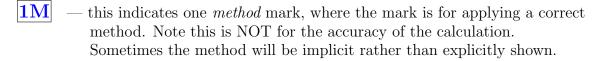
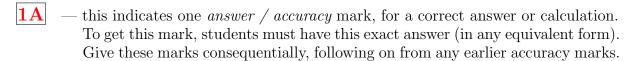
MATH1019 MID-SEMESTER TEST

Solutions and marking scheme

2018 Semester 1

The marking scheme includes marks of the following different types:





Half marks should generally only be given in certain circumstances (usually indicated in the marking scheme) where there are multiple elements in the calculation, and there is only a small numerical error in a small part of this. 1. Consider the following system of equations:

$$x - 3y + 2z = 1$$
$$2x - 5y + 6z = 5$$
$$-x + 5y + 2z = 5$$

- (a) Write the system as an augmented matrix, and use the Gauss-Jordan method to manipulate the augmented matrix into reduced row echelon form.
- (b) State the number of solutions to the system. Find all solutions, or justify why there are none.

[8 marks]

$$\begin{bmatrix} 1 & -3 & 2 & 1 \\ 2 & -5 & 6 & 5 \\ -1 & 5 & 2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \end{bmatrix} (R_2 - 2R_1) \\ \sim \begin{bmatrix} 1 & 0 & 8 & 10 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} (R_1 + 3R_2) \\ (R_3 - 2R_2)$$

Since the column corresponding to z is without a leading entry, we let $z = t, t \in \mathbb{R}$.

$$x + 8z = 10 \implies x = 10 - 8t$$

$$y + 2z = 3 \implies y = 3 - 2t$$

2. If k is a real constant, then we can define a system by

$$\begin{array}{rclrcr}
2x & -4y & +4z & = & 12 \\
3x & + & y & -8z & = & 4 \\
-5x & +11y & +kz & = & -32
\end{array}$$

- (a) Write the system as an augmented matrix, and use Gaussian elimination to reduce the matrix to row-echelon form.
- (b) Briefly justify for which value(s) of k, if any, the system has:
 - (i) a unique solution
 - (ii) no solution
 - (iii) infinitely many solutions

You do **not** have to solve the system in any of these cases.

(a)
$$\begin{bmatrix} 2 & -4 & 4 & | & 12 \\ 3 & 1 & -8 & | & 4 \\ -5 & 11 & k & | & -32 \end{bmatrix} R_1 = \frac{1}{2}R_1 \\ \sim \begin{bmatrix} 1 & -2 & 2 & | & 6 \\ 3 & 1 & -8 & | & 4 \\ -5 & 11 & k & | & -32 \end{bmatrix} R_2 = R_2 - 3R_1 \\ \sim \begin{bmatrix} 1 & -2 & 2 & | & 6 \\ 0 & 7 & -14 & | & -14 \\ 0 & 1 & k + 10 & | & -2 \end{bmatrix} R_2 = \frac{1}{7}R_2 \\ \sim \begin{bmatrix} 1 & -2 & 2 & | & 6 \\ 0 & 1 & | & k + 10 & | & -2 \end{bmatrix} R_3 = R_3 - R_2 \\ \sim \begin{bmatrix} 1 & -2 & 2 & | & 6 \\ 0 & 1 & | & -2 & | & -2 \\ 0 & 0 & k + 10 & | & -2 \end{bmatrix} R_3 = R_3 - R_2$$

(b) For $k \neq -12$ the number of non-zero rows equals the number of variables, so there is a unique solution. For k = -12 the number of non-zero rows is less than the number of variables, so there are an infinite number of solutions.

[10 marks]

3. Consider the system of equations

$$x - 2y - 4z = 1$$
$$2x - 3y - 6z = 2$$
$$-3x + 6y + 15z = 3$$

- (a) Express the system in the form $A\mathbf{x} = \mathbf{b}$, where A is the matrix of coefficients, \mathbf{x} is the column vector of variables, and \mathbf{b} is the column vector of constants.
- (b) Use the inverse of A to solve the system of equations, or justify why there are no solutions.

[10 marks]

(a)
$$\begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}. \text{ So } A = \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix}$$

(b)
$$\begin{bmatrix} 1 & -2 & -4 & 1 & 0 & 0 \\ 2 & -3 & -6 & 0 & 1 & 0 \\ -3 & 6 & 15 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 3 & 3 & 0 & 1 \end{bmatrix} (R_2 - 2R_1) (R_3 + 3R_1)$$
$$\sim \begin{bmatrix} 1 & 0 & 0 & -3 & 2 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} \end{bmatrix} (R_1 + 2R_2)$$
$$\sim \begin{bmatrix} 1 & 0 & 0 & -3 & 2 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} \end{bmatrix} (R_2 - 2R_3)$$
$$\sim \begin{bmatrix} 1 & 0 & 0 & -3 & 2 & 0 \\ 0 & 1 & 0 & -4 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} \end{bmatrix} (R_2 - 2R_3)$$

So
$$A^{-1} = \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix}$$

 $X = A^{-1}B = \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3+4+0 \\ -4+2-2 \\ 1+0+1 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}.$
So $x = 1, \ y = -4, \ z = 2$

4. Calculate the rank and determinant of each of the following matrices

(a)
$$A = \begin{bmatrix} 1 & -3 & 7 \\ -3 & 9 & -21 \\ 2 & -6 & 14 \end{bmatrix}$$

$$\det(A) = 1 \begin{vmatrix} 9 & -21 \\ -6 & 14 \end{vmatrix} - (-3) \begin{vmatrix} -3 & -21 \\ 2 & 14 \end{vmatrix} + 7 \begin{vmatrix} -3 & 9 \\ 2 & -6 \end{vmatrix}$$

$$= 1 \left(9 \times 14 - (-21) \times (-6) \right) + 3 \left(-3 \times 14 - (-21) \times 2 \right) + 7 \left((-3) \times (-6) - 9 \times 2 \right)$$

$$= (126 - 126) + 3(-42 + 42) + 7(18 - 18) = 0$$

(b)
$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

rank is 3, det is -8

[2 marks]

5. Use Cramer's rule to solve the following system:

$$3x_1 - 2x_2 = 6$$
$$-5x_1 + 4x_2 = 8$$

[7 marks]

$$A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}, \ A_1 = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix}, \ A_2 = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix}$$
$$\det(A) = (3)(4) - (-2)(-5) = 12 - 10 = 2$$
$$\det(A_1) = (6)(4) - (-2)(8) = 24 + 16 = 40$$
$$\det(A_2) = (3)(8) - (6)(-5) = 24 + 30 = 54$$
$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{20}{2} = 20$$
$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{54}{2} = 27$$

(A total of 7 marks for Question 5)