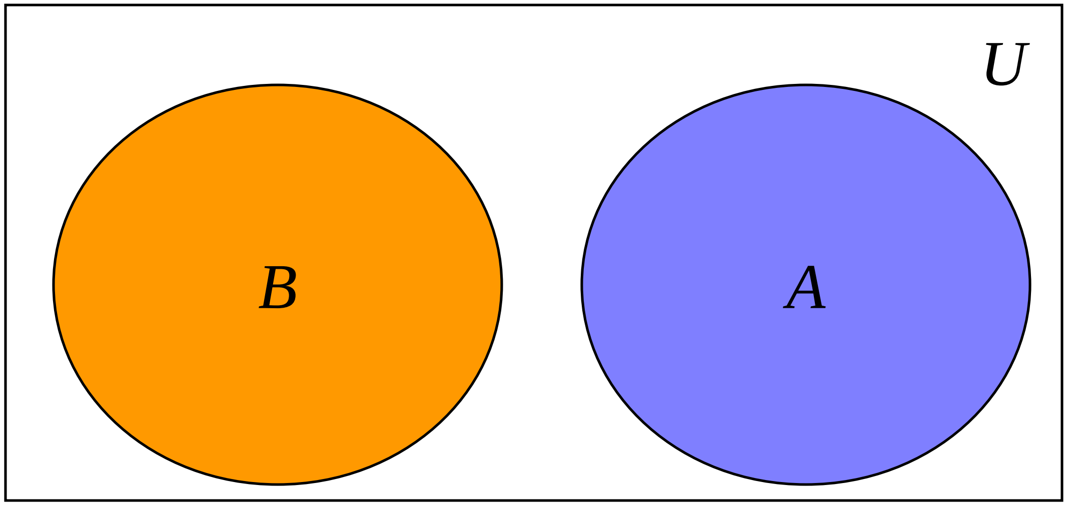


CS Students ∩ EE Students

= CS and EE Students

Disjoint Sets

Venn diagram representing the disjoint sets *A* and *B*



Disjoint Sets

Definition 3: When $A \cap B = \varnothing$ the two sets A and B are called disjoint

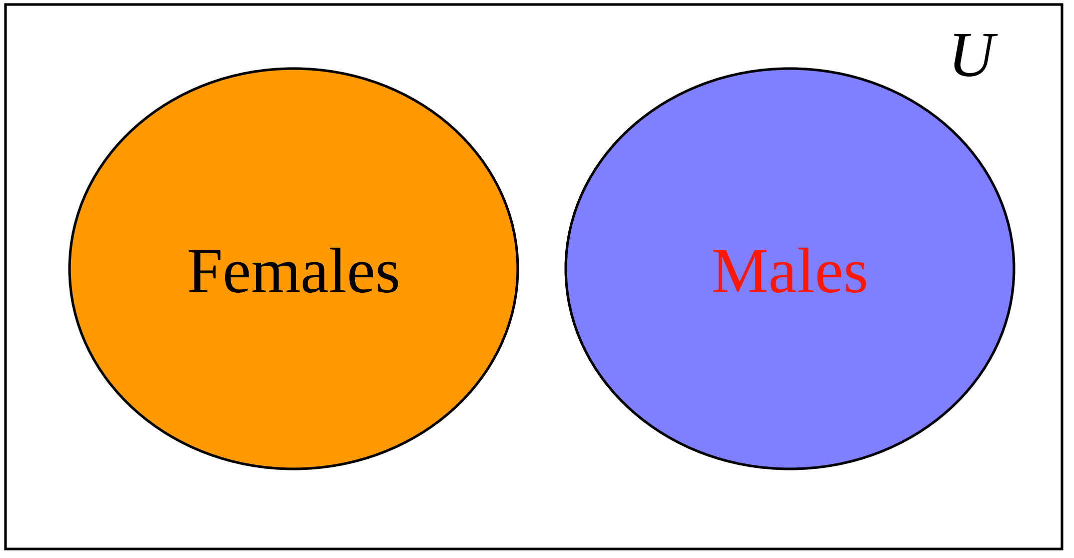
Therefore,

$$|A \cup B| = |A| + |\boldsymbol{B}| - \cancel{|A \cap B|}$$

Since,

$$A \cap B = \varnothing$$

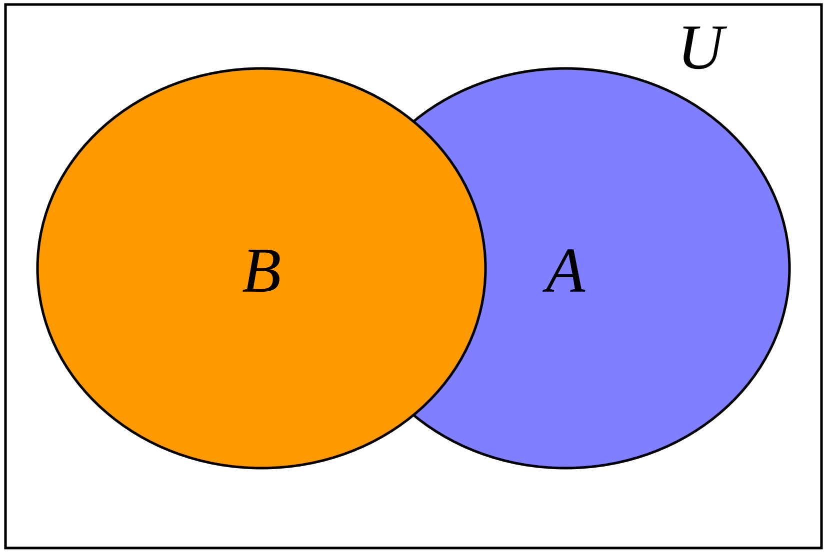
Example 3



$$\text{Females} \cap \text{Males} = \varnothing$$

Difference Between Two Sets

Venn diagram representing the Difference of *A* and *B*



The $A - B$ is shown by the blue area

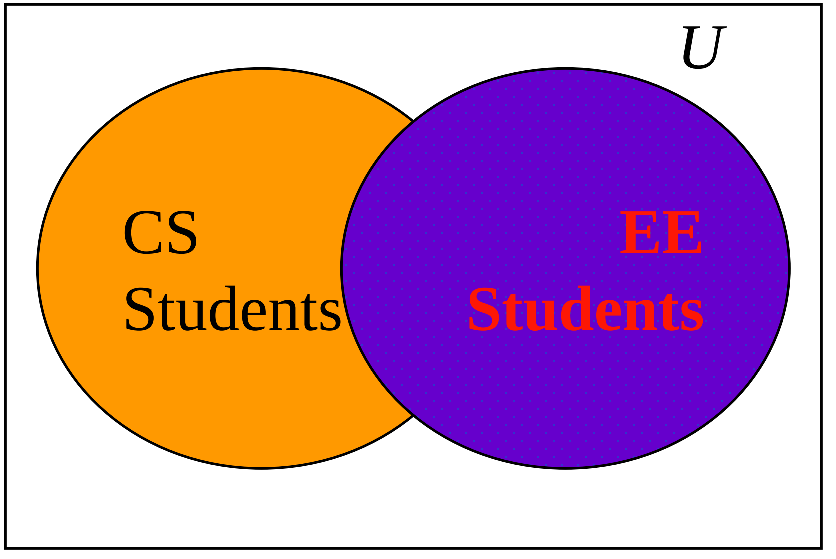
Difference Between Two Sets

Definition 4: The difference of A and B is the set containing those elements that are in A but not in B.

The difference of *A* and *B* is also called *the complement of B with respect to A*

$$A - B = \{x \mid x \in A \land x \notin B\}$$

Example 4

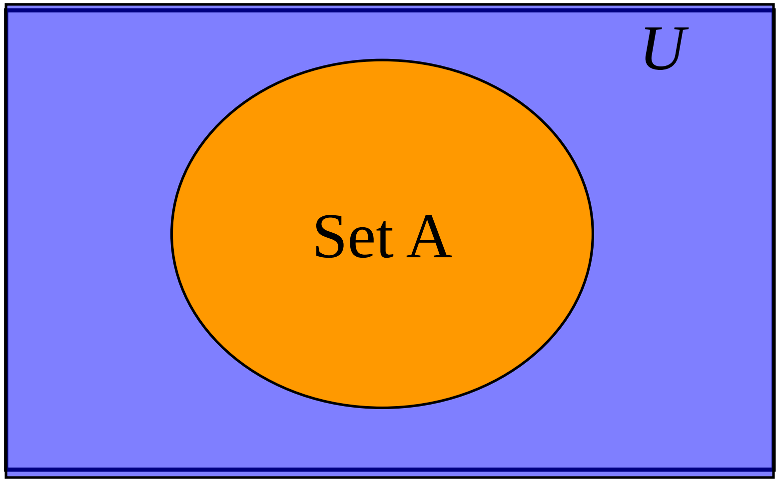


CS Students – EE Students

= CS Students who are not EE Students

Complement of a Set

Venn diagram representing the complement of the set A



$\overline{A}$ is shown by the blue area

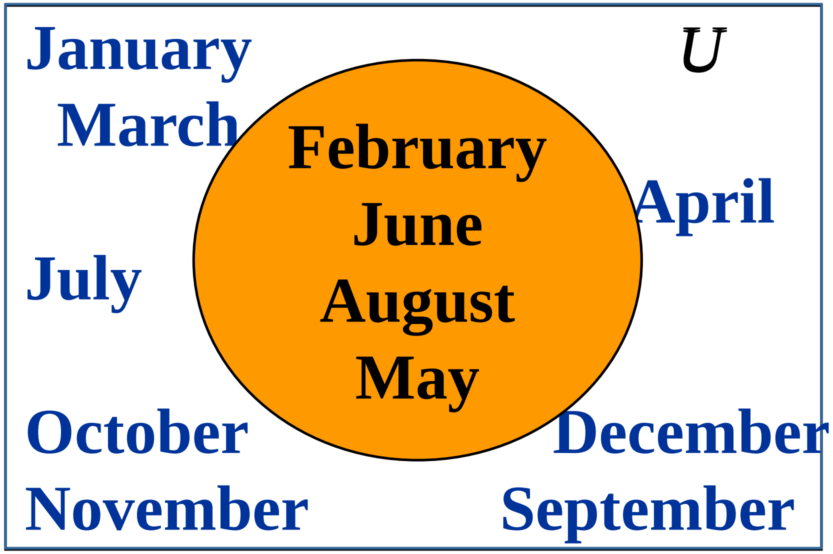
Complement of a Set

Definition 5: The complement of A is the complement of A with respect to U.

$\overline{A} = U - A.$

$$\overline{A} = \{x \mid x \notin A\}$$

Example 5



Let A = {February, June, August, May}

$\overline{A}$ = {January, March, July, October, November, April, September, December}

Set Identities

Identity	Name
$A \cup \varnothing = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \varnothing = \varnothing$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation laws

Set Identities

Identity	Name
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws

Set Identities - De Morgan's Laws Proof

Show that: $\overline{A \cap B} = \overline{A} \cup \overline{B}$ by showing that each set is a subset of the other

$x \in \overline{A \cap B} \leftrightarrows x \notin A \cap B$ **(definition of complement)**

$\leftrightarrows \neg(x \in A \wedge x \in B)$ **(definition of intersection)**

$\leftrightarrows \neg(x \in A) \vee \neg(x \in B)$ **(De Morgan's law)**

$\leftrightarrows x \notin A \vee x \notin B$ **(definition of $\notin$)**

$\leftrightarrows x \in \bar{A} \vee x \in \bar{B}$ **(definition of complement)**

$\leftrightarrows x \in \overline{A} \cup \overline{B}$ **(definition of union of sets)**

Set Identities - De Morgan's Laws Proof

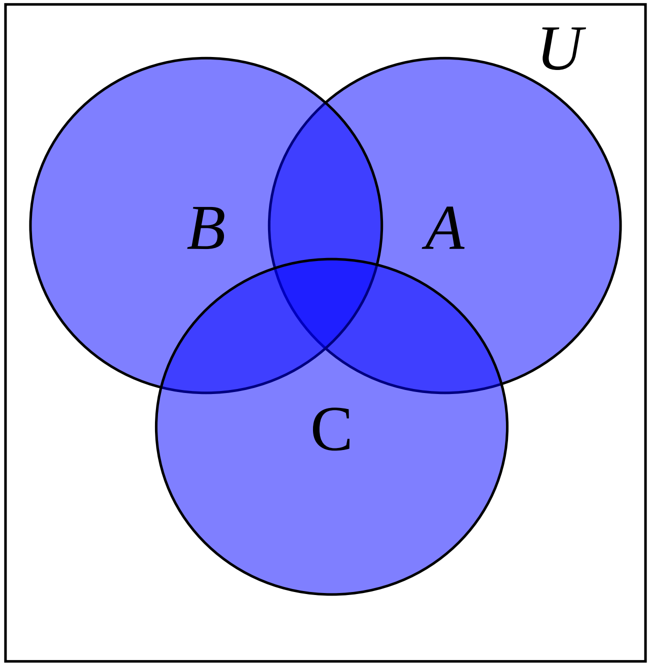
Show that: $\overline{A \cap B} = \overline{A} \cup \overline{B}$ using the definition of set

$$\overline{A \cap B} = \{x \mid x \notin A \cap B\}$$
$$= \{x \mid \neg (x \in (A \cap B))\}$$
$$= \{x \mid \neg (x \in A \wedge x \in B)\}$$
$$= \{x \mid \neg x \in A \vee \neg x \in B\}$$
$$= \{x \mid x \notin A \vee x \notin B\}$$
$$= \{x \mid x \in \overline{A} \vee x \in \overline{B}\}$$
$$= \{x \mid x \in \overline{A} \cup \overline{B}\}$$

Membership Table

A B C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1 1 1	1	1	1	1	1
1 1 0	1	1	1	0	1
1 0 1	1	1	0	1	1
1 0 0	0	0	0	0	0
0 1 1	1	0	0	0	0
0 1 0	1	0	0	0	0
0 0 1	1	0	0	0	0
0 0 0	0	0	0	0	0

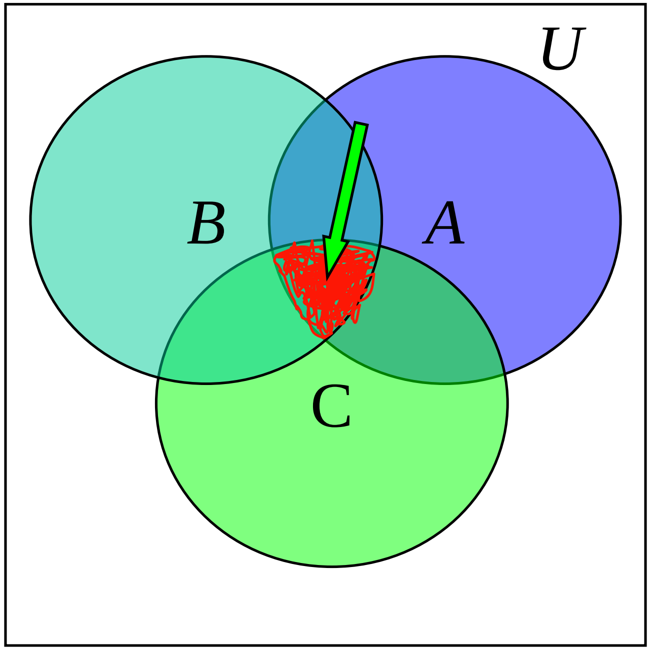
Generalised Unions



A ∪ *B* ∪ C is shown by the blue area

Definition 6:
The *union* of a collection of sets is the set that contains those elements that are members of at least one set in the collection.

Generalised Intersections



$A \cap B \cap C$ is shown by the red area

Definition 7:
The *intersection* of a collection of sets is the set that contains those elements that are members of all sets in the collection.

Example 7

A = {0, 2, 4, 6, 8} B = {0, 1, 2, 3, 4} C = {0, 3, 6, 9}

Therefore:

$A \cup B \cup \mathrm{C} = \{0, 1, 2, 3, 4, 6, 8, 9\}$

And:

$A \cap B \cap \mathrm{C} = \{0\}$

Computer Representation of Sets

Let:

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

Bit string representing the set of odd integers in U = 1010101010

Bit string representing the set of even integers in U = 0101010101

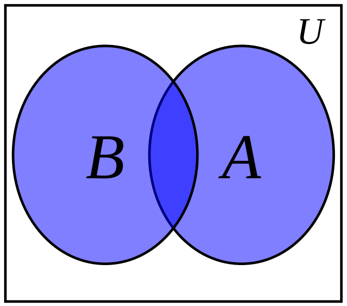
Bit string representing the set of all integers less than 6 in U

= 1111100000

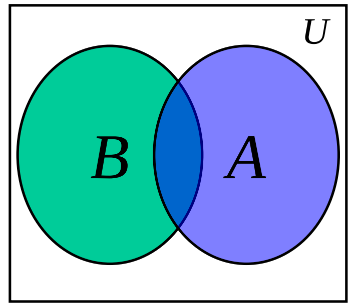
Summary

- What is a set?
- Equality of sets
- Membership in sets
- Cardinality of sets
- Cartesian products

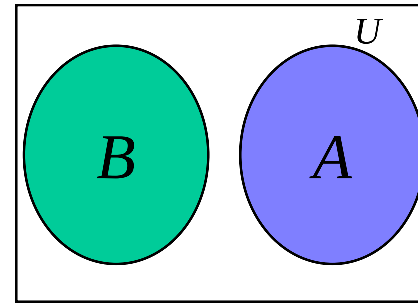
Summary: Set operations



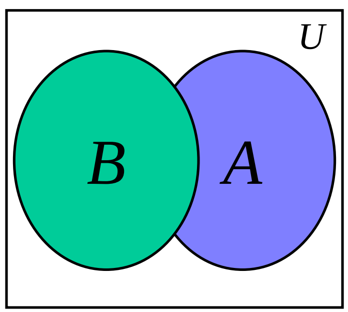
$A \cup B$



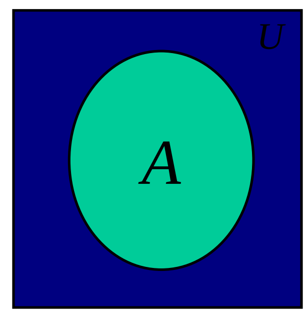
$A \cap B$



$A \cap B = \varnothing$



$A - B$



$\overline{A}$