WORKSHOP 11a SOLUTIONS

1. (i) $\boldsymbol{a} + 2\boldsymbol{b} = [1, 2, 0, 2] + 2[-2, 0, 1, 1] = [1, 2, 0, 2] + [-4, 0, 2, 2]$
$= [1 + (-4), 2 + 0, 0 + 2, 2 + 2] = [-3, 2, 2, 4]$

(ii) $||\boldsymbol{b}|| = \sqrt{(-2)^2 + (0)^2 + (1)^2 + (1)^2} = \sqrt{4 + 0 + 1 + 1} = \sqrt{6}$
$\hat{\boldsymbol{b}} = \frac{\boldsymbol{b}}{||\boldsymbol{b}||} = \frac{[-2, 0, 1, 1]}{\sqrt{6}} = \left[\frac{-2}{\sqrt{6}}, 0, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right]$

(iii) $||\boldsymbol{a}||\,\hat{\boldsymbol{b}} = \sqrt{(1)^2 + (2)^2 + (0)^2 + (2)^2}\left[\frac{-2}{\sqrt{6}}, 0, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right] = \sqrt{9}\left[\frac{-2}{\sqrt{6}}, 0, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right]$
$= 3\left[\frac{-2}{\sqrt{6}}, 0, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right] = \left[\frac{-6}{\sqrt{6}}, 0, \frac{3}{\sqrt{6}}, \frac{3}{\sqrt{6}}\right]$

2. $\overrightarrow{AB} = [3 - 2, 1 - 4, 1 - 3, 0 - (-1), -2 - 1] = [1, -3, -2, 1, -3]$
Distance $= \|\overrightarrow{AB}\| = \sqrt{(1)^2 + (-3)^2 + (-2)^2 + (1)^2 + (-3)^2} = \sqrt{24}$

3. $\boldsymbol{a} \cdot \boldsymbol{b} = [4, 1, -2, 2] \cdot [1, 0, 3, 2]$
$= (4)(1) + (1)(0) + (-2)(3) + (2)(2) = 4 + 0 - 6 + 4 = 2$
$\|\boldsymbol{b}\| = \sqrt{(1)^2 + (0)^2 + (3)^2 + (2)^2} = \sqrt{14}$
scalar projection, $p = \boldsymbol{a} \cdot \hat{\boldsymbol{b}} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{||\boldsymbol{b}||} = \frac{2}{\sqrt{14}}$
vector projection, $\boldsymbol{p} = p\,\hat{\boldsymbol{b}} = p\,\frac{\boldsymbol{b}}{||\boldsymbol{b}||} = \frac{2}{\sqrt{14}}\frac{[1, 0, 3, 2]}{\sqrt{14}} = \left[\frac{2}{14}, 0, \frac{6}{14}, \frac{4}{14}\right] = \left[\frac{1}{7}, 0, \frac{3}{7}, \frac{2}{7}\right]$

4. The normal vectors to the planes are $\boldsymbol{n}_1 = [2, -1, -2, 1]$ and $\boldsymbol{n}_2 = [1, 3, 0, -1]$
The angle between the normal vectors and hence the planes is,
$\theta = \cos^{-1}\left(\frac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{||\boldsymbol{n}_1||\,||\boldsymbol{n}_2||}\right) = \cos^{-1}\left(\frac{[2, -1, -2, 1] \cdot [1, 3, 0, -1]}{\sqrt{(2)^2 + (-1)^2 + (-2)^2 + (1)^2}\,\sqrt{(1)^2 + (3)^2 + (0)^2 + (-1)^2}}\right)$
$\therefore\ \theta = \cos^{-1}\left(\frac{-2}{\sqrt{10}\,\sqrt{11}}\right) = 100.99^\circ$

5. (i) V consists of all elements of $\mathbb{R}^3$ for which the first component subtract twice the second component equals zero. For example, the vector $[4, 2, 3]$ is in V since $4 - 2(2) = 0$ (i.e., the first component subtract twice the second component equals zero). Whereas the vector $[3, 4, 1]$ is not in V since $3 - 2(4) = -5 \neq 0$ (i.e., the first component subtract twice the second component does not equal zero).
To demonstrate closure under addition and multiplication, we need to show that these hold for general vectors within the set.
Let $\boldsymbol{u} = [u_1, u_2, u_3]$ and $\boldsymbol{v} = [v_1, v_2, v_3]$ be in V and let s be any scalar. Then $\boldsymbol{u}, \boldsymbol{v} \in V$ means $u_1 - 2u_2 = 0$ and $v_1 - 2v_2 = 0$.

Now consider $\boldsymbol{u}+\boldsymbol{v}=[u_1+v_1,u_2+v_2,u_3+v_3]$. Notice that, $(u_1+v_1)-2(u_2+v_2)=u_1+v_1-2u_2-2v_2=(u_1-2u_2)+(v_1-2v_2)=0+0=0$ which shows that $\boldsymbol{u}+\boldsymbol{v}\in U$, (i.e., $\boldsymbol{u}+\boldsymbol{v}$ is in V since the first component subtract twice the second component equals zero) so U is closed under addition. Next, consider $s\boldsymbol{u}=[su_1,su_2,su_3]$. Note that, $su_1-2su_2=s(u_1-2u_2)=s(0)=0$ which shows that $s\boldsymbol{u}\in U$, (i.e., $s\boldsymbol{u}$ is in U since the first component subtract twice the second component equals zero) so U is therefore closed under scalar multiplication.
As U is both closed under addition and closed under scalar multiplication U must be a subspace of $\mathbb{R}^3$.

(ii) U consists of all elements of $\mathbb{R}^3$ for which the first component squared is equal to twice the second component. For example, $\boldsymbol{a}=[2,2,3]$ and $\boldsymbol{b}=[4,8,1]$ are both in U since their first component squared is equal to twice the second component. However when we add these two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ together we get $\boldsymbol{a}+\boldsymbol{b}=[6,10,4]\notin U$, (i.e., $\boldsymbol{a}+\boldsymbol{b}$ is not in U since the first component squared is not equal to twice the second component, $(6)^2=36\neq 2(10)=20$) so the set is not closed under addition and it is thus not a subspace.

(iii) W consists of all elements of $\mathbb{R}^3$ for which the first component is equal to 2. For example, the vector $[2,4,3]$ is in W since the first component is equal to 2. However, note that $\mathbf{0}=[0,0,0]$ does not have its first component equal to 2, *i.e.* it is not in W. Thus, since W does not contain the zero vector, it cannot be a subspace of $\mathbb{R}^3$.
Alternatively, $\boldsymbol{a}=[2,1,1]$ and $\boldsymbol{b}=[2,3,-4]$ are both in W (i.e., their first component is equal to 2) but when we add the vectors together we get $\boldsymbol{a}+\boldsymbol{b}=[4,4,-3]\notin W$, (i.e., $\boldsymbol{a}+\boldsymbol{b}$ is not in W since the first component is not equal to 2) so the set is not closed under addition and it is thus not a subspace.

6. Need to solve $c_1\boldsymbol{v}_1+c_2\boldsymbol{v}_2+c_3\boldsymbol{v}_3=\boldsymbol{w}$

$$c_1\begin{bmatrix}1\\1\\1\end{bmatrix}+c_2\begin{bmatrix}1\\0\\1\end{bmatrix}+c_3\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}-1\\1\\3\end{bmatrix}$$

i.e.

$$\begin{aligned}c_1+c_2&=-1\\c_1+c_3&=1\\c_1+c_2+c_3&=3\end{aligned}$$

Now we solve the augmented matrix:

$$\left[\begin{array}{ccc|c}1&1&0&-1\\1&0&1&1\\1&1&1&3\end{array}\right]\begin{array}{l}\\R_2=R_2-R_1\\R_3=R_3-R_1\end{array}\sim\left[\begin{array}{ccc|c}1&1&0&-1\\0&-1&1&2\\0&0&1&4\end{array}\right]$$

STOP: $r(A)=r(A|b)=n=3\ \Rightarrow$ Unique solution.

Row 3: $\boxed{c_3=4}$

Row 2: $-c_2+c_3=2\ \Rightarrow\ c_2=c_3-2\ \Rightarrow\ c_2=4-2\ \Rightarrow\boxed{c_2=2}$

Row 1: $c_1+c_2=-1\ \Rightarrow\ c_1=-1-c_2\ \Rightarrow\ c_1=-1-2\ \Rightarrow\boxed{c_1=-3}$
$\Rightarrow -3\boldsymbol{v}_1+2\boldsymbol{v}_2+4\boldsymbol{v}_3=\boldsymbol{w}\ \therefore\ \boldsymbol{w}$ is a linear combination of $\boldsymbol{v}_1,\boldsymbol{v}_2\ \&\ \boldsymbol{v}_3$.

7. Need to solve $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 = \boldsymbol{w}$

$$c_1\begin{bmatrix}1\\1\\0\end{bmatrix} + c_2\begin{bmatrix}-1\\2\\2\end{bmatrix} + c_3\begin{bmatrix}2\\-1\\-2\end{bmatrix} = \begin{bmatrix}3\\-3\\1\end{bmatrix}$$

i.e.

$$\begin{aligned}c_1 - c_2 + 2c_3 &= 3\\ c_1 + 2c_2 - c_3 &= -3\\ 2c_2 - 2c_3 &= 1\end{aligned}$$

Now we solve the augmented matrix:

$$\left[\begin{array}{ccc|c}1&-1&2&3\\1&2&-1&-3\\0&2&-2&1\end{array}\right] R_2 = R_2 - R_1 \sim \left[\begin{array}{ccc|c}1&-1&2&3\\0&3&-3&-6\\0&2&-2&1\end{array}\right] R_3 = 3R_3 - 2R_2$$

$$\sim \left[\begin{array}{ccc|c}1&-1&2&3\\0&3&-3&-6\\0&0&0&15\end{array}\right]$$

STOP: $r(A) = 2 < r(A|b) = 3 \;\Rightarrow$ No solution. $\therefore$ $\boldsymbol{w}$ is not a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2$ & $\boldsymbol{v}_3$.

8. (i) Is $\boldsymbol{v}_1 = m\boldsymbol{v}_2$?

$$\begin{bmatrix}-10\\15\end{bmatrix} = m\begin{bmatrix}4\\-6\end{bmatrix} \Rightarrow \begin{bmatrix}-10\\15\end{bmatrix} = \begin{bmatrix}4m\\-6m\end{bmatrix} \quad \begin{aligned}&\Rightarrow -10 = 4m \;\Rightarrow\; m = \tfrac{-10}{4} = -2.5\\ &\Rightarrow 15 = -6m \;\Rightarrow\; m = \tfrac{15}{-6} = -2.5\end{aligned}$$

$\therefore$ $\boldsymbol{v}_1 = -2.5\boldsymbol{v}_2$ Thus linearly dependent.

(ii) Is $\boldsymbol{v}_1 = m\boldsymbol{v}_2$?

$$\begin{bmatrix}7\\3\end{bmatrix} = m\begin{bmatrix}21\\12\end{bmatrix} \Rightarrow \begin{bmatrix}7\\3\end{bmatrix} = \begin{bmatrix}21m\\12m\end{bmatrix} \quad \begin{aligned}&\Rightarrow 7 = 21m \;\Rightarrow\; m = \tfrac{7}{21} = \tfrac{1}{3}\\ &\Rightarrow 3 = 12m \;\Rightarrow\; m = \tfrac{3}{12} = \tfrac{1}{4}\end{aligned}$$

$\therefore$ $\boldsymbol{v}_1 \neq m\boldsymbol{v}_2$ Thus linearly independent.

(iii) There are more vectors ($m = 4$) than space ($n = 3$), so they must be linearly dependent.

(iv) Let $A = [\,\boldsymbol{v}_1 \,|\, \boldsymbol{v}_2 \,|\, \boldsymbol{v}_3\,]$. Find $\det(A)$. Using cofactor expansion along 1st row:

$$\det(A) = \begin{vmatrix}1^{+} & -1^{-} & 0^{+}\\ -1 & 0 & 1\\ 0 & 1 & -1\end{vmatrix} \quad \text{Cofactor expansion along 1st row:}$$

$$\det(A) = (1)\begin{vmatrix}0&1\\1&-1\end{vmatrix} - (-1)\begin{vmatrix}-1&1\\0&-1\end{vmatrix} + (0)$$

$= 1(0-1) + 1(1-0) + 0 = -1 + 1 + 0 = 0.$

Since $\det(A) = 0$, they are linearly dependent.

(v) Let $A = [\,\boldsymbol{v}_1 \,|\, \boldsymbol{v}_2 \,|\, \boldsymbol{v}_3\,]$. Find $\det(A)$. Using cofactor expansion along 1st row:

$$\det(A) = \begin{vmatrix} 1^{+} & 2^{-} & 3^{+} \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} \quad \text{Cofactor expansion along 1st row:}$$

$$\det(A) = (1)\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} - (2)\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + (3)\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix}$$

$= 1(6-1) - 2(4-3) + 3(2-9) = 1(5) - 2(1) + 3(-7) = 5 - 2 - 21 = -18.$

Since $\det(A) = -18 \neq 0$, they are linearly independent.

(vi) Need to solve $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 = \mathbf{0}$

$$c_1 \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ -1 \\ 4 \\ -4 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ -3 \\ 6 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e.

$$\begin{aligned} &c_1 + 3c_2 + 2c_3 = 0 \\ &-c_1 - c_2 - 3c_3 = 0 \\ &2c_1 + 4c_2 + 6c_3 = 0 \\ &-2c_1 - 4c_2 - 2c_3 = 0 \end{aligned}$$

Now we solve the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ -1 & -1 & -3 & 0 \\ 2 & 4 & 6 & 0 \\ -2 & -4 & -2 & 0 \end{array}\right] \begin{array}{l} \\ R_2 = R_2 + R_1 \\ R_3 = R_3 - 2R_1 \\ R_4 = R_4 + 2R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3 = R_3 + R_2 \\ R_4 = R_4 - R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 = R_4 - 3R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

STOP: $r(A) = r(A|\mathbf{0}) = \mathbf{n} = \mathbf{3} \;\Rightarrow$ Unique solution, i.e., trivial solution $\boldsymbol{c}_1 = 0, \boldsymbol{c}_2 = 0, \boldsymbol{c}_3 = 0$ $\therefore$ They are linearly independent.