CURTIN UNIVERSITY
Department of Mathematics and Statistics

Linear Algebra and Statistics for Engineers

MID-SEMESTER TEST

Semester 2, 2017

INSTRUCTIONS: Answer all questions in the spaces provided.

To obtain full marks for a question you must **clearly** show appropriate working.

TIME ALLOWED: 55 minutes.

TOTAL MARKS: 40

AIDS ALLOWED:
1. Scientific Calculator.
2. A4 Sheet of handwritten or typed notes (both sides).

SOLUTIONS

Last Name: ______________________

Given Name: ______________________

Student Number: ______________________

Tutors Name: ______________________

Workshop Day: ____________ Workshop Time: ____________

Question 1

Solve the following systems of linear equations by first writing it in the form of an augmented matrix $[A|\boldsymbol{b}]$ and then using the Gaussian Elimination method. Make sure you state the rank of A and the rank of $[A|\boldsymbol{b}]$.

$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ 2x_1 - x_2 - x_3 &= -2 \\ 4x_1 + x_2 - 3x_3 &= 5 \end{aligned}$$

(6 marks)

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & -1 & -1 & -2 \\ 4 & 1 & -3 & 5 \end{array}\right] \quad (1) \quad \begin{array}{l} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 4R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -3 & 1 & -2 \\ 0 & -3 & 1 & 5 \end{array}\right] \quad (2) \quad R_3 = R_3 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -3 & 1 & -2 \\ 0 & 0 & 0 & 7 \end{array}\right] \quad (1)$$

$r(A) = 2$, $(\frac{1}{2})$ $r(A|b) = 3$, $(\frac{1}{2})$ No Solution (1)

(since $r(A) \neq r(A|b)$)

Question 2

Solve the following homogeneous system of linear equations by first writing it in the form of an augmented matrix $[A|\mathbf{0}]$ and then using the Gaussian Elimination method. Make sure you state the rank of A.

$$2x_1 - x_2 + 2x_3 = 0$$
$$-x_1 + x_2 + x_3 = 0$$
$$-x_1 - 3x_3 = 0$$

(8 marks)

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & 0 & -3 & 0 \end{array}\right] \quad 1 \quad \begin{array}{l} R_2 = 2R_2 + R_1 \\ R_3 = 2R_3 + R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -1 & -4 & 0 \end{array}\right] \quad 2 \quad R_3 = R_3 + R_1$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad 1$$

$r(A) = 2$ 1, $n = 3$ $\therefore$ Infinite Solutions (since $r(A) < n$)

Let $x_3 = t$ 1, $t \in \mathbb{R}$

Row 2: $x_2 + 4x_3 = 0 \Rightarrow x_2 = -4t$ 1

Row 1: $2x_1 - x_2 + 2x_3 = 0 \Rightarrow 2x_1 = -4t - 2t$

$2x_1 = -6t \Rightarrow x_1 = -3t$ 1

$$\therefore \underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3t \\ -4t \\ t \end{bmatrix} = t \begin{bmatrix} -3 \\ -4 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

Question 3 is on the next page...

Question 3

Find the inverse of the matrix,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$$

(6 marks)

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 6 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 = R_2 - R_1 \\ R_3 = R_3 - R_1 \end{array} \quad 1$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 5 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ R_3 = R_3 - 2R_2 \end{array} \quad 1$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right] \begin{array}{l} R_1 = R_1 - R_3 \\ R_2 = R_2 - 2R_3 \\ \end{array} \quad 1$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & -3 & 5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right] \begin{array}{l} R_1 = R_1 - R_2 \\ \\ \end{array} \quad 1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -3 & 5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right] \quad 1$$

$$\therefore A^{-1} = \begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix} \quad 1$$

Question 4 is on the next page...

Question 4

Solve the following system of linear equations by using the inverse of the coefficient matrix,

$$7x_1 + 3x_2 = -8$$
$$8x_1 + 4x_2 = -8$$

(6 marks)

$A = \begin{bmatrix} 7 & 3 \\ 8 & 4 \end{bmatrix}$ $\underset{\sim}{b} = \begin{bmatrix} -8 \\ -8 \end{bmatrix}$

$A^{-1} = \frac{1}{7(4)-(3)(8)} (1) \begin{bmatrix} 4 & -3 \\ -8 & 7 \end{bmatrix} (1) = \frac{1}{4}\begin{bmatrix} 4 & -3 \\ -8 & 7 \end{bmatrix} = \begin{bmatrix} 1 & -3/4 \\ -2 & 7/4 \end{bmatrix} (1)$

$\underset{\sim}{x} = A^{-1}\underset{\sim}{b} = \begin{bmatrix} 1 & -3/4 \\ -2 & 7/4 \end{bmatrix}\begin{bmatrix} -8 \\ -8 \end{bmatrix} (2) = \begin{bmatrix} -8+6 \\ 16-14 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} (1)$

$\therefore x = -2, \ y = 2$

Question 5

Calculate the determinant $|B|$ of the matrix $B = \begin{bmatrix} 1 & 0 & 3 \\ 5 & 6 & 2 \\ 3 & -2 & 1 \end{bmatrix}$. From this determinant value, does B have an inverse? Give a reason for your decision. (Note: you do not have to calculate the inverse matrix if it exists)

(8 marks)

$\det(B) = \begin{vmatrix} 1 & 0 & 3 \\ 5 & 6 & 2 \\ 3 & -2 & 1 \end{vmatrix}$ Cofactor expansion along 1st row

$= 1\begin{vmatrix} 6 & 2 \\ -2 & 1 \end{vmatrix} (1) - 0 (1) + 3\begin{vmatrix} 5 & 6 \\ 3 & -2 \end{vmatrix} (1)$

$= 1(6-(-4)) (1) + 3(-10-18) (1)$

$= 1(10) + 3(-28) = 10 - 84 = -74 \ (1)$

Since $\det(B) \neq 0$ (1) $\therefore$ Inverse exists (1)

Question 6

Use Cramer's rule to solve the following system for x_1,

$$2x_1 + 4x_2 = -2$$
$$-3x_1 + x_2 = -11$$

(6 marks)

$A = \begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix}$ 1 $\underset{\sim}{b} = \begin{bmatrix} -2 \\ -11 \end{bmatrix}$

$A_1 = \begin{bmatrix} -2 & 4 \\ -11 & 1 \end{bmatrix}$ 1

$\det(A) = (2)(1) - (4)(-3) = 2 + 12 = 14$ 1

$\det(A_1) = (-2)(1) - (4)(-11) = -2 + 44 = 42$ 1

$\therefore x_1 = \dfrac{\det(A_1)}{\det(A)}$ 1 $= \dfrac{42}{14} = 3$ 1