Lecture 1. Propositional Logics

Ref.: K. H. Rosen, Section 1.1 & 1.2

Definition

Definition:
A Proposition is a declarative statement that is either true or false but not both.

True

False

Examples

Propositions:

"Qilin is a CS lecturer." **T**

"1 + 1 = 3" **F**

Non-Propositions:

"Is he a CS student?"

"X + 1 = 3"

"This sentence is false." a paradox

Propositions cannot contain undefined variables!

Denotations

Letters (p, q, r, s…) are used to denote propositions

E.g., *p* is defined as "Canberra is the capital of Australia "

"Canberra is the capital of Australia " **T**

p **T**

How to denote "Canberra is the capital of Australia and Canberra is located at the southeast of Australia" ?

Compound Propositions

New propositions can be generated by combining existing propositions using logical operators, and they are called compound propositions.

Negation

"Today is Tuesday"

"Today is not Tuesday"

Definition :

Let p be a proposition. The statement

"It is not the case that p"

is another proposition, called the negation of p.

The negation of p is denoted by $\neg p$.

The proposition $\neg p$ is read "not p"

Negation: Truth table

p	$\neg p$
T	F
F	T

- A truth table displays the truth value of complex propositions ($\neg p$) corresponding to the truth value of elementary propositions (p)

Conjunction

"Today is Tuesday" (p) "It is raining today" (q)

"Today is Tuesday *and* it is raining today"

p and q

- We also use the symbol $\wedge$ to represent *and*.

$p \wedge q$

Conjunction

Definition:

Let p and q be propositions. The propositions "p and q" denoted by $p \wedge q$, is true when p and q are both true and is false otherwise.

The proposition $p \wedge q$ is called the Conjunction of p and q.

Conjunction

p	q	p ^ q
T	T	T
T	F	F
F	T	F
F	F	F

Given n elementary propositions, the number of rows equals to 2^n!

Disjunction

"Today is Tuesday" (p) "It is raining today"(q)

"Today is Tuesday *or* it is raining today"

(both might be true)

p or q

- When using logical *or* it is represented by the symbol $\vee$

$$p \vee q$$

Disjunction

Definition:
Let p and q be propositions. The proposition "p or q" denoted by $p \vee q$, is the proposition that is false when p and q are both false and true otherwise.
The proposition $p \vee q$ is called the Disjunction of p and q.

Disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Exclusive *or*

"You may have ice cream"(p) "You may have cake" (q)

"As desert you may have either ice cream *or* cake" (but not both!)

- We use the symbol $\oplus$ to represent the *exclusive or*

$$p \oplus q$$

Exclusive *or*

Definition:
Let p and q be propositions. The exclusive or of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.

Exclusive *or*

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Implication

"It is sunny today" (p) "We go to the beach" (q)

If p, then q

"If it is sunny today then we go to the beach"

- When using logical *implication* it is represented by the symbol $\rightarrow$

$$p \rightarrow q$$

Implication

Definition:
Let p and q be propositions. The implication $p \rightarrow q$ is the proposition that is false when p is true and q is false and true otherwise.
In this implication p is called the hypothesis and q is called the conclusion.

Implication

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Implication

- Think of implication as a rule

"If Tom is a cat (p) then Tom has four legs (q)"

- Tom is a cat and Tom has four legs
- Tom is a cat and Tom doesn't have four legs
- Tom is not a cat and Tom has four legs
- Tom is not a cat and Tom doesn't have four legs

Sounds good

Something is wrong!

No problem

Implication

- Think of implication as a promise

"If you score 80% or above (p) then I will give you a dollar (q)"

- If you did score above 80%, and I gave you a dollar, the promise is kept (p is True, q is True, $p \rightarrow q$ is True)
- If you did score above 80%, and I didn't give you a dollar, the promise is broken (p is True, q is False, $p \rightarrow q$ is False)
- If you did **not** score above 80%, no matter I give you a dollar or not, you cannot complain the promise is broken (p is False, $p \rightarrow q$ is True, no matter q)

A false statement implies anything.

Implication

Meaning of $p \to q$:

If p then q

p implies q

p only if q

p is sufficient for q

q if p

q whenever p

q is necessary for p

Implication

Meaning of $p \to q$: *Example*

if $x=1+3$, *then* $x=4$

If p then q:	If $x=1+3$ then $x=4$
p implies q:	$x=1+3$ implies $x=4$
p only if q:	$x=1+3$ only if $x=4$
p is sufficient for q:	$x=1+3$ is sufficient for $x=4$
q if p:	$x=4$ if $x=1+3$
q whenever p:	$x=4$ whenever $x=1+3$
q is necessary for p:	$x=4$ is necessary for $x=1+3$

Implication

Meaning of $p \rightarrow q$: *Example*

if you finish all exercises in the textbook,

then you'll pass the exam.

If p then q:

p implies q:

p only if q:

p is sufficient for q:

q if p:

q whenever p:

q is necessary for p:

Biconditional

"The polygon has exactly 3 sides" (p) "The polygon is a triangle" (q)

"The polygon has exactly 3 sides
if and only if
the polygon is a triangle"

p if and only if q

- When using logical *biconditional* it is represented by the symbol $\leftrightarrow$

$$p \leftrightarrow q$$

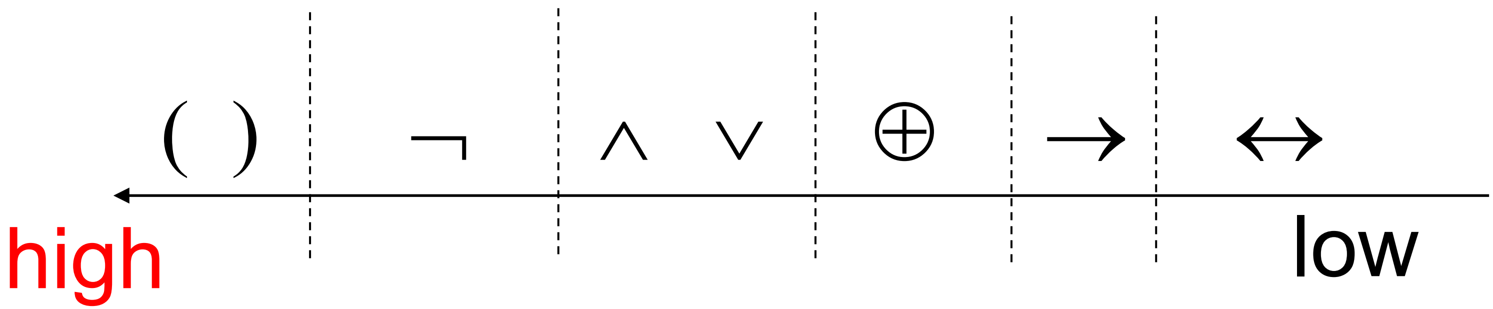
Biconditional

Definition:
Let p and q be propositions. The biconditional $p \leftrightarrow q$ is the proposition that is true when p and q have the same truth values and is false otherwise.

Biconditional

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Precedence of connectives



$\wedge$ and $\vee$ have co-equal priority, hence it is necessary to include parentheses to avoid ambiguity in some expression;

e.g. $(p \wedge q) \vee r$ vs. $p \wedge (q \vee r)$

Precedence of connectives (cont.)

- when more than 2 instances of binary connectives of equal priority are not separated by ( ), the leftmost one has the highest priority;

  E.g. $p \to q \to r \equiv (p \to q) \to r$

- when more than 2 instances of $\neg$ are not separated by ( ), the right most one has precedence;

  E.g., $\neg \neg \neg \neg p \equiv \neg(\neg(\neg(\neg p)))$

A Few Terminologies

- A compound proposition that is always true, no matter what the truth values of the propositions that occur in it, is called a tautology.
- A compound proposition that is always false is called a contradiction.
- Finally, a proposition that is neither a tautology nor a contradiction is called a contingency.

Examples

$p \lor \neg p$ is a tautology

$p \land \neg p$ is a contradiction

p	$\neg p$	$p \lor \neg p$	$p \land \neg p$
T	F	T	F
F	T	T	F

Logical equivalence

- Two statement forms are logically equivalent iff they have identical truth values for all possible combinations of truth values of their propositional symbols

  $p \Leftrightarrow q$ denotes that p and q are logically equivalent. Sometimes $p \equiv q$.

Remember biconditional?

Definition:
Let p and q be propositions. The biconditional $p \leftrightarrow q$ is the proposition that is true when p and q have the same truth values and is false otherwise.

Hence, the propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology.

Example

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	**F**	F	F	**F**
T	F	T	**F**	F	T	**F**
F	T	T	**F**	T	F	**F**
F	F	F	**T**	T	T	**T**

$\neg(p \vee q)$ is equivalent to $\neg p \wedge \neg q$

$\neg(p \vee q) \leftrightarrow \neg p \wedge \neg q$ is a tautology

Example

p q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T T	T	**F**	F	F	**F**
T F	F	**T**	F	T	**T**
F T	F	**T**	T	F	**T**
F F	F	**T**	T	T	**T**

$\neg(p \wedge q)$ is equivalent to $\neg p \vee \neg q$

Example

- Negation of "it is sunny but it is not hot"

  $\equiv$ 'it is not sunny' or 'it is hot' or 'both'

- $p \equiv$ "John is clever and he is rich" $\equiv a \wedge r$
- $p' \equiv$ "John is clever and rich" $\equiv a \wedge r$
- $\neg p' \equiv$ "John is not 'clever and rich'"

$\neg(a \wedge r) \equiv \neg a \vee \neg r$? vs. $\neg a \wedge \neg r$?

Equivalent form for implication

p	q	$p \to q$	$\neg p \vee q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

"If it is sunny then we go to the beach"

$\Leftrightarrow$

"It is not sunny or we go to the beach"

Negation of implication

$$\neg (p \rightarrow q) \equiv \neg(\neg p \vee q)$$
$$\equiv \neg (\neg p) \wedge \neg q$$
$$\equiv p \wedge \neg q$$

E.g. Negate “If it is sunny then we go to the beach”

$\Leftrightarrow$

”It is sunny and we do not go to the beach"

Example

Show that $(p \wedge q \rightarrow p)$ and $(p \rightarrow q \vee p)$ are tautologies.

Method 1: use truth table:

p	q	$p\wedge q$	$q\vee p$	$p\wedge q\rightarrow p$	$p\rightarrow q\vee p$
T	T	T	T	T	T
T	F	F	T	T	T
F	T	F	T	T	T
F	F	F	F	T	T

Example

Method 2:

$$(p \wedge q \rightarrow p) \equiv \neg(p \wedge q) \vee p$$
$$\equiv (\neg p \vee \neg q) \vee p$$
$$\equiv (\neg p \vee p) \vee \neg q$$
$$\equiv t \vee \neg q \quad \equiv t$$

$$(p \rightarrow q \vee p) \equiv \neg p \vee (q \vee p)$$
$$\equiv (\neg p \vee p) \vee q$$
$$\equiv t \vee q \quad \equiv t$$

Example

Show: $p \vee q \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$

Method 1: use truth table;

Method 2:

$$
\begin{aligned}
p \vee q \rightarrow r &\equiv (p \vee q) \rightarrow r \\
&\equiv \neg(p \vee q) \vee r \\
&\equiv (\neg p \wedge \neg q) \vee r \\
&\equiv (\neg p \vee r) \wedge (\neg q \vee r) \\
&\equiv (p \rightarrow r) \wedge (q \rightarrow r)
\end{aligned}
$$

Three terms about $p \to q$

- The **contraposititive** of $p \to q$ is $\neg q \to \neg p$

  In fact, $\boxed{\neg p \to \neg q \equiv q \to p}$

- The **converse** of $p \to q$ is $q \to p$

  $\cancel{q \to p \equiv p \to q\,?}$

- The **inverse** of $p \to q$ is $\neg p \to \neg q$

  $\cancel{\neg p \to \neg q \equiv p \to q\,?}$

- Clearly, $\neg p \to \neg q \equiv q \to p$

Example

E.g. “If it is sunny then we go to the beach.”

Contrapositive:

“If we do not go to the beach then it is not sunny.”

Converse:

“If we go to the beach then it is sunny.”

Inverse:

“If it is not sunny then we do not go to the beach.”

Biconditional

$p \leftrightarrow q \equiv p$ if, and only if, q

$\equiv p$ if q, and, p only if, q

p q	p → q	q → p	p ↔ q
T T	T	T	**T**
T F	F	T	**F**
F T	T	F	**F**
F F	T	T	**T**

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (\neg p \rightarrow \neg q) \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

Common logical equivalence

- **De Morgan's laws**

$$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$$

$$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$$

- **Commutative**

$$p \wedge q \Leftrightarrow q \wedge p$$

$$p \vee q \Leftrightarrow q \vee p$$

Common logical equivalence

- **Associative**

$$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$$

$$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$$

- Distributive

$$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$$

$$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$$

[Remember? a(b+c)=ab+ac]

Common logical equivalence

- Identity (t is tautology, c is contradiction)

$$p \wedge t \Leftrightarrow p \qquad p \vee c \Leftrightarrow p$$

- Universal bound

$$p \vee t \Leftrightarrow t \qquad p \wedge c \Leftrightarrow c$$

- Negation

$$p \wedge \neg p \Leftrightarrow c \qquad p \vee \neg p \Leftrightarrow t$$

Common logical equivalence

- Double Negation

$$\neg(\neg p) \Leftrightarrow p$$

- Idempotent

$$p \wedge p \Leftrightarrow p \qquad p \vee p \Leftrightarrow p$$

- Absorption

$$p \vee (p \wedge q) \Leftrightarrow p \qquad p \wedge (p \vee q) \Leftrightarrow p$$

Example 1: Proof of Distributive Law

$$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$$

p q r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T T T	T	**T**	T	T	**T**
T T F	F	**T**	T	T	**T**
T F T	F	**T**	T	T	**T**
T F F	F	**T**	T	T	**T**
F T T	T	**T**	T	T	**T**
F T F	F	**F**	T	F	**F**
F F T	F	**F**	F	T	**F**
F F F	F	**F**	F	F	**F**

Example 2:

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent

$\neg(p \vee (\neg p \wedge q))$	$\Leftrightarrow \neg p \wedge \neg(\neg p \wedge q)$	De Morgan
	$\Leftrightarrow \neg p \wedge (\neg(\neg p) \vee \neg q)$	De Morgan
	$\Leftrightarrow \neg p \wedge (p \vee \neg q)$	Double negation
	$\Leftrightarrow (\neg p \wedge p) \vee (\neg p \wedge \neg q)$	Distributive
	$\Leftrightarrow F \vee (\neg p \wedge \neg q)$	known contradiction
	$\Leftrightarrow (\neg p \wedge \neg q) \vee F$	Commutative
	$\Leftrightarrow \neg p \wedge \neg q$	Identity

Example

- Proof of Absorption

$$p \vee (p \wedge q) \Leftrightarrow p \quad p \wedge (p \vee q) \Leftrightarrow p$$

p q	$p \wedge q$	$p \vee (p \wedge q)$	$p \vee q$	$p \wedge (p \vee q)$
T T	T	**T**	T	**T**
T F	F	**T**	T	**T**
F T	F	**F**	T	**F**
F F	F	**F**	F	**F**

Translating English Sentences

Statement 1: John is short or Marry is pretty, and, John is not short or Marry is not pretty

Statement 2: It is not the case that both John is short and Marry is pretty, or, both John is not short and Marry is not pretty

Logical Forms:

1. $(P \lor Q) \land (\neg P \lor \neg Q)$

2. $\neg((P \land Q) \lor (\neg P \land \neg Q))$

Translating English Sentences

They are equivalent:

$$\neg((P \wedge Q) \vee (\neg P \wedge \neg Q))$$
$$\Leftrightarrow \neg(P \wedge Q) \wedge \neg(\neg P \wedge \neg Q)$$
$$\Leftrightarrow (\neg P \vee \neg Q) \wedge (P \vee Q)$$
$$\Leftrightarrow (P \vee Q) \wedge (\neg P \vee \neg Q)$$

Use the Truth Table:

Translating English Sentences

$p \quad q$	$\neg p \quad \neg q$	$p \vee q$	$\neg p \vee \neg q$	$(p \vee q) \wedge (\neg p \vee \neg q)$	$p \oplus q$
T T	F F	T	F	F	F
T F	F T	T	T	T	T
F T	T F	T	T	T	T
F F	T T	F	T	F	F

Hence the two confusing sentences are logically equivalent to the more legible form:

Either John is short or Marry is pretty, but not both.

Only If

"Emma eats dinner only if she is hungry"

$\equiv$ "If she is not hungry
then Emma does not eat dinner"

$\equiv$ "If Emma eats dinner then she is hungry"

$$p \textit{ only if } q$$
$$\equiv (\neg q \rightarrow \neg p)$$
$$\equiv p \rightarrow q$$

Sufficient condition

"Vandalizing others' property is a sufficient condition for Michael to be fined"

$\equiv$ "If Michael vandalizes others' property then he will be fined"

p is a sufficient condition for q $\equiv$ $p \rightarrow q$

Necessary condition

"Being over 16 is a necessary condition for a person to get a driver's licence"

$\equiv$ "If a person has not turned 16 then he/she cannot get a driver's licence"

$\equiv$ "If a person get his/her driver's licence then he/she is over 16"

p is a necessary condition for q $\equiv \neg p \rightarrow \neg q$

$\equiv q \rightarrow p$

Example

Are the following 3 statements logically equivalent?

- "You fix my ceiling or I won't pay my rent"

  x: $P \vee \neg Q$

- "If you do not fix my ceiling then I won't pay my rent"

  y: $\neg P \rightarrow \neg Q$

- "I will pay my rent only if you fix my ceiling"

  z: $Q \rightarrow P$

Example

Rewrite ‘if n is prime, then n is odd or n is 2’ in a few other (logically equivalent) ways

Example

A prestige company wrote: 'a person will be hired only if he majors in mathematics or computer science, get a *B* average or better, and take accounting'; an applicant with such qualifications was turned down; did the company lie?

No. They stated the necessary conditions, not sufficient condition. You will be considered only with the qualification , but not guaranteed to be hired.

Summary

- Definition of Proposition
- Truth table of propositions
- Logical Connectives of proposition:
  - *not* $\neg$, *and* $\wedge$, *or* $\vee$, exclusive or $\oplus$, implication $\rightarrow$, and biconditional $\leftrightarrow$
- Translation to and from English
- Tautology and Contradiction

Summary

- Logical Equivalence
- Common Logical Laws
- Negation of $p \to q$
- Contrapositive, Converse and inverse of $p \to q$
- Sufficient condition, necessary condition, bicondition.