Lecture 7

Systems of Linear Equations

& Gaussian Elimination

Linear Systems of Equations

A general linear system of m equations in n variables is written as

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n &= b_3 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn} &= b_m
\end{aligned}$$

where the variables are x_1, x_2, . . . , x_n and the coefficients a_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n$, as well as the right hand side values b_i, $i = 1, m$, are given.

A system of linear equations can have a unique solution, infinitely many solutions or no solution at all. It is not possible for it to have a finite (*e.g.* 2 or 3) number of solutions. If solutions exists, there's either only one of them or infinitely many of them.

Let us start with an example of 3 equations in three unknowns.

$$\begin{aligned} 2x_1 + 4x_2 - 3x_3 &= 1 \\ x_1 + x_2 + 2x_3 &= 9 \\ 3x_1 + 6x_2 - 5x_3 &= 0 \end{aligned}$$

The process we used in solving these equations consisted of three basic operations:

(i) We could multiply an equation by a nonzero constant.

(ii) We could add a multiple of one equation to another.

(iii) We could interchange two equations.

Note that none of these operations actually changed the solution set of the system of equations.

Better to work with the *augmented matrix*.

Firstly, notice that the general system of equations

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n &= b_3 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn} &= b_m
\end{aligned}$$

can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix},$$

i.e. $A\boldsymbol{x} = \boldsymbol{b}$, where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$,

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}.$$

The augmented matrix of the system then is

$$[A|\,\boldsymbol{b}] = \left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right],$$

Ex: Find the augmented matrix for the following system of linear equations.

$$\begin{array}{cccccc} 2x_1 & +3x_2 & -x_3 & = & 7 \\ -x_1 & +4x_3 & & = & 5 \\ 6x_1 & -3x_2 & & = & 1 \end{array}$$

If $[A|\boldsymbol{b}]$ is in *row echelon form*, it will be easy to determine the solution(s) . . .

For any matrix, the first non zero entry in a particular row is called the leading entry of that row. *e.g.*

$$\begin{bmatrix} 2 & 1 & 4 & 7 & 1 \\ 0 & 0 & 5 & 3 & 2 \\ 0 & 7 & 4 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A matrix is said to be in row echelon form if the leading entry in each row lies to the right of that in the preceding row. Any row of zeros must appear below the non zero rows.

$$e.g. \quad \begin{bmatrix} 4 & 2 & 3 & 4 \\ 0 & 6 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

are in row echelon form, while

$$\begin{bmatrix} 2 & 1 & 4 & 7 & 1 \\ 0 & 0 & 5 & 3 & 2 \\ 0 & 7 & 4 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 3 & 6 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 5 & 0 & 3 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & -3 & 2 & 0 & 1 \\ 0 & 0 & 3 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

are not.

What is so important about a row echelon form?

Ex: The following augmented matrix is in row echelon form. Solve the corresponding linear system.

$$[A|\boldsymbol{b}] = \left[\begin{array}{ccc|c} -1 & -2 & 1 & -9 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 2 & -4 \end{array}\right].$$

This process of determining the variables is known as back substitution, but it is only possible if $[A|\boldsymbol{b}]$ is in row echelon form.

$[A|\boldsymbol{b}]$ is not always in row echelon form. Need *Gaussian elimination* . . .

Gaussian Elimination: Reduce $[A|\,\boldsymbol{b}]$ to row echelon form.

How? We are allowed to use three of types of elementary row operations (e.r.o.'s):

(i) We can multiply a row by a non zero constant.

(ii) We can add a multiple of one row to another.

(iii) We can interchange two rows.

These do not change the solutions (or non-solvability) of the underlying system!

<u>Ex:</u> Solve the system

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 6 \\ x_1 + 2x_2 + 4x_3 &= 9 \\ 2x_1 + x_2 + 6x_3 &= 11 \end{aligned}$$

Rank of a matrix

The rank of a matrix A, denoted r(A), is simply the number of linearly independent rows in A.

If A has just two rows then by inspection it's relatively easy to determine r(A) by inspection.

Ex: Find the rank of A and B.

$$A = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}$$

If there's more than 2 rows then we need a systematic way to determine ranks of matrices.

Can show that

- The rank of a matrix in row echelon form is equal to the number of nonzero rows.

- Elementary row operations do not change the rank of a matrix.

Ex: Find the rank of

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 & 2 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For a matrix which is not in row echelon form, we first use e.r.o.'s to reduce it to row echelon form.

Ex: Find the rank of

$$B = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 5 & 3 \\ 3 & 8 & 4 \end{bmatrix}$$

Soln:

$$B = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 5 & 3 \\ 3 & 8 & 4 \end{bmatrix} \begin{array}{l} \\ \mathsf{R}_2 \to \mathsf{R}_2 - 2\mathsf{R}_1 \\ \mathsf{R}_3 \to \mathsf{R}_3 - 3\mathsf{R}_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{bmatrix} \begin{array}{l} \\ \\ \mathsf{R}_3 \to \mathsf{R}_3 - 2\mathsf{R}_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

i.e. $\mathsf{r}(B) = 2$.

Recognizing the Nature of Solutions

Consider a general system of m linear equations in n unknowns, $A\boldsymbol{x} = \boldsymbol{b}$, *i.e.* A is $m \times n$, $\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$. Let $[A|\,\boldsymbol{b}]$ be the augmented matrix for the system.

Then, one of the following will always occur:

(i) If $\mathrm{r}(A) \neq \mathrm{r}([A|\,\boldsymbol{b}])$, then the equations are inconsistent and there are no solutions.

(ii) If $\mathrm{r}(A) = \mathrm{r}([A|\,\boldsymbol{b}]) = n$, then the equations are consistent and have a unique solution.

(iii) If $\mathrm{r}(A) = \mathrm{r}([A|\,\boldsymbol{b}]) = r < n$, then the equations are consistent and have infinitely many solutions, which we can express in terms of $n - r$ parameters . . . see below.

Ex: Solve, if possible:

$$\begin{array}{rrrcr} 2x_1 & +x_2 & +2x_3 & = & -1 \\ 4x_1 & +3x_2 & +5x_3 & = & 1 \\ 6x_1 & +5x_2 & +5x_3 & = & -3 \end{array}$$

Ex: Solve, if possible:

$$\begin{array}{rrrcr} x_1 & +2x_2 & +3x_3 & = & 1 \\ x_1 & +3x_2 & +4x_3 & = & 3 \\ x_1 & +4x_2 & +5x_3 & = & 4 \end{array}$$

Ex: Solve, if possible:

$$\begin{array}{rrrcr} 2x_1 & +3x_2 & +x_3 & = & 1 \\ x_1 & +x_2 & +x_3 & = & 3 \\ 3x_1 & +4x_2 & +2x_3 & = & 4 \end{array}$$

Here, we have 3 variables, *i.e.* $n = 3$. Also, from the final augmented matrix, we can see that $\mathrm{r}(A) = \mathrm{r}([A|\,\boldsymbol{b}]) = r = 2 < n$, *i.e.* the equations are consistent, but the system has infinitely many solutions.

Comments on Gaussian Eliminations

- Interchanging of rows is usually only needed when we want a convenient leading entry (typically a 1) to appear in the row which we use to manipulate the remaining rows. This leading entry is often called a pivot.

- We can apply more than one e.r.o. at a time, as long as two different rows are being manipulated. Beware of performing two e.r.o.'s which change the same row (*e.g.* adding a multiple of one row to another whilst swapping that row with a third row).

- Work on one column at a time, reducing all entries below a leading entry in that column to zero. Once done, don't come back to that column later, but move to the next column on the right. This way, you won't end up applying e.r.o.'s which get you away from the row echelon form.

- **Never use elementary column operations!** Unlike e.r.o.'s, these will change the solution set of your system of equations. Computer software solving linear systems of equations does actually use column swaps to improve the numerical stability of the solution process, but it also keeps track of the corresponding swaps in variables (which is too tedious for hand calculations).