2017 Sem 2 Exam Solutions

Q1)(a).(i). $-\underset{\sim}{a} = -[-3, 4, 1] = [3, -4, -1]$ (1 mark)

(ii). $4\hat{\underset{\sim}{b}} = \frac{4\underset{\sim}{b}}{\|\underset{\sim}{b}\|} = \frac{4[2,-2,-1]}{\sqrt{(2)^2+(-2)^2+(-1)^2}}$ (1) $= \frac{4[2,-2,-1]}{\sqrt{9}}$

$= \left[\frac{8}{3}, \frac{-8}{3}, \frac{-4}{3}\right]$ (1)

(iii). $\theta = \cos^{-1}\left(\frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\|\underset{\sim}{a}\|\|\underset{\sim}{b}\|}\right) = \cos^{-1}\left(\frac{[-3,4,1]\cdot[2,-2,-1]}{\sqrt{(-3)^2+4^2+1^2}\sqrt{2^2+(-2)^2+(-1)^2}}\right)$ (1)

$= \cos^{-1}\left(\frac{-6-8-1}{\sqrt{26}\sqrt{9}}\right)$ (1) $= \cos^{-1}\left(\frac{-15}{3\sqrt{26}}\right) \approx 168.7°$ (1)

(iv). Scalar proj $p = \underset{\sim}{a}\cdot\hat{\underset{\sim}{b}} = \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\|\underset{\sim}{b}\|} = \frac{-15}{3} = -5$ (1)

Vector proj $\underset{\sim}{p} = p\hat{\underset{\sim}{b}} = p\frac{\underset{\sim}{b}}{\|\underset{\sim}{b}\|} = -5\frac{[2,-2,-1]}{3} = \left[\frac{-10}{3}, \frac{10}{3}, \frac{5}{3}\right]$ (1)

(v). $\underset{\sim}{b}\cdot\underset{\sim}{c} = 0 \Rightarrow [2,-2,-1]\cdot[x,4,6] = 0$ (1)

$2x - 2(4) - 1(6) = 0 \Rightarrow 2x = 14 \Rightarrow x = 7$ (1)

(b). Coplanar if $\underset{\sim}{a}\cdot(\underset{\sim}{b}\times\underset{\sim}{c}) = 0$

$$\underset{\sim}{b}\times\underset{\sim}{c} = \begin{matrix} i & j & k & i & j \\ -3 & 2 & 1 & -3 & 2 \\ 2 & 4 & 0 & 2 & 4 \end{matrix}$$

$= \underset{\sim}{i}(0-4) + \underset{\sim}{j}(2-0) + \underset{\sim}{k}(-12-4)$ (2) $= [-4, 2, -16]$ (1)

$\underset{\sim}{a}\cdot(\underset{\sim}{b}\times\underset{\sim}{c}) = [-3,0,1]\cdot[-4,2,-16]$ (1)

$= -3(-4) + 0(2) + 1(-16) = -4 \neq 0$ (1) $\therefore$ Not coplanar (1)

(c). $\underset{\sim}{s} = [5,2,3] - [1,4,-1] = [4,-2,4]$ (1)

$W = \underset{\sim}{F}\cdot\underset{\sim}{s} = [3,-4,5]\cdot[4,-2,4]$ (1)

$= 3(4) - 4(-2) + 5(4) = 40\text{ J}$ (2)

Q2). (a). Point $(x_0, y_0, z_0) = (-3, 4, 1)$ ½

Direction of line $\underset{\sim}{a} = [4, 3, -2]$ 1

Parametric equations: $x = -3 + 4t$, ½ $y = 4 + 3t$, ½ $z = 1 - 2t$ ½

(b). Directions. Line L_1, $d_1 = [-1, 1, 2]$ ½

Line L_2, $d_2 = [1, 2, 2]$ ½

Since $\underset{\sim}{d_1} \neq m\underset{\sim}{d_2}$ $\therefore$ Not parallel 1

$$\left.\begin{aligned} -1 - t &= 2 + \tau \\ 4 + t &= 7 + 2\tau \\ -2 + 2t &= 2\tau \end{aligned}\right\} \text{1}$$

$\rightarrow t = -1 - 2 - \tau \Rightarrow t = -3 - \tau$

$4 + (-3 - \tau) = 7 + 2\tau$

$1 - \tau = 7 + 2\tau \Rightarrow -3\tau = 6 \quad \therefore \tau = -2$ 1

$\therefore t = -3 - (-2) = -1$ 1

Test z: $-2 + 2(-1) = 2(-2)$

$-4 = -4 \quad \therefore$ Intersect 1

Point of intersection: $(-1-(-1), 4+(-1), -2+2(-1)) = (0, 3, -4)$ 1

(c). Direction of line $\underset{\sim}{a} = [-1, 1, 2]$ 1

Point on line $Q(2, 4, -3)$

Vector $\vec{PQ} = [2-1, 4-(-1), -3-3] = [1, 5, -6]$ 1

$$\text{Distance} = \sqrt{\|\vec{PQ}\|^2 - \left(\frac{\vec{PQ}\cdot\underset{\sim}{a}}{\|\underset{\sim}{a}\|}\right)^2} \text{ 1 } = \sqrt{\left(\sqrt{1^2+5^2+(-6)^2}\right)^2 - \left(\frac{[1,5,-6]\cdot[-1,1,2]}{\sqrt{(-1)^2+1^2+2^2}}\right)^2}$$

$$= \sqrt{62 - \left(\frac{-8}{\sqrt{6}}\right)^2} \text{ 1 } = \sqrt{62 - \frac{64}{6}} \approx 7.16 \text{ 1}$$

(d). Vector $\vec{PQ} = [3-2, 0-(-1), 4-(-1)] = [1, 1, 5]$ ½

Vector $\vec{PR} = [-2-2, -1-(-1), 2-(-1)] = [-4, 0, 3]$ ½

$$\underset{\sim}{n} = \vec{PQ} \times \vec{PR} \text{ 1 } = \begin{array}{ccccc} i & j & k & i & j \\ 1 & 1 & 5 & 1 & 1 \\ -4 & 0 & 3 & -4 & 0 \end{array}$$

$= \underset{\sim}{i}(3-0) + \underset{\sim}{j}(-20-3) + \underset{\sim}{k}(0-(-4))$ 1 $= [3, -23, 4]$ 1 $= [a, b, c]$

Point $(x_0, y_0, z_0) = (2, -1, -1)$

Eqn. of plane: $3(x-2) - 23(y-(-1)) + 4(z-(-1)) = 0$ 1

$3x - 23y + 4z = 25$

Q3). (a).

$$\left[\begin{array}{ccc|ccc} 2 & 4 & 1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 \\ 1 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 = 2R_2 + R_1 \\ R_3 = 2R_3 - R_1 \end{array} \quad 1$$

$$\left[\begin{array}{ccc|ccc} 2 & 4 & 1 & 1 & 0 & 0 \\ 0 & 6 & -1 & 1 & 2 & 0 \\ 0 & 4 & -1 & -1 & 0 & 2 \end{array}\right] \begin{array}{l} \\ 1 \\ R_3 = 6R_3 - 4R_2 \end{array} \quad \left[\begin{array}{ccc|ccc} 2 & 4 & 1 & 1 & 0 & 0 \\ 0 & 6 & -1 & 1 & 2 & 0 \\ 0 & 0 & -2 & -10 & -8 & 12 \end{array}\right] \begin{array}{l} \\ 1 \\ R_3 \div (-2) \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & 4 & 1 & 1 & 0 & 0 \\ 0 & 6 & -1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 5 & 4 & -6 \end{array}\right] \begin{array}{l} R_1 = R_1 - R_3 \\ R_2 = R_2 + R_3 \\ 1 \end{array} \quad \left[\begin{array}{ccc|ccc} 2 & 4 & 0 & -4 & -4 & 6 \\ 0 & 6 & 0 & 6 & 6 & -6 \\ 0 & 0 & 1 & 5 & 4 & -6 \end{array}\right] \begin{array}{l} 1 \\ R_2 \div (6) \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & 4 & 0 & -4 & -4 & 6 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 5 & 4 & 6 \end{array}\right] \begin{array}{l} R_1 = R_1 - 4R_2 \\ 1 \\ \end{array} \quad \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -8 & -8 & 10 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 5 & 4 & 6 \end{array}\right] \begin{array}{l} R_1 \div (2) \\ 1 \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & -4 & 5 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 5 & 4 & 6 \end{array}\right] \qquad \therefore A^{-1} = \begin{bmatrix} -4 & -4 & 5 \\ 1 & 1 & -1 \\ 5 & 4 & 6 \end{bmatrix} \quad 1$$

(b). Cofactor expansion along 2nd column,

$$|B| = -0 + 0 - 3\begin{vmatrix} -4 & 6 \\ -1 & 5 \end{vmatrix} \quad 2$$

$$= -3(-20-(-6)) \quad 1 \quad = -3(-14) \quad = 42 \quad 1$$

(c).

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 3 & -1 & -1 & -1 \\ -2 & 2 & 1 & 3 \end{array}\right] \begin{array}{l} 2 \\ R_2 = R_2 - 3R_1 \\ R_3 = R_3 + 2R_1 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & -4 & -4 & -16 \\ 0 & 4 & 3 & 13 \end{array}\right] \quad 1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & -4 & -4 & -16 \\ 0 & 0 & -1 & -3 \end{array}\right] \quad 1$$

STOP $r(A) = 3 \overset{1/2}{=} r(A|\underset{\sim}{b}) = 3 \overset{1/2}{=} n = 3$

$\therefore$ Unique Solution

Row 3: $-1x_3 = -3 \Rightarrow x_3 = 3$ 1

Row 2: $-4x_2 - 4x_3 = -16 \Rightarrow x_2 + x_3 = 4 \Rightarrow x_2 + 3 = 4 \quad \therefore x_2 = 1$ 1

Row 1: $x_1 + x_2 + x_3 = 5 \Rightarrow x_1 + 1 + 3 = 5 \Rightarrow x_1 = 1$ 1

Q4). (a). (i). Since there are more vectors (4) than space (3) 1
$\therefore$ Set is linearly dependent 1

(ii). Since $\underset{\sim}{u}_2 = -3\underset{\sim}{u}_1$ 1 (i.e. vectors are a multiple of each other)
$\therefore$ Set is linearly dependent 1

(b). $A = \begin{bmatrix} 1 & -3 \\ 5 & -1 \end{bmatrix}$ ½ $\underset{\sim}{b} = \begin{bmatrix} -3 \\ 13 \end{bmatrix}$ $A_1 = \begin{bmatrix} -3 & -3 \\ 13 & -1 \end{bmatrix}$ 1 $A_2 = \begin{bmatrix} 1 & -3 \\ 5 & 13 \end{bmatrix}$ 1

$\det(A) = 1(-1) - (-3)(5) = -1 + 15 = 14$ ½
$\det(A_1) = -3(-1) - (-3)(13) = 3 + 39 = 42$ ½
$\det(A_2) = 1(13) - (-3)(5) = 13 + 15 = 28$ ½

$\therefore x_1 = \dfrac{\det(A_1)}{\det(A)} = \dfrac{42}{14} = 3$ 1 $x_2 = \dfrac{\det(A_2)}{\det(A)} = \dfrac{28}{14} = 2$ 1

Q4(c).(i) $P(X \leq 91) = P\left(Z \leq \frac{91-73}{8}\right)$ (1)

$= P(Z \leq 2.25)$ (1)

$= 0.9878$ (1)

(ii) $P(X > x) = P\left(Z > \frac{x-\mu}{\sigma}\right) = 0.1$

i.e. $P\left(Z < \frac{x-\mu}{\sigma}\right) = 0.9$ (1)

Case 1: $\mu = 73, \sigma = 8 \rightarrow \frac{x-73}{8} = 1.28$

i.e. $x = 83.24$; so student gets "A" if score > 83 (1)

Case 2: $\mu = 62, \sigma = 3 \rightarrow \frac{x-62}{3} = 1.28$

i.e. $x = 65$, so student gets "A" if score > 65 (1)

$\therefore$ In case 2 student gets "A", not in case 1. (1)

(d) X = the number of chips (out of 15) with thick enough coatings

$X \sim$ Binomial $(n = 15, p = 0.7)$ (1)

$\therefore P(X \geq 12) = 1 - P(X \leq 11)$ (1)

$= 1 - 0.7031 = 0.2969$ (1)

Q5 (a)(i) $\bar{x} \pm t_{(\alpha/2,\, n-1)} \frac{s}{\sqrt{n}} = \bar{x} \pm t_{(0.01,\, 9)} \frac{s}{\sqrt{n}}$ 1

$= 84.1 \pm 2.821 \left(\frac{6.806043}{\sqrt{10}} \right)$ 2

$= (78.0285,\ 90.1715)$ 1

(ii) Assumptions

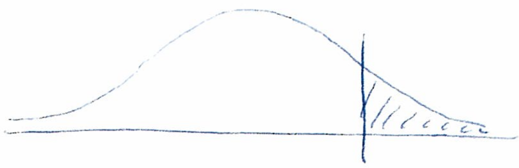
1. Observations are SRS 1/2
2. Population is approx. normally distributed. 1/2
3. σ is unknown. 1/2

Test Statistic

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{84.1 - 78}{6.806043/\sqrt{10}} = 2.83423$$

1 1 1/2

Critical Region



$t_{0.05,9} = 1.833$ 1

Conclusion

$t > t_{0.05,9}$, i.e. t is within the critical region. Therefore, we have sufficient evidence to reject H_0 at the 5% level of significance. 1

Q5 (b) $P(\bar{x} > 14) = 1 - P(\bar{x} \leq 14)$ (½)
(½)

$$= 1 - P\left(Z \leq \frac{14-12}{9/10}\right) \quad (1)$$

$$= 1 - P(Z \leq 2.22) \quad (1)$$

$$= 1 - 0.9868$$

$$= 0.0132 \quad (1)$$

(c) (i) 25, 27, 34, 36, 48, 51, 75, 99 (½)

Min = 25 (½)

$Q_1: (n+1)p = 9/4 = 2\frac{1}{4}$

Hence, $Q_1 = y_2 + \frac{1}{4}(y_3 - y_2)$

$= 27 + \frac{1}{4}(34 - 27)$ (1)

$= 28.75$

$Q_2 = \frac{36+48}{2} = 42$ (½)

$Q_3: (n+1)p = 9\left(\frac{3}{4}\right) = 6\frac{3}{4}$

Hence, $Q_3 = y_6 + \frac{3}{4}(y_7 - y_6)$

$= 51 + \frac{3}{4}(75 - 51)$

$= 69$ (1)

Max = 99 (½)

(ii) $IQR = Q_3 - Q_1 = 69 - 28.75 = 40.25$ (½)

$Q_1 - 1.5\,IQR = 28.75 - 1.5(40.25) = -31.625$ (½)

$Q_3 + 1.5\,IQR = 69 + 1.5(40.25) = 129.375$ (½)

All observations are between -31.625 & 129.375 ∴ No outliers. (½)