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Theoretical Foundations of Computer Science

Lecture 3b

The Pumping Lemma

Context-Free Grammars

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Topics

- Non-regular expressions
  - ➢ Pumping lemma

Unit Learning Outcome

- Synthesize FA, PDA, CFG, and TMs with specific properties, and to relate and convert from one form to another.

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Assessment Criteria

- **Use** the pumping lemma to prove a language to be a non-regular language.

Non-regular languages

- Limitations of finite automata
  - certain languages cannot be recognized by any finite automaton
- Example: Language $B = \{0^n 1^n \mid n \geq 0\}$
  - Claim:
    - a machine recognizing B need to remember how many 0s have been seen so far as it reads input
    - an unlimited number of states needed for this
    - Thus non-regular

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Failure of Intuition

- Another non-regular language:
  - $C = \{w \mid w$ has an equal number of 0s and 1s$\}$
- However, the following language is regular:
  - $D = \{w \mid w$ has an equal number of occurrences of 01 and 10 as sub-strings$\}$
- Thus claims that a language is regular (or not) must be proven.

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Pumping lemma

- All regular languages have a special property
  - ➢ If we can show that a language does not have this property, then it is guaranteed to be non-regular
- The property is that all strings in the language can be "pumped" if they are longer than a certain special value, called the *pumping length*.
  - ➢ Each such string contains a section that can be repeated any number of times and the resulting strings will remain in the given language.

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Pumping lemma

- If A is a regular language, there is a pumping length p such that if s is any string in A of at least length p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:
    1. for each $i \geq 0$, $xy^iz \in A$,
    2. $|y| > 0$, and
    3. $|xy| \leq p$.
- Note on notation:
    - y^i means i copies of y concatenated together ($y^0 = \varepsilon$)
    - either x or z may be ε, but y cannot be ε

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Using the Lemma

- To show that a language *C* is not regular,
  - first assume it is regular
  - That is: there is a pumping length *p* for *C*
    - WE DO NOT KNOW A VALUE FOR P
  - Select a pattern typically at start of string
    - EXCLUDE something that is required
  - Concatenate start string p times.
  - Now complete the string so it is part of language

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Using the Lemma

- To show that a language *C* is not regular,
  - We have a string of length greater than p
  - Such that strings of length *p* or greater can be pumped
  - Aim to show our suitably chosen string *s* in *C* in fact cannot be pumped, thus contradicting the assumption that *C* is regular.
  - Pumping means we can add (subtract) portion of start.
    - Our tail is unaltered – it contains essential element
    - Some rule connecting start with tail should break

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Example

- Example: Language $B = \{0^n 1^n \mid n \geq 0\}$
  - ➢ Assume regular => exist p
  - ➢ Start consists of 0, but 1's are important
    - – Let 0 be base starting pattern
    - – Use p by raising starting pattern to power p – 0^p
    - – Complete an element of the language: $0^p 1^p$
    - – Now have trial string
  - ➢ Pumping
    - – y of the lemma must be sequence of 0
    - – Pumping means adding 0
    - – But no 1's added
    - – So new string will not have equal 0's and 1's

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Proof

Let $s=0^p1^p$

By PL:

1) $s=xyz$

2) $|xy| <= p$

3) $|y| > 0$

4) $xy^kz \in B$

We can choose k:

Usually 0 or 2.

Say 2.

By 2) xy must be within the p letters of s

But first p letters of s are all 0's.

So from 1), there is i, j such that

$x = 0^i$ $\quad 0 <= i < p$ *by 2) 3)*

$y = 0^j$ $\quad 1 <= j <= p-1$ *by 2) 3)*

$z = 0^{(p-i-j)}1^p$

$xy^2z = 0^i0^{2j}0^{(p-i-j)}1^p = 0^{(p+j)}1^p$

As (4) this is in **B**, so $p+j = p$ that is $j=0$

BUT (3) says $j>= 1$, CONTRADICTION

Hence B is not regular.

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Summary

- Non-regular expressions
  - ➢ Pumping lemma
    - Know the theorem
    - <ULO> Use the theorem

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