EXAM SOLUTIONS

MATH1019
SEM 1, 2019

Q1). (a). (i). $\underset{\sim}{a}+3\underset{\sim}{b} = [1,2,-3]+3[2,1,1] = [1,2,-3]+[6,3,3]$ (1)
$= [7,5,0]$ (1)

(ii). $\|\underset{\sim}{a}\| = \sqrt{(1)^2+(2)^2+(-3)^2} = \sqrt{1+4+9} = \sqrt{14}$ (1)
$\|\underset{\sim}{b}\| = \sqrt{(2)^2+(1)^2+(1)^2} = \sqrt{4+1+1} = \sqrt{6}$ (1)
$2\|\underset{\sim}{a}\|\hat{\underset{\sim}{b}} = 2\|\underset{\sim}{a}\|\dfrac{\underset{\sim}{b}}{\|\underset{\sim}{b}\|} = 2(\sqrt{14})\dfrac{[2,1,1]}{\sqrt{6}}$ (1) $= 2\sqrt{7/3}[2,1,1] = [4\sqrt{7/3}, 2\sqrt{7/3}, 2\sqrt{7/3}]$

(iii). $\|(\underset{\sim}{c}.\underset{\sim}{a})\hat{\underset{\sim}{b}}\| = \|([2,1,0].[1,2,-3])[2,1,1]\| = \|(2+2+0)[2,1,1]\|$
$= \|4[2,1,1]\|$ (1) $= \|[8,4,4]\|$ (1) $= \sqrt{8^2+4^2+4^2} = \sqrt{96}$ (1)

(iv). $p = \underset{\sim}{a}.\hat{\underset{\sim}{b}} = \dfrac{\underset{\sim}{a}.\underset{\sim}{b}}{\|\underset{\sim}{b}\|}$ (1/2) $= \dfrac{[1,2,-3].[2,1,1]}{\sqrt{6}}$ (1/2) $= \dfrac{2+2-3}{\sqrt{6}} = \dfrac{1}{\sqrt{6}}$ (1)

(v). $\underset{\sim}{a}\times\underset{\sim}{c} = \begin{vmatrix} i & j & k \\ 1 & 2 & -3 \\ 2 & 1 & 0 \end{vmatrix} = \underset{\sim}{i}(0+3) - \underset{\sim}{j}(0+6) + \underset{\sim}{k}(1-4)$ (1)
$= [3,-6,-3]$ (1)

Area $= \|\underset{\sim}{a}\times\underset{\sim}{c}\| = \sqrt{3^2+(-6)^2+(-3)^2}$ (1) $= \sqrt{9+36+9} = \sqrt{54}$ (1)

(b). $\vec{AB} = [3-3, 0-2, -1-1] = [0,-2,-2]$ (1/2)
$\vec{AC} = [2-3, 2-2, -3-1] = [-1,0,-4]$ (1/2)
$\vec{AD} = [0-3, 4-2, 1-1] = [-3,2,0]$ (1/2)

or $\begin{vmatrix} -1 & 0 & 4 \\ 0 & -2 & -2 \\ -3 & 2 & 0 \end{vmatrix}$ then take determinant

$\vec{AC}\times\vec{AD} = \begin{vmatrix} i & j & k \\ -1 & 0 & -4 \\ -3 & 2 & 0 \end{vmatrix} = \underset{\sim}{i}(0+8) - \underset{\sim}{j}(0-12) + \underset{\sim}{k}(-2-0)$ (1/2)
$= [8, 12, -2]$ (1)

$\vec{AB}.(\vec{AC}\times\vec{AD}) = [0,-2,-2].[8,12,-2]$
$= 0-24+4$ (1/2) $= -20$ (1/2) $\neq 0$ (1) $\therefore$ Not coplanar (1)

Q2). (a). (i). $B-A = \begin{bmatrix} 0 & 2 \\ 4 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 4 & 1 \\ -1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 5 & -3 \\ 1 & -3 \end{bmatrix}$ (1)

(ii). $B^2 =$ d.n.e (1) Since B is not a square matrix (1)

(iii). $AC = \begin{bmatrix} 4 & 1 \\ -1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 16-2 & 12+1 \\ -4-4 & -3+2 \\ 0-6 & 0+3 \end{bmatrix} = \begin{bmatrix} 14 & 13 \\ -8 & -1 \\ -6 & 3 \end{bmatrix}$ (1/2) for each correct entry

(iv). $3I_2C = 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}$ (1) $= 3\begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}$ (1) $= \begin{bmatrix} 12 & 9 \\ -6 & 3 \end{bmatrix}$ (1)

or $= \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}$ (1) $= \begin{bmatrix} 12 & 9 \\ -6 & 3 \end{bmatrix}$ (1)

or $3I_2C = 3C$ (2) $= 3\begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 9 \\ -6 & 3 \end{bmatrix}$ (1)

(v). $C^{-1} = \frac{1}{(4)(1)-(3)(-2)}$ (1) $\begin{bmatrix} 1 & -3 \\ -(-2) & 4 \end{bmatrix}$ (1) $= \frac{1}{10}\begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}$ (1)

(vi). $\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 2 & 2 & -3 & 0 & 1 & 0 \\ -1 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$ (1) $\begin{matrix} R_2 = R_2 - 2R_1 \\ R_3 = R_3 + R_1 \end{matrix} \sim \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right]$ (2) Swap $R_1 \leftrightarrow R_2$

$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right] \begin{matrix} R_3 = R_3 - R_1 \\ R_2 = R_2 + R_1 \end{matrix}$ (1) $\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 3 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right] \begin{matrix} R_1 = R_1 - R_2 \\ R_3 \times (-1) \end{matrix}$ (2)

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -2 & -1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$ (1) $\therefore D^{-1} = \begin{bmatrix} 4 & -2 & -1 \\ -1 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix}$ (1)

Q3). (a). Direction L_1 : $d_1 = [4, 3, 1]$ (½)

Direction L_2 : $\underset{\sim}{d_2} = [1, 2, 1]$ (½)

Since $\underset{\sim}{d_1} \neq m\underset{\sim}{d_2}$ (1) $\therefore$ Not parallel (1)

x: $3 + 4t = \tau \quad \sim(1)$

y: $10 + 3t = -1 + 2\tau \quad \sim(2)$ (1)

z: $1 + t = 2 + \tau \quad \sim(3)$

Sub (1) into (2) and solve for t

$10 + 3t = -1 + 2(3 + 4t) \Rightarrow 10 + 3t = -1 + 6 + 8t$

$\Rightarrow 5 = 5t \Rightarrow t = 1$ (1)

$\therefore \tau = 3 + 4(1) = 7$

Test (3) z: $1 + 1 = 2 + 7 \Rightarrow 2 \neq 9$ (1) $\therefore$ Don't intersect

$\therefore$ Skew Lines (1)

(b). Normal vector $\underset{\sim}{n} = [4, -2, 1]$ (1)

Point on plane if $x = y = 0 \therefore z = -8 \quad \therefore A(0, 0, -8)$ (1)

Vector $\vec{AP} = [0-0, 3-0, 2-(-8)] = [0, 3, 10]$ (1)

Distance $= |\vec{AP} \cdot \hat{\underset{\sim}{n}}| = \left| \frac{[0,3,10] \cdot [4,-2,1]}{\sqrt{4^2 + (-2)^2 + 1^2}} \right|$ (1)

$= \left| \frac{0 - 6 + 10}{\sqrt{16 + 4 + 1}} \right| = \left| \frac{4}{\sqrt{21}} \right| = \frac{4}{\sqrt{21}}$ (1)

(c). Sub line into plane, solve for t

$(2+t)+2(1-t)-(-4t)=10$ (1)

$2+t+2-2t+4t=10$ (½) $\Rightarrow 3t=6 \Rightarrow t=2$ (1)

sub $t=2$ into line

$x=2+2=4$, (½) $y=1-2=-1$, (½) $z=-4(2)=-8$ (½)

$\therefore$ Point $(4,-1,-8)$

(d). Normal vector of P_1: $\underset{\sim}{n_1}=[-2,1,-1]$
Normal vector of P_2: $\underset{\sim}{n_2}=[6,-3,3]$
Normal vector of P_3: $\underset{\sim}{n_3}=[4,5,-3]$
(1 mark)

(i). Since $\underset{\sim}{n_1}=m\underset{\sim}{n_2}$ (½) for $m=-1/3$ (½)

$\therefore$ Parallel

(ii). $\underset{\sim}{n_1}\cdot\underset{\sim}{n_3}=[-2,1,-1]\cdot[4,5,-3]$ (1)

$=-8+5+3=0$ (1) $\therefore$ Perpendicular

Q4). (a)
$$\begin{vmatrix} 1^{+} & 2^{-} & -3^{-} \\ 8 & 4 & 3 \\ -1 & 0 & -2 \end{vmatrix}$$
Cofactor expansion along 1st row

$$\det(A)=1\begin{vmatrix}4 & 3\\0 & -2\end{vmatrix}-2\begin{vmatrix}8 & 3\\-1 & -2\end{vmatrix}-3\begin{vmatrix}8 & 4\\-1 & 0\end{vmatrix}$$ (2)

$=1(-8-0)-2(-16+3)-3(0+4)$ (1)

$=1(-8)-2(-13)-3(4)=-8+26-12=16$ (1) $\neq 0$ (1) $\therefore$ Non-singular (1)

(b). $A=\begin{bmatrix}3 & 2\\-1 & 1\end{bmatrix}$ (½) $\quad \underset{\sim}{b}=\begin{bmatrix}4\\-3\end{bmatrix}$

$\det(A)=(3)(1)-(2)(-1)=3+2=5$ (½)

$A_1=\begin{bmatrix}4 & 2\\-3 & 1\end{bmatrix}$ (1)

$\det(A_1)=(4)(1)-(2)(-3)=4+6=10$ (½)

$A_2=\begin{bmatrix}3 & 4\\-1 & -3\end{bmatrix}$ (1)

$\det(A_2)=(3)(-3)-(4)(-1)=-9+4=-5$ (½)

$\therefore x_1=\dfrac{\det(A_1)}{\det(A)}=\dfrac{10}{5}=2$ (1) $\quad x_2=\dfrac{\det(A_2)}{\det(A)}=\dfrac{-5}{5}=-1$ (1)

(c). Set up $[\underset{\sim}{u_1} \mid \underset{\sim}{u_2} \mid \underset{\sim}{u_3} \,\vdots\, \underset{\sim}{w}]$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 3 & -2 & 1 & 3 \\ 2 & 7 & -11 & 3 \end{array}\right] \begin{array}{l} R_2 = R_2 - 3R_1 \\ R_3 = R_3 - 2R_1 \end{array} \; 2 \sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & -5 & 7 & 0 \\ 0 & 5 & -7 & 1 \end{array}\right] \; 2 \quad R_3 = R_3 + R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & -5 & 7 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \; 1$$

$r(A) = 2 \neq r(A|b) = 3$ 1 $\therefore$ No Solution 1

$\therefore$ Not a l.c. 1

Q5). (a).

(i). $$\left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{array}\right] R_3 = R_3 + R_1 \; 1 \sim \left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{array}\right] \begin{array}{l} R_3 = R_3 - 2R_2 \\ R_4 = R_4 - R_2 \end{array} \; 1$$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{array}\right] R_4 = R_4 - R_3 \; 2 \sim \left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} R_1 = R_1 + R_4 \\ R_2 = R_2 - R_4 \\ R_3 = R_3 + 3R_4 \end{array} \; 1$$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] R_1 = R_1 + R_2 \; 3 \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \; 1$$

(ii). $r(A) = 4 = n = 4$ 1 $\therefore$ Unique Solution 1

$\therefore x_1 = 0, \; x_2 = 0, \; x_3 = 0 \;\&\; x_4 = 0$ 1

(b). $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ -3 & 2 \\ -1 & -4 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 1 & 2 & -3 & -1 \\ 1 & 1 & 2 & -4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ -3 & 2 \\ -1 & -4 \end{bmatrix} = \begin{bmatrix} 1+4+9+1 & 1+2-6+4 \\ 1+2-6+4 & 1+1+4+16 \end{bmatrix} = \begin{bmatrix} 15 & 1 \\ 1 & 22 \end{bmatrix} \; 2$$

1

$$(A^TA)^{-1} = \frac{1}{(15)(22)-(1)(1)}\begin{bmatrix} 22 & -1 \\ -1 & 15 \end{bmatrix}$$ 1 $$= \frac{1}{329}\begin{bmatrix} 22 & -1 \\ -1 & 15 \end{bmatrix}$$ 1

2

$$\text{pinv}(A) = (A^TA)^{-1}A^T = \frac{1}{329}\begin{bmatrix} 22 & -1 \\ -1 & 15 \end{bmatrix}\begin{bmatrix} 1 & 2 & -3 & -1 \\ 1 & 1 & 2 & -4 \end{bmatrix} = \frac{1}{329}\begin{bmatrix} 21 & 43 & -68 & -1 \\ 14 & 13 & 33 & -5 \end{bmatrix}$$

$\therefore$ Solution,

$$\hat{x} = \text{pinv}(A)\,\underset{\sim}{b} = \frac{1}{329}\begin{bmatrix} 21 & 43 & -68 & -18 \\ 14 & 13 & 33 & -59 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} = \frac{1}{329}\begin{bmatrix} 28 \\ -81 \end{bmatrix} \approx \begin{bmatrix} 0.0851 \\ -0.2462 \end{bmatrix}$$ 1