Lecture 7. Counting

Ref.: Rosen Section 4.3

Basic Counting Principles

- The Sum Rule
- The Product Rule
- The Principle of Inclusion/Exclusion

The Sum Rule

If A and B are <u>disjoint</u> sets then

$$|A \cup B| = |A| + |B|$$

The Sum Rule: Example

Suppose statement labels in a programming language must be a single letter or a single decimal digit.

Since a label cannot be both at the same time,

the number of labels
= the number of letters + the number of decimal digits
= 26 + 10 = 36.

The Product Rule

If A and B are sets then

$$|A \times B| = |A| * |B|$$

Examples

Count the number of bit strings of length 4.

Apply the rule of product to get

$$2^4.$$

Examples

Find the number of three-letter initials where none of the letters is repeated.

Apply the rule of product remembering that a letter cannot appear twice to get

$$(26)(25)(24).$$

Examples

Count the number of bit strings of length 4 or less.

Apply the rule of sum to get the disjoint subsets of length 1, 2, 3 and 4.

Then apply the rule of product to count each subset to get

$$2 + 4 + 8 + 16 = 2^1 + 2^2 + 2^3 + 2^4$$

Examples:

Statement labels in Basic can be either

- a single letter or
- a letter followed by a digit.

Find the number of possible labels.

Examples:

We can partition the set of all labels *L* into the disjoint subsets consisting of

- the set of single letter labels *S*

and

- the set of single letters followed by a digit *D*

Examples:

Use the Sum Rule to compute the cardinality of *L* if we can compute the cardinality of *D*.

- The elements of D are ordered pairs of the form [*a, d*] where a is an alphabetic character and d is a digit.
- By the Product Rule the cardinality of D is the product of the cardinality of the two sets:

(the alphabetic characters)(the decimal digits)

= (26)(10) = 260.

The cardinality of *L* is 26 + 260 = 286.

The Principle of Inclusion-Exclusion

If A and B are <u>**not** disjoint</u> sets then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

<u>Don't count objects in the intersection of two sets more than once!</u>

The Principle of Inclusion-Exclusion

Given any 3 sets A, B, and C:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Examples:

Count the number of bit strings of length 4 which begin with a 1 or end with a 00.

The set can be expressed as the union of

- the subset S of strings which begin with 1

and

- the subset O that end in 00.

Examples:

Unfortunately the two subsets overlap.

- The cardinality of S is 8 (why?)
- The cardinality of O is 4 (why?).

Hence, by the exclusion-inclusion principle, the cardinality of the union is 12 minus the cardinality of the intersection.

Examples:

How many strings are in the intersection?
Those strings that begin with 1 and end in 00 are such strings. There are 2 such strings.

The total number is 10 = 8 + 4 - 2.

Check:

- Strings in S that begin with 1:

1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111

- Strings in O that end with 00:

0000, 0100, 1000, 1100

- 1000 and 1100 appear in both sets.
  Count them once.

Examples

Count the set S of 3 digit numbers which begin or end with an even digit.

Assume that 0 is even but a number cannot begin with 0.

The set is the union of the two subsets:

- The set B of three digit numbers that begin with 2, 4, 6 or 8.

  This set has cardinality: (4)(10)(10).

- The set C of three digit numbers that end with 0, 2, 4, 6, or 8 and do not begin with 0.

  This set has cardinality: (5)(9)(10).

Examples

Now we use the inclusion-exclusion principle to eliminate the overlap of sets B and C.

Their intersection:

The 3 digit numbers that begin with 2, 4, 6, or 8 and end with 0, 2, 4, 6, or 8.

The intersection has the cardinality

$$(4)(10)(5)$$

Hence the cardinality is

$$400 + 450 - 200 = 650.$$

Example:

In a class of 50 students, 30 know Pascal, 18 know Fortran, 26 know COBOL, 9 know both Pascal and Fortran, 16 know Pascal and COBOL, 8 know both Fortran and COBOL, 47 know at least one of the three languages.

(a) How many students know none of the three languages?

b) How many students know all three languages?

c) How many students know Pascal and Fortran but not COBOL?

d) How many students know Pascal, but neither Fortran nor COBOL?

Example:

P = set of students who know Pascal
C = set of students who know COBOL
F = set of students who know Fortran
U = All students

$$|P \cup C \cup F| = |P| + |C| + |F| - |P \cap C| - |P \cap F| - |C \cap F| + |P \cap C \cap F|$$

$$47 = 30 + 26 + 18 - 16 - 9 - 8 + |P \cap C \cap F|$$

Thus,

$$|P \cap C \cap F| = 6$$

Example:

Therefore:

(a) **How many students know none of the three languages?**

$$|U| - |P \cup C \cup F| = 50 - 47 = 3$$

(b) **How many students know all three languages?**

$$|P \cap C \cap F| = 6$$

(c) **How many students know Pascal and Fortran but not COBOL?**

$$|P \cap F| - |P \cap C \cap F| = 9 - 6 = 3$$

(d) **How many students know Pascal, but neither Fortran nor COBOL?**

$$|P| - |P \cap F| - |P \cap C| + |P \cap C \cap F| = 30 - 9 - 16 + 6 = 11$$

Urn models

Given set of n objects in an urn (don’t ask why it’s called an “urn” - probably due to some statistician years ago) .

Pick (select) r objects from the urn in sequence.

After an object is chosen

- it can be replaced-(*selection with replacement*)
- or not -(*selection without replacement*).

If r objects are chosen, how many different possible sequences of r objects are there?

Does the order of the objects matter or not?

Permutations

Selection without replacement of r objects from the urn with n objects.

A *permutation* is an arrangement.

Order matters.

After selecting the objects, two different orderings or arrangements constitute different permutations.

Permutations

- Choose the first object n ways,
- Choose the second object (since selection is without replacement) (n - 1) ways,

……

- the rth object (n - r + 1) ways.

By the rule of product:

The number of permutations of n things taken r at a time

$$P(n,r) = n(n - 1)(n - 2) \ldots (n - r + 1)$$

Permutations

$$P(n, r) = n\,(n-1)\,(n-2)\ \dots\ (n-r+1)$$

$$= n!\,/\,(n-r)!$$

Corollary: $P(n, n) = n!\,/\,(n-n)! = n!$

Permutations: Example

(a) How many different ways can three of the letters of the word BYTES be chosen and written in a row?

(b) How many different ways can this be done if the first letter must be “B”?

(a) The problem is a 3-permutation from a set of 5 objects.

Answer: $P(5, 3) = 5! / 2! = 60$

(b) 2 step process (Product rule)

step 1: Assign B to the first position: 1 way

step 2: Assign 4 letters to the remaining 2 positions; $P(4, 2)$ ways, regardless of step 1.

Answer: $1 \times P(4, 2) = 12$

Circular Permutations

The number of ways of permuting r objects from a set of n objects in a circle (in which two arrangements are the same when one is a rotation of the other) is $P(n, r) / r$.

When $n = r$, we have $P(n, n) / n = (n – 1)!$

Example

(a) How many ways can a group of 6 people be seated around a table?

(b) How many ways can a group of 6 people be seated around a table when two of them cannot sit next to each other?

Answer:

(a) P(6, 6) / 6 = 5! (circular permutation)

(b) By difference rule:

Number of ways a group of 6 can sit around a table - Number of ways in which the 2 enemies sit next to each other = 5! – 2 x P(5, 5)/ 5 (treat the 2 enemies as 1 person) = 5! – 2 x 4!

Permutations: Example

Let A and B be finite sets and $|A| \leq |B|$
Count the number of injections from A to B.

Note there are no injections if $|A| > |B|$ (why?)
There are $P(|B|, |A|)$ injections:
We order the elements of A, {a1, a2, . . . } and assume the urn contains the set B.

- There are $|B|$ ways to choose the image of a1,
- $|B| - 1$ ways to choose the image of a2,

and so forth.
Selection is without replacement. Otherwise we do not construct an injection.

Combinations

Selection is without replacement but

Order does not matter.

It is equivalent to selecting subsets of size r from a set of size n.

Combinations

An r-combination of a set of n elements is a subset of r elements taken from the set of n elements. The number of r-combinations of a set of n elements is denoted as $C(n, r)$.

$$C(n, r) = P(n, r) / r! = n! / (r! (n - r)!)$$

Combinations: Example

You are to select five members from a group of twelve to form a team.

(a) How many distinct five-person teams can be selected?

(b) If two of them insist on working together as a pair, such that any team must either contain both or neither. How many five-person teams can be formed?

Combinations: Example

Answer:

(a) C(12, 5)

(b) Those teams which involve the 2 friends + those teams which do not involve the 2 friends = 1 x C(10, 3) + C(10, 5)

Combinations: Example

10 people are to sit around two tables in a hawker center. The first table has 6 chairs, and the second table has 4 chairs.

(a) How many ways can this be done?

(b) How many ways can this be done if two of them need to sit in a same table?

Combinations: Example

Answer to (a) (version #1)

Step 1: Choose 6 people to sit on the first table: C(10, 6)

Step 2: Arrange the 6 people on the first table: 5!
(Circular Permutation)

Step 3: Arrange the remaining 4 on the second table: 3!
(Circular Permutation)

By multiplication rule = C(10,6) x 5! X 3!

(version #2)

Step 1: Permute 6 people from 10 circularly around the first table P(10, 6)/6

Step 2: Arrange the remaining 4 on the second table: 3!
(Circular Permutation)

By multiplication rule = P(10,6) / 6 X 3!

Combinations: Example

Answer to (b)

Number of sitting arrangement where the two sit on the 1st table:

By multiplication rule

Step 1: Put the two on the first table: 1 way

Step 2: Select 4 more to join them: C(8,4)

Step 3: Permute them around the table: 5!

Step 4: Permute the remaining 4 circularly on 2nd table: 3!

Combinations: Example

Number of sitting arrangement where the two sit on the 2nd table

By multiplication rule

Step 1: Put the two on the second table: 1 way

Step 2: Select 2 more to join them: C(8.2)

Step 3: Permute them around the table: 3!

Step 4: Permute remaining 6 circularly on 1st table: 5!

By addition rule, we have

$$C(8, 4) \times 5! \text{ X } 3! + C(8, 2) \times 3! \text{ X } 5!$$

Algebra of Combinations

1. $C(n, r) = C(n, n-r)$

2. Pascal's Formula:

$$C(n+1, r) = C(n, r-1) + C(n, r)$$

Proof of (1)

$$C(n, r) = n! / (r! \times (n-r)!)$$
$$= n! / ((n-r)! \times (n-(n-r))!)$$
$$= C(n, n-r)$$

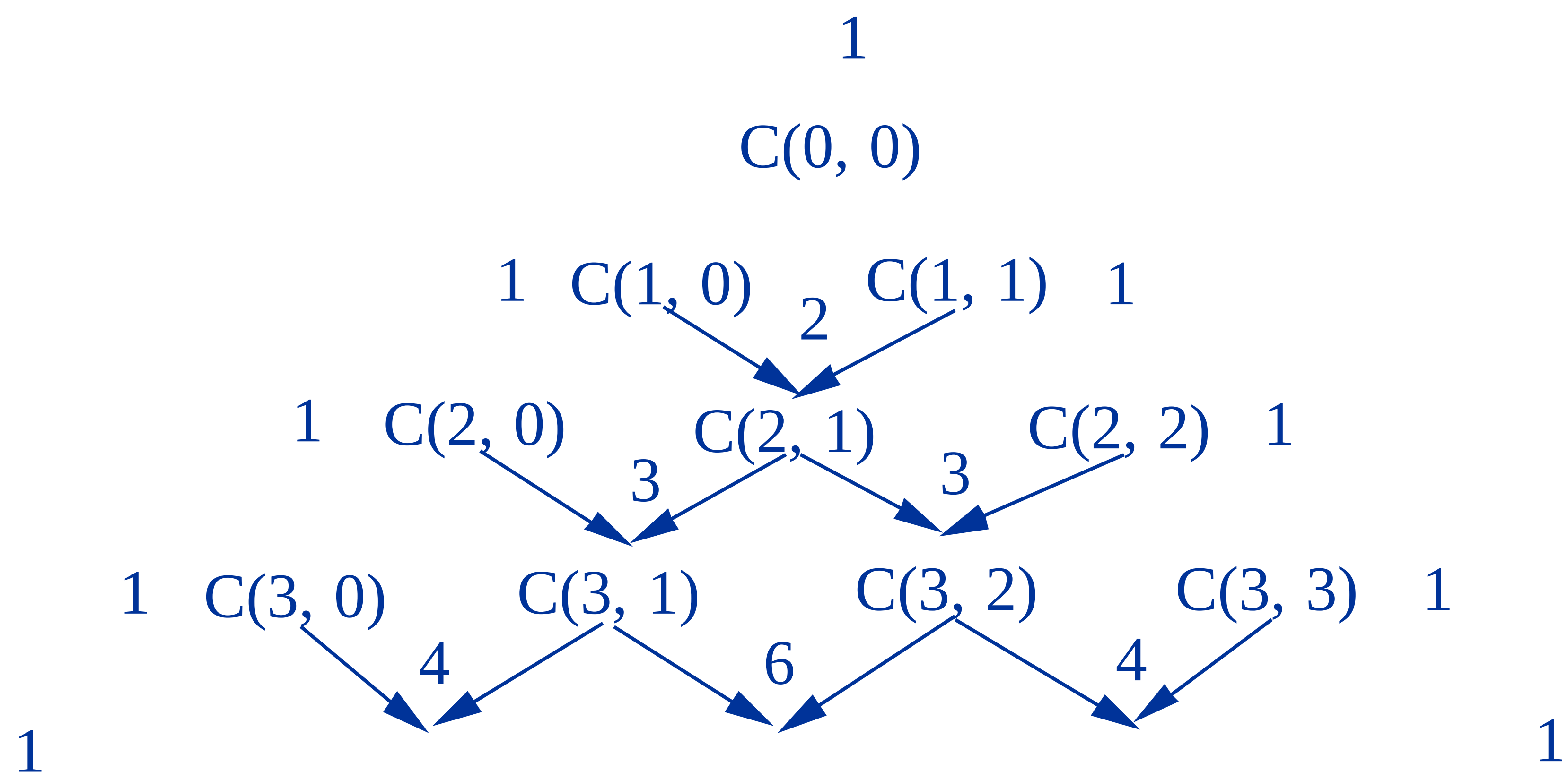
Proof of (2)

To pick r elements from a set of n+1 element, we can do by:

- pick the (n+1)th element, pick the remaining r – 1 from the n elements. 1 x C(n, r – 1)
- exclude the (n+1)th element, pick the r elements from the n elements. C(n, r)

So, total is C(n+1, r) = C(n, r – 1) + C(n, r)

Pascal's triangle



Example:

How many bit strings of length 4 have exactly 2 ones (or exactly 2 zeros)?

We solve the problem by determining the positions of the two ones in the bit string.

- **place the first one - 4 possibilities**
- **place the second one - 3 possibilities**

Hence it appears that we have (4)(3)= 12 possibilities.

We enumerate them to make sure:

0011, 0101, 1001, 0110, 1010, 1100.

There are actually only 6 possibilities. What is wrong?

Example:

The answer would be correct if we had two <u>different</u> objects to place in the string.
For example, if we were going to place an ‘a’ and a ‘b’ in the string we would have
00ab, 00ba, 0a0b, 0b0a, a00b, b00a, ……
and so forth for a total of 12.

But......the objects (1 and 1) are the same <u>so the order is not important</u>!
Divide through by the number of orderings = 2! = 2.
Therefore the answer is 12/2 = 6.

Summary

- Basic Counting principles:
  - Sum Rule
  - Product Rule
  - Principle of Inclusion/Exclusion
- Permutations
- Combinations