Question 1

(a) (i) $\hat{\underset{\sim}{\boldsymbol{a}}} = \dfrac{\underset{\sim}{\boldsymbol{a}}}{||\underset{\sim}{\boldsymbol{a}}||} \;=\; \dfrac{1}{\sqrt{2^2+(-3)^2+6^2}}[2,-3,6] \;=\; \dfrac{1}{7}[2,-3,6]$

(1 mark)

(ii) $\underset{\sim}{\boldsymbol{b}} \cdot \hat{\underset{\sim}{\boldsymbol{a}}} = [1,2,3] \cdot \dfrac{1}{7}[2,-3,6] = \dfrac{1}{7}(2-6+18) = 2$

(2 marks)

(iii) $$\underset{\sim}{\boldsymbol{a}} \times \underset{\sim}{\boldsymbol{b}} = [2,-3,6] \times [1,2,3] = \begin{vmatrix} \underset{\sim}{\boldsymbol{i}} & \underset{\sim}{\boldsymbol{j}} & \underset{\sim}{\boldsymbol{k}} \\ 2 & -3 & 6 \\ 1 & 2 & 3 \end{vmatrix} \;=\; \underset{\sim}{\boldsymbol{i}}(-9-12) - \underset{\sim}{\boldsymbol{j}}(6-6) + \underset{\sim}{\boldsymbol{k}}(4+3)$$
$$= \; -21\underset{\sim}{\boldsymbol{i}} + 0\underset{\sim}{\boldsymbol{j}} + 7\underset{\sim}{\boldsymbol{k}} \;=\; [-21,0,7]$$

(4 marks)

(iv) The vectors will be parallel if $\underset{\sim}{\boldsymbol{c}} = k\underset{\sim}{\boldsymbol{a}}$ for some $k \in \mathbb{R}$.
So we want to find x where $[1,x,3] = k[2,-3,6]$.
From the first and third components, $k = \frac{1}{2}$.
From the second component, this gives $x = -\frac{3}{2}$.

(2 marks)

(v) Since there are only two vectors, and they are not parallel, the vectors are linearly independent.
Alternatively, students may show that the only solution to $c_1\underset{\sim}{\boldsymbol{a}} + c_2\underset{\sim}{\boldsymbol{b}} = \underset{\sim}{\boldsymbol{0}}$ is the trivial solution $c_1 = 0,\ c_2 = 0$.

(2 marks)

(b) The volume is given by the absolute value of the scalar triple product, $\left|\underset{\sim}{\boldsymbol{a}} \cdot (\underset{\sim}{\boldsymbol{b}} \times \underset{\sim}{\boldsymbol{c}})\right|$

$$\underset{\sim}{\boldsymbol{a}} \cdot (\underset{\sim}{\boldsymbol{b}} \times \underset{\sim}{\boldsymbol{c}}) = \begin{vmatrix} 2 & -1 & 3 \\ 4 & 1 & 0 \\ 3 & 2 & 2 \end{vmatrix}$$
$$= -4\begin{vmatrix} -1 & 3 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix}$$
$$= -4(-2-6) + (4-9)$$
$$= 27$$

So the volume is $\left|\underset{\sim}{\boldsymbol{a}} \cdot (\underset{\sim}{\boldsymbol{b}} \times \underset{\sim}{\boldsymbol{c}})\right| = 27$ (6 marks)

(c) $W = \underset{\sim}{\boldsymbol{F}} \cdot \underset{\sim}{\boldsymbol{s}} = ||\underset{\sim}{\boldsymbol{F}}||\,||\underset{\sim}{\boldsymbol{s}}||\,\cos\theta = (10)(7)\cos(45°) = 35\sqrt{2} \;\approx\; 49.5$ J

(3 marks)

(A total of 20 marks for this question.)

Question 2

(a) A vector in the direction of the line is $\underset{\sim}{\boldsymbol{v}} = [7,-2,4] - [5,-5,3] = [2,3,1]$.
So the vector equation of line is

$$\underset{\sim}{\boldsymbol{r}} = [7,-2,4] + t[2,3,1],\ t \in \mathbb{R}$$

Parametric equations of line are:

$$\begin{aligned} x &= 7 + 2t \\ y &= -2 + 3t \\ z &= 4 + t \end{aligned}$$

Rearranging for t and equating, we get the cartesian equations:

$$\frac{x-7}{2} = \frac{y+2}{3} = z - 4$$

(4 marks)

(b) Since the plane contains the line, it contains the vector $\underset{\sim}{\boldsymbol{v}} = [-1,-1,1]$ and the point $Q = (1,2,0)$. The plane also contains the point $P = (1,3,-2)$.
We find a second vector on the plane $\underset{\sim}{\boldsymbol{u}} = \overrightarrow{PQ} = [0,-1,2]$.
So the vector equation of line is

$$\underset{\sim}{\boldsymbol{r}} = [1,3,-2] + t[-1,-1,1] + s[0,-1,2],\ t \in \mathbb{R},\ s \in \mathbb{R}$$

Take the cross product of two vectors on the plane to find the normal of the plane:

$$\underset{\sim}{\boldsymbol{n}} = \underset{\sim}{\boldsymbol{u}} \times \underset{\sim}{\boldsymbol{v}} = \begin{vmatrix} \underset{\sim}{\boldsymbol{i}} & \underset{\sim}{\boldsymbol{j}} & \underset{\sim}{\boldsymbol{k}} \\ 0 & -1 & 2 \\ -1 & -1 & 1 \end{vmatrix} = \underset{\sim}{\boldsymbol{i}}(-1+2) - \underset{\sim}{\boldsymbol{j}}(0+2) + \underset{\sim}{\boldsymbol{k}}(0-1) = [1,-2,-1]$$

$$\underset{\sim}{\boldsymbol{n}} \cdot \underset{\sim}{\boldsymbol{r}} = \underset{\sim}{\boldsymbol{n}} \cdot \overrightarrow{OP} \Rightarrow [1,-2,-1]\cdot[x,y,z] = [1,-2,-1]\cdot[1,3,-2] \Rightarrow \boxed{x - 2y - z = -3}$$

(8 marks)

(c)

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 4 & 5 & 6 & 52 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & -3 & -6 & 12 \end{array}\right] (R_2 - 4R_1) \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 1 & 2 & -4 \end{array}\right] (-\tfrac{1}{3}R_2)$$

Let $z = t,\ t \in \mathbb{R}$. From R_2, we have $y + 2z = -4 \Rightarrow y = -4 - 2t$
From R_3, we have $x + 2y + 3z = 10 \Rightarrow x = 10 - 3t - 2(-4-2t) = 18 + t$
So the intersection is $\boxed{[x,y,z] = [18,-4,0] + t\,[1,-2,1],\ t \in \mathbb{R}}$
This is a line in the direction of the vector $[1,-2,1]$, passing through the point $(18,-4,0)$.

(8 marks)

(A total of 20 marks for this question.)

Question 3

(a)

$$\left[\begin{array}{rrr|r} 1 & -3 & 2 & 1 \\ 2 & -5 & 6 & 5 \\ -1 & 5 & 2 & 6 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & -3 & 2 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 7 \end{array}\right] \begin{array}{l} \\ (R_2 - 2R_1) \\ (R_3 + R_1) \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & 8 & 10 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} (R_1 + 3R_2) \\ \\ (R_3 - 2R_2) \end{array}$$

The system is inconsistent (may make an argument about rank or similar), so there are no solutions.

(6 marks)

(b)

$$\left[\begin{array}{rrr|rrr} 1 & -2 & -4 & 1 & 0 & 0 \\ 2 & -3 & -6 & 0 & 1 & 0 \\ -3 & 6 & 15 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 3 & 3 & 0 & 1 \end{array}\right] \begin{array}{l} \\ (R_2 - 2R_1) \\ (R_3 + 3R_1) \end{array}$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -3 & 2 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} \end{array}\right] \begin{array}{l} (R_1 + 2R_2) \\ \\ (\frac{1}{3}R_3) \end{array}$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -3 & 2 & 0 \\ 0 & 1 & 0 & -4 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} \end{array}\right] (R_2 - 2R_3)$$

So $A^{-1} = \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix}$

(7 marks)

(c)

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 2 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & -5 & -3 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ -5 & -3 \end{vmatrix} = -9 + 10 = 1$$

$$|A_2| = \begin{vmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 2 & 1 & -1 \end{vmatrix} = -\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = -(1+1) = -2$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{-2}{1} = -2$$

(7 marks)

(A total of 20 marks for this question.)

Question 4

(a)
- **Assumptions**
  1. The eight shells constitute an SRS.
  2. The muzzle velocities can be reasonably modelled by a normal probability distribution.
  3. σ is unknown.

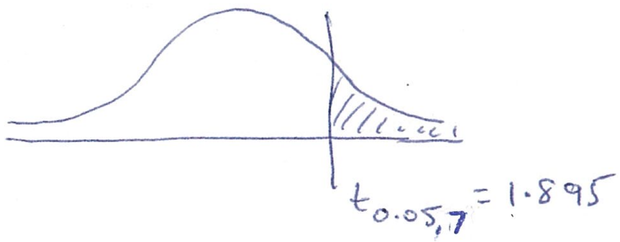
  $H_0 : \mu = 3000$

  $H_A : \mu > 3000$

- **Test Statistic**

$$t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} = \frac{2959 - 3000}{\frac{39.4}{\sqrt{8}}} = -2.943$$

- **Critical Region**



- **Conclusion**

  Since $t < t_{0.05}$, we accept the null hypothesis at the 5% level of significance. There appears to be sufficient evidence to doubt the manufacturer's claims.

(10 marks)

(b) (i)
$$P(1 < \bar{X} < 5) = P\left(\frac{1-4}{\frac{0.8}{\sqrt{30}}} < Z < \frac{5-4}{\frac{0.8}{\sqrt{30}}}\right)$$
$$= P(-20.5396 < Z < 6.8465)$$
$$= 1$$

(4 marks)

(ii)
$$P\left(\sum_{i=1}^{30} X_i < 115\right) = P(30\bar{X} < 115)$$
$$= P\left(\bar{X} < \frac{115}{30}\right)$$
$$= P\left(\bar{Z} < \frac{\frac{115}{30} - 4}{\frac{0.8}{\sqrt{30}}}\right)$$
$$= P(\bar{Z} < -1.1411)$$
$$= 0.1271$$

(6 marks)

(A total of 20 marks for this question.)

Question 5

(a) $n = 15,\ p = 0.05$

$$\begin{aligned}
P(X \geq 2) &= 1 - P(X \leq 1) \\
&= 1 - \sum_{x=0}^{1} \binom{15}{x} (0.05)^x (1-0.05)^{15-x} \\
&= 1 - 0.8290 \\
&= 0.1710
\end{aligned}$$

(5 marks)

(b) CI: $\bar{x} \pm z_{\frac{\alpha}{2}} \dfrac{\sigma}{\sqrt{n}}$

$$\begin{aligned}
&= 477 \pm 1.645 \left(\frac{13}{\sqrt{150}} \right) \\
&= (475.25,\ 478.75)
\end{aligned}$$

Since sample size is large, the only assumption is that the sample is SRS.

(6 marks)

(c) Ordered set: $19,\ 31,\ 34,\ 43,\ 49,\ 56,\ 69,\ 77,\ 81,\ 85,\ 97$

Min $= 19$

$Q_1 : (n+1)\,p = \frac{12}{4} = 3 \quad \Rightarrow \quad Q_1 = 34$

$Q_2 : (n+1)\,p = \frac{12}{2} = 6 \quad \Rightarrow \quad Q_2 = 56$

$Q_3 : (n+1)\,p = 12\left(\frac{3}{4}\right) = 9 \quad \Rightarrow \quad Q_3 = 81$

Max $= 97$

(6 marks)

(c) The correct statement is (iii), the mean is greater than the median.

Students should sketch an example of a distribution with a long right tail, and justify why the mean is greater than the median (for example, by sketching a rough location for each, or by explaining in words).

(3 marks)

(A total of 20 marks for this question.)

END OF EXAMINATION