Lecture 10. Graphs

References:

Rosen, Chapter 8

Lecture Outline

1. Simple Graphs

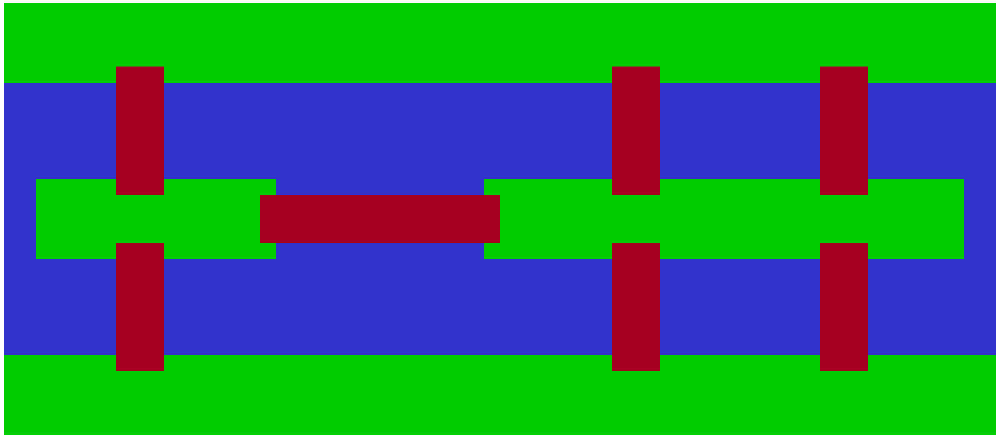
2. Multi-graphs

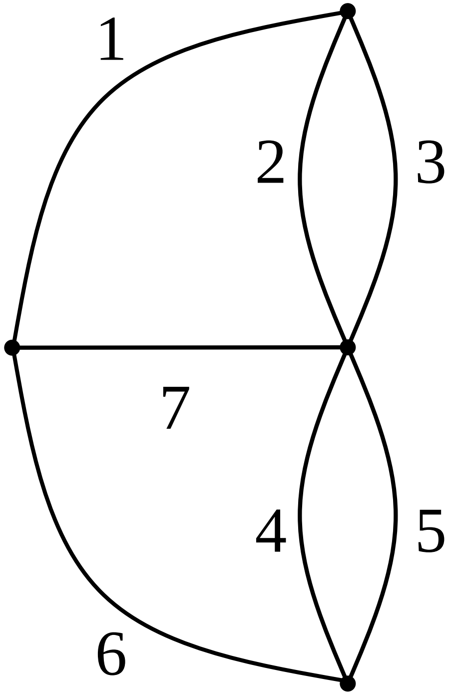
EXAMPLE

4. Directed Graphs & Multigraphs

3. Pseudo-graphs

Introduction



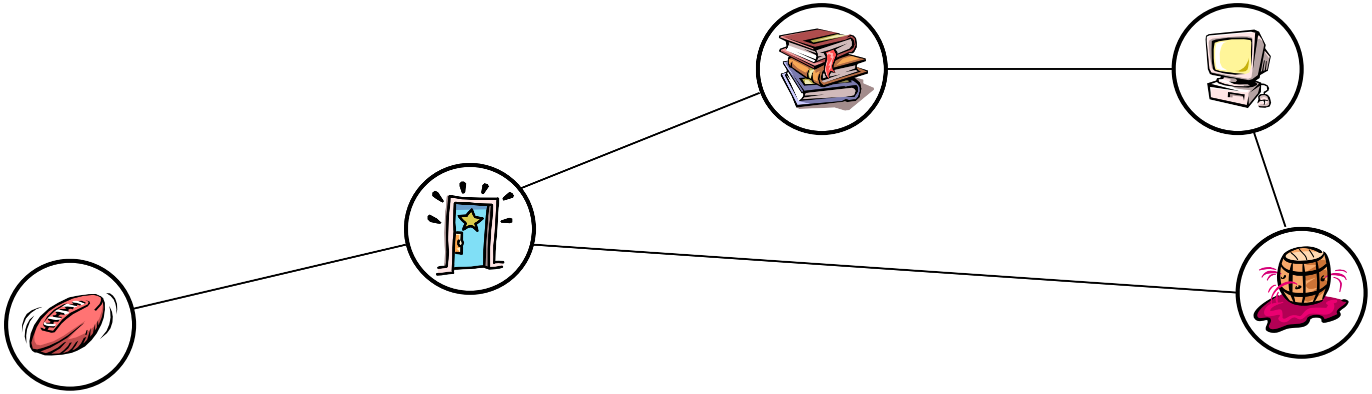


Leonhard Euler **introduced Graphs in 1736 to solve the Königsberg Bridge problem**

Königsberg is divided into 4 section by the 2 branches of the Pregel river. Sections are connected by 7 bridges.

Is it possible to walk across every bridge without crossing any bridge more than once?

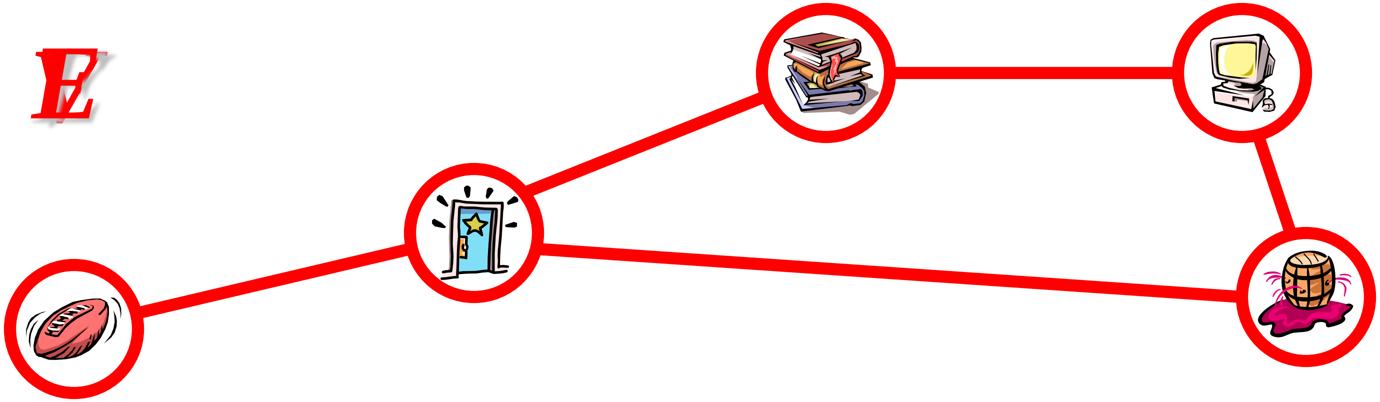
Simple Graphs



- Library
- 314
- Guild
- Canteen
- Sports

- **Design a new network for the campus**
- **This model is called a Simple Graph**
  - ***there is only one line between computers***
  - ***each line operates in both directions***
  - ***no computer has a telephone line to itself***

Simple Graphs

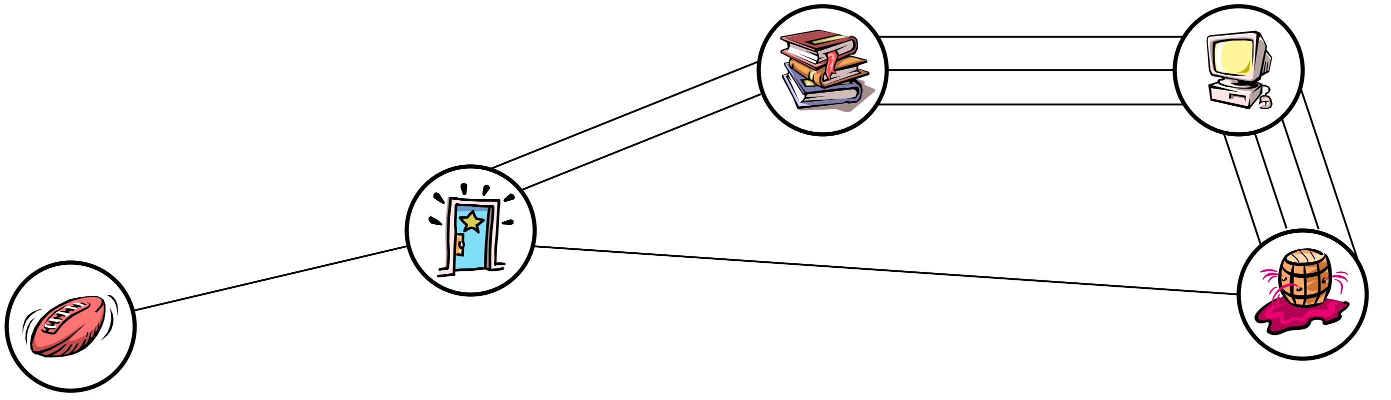


A **Simple Graph G=(V,E)** consists of **V**, a nonempty set of vertices, and **E**, a set of unordered pairs of distinct elements of **V** called edges

V = {Lib, Computing, Guild, Canteen, Sports}

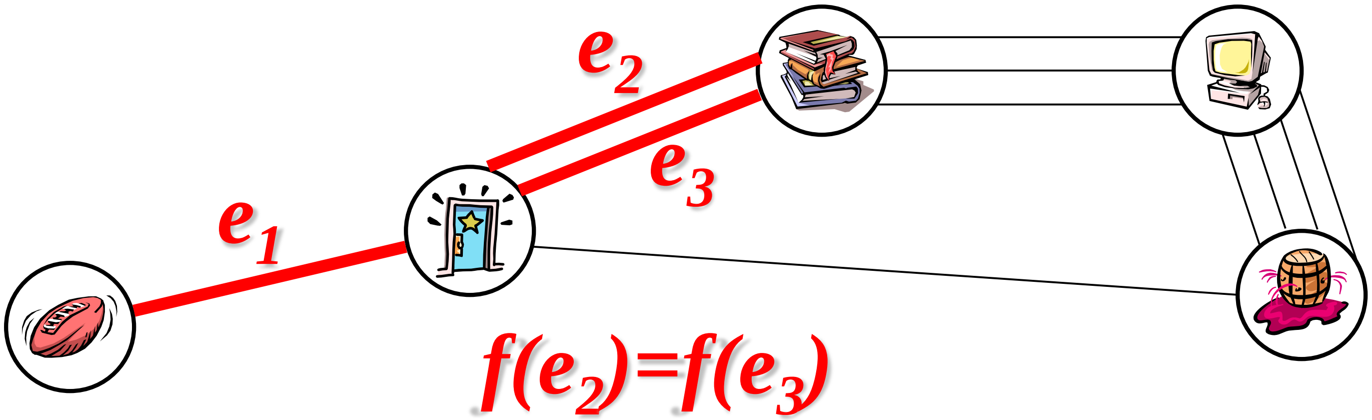
E = {{Sports,Canteen}, {Canteen,Lib}, {Canteen,Guild}, {Lib, 314}, {314,Guild}}

Multigraphs



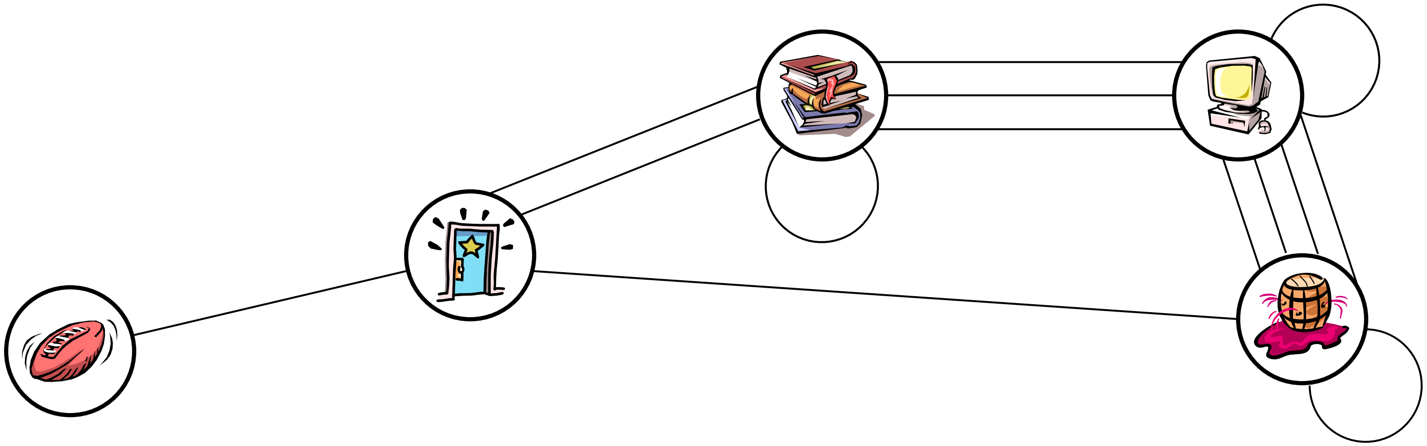
- **If computer traffic is heavy there may need to be multiple phone lines between certain computers...**
- **This model is called a Multigraph**
  - ***it consists of vertices and undirected edges between these vertices***
  - ***loops back to a vertex are not allowed***

Multigraphs



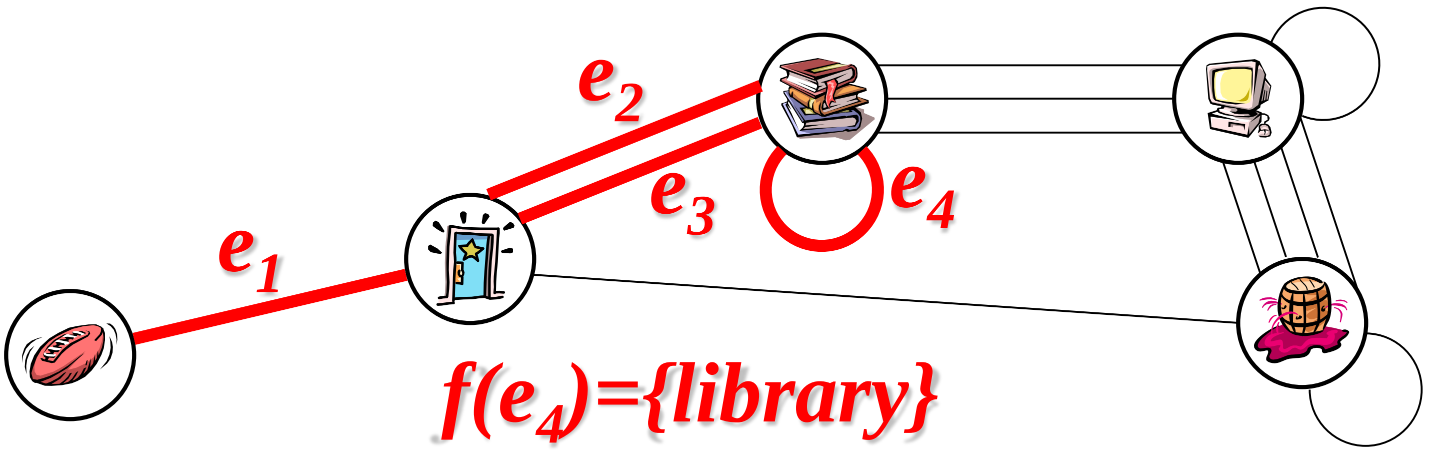
A **Multigraph G=(V,E)** consists of a set **V** of vertices, a set **E** of edges and a function **f** from **E** to **$\{\{u,v\} | u,v \in V, u \neq v\}$**. The edges e_1 and e_2 are called **multiple or parallel edges** if $f(e_1)=f(e_2)$

Pseudographs



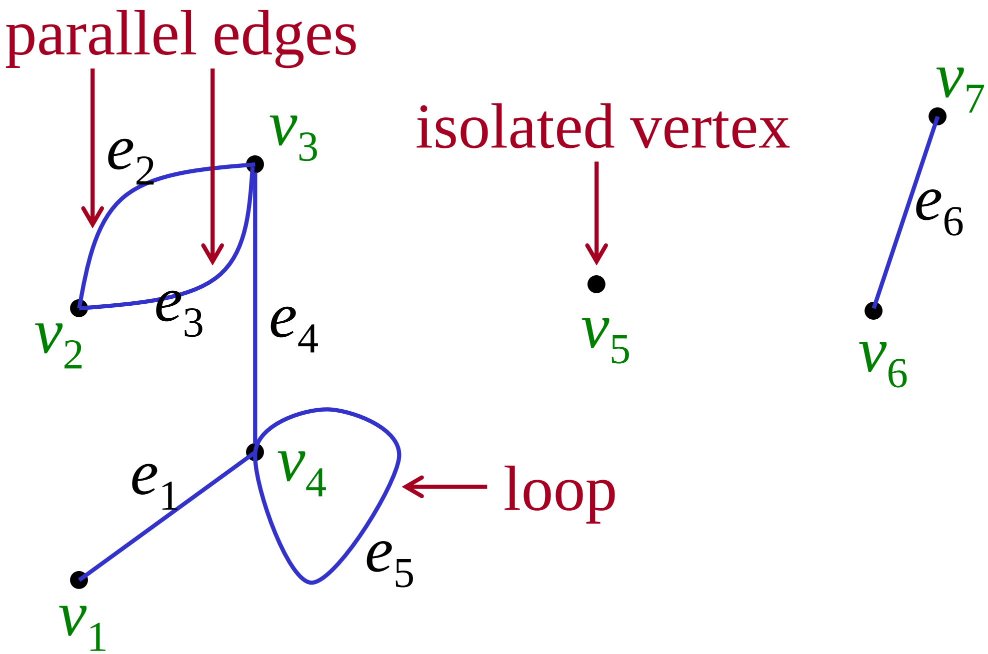
- **Computers can have phone lines to themselves...**
- **This model is called a Pseudograph**
  – ***psuedographs are more general than multigraphs***
  – ***edges can connect vertices to themselves***

Pseudographs



A Pseudograph G=(V,E) consists of a set V of vertices, a set E of edges and a function f from E to $\{\{u,v\}|u,v \in V\}$. An edge e is a loop if f(e)= {u,u} = {u} for some $u \in V$.

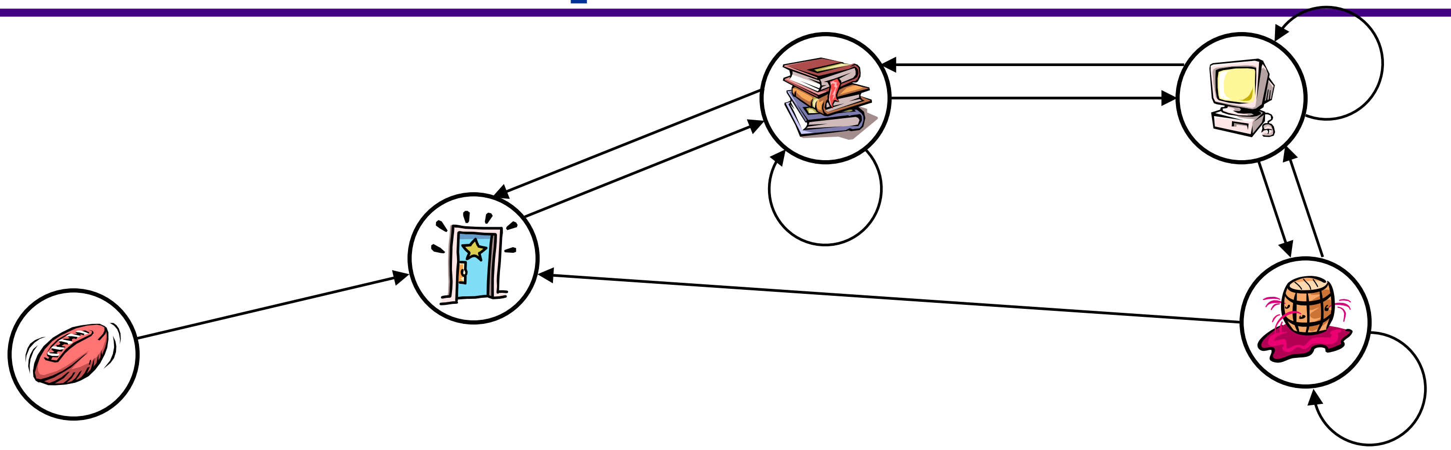
Pseudographs



$$V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\},\ E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$$

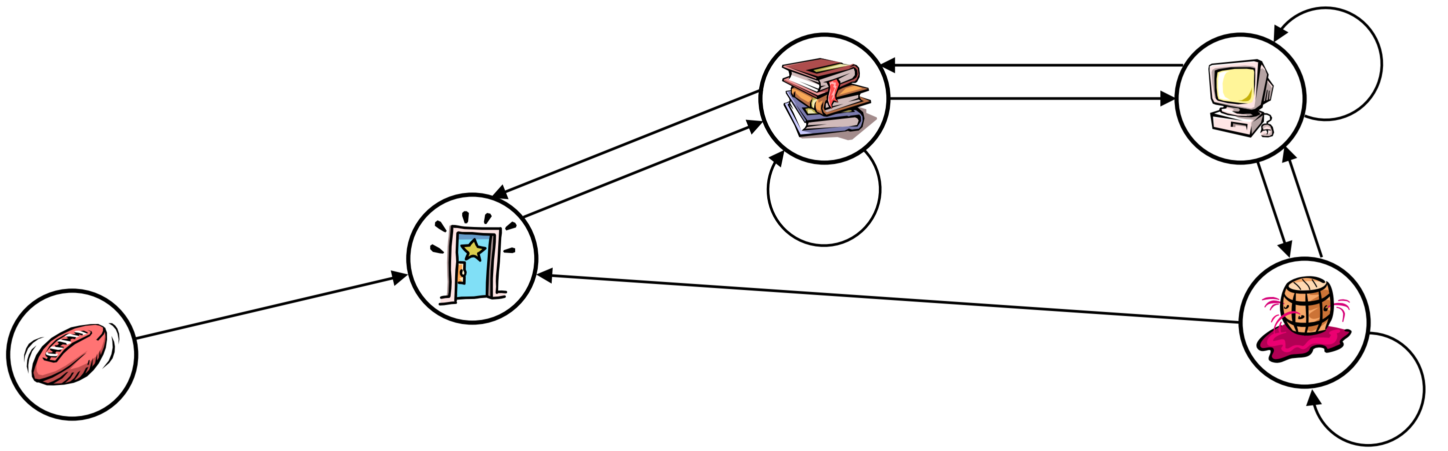
f(e_1)={v_1, v_4}, f(e_2)={v_2, v_3}, f(e_3)={v_2, v_3}, f(e_4)={v_3, v_4}, f(e_5)={v_4}, f(e_6)={v_6, v_7}

Directed Graphs



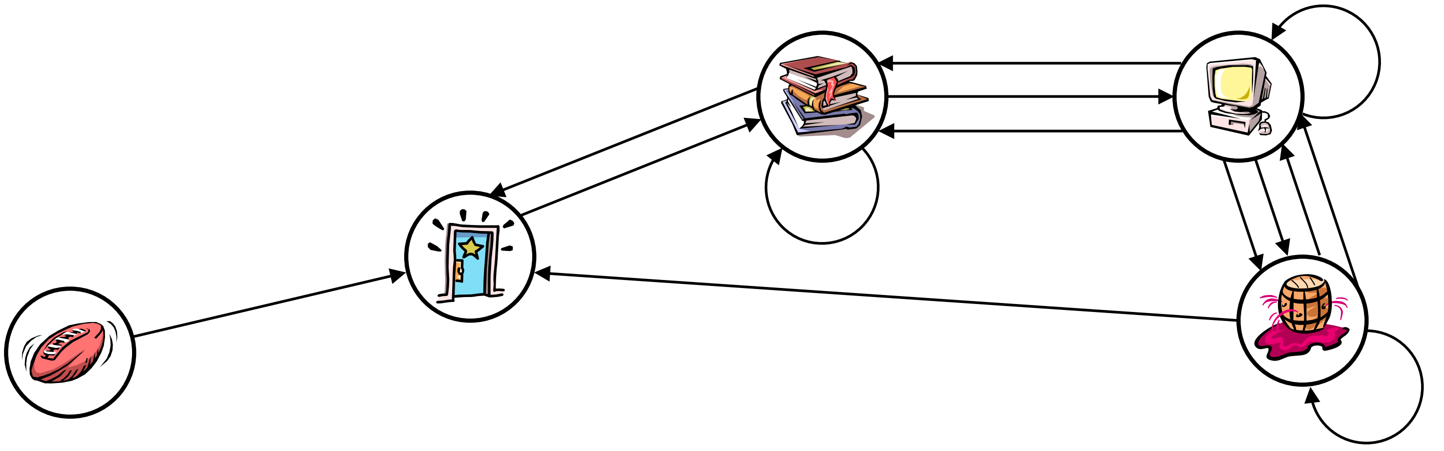
- **The phone lines in the computer network may or may not operate in both directions...**
- **This model is called a Directed Graph**
  - ***all edges on the graph now display direction***
  - ***multiple edges in the same direction are not allowed***

Directed Graphs



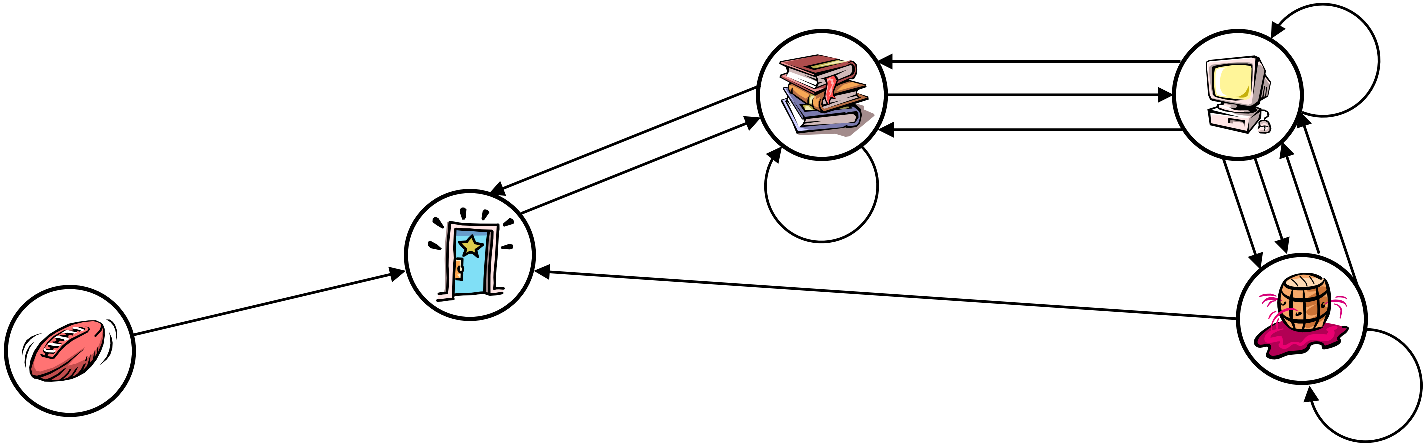
A Directed Graph G=(V,E) consists of a set of vertices V and a set of edges E that are ordered pairs of elements of V

Directed Multigraphs



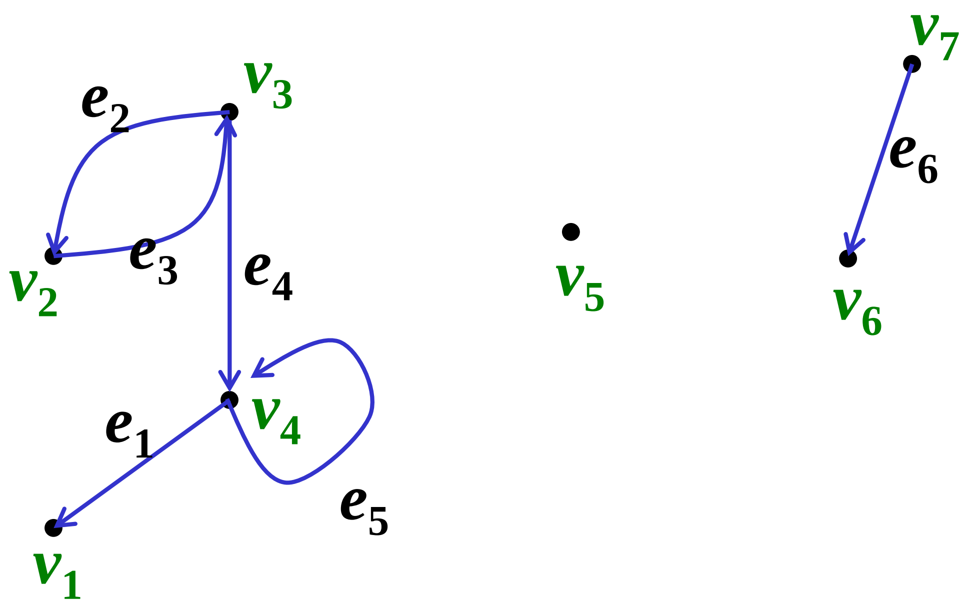
- **Networks can contain both one way lines and more than one line back to each remote computer from the host...**
- **This model is called a Directed Multigraph**
  - *multiple directed edges from a vertex to a second (possibly the same) vertex are now allowed*
  - *this graph has it all!*

Directed Multigraphs



A Directed Multigraph G=(V,E) consists of a set V of vertices, a set E of edges and a function f from f from E to $\{(u,v) | u,v \in V\}$. The edges e_1 and e_2 are multiple edges if $f(e_1) = f(e_2)$

Directed Multigraphs



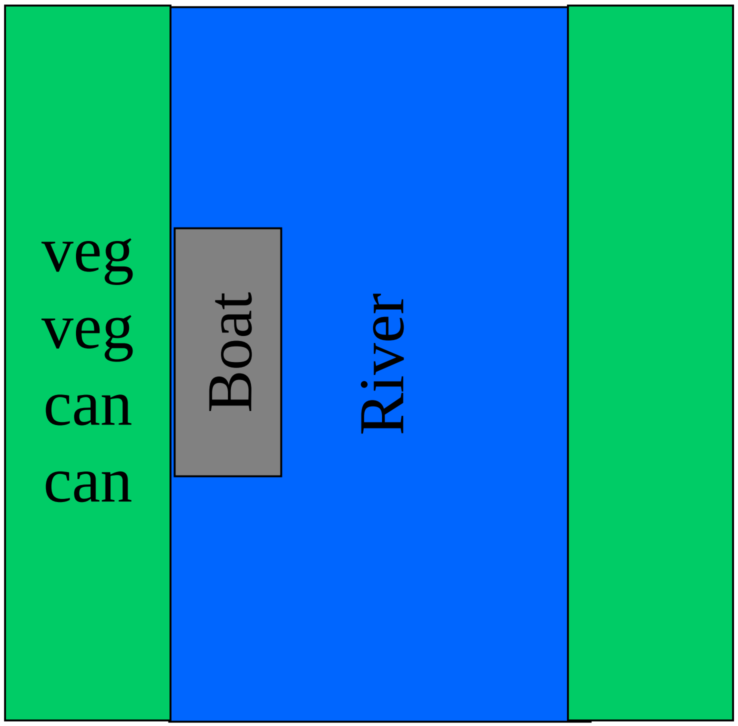
$V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$

$f(e_1)=(v_4,v_1)$, $f(e_2)=(v_3,v_2)$, $f(e_3)=(v_2,v_3)$, $f(e_4)=(v_4,v_3)$, $f(e_5)=(v_4,v_4)$, $f(e_6)=(v_7,v_6)$,

Summary

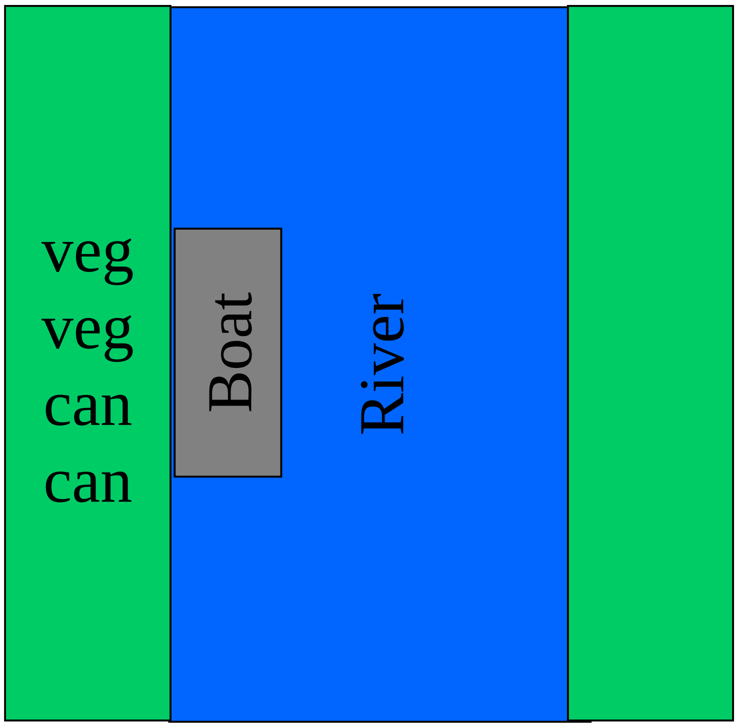
Type	Edges	Mutliple Edges Allowed?	Loops Allowed?
Simple Graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Directed Graph	Directed	No	Yes
Directed Multigraph	Directed	Yes	Yes

Example: Vegetarians and Cannibals



2 vegetarians and 2 cannibals are on the left bank of a river.
With them is a boat that can hold a maximum of 2 people.
Find a way to transport all cannibals and all vegetarians to the right bank of the river by using a graph.
At no time should the number of cannibals on either side outnumber the number of vegetarians !!!

Example: Vegetarians and Cannibals



Notation:

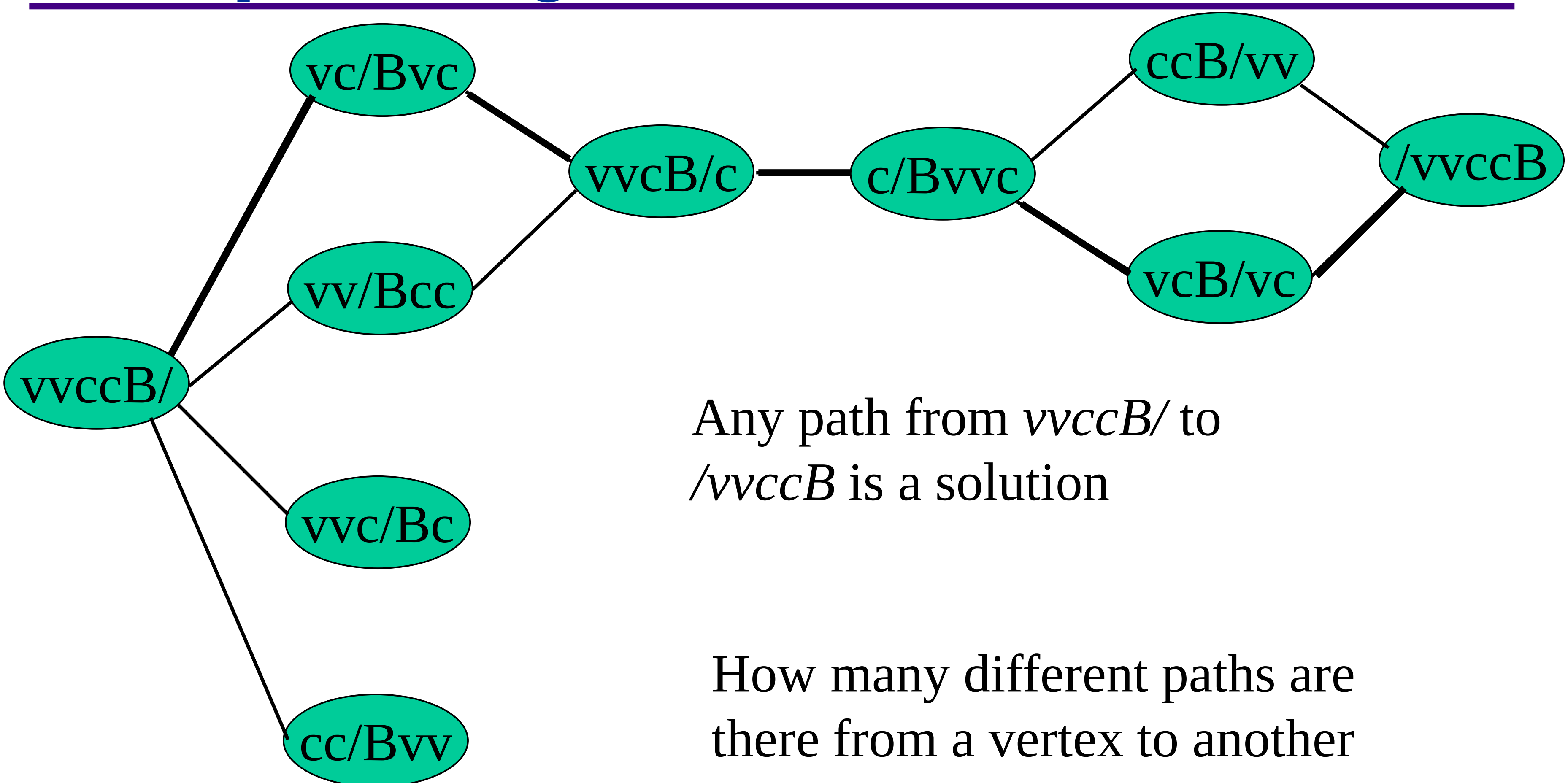
(vvc/Bc): two vegetarians and 1 cannibal on the left side and one cannibal and one boat on the right side.

Initial situation: (vvccB/)

aim: (/Bvvcc)

Construct a graph whose vertices are the various arrangements that can be reached by a sequence of legal moves starting from (vvccB/)

Example: Vegetarians and Cannibals

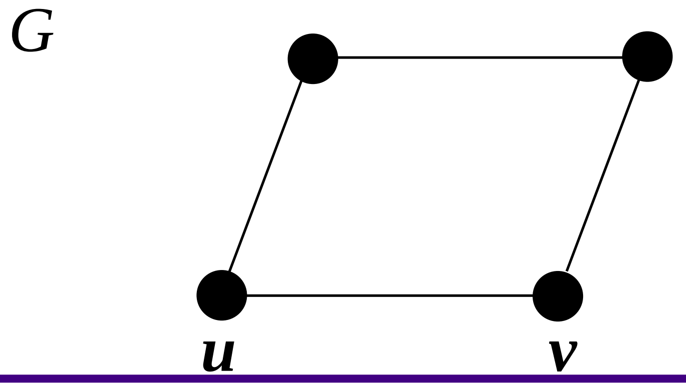


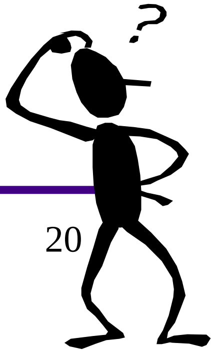
Any path from *vvccB/* to */vvccB* is a solution

How many different paths are there from a vertex to another vertex?

Concept of the Degree

Two vertices *u* and *v* in an undirected graph *G* are called <u>adjacent</u> (or neighbors) in *G* if {*u*,*v*} is an edge of *G*.
If *e* = {*u*,*v*}, the edge *e* is called <u>incident</u> with the vertices *u* and *v*. The edge *e* is also said to connect *u* and *v*. The vertices *u* and *v* are called endpoints of the edge {*u*,*v*}.





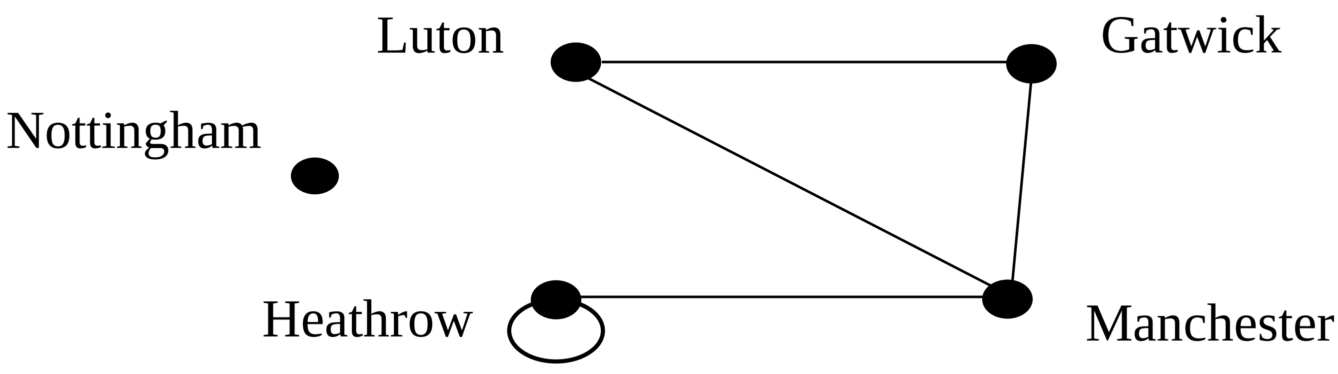
Concept of the Degree

The *degree* of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex.

The degree of the vertex *v* is denoted by *deg(v)*

Example

What are the degrees of the vertices in the graph displayed below?



deg(Luton) = 2 deg(Gatwick) = 2 deg(Manchester) = 3
deg(Heathrow) = 3 deg(Nottingham) = 0

- **Nottingham is not adjacent to any vertex**
- **A vertex with degree 0 is called <u>isolated</u>**

Total Degree

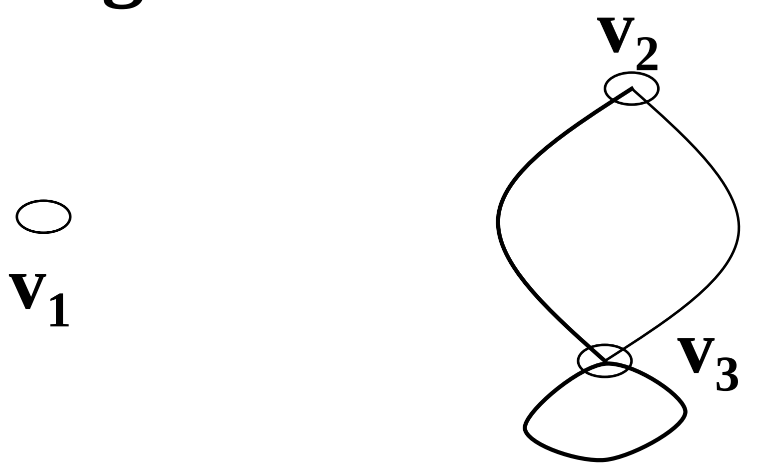
Let *G* be an undirected graph.

The *total degree of G* is the sum of the degrees of all the vertices of G,

i.e. total degree of G = $\sum_{v \in V} deg(v)$

Example

Find the total degree of the following graph:



$$\textbf{total degree} = \deg(v_1) + \deg(v_2) + \deg(v_3)$$
$$= 0 + 2 + 4 = \mathbf{6}$$

This equals twice the number of edges! Is this true in general?

The Handshaking Theorem

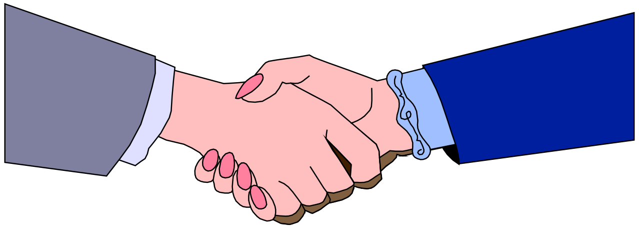
Theorem 1:

Let *G* = *(V,E)* be an undirected graph with *e* edges. Then

$$2e = \sum_{v \in V} deg(v)$$

(Note that this even applies if multiple edges and loops are present.)

Why ? Each edge contributes 2 to the sum of the degrees of the vertices since an edge is incident with exactly two (possibly equal vertices)

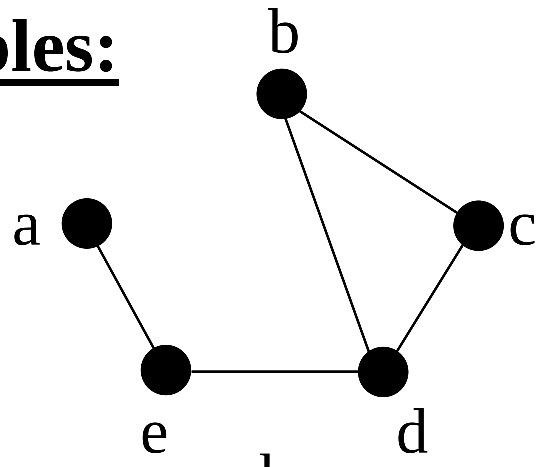


Concept of the Degree

How many vertices of odd degree are in an undirected graph?

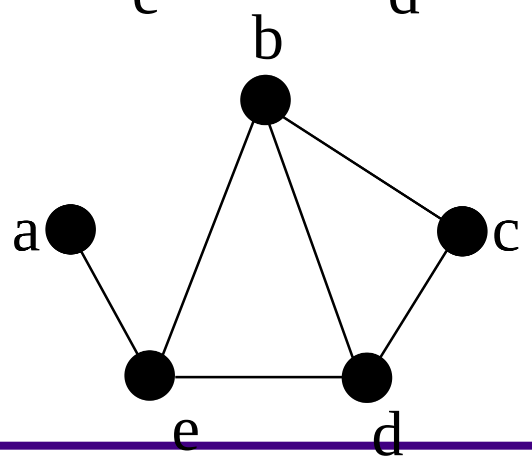
Even!

Examples:



deg(a) = 1,
deg(b) = deg(c) = deg(e) = 2,
deg(d) = 3

even = 3 odd = 2



deg(a) = 1
deg(c) = 2
deg(e) = deg(b) = deg(d) = 3

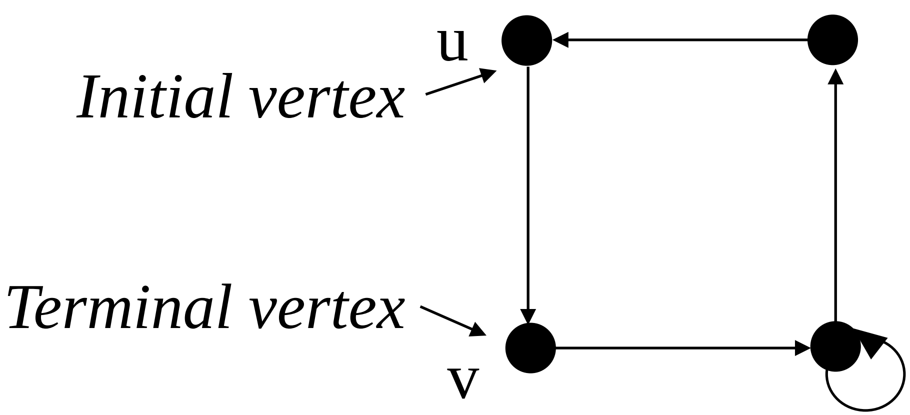
even = 1 odd = 4

Theorem 2

An undirected graph has an even number of vertices of odd degree

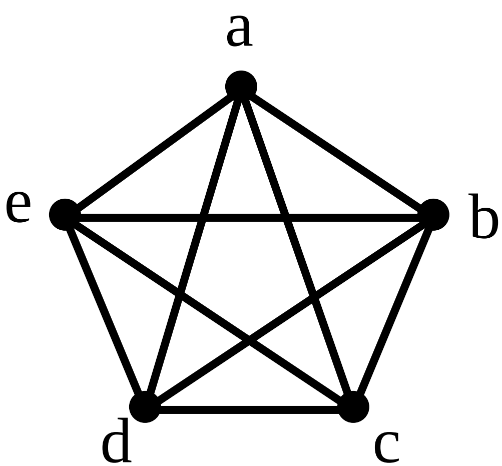
Concept of the Degree

When *(u,v)* is an edge of the graph *G* with directed edges, *u* is said to be *adjacent to v* and *v* is said to be *adjacent from u*.
The vertex *u* is called the *initial vertex* of *(u,v)*, and *v* is called the *terminal* or *end vertex* of (*u,v*).
The initial vertex and terminal vertex in a loop are the same.

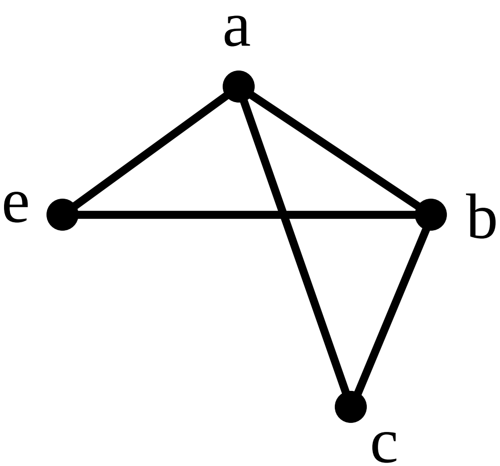


Subgraphs

A ***subgraph*** **of a graph** $G = (V,E)$ **is a graph** $H = (W,F)$ **where** $W \subseteq V$ **and** $F \subseteq E$**.**

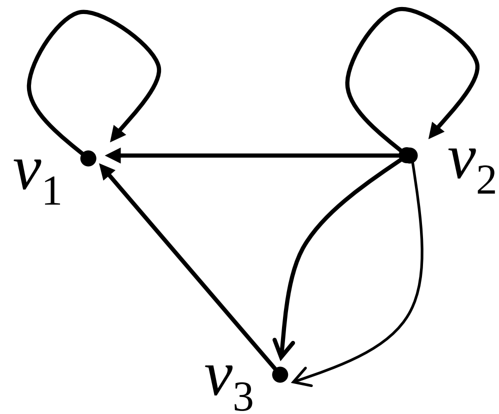


K_5



Subgraph of K_5

Adjacency matrices



This graph can be represented by a matrix $A = (a_{ij})$ with a_{ij} = the number of arrows from v_i to v_j

$$A = \begin{array}{c|ccc} & v_1 & v_2 & v_3 \\ \hline v_1 & 1 & 0 & 0 \\ v_2 & 1 & 1 & 2 \\ v_3 & 1 & 0 & 0 \end{array}$$

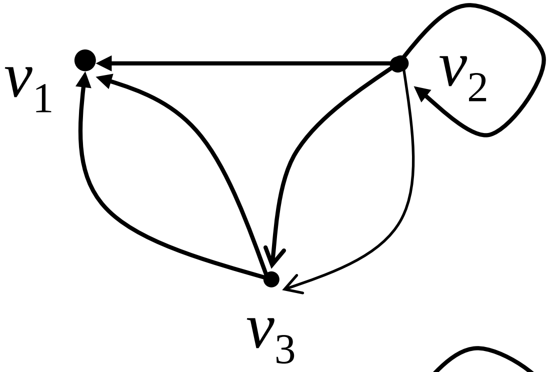
Adjacency matrices

Let $G=(V,E)$ be a <u>directed graph</u> with n vertices.
Suppose the vertices are ordered as $v_1,v_2,\ldots,v_n$
The <u>adjacency matrix</u> of G is the $n \times n$ matrix $A = (a_{ij})$ over the set of non-negative integers such that

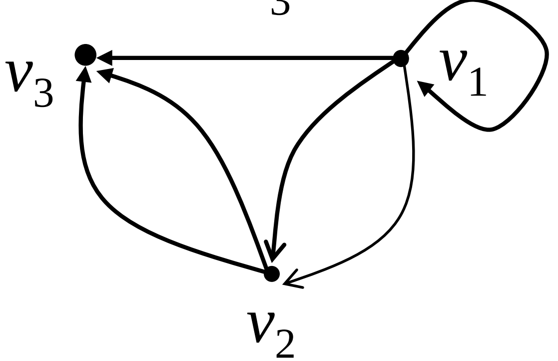
$$a_{ij} = \text{No. of edges from } v_i \text{ to } v_j$$

for all $i, j = 1, 2, \ldots, n$

Adjacency matrices



$$A_1 = \begin{array}{c|ccc} & v_1 & v_2 & v_3 \\ \hline v_1 & 0 & 0 & 0 \\ v_2 & 1 & 1 & 2 \\ v_3 & 2 & 0 & 0 \end{array}$$



$$A_2 = \begin{array}{c|ccc} & v_1 & v_2 & v_3 \\ \hline v_1 & 1 & 2 & 1 \\ v_2 & 0 & 0 & 2 \\ v_3 & 0 & 0 & 0 \end{array}$$

These two graphs differ only in the ordering of vertices:

If the vertices of a graph are reordered, then the entries in the rows and columns of the corresponding adjacency matrix are moved around.

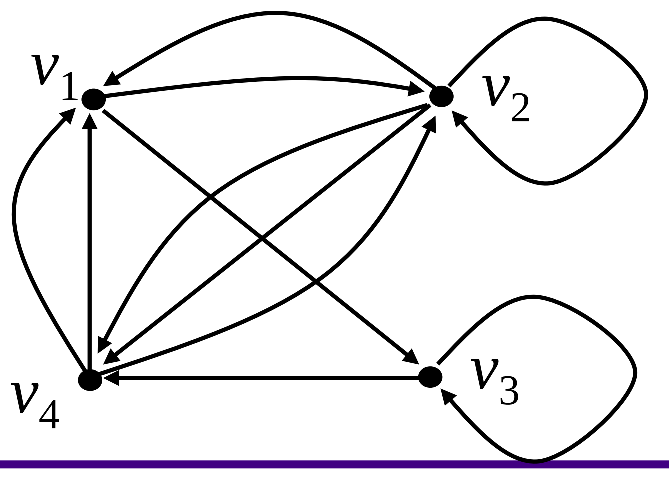
Adjacency matrices

Obtaining a graph from a matrix:

$$A = \begin{matrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \end{matrix}$$

Let v_1, v_2, v_3, v_4 be the vertices of the graph. Label A across the top and down the left side with these vertex names :

$$A = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 1 & 1 & 0 \\ v_2 & 1 & 1 & 0 & 2 \\ v_3 & 0 & 0 & 1 & 1 \\ v_4 & 2 & 1 & 0 & 0 \end{array}$$



Adjacency matrices

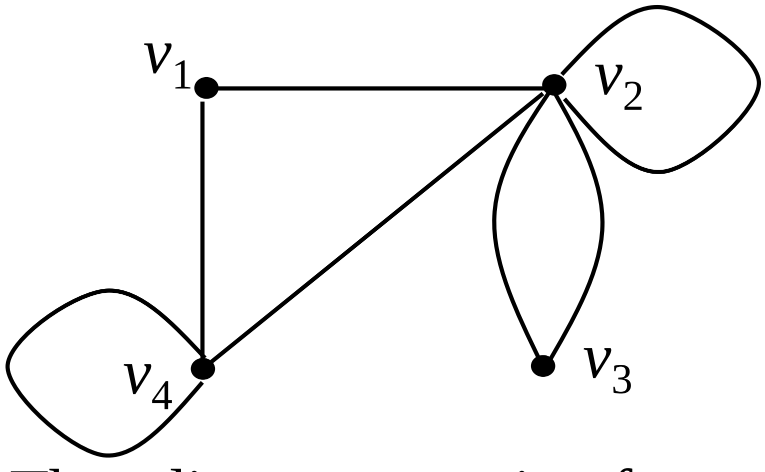
Let G=(V,E) be a undirected graph with n vertices.
Suppose the vertices are ordered as $v_1, v_2, \ldots, v_n$
The adjacency matrix of G is the $n \times n$ matrix $A = (a_{ij})$ of G over the set of non-negative integers such that

$$a_{ij} = \text{No. of edges connecting } v_i \text{ and } v_j$$

for all $i, j = 1, 2, \ldots, n$

Adjacency matrices

Find the adjacency matrix for the following graph:



$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{array}$$

The adjacency matrix of an undirected graph is symmetric, i.e. $a_{ij} = a_{ji}$ for all $i, j = 1, 2, \ldots, n$

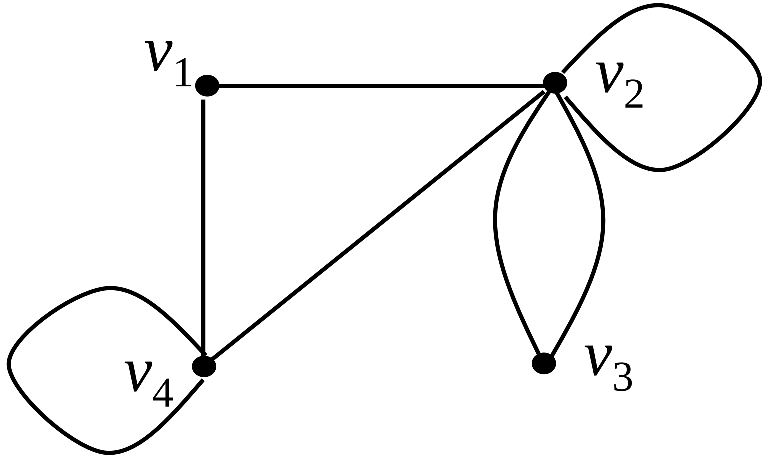
The adjacency matrix of a simple graph is a 0-1 matrix.

Furthermore, since a simple graph has no loops each entry on the main diagonal is 0, i.e. $a_{ii} = 0$ for all $i = 1, 2, \ldots, n$

Adjacency matrices

What is the sum of the entries in a row of the adjacency matrix for an undirected graph?

deg(v) – number of loops at v

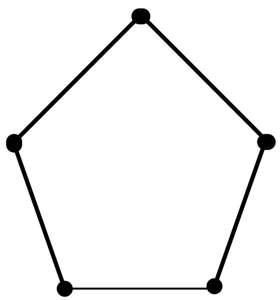


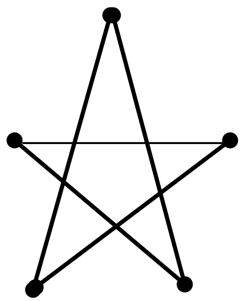
$$\begin{array}{c|cccc|l|l|l}
 & v_1 & v_2 & v_3 & v_4 & & & \\
v_1 & 0 & 1 & 0 & 1 & \sum=2 & \deg(v_1)=2 & \deg(v_1)\text{–loops at } v_1 = 2 \\
v_2 & 1 & 1 & 2 & 1 & \sum=5 & \deg(v_2)=6 & \deg(v_2)\text{–loops at } v_2 = 5 \\
v_3 & 0 & 2 & 0 & 0 & \sum=2 & \deg(v_3)=2 & \deg(v_3)\text{–loops at } v_3 = 2 \\
v_4 & 1 & 1 & 0 & 1 & \sum=3 & \deg(v_4)=4 & \deg(v_4)\text{–loops at } v_4 = 3
\end{array}$$

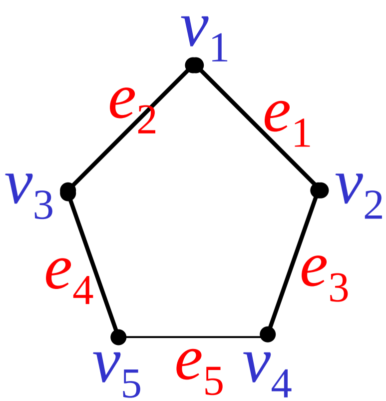
Does the same hold for the sum of the entries in a column of the adjacency matrix for an undirected graph?

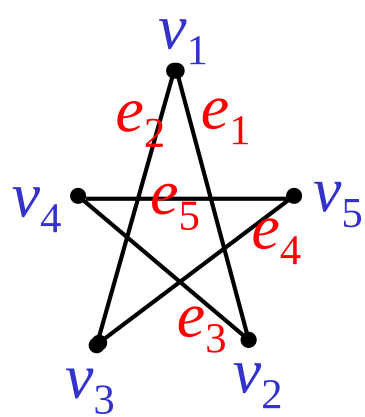
Isomorphism

Labeling drawings to show they represent the same graph.
Can you label the vertices and edges in such a way that both drawings represent the same graph?

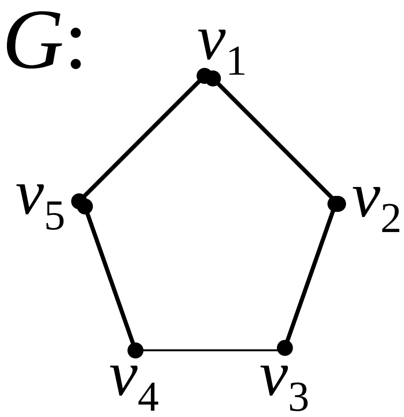






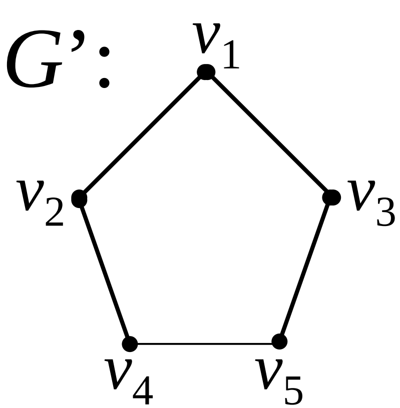


Isomorphism

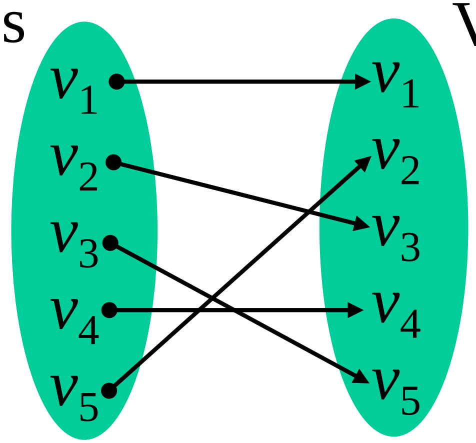


G **and** ***G'*** **are different (for instance, {v_1 ,v_5} is an edge in G, but not in in** ***G'*****)**

But the vertices of ***G'*** **can be relabeled by the following functions, then** ***G'*** **becomes the same to** ***G*****.**



Vertices of *G*



Vertices of *G'*

One-to-one?

Yes!

Onto?

Yes!

Isomorphism of Graphs

The simple graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are isomorphic if there is a *one-to-one* and *onto* function f from V_1 to V_2 with the property that

for all vertices $a, b \in V_1$: $\{a, b\}$ is an edge in $G_1 \Leftrightarrow$

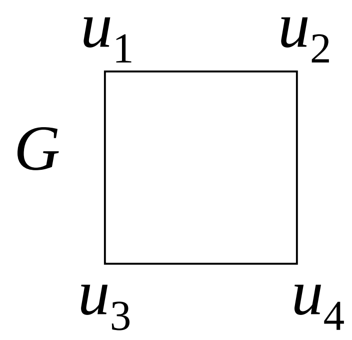
$\{f(a), f(b)\}$ is an edge in G_2

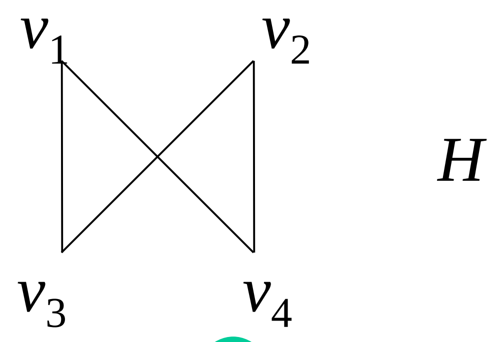
Such a function f is called an isomorphism.

In other words, two simple graphs are isomorphic, if there is a one-to-one correspondence (bijection) between the vertices of the two graphs that preserves the adjacency relationship.

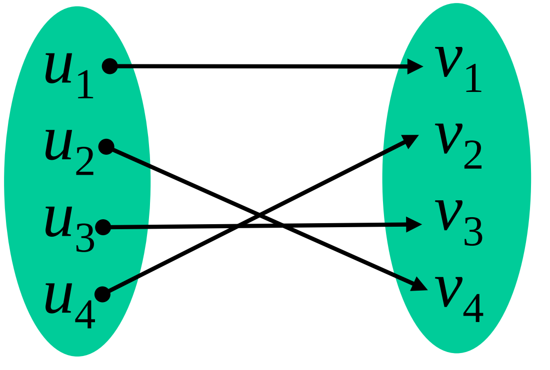
Example

Show that these graphs are isomorphic:





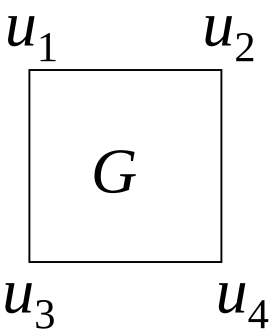
Find one-to-one correspondence f between the vertices:



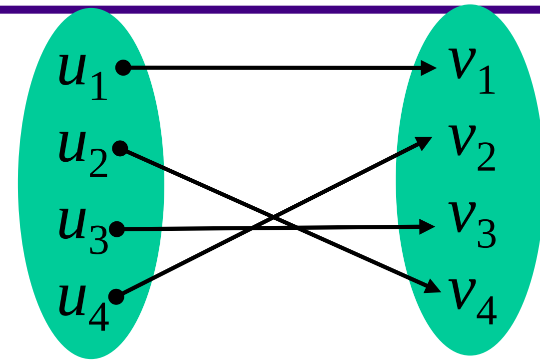
Show that f preserves adjacency:

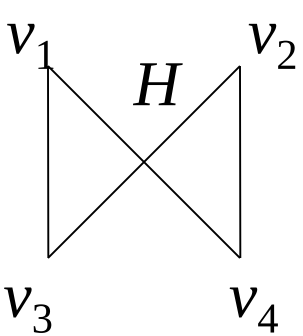
$$\{u_1, u_2\} \Leftrightarrow \{f(u_1), f(u_2)\} = \{v_1, v_4\}$$
$$\{u_1, u_3\} \Leftrightarrow \{f(u_1), f(u_3)\} = \{v_1, v_3\}$$
$$\{u_2, u_4\} \Leftrightarrow \{f(u_2), f(u_4)\} = \{v_4, v_2\}$$
$$\{u_3, u_4\} \Leftrightarrow \{f(u_3), f(u_4)\} = \{v_3, v_2\}$$

Example



Find one-to-one correspondence f between the vertices:



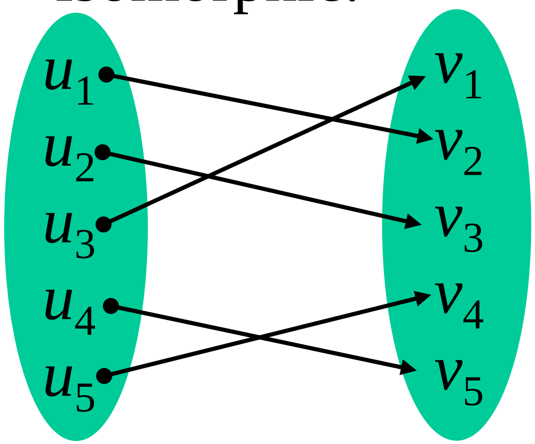


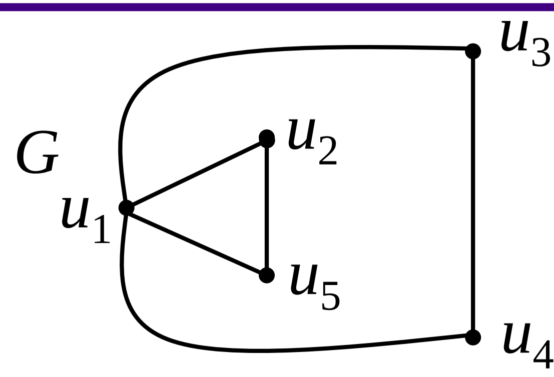
$$A_G = \begin{array}{c|cccc} & u_1 & u_2 & u_3 & u_4 \\ \hline u_1 & 0 & 1 & 1 & 0 \\ u_2 & 1 & 0 & 0 & 1 \\ u_3 & 1 & 0 & 0 & 1 \\ u_4 & 0 & 1 & 1 & 0 \end{array} \qquad A_H = \begin{array}{c|cccc} & v_1 & v_4 & v_3 & v_2 \\ \hline v_1 & 0 & 1 & 1 & 0 \\ v_4 & 1 & 0 & 0 & 1 \\ v_3 & 1 & 0 & 0 & 1 \\ v_2 & 0 & 1 & 1 & 0 \end{array}$$

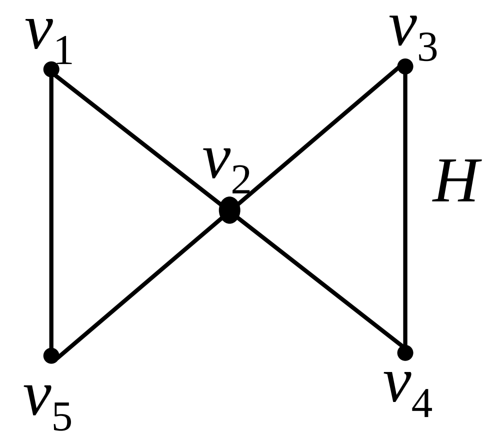
Since $A_G = A_H$, it follows that f preserves adjacency.

Example

Show that these graphs are isomorphic:





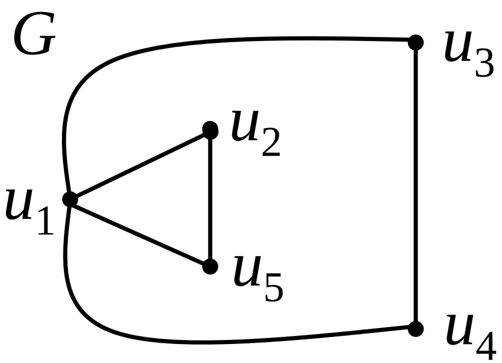


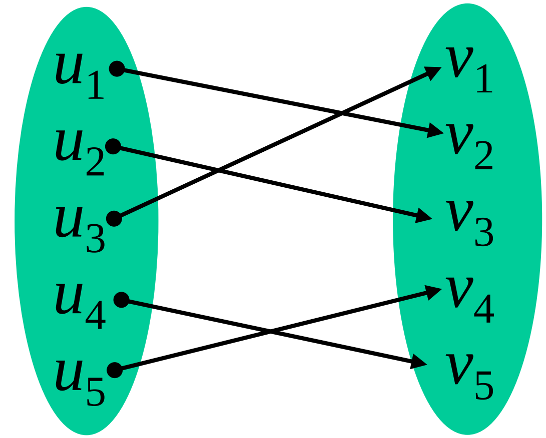
$\deg(u_1) = 4$ and v_2 is the only vertex in H with degree 4 $\Rightarrow f(u_1)=v_2$

adjacent u_2 and u_5 are adjacent to $u_1 \Rightarrow f(u_2)=v_3$ and $f(u_5)=v_4$

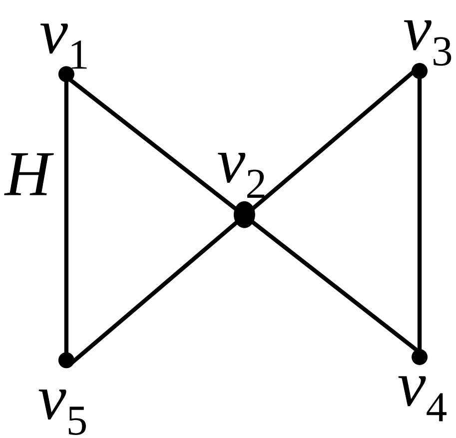
adjacent u_3 and u_4 are adjacent to $u_1 \Rightarrow f(u_3)=v_1$ and $f(u_4)=v_5$

Example





Compare adjacency matrices:

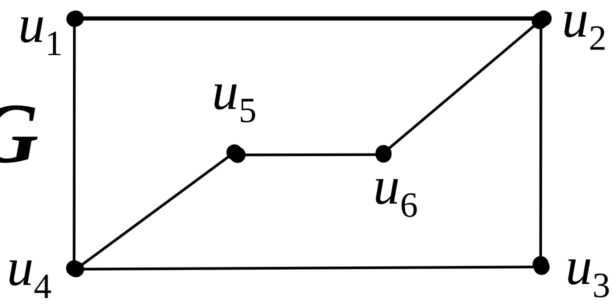


$$A_G = \begin{array}{c|ccccc} & u_1 & u_2 & u_3 & u_4 & u_5 \\ \hline u_1 & 0 & 1 & 1 & 1 & 1 \\ u_2 & 1 & 0 & 0 & 0 & 1 \\ u_3 & 1 & 0 & 0 & 1 & 0 \\ u_4 & 1 & 0 & 0 & 1 & 0 \\ u_5 & 1 & 0 & 0 & 0 & 1 \end{array} \qquad A_H = \begin{array}{c|ccccc} & v_2 & v_3 & v_1 & v_5 & v_4 \\ \hline v_2 & 0 & 1 & 1 & 1 & 1 \\ v_3 & 1 & 0 & 0 & 0 & 1 \\ v_1 & 1 & 0 & 0 & 1 & 0 \\ v_5 & 1 & 0 & 0 & 1 & 0 \\ v_4 & 1 & 0 & 0 & 0 & 1 \end{array}$$

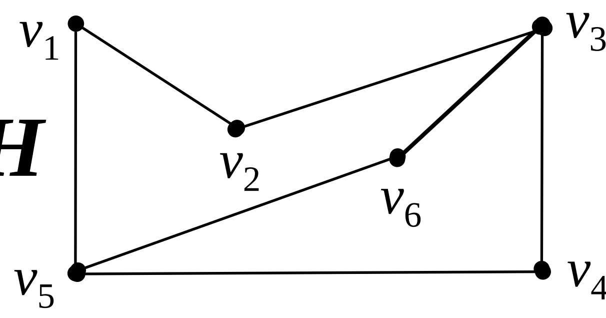
Example

Show that these graphs are isomorphic:

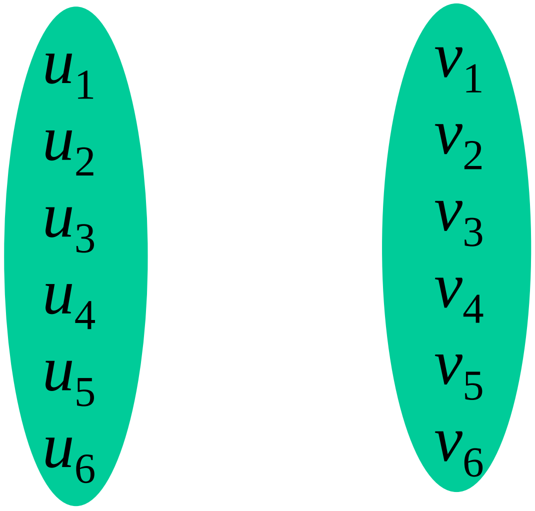
G



H



Find one-to-one correspondence f between the vertices which preserves adjacency :



Try it yourself!

Isomorphism

Problem: Determine whether two simple graphs with *n* vertices are isomorphic.

$\Rightarrow$ *n*! possible one-to-one correspondences.

$\Rightarrow$ Testing each of them to see whether it preserves adjacency (worst case)

$\Rightarrow$ Impractical for large *n*

$\Rightarrow$ Best known algorithms have exponential worst-case time complexity

Isomorphism

A property P is called <u>isomorphic invariant</u>
if and only if
given any simple graphs *G* and *H*, if *G* has property P and *H* is isomorphic to *G*, then H has property P.

Examples of isomorphic invariants (more later):

1. **Same number of vertices**
2. **Same number of edges**
3. **Same degrees of vertices (that is, a vertex *v* of degree *d* in *G* must correspond to a vertex *f*(*v*) of degree *d* in *H*)**

Isomorphism

Using isomorphic invariants to show that two simple graphs *G* and *H* are not isomorphic:

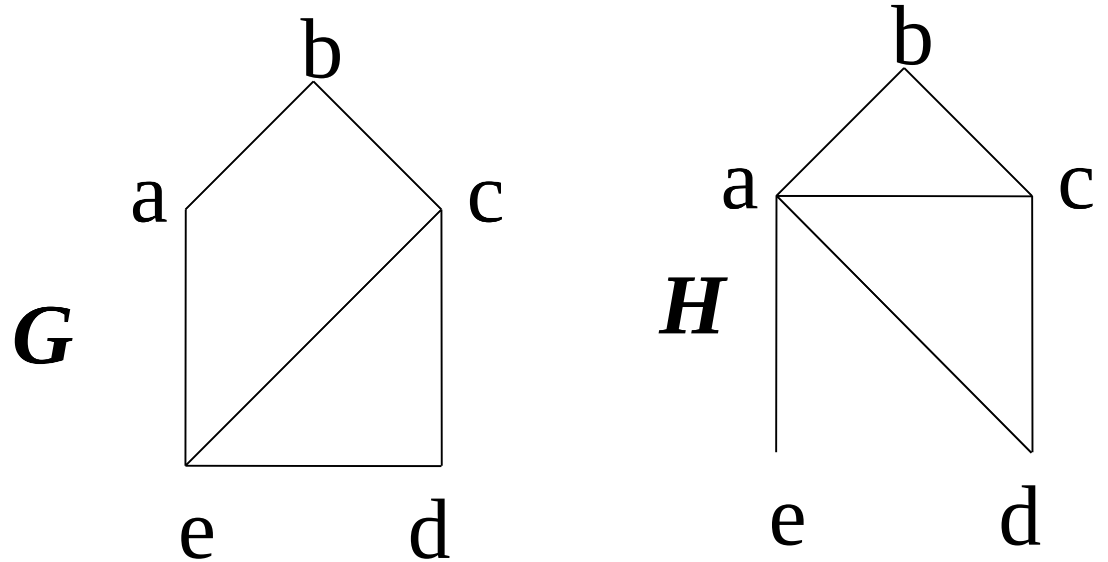
(a) if *G* has 16 vertices and *H* has 17 vertices, then *G* and *H* are not isomorphic

(b) if *G* has 20 edges and and *H* has 18 edges, then *G* and *H* are not isomorphic

(c) if *G* has a vertex with degree 5 and *H* has not, then *G* and *H* are not isomorphic

Example

Show that *G* and *H* are not Isomorphic



- Number of vertices: both 5
- Number of edges: both 6
- Degrees of vertices: deg(*e*)=1 in *H*, but *G* has no vertex of degree 1 !

Example

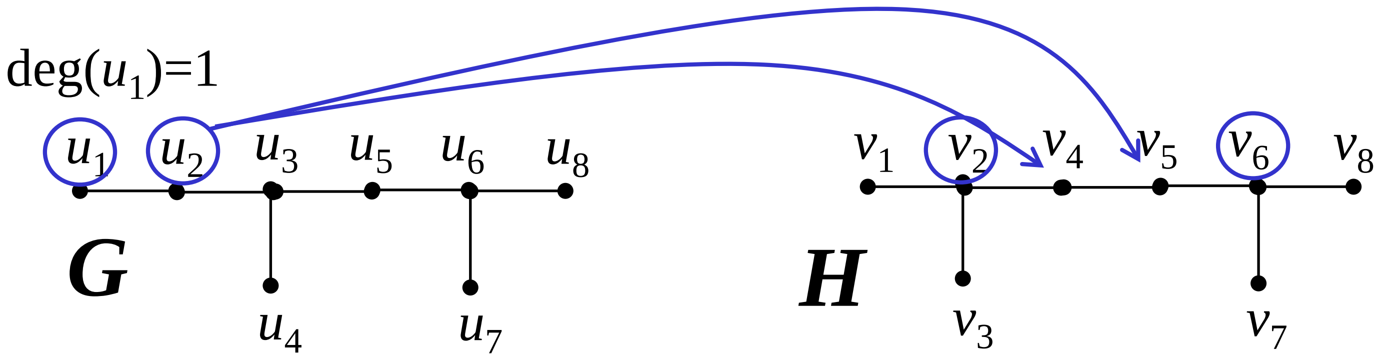
Are these Graphs Isomorphic?

Check Invariants:

- Number of vertices: both 8 ☑
- Number of edges: both 7 ☑
- Degrees of vertices: 4× deg=1, 2× deg=2, 2× deg=3 ☑

Example

Although Invariants are the same, these graphs are <u>not isomorphic</u> !



- since deg(u_2)=2 in G, u_2 must correspond to v_4 or v_5 in H, since these are the vertices of degree 2 in H
- u_2 is adjacent to u_1, a vertex of degree 1
- but neither v_4 or v_5 is adjacent to a vertex of degree 1 in H

Special Graphs

- Complete Graph
- Cycle
- Wheel
- Complete Bipartite Graph

Complete Graphs

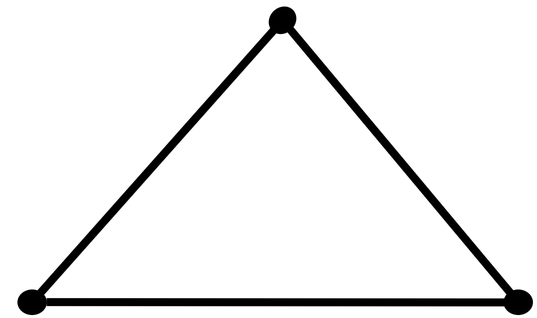
The <u>**c*omplete graph* on *n* vertices**</u>, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices.

Complete Graphs

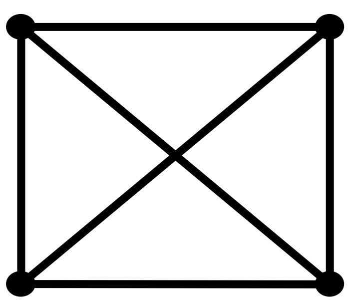
Contains exactly one edge between each pair of distinct vertices

K_1

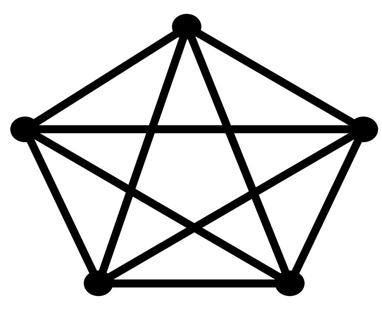
K_2



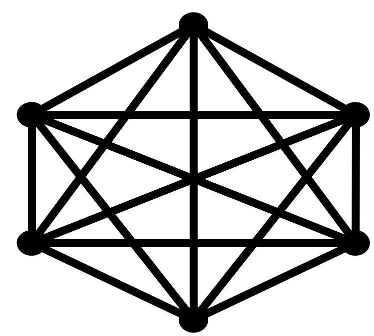
K_3



K_4



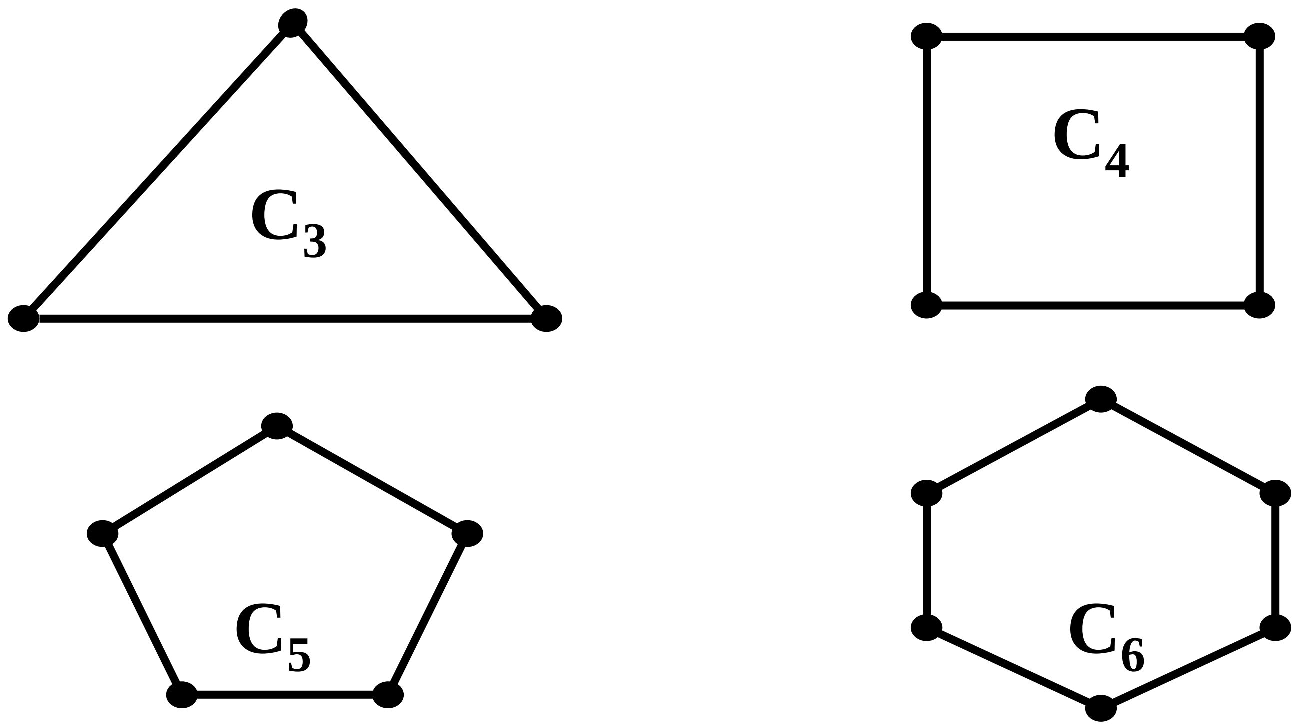
K_5



K_6

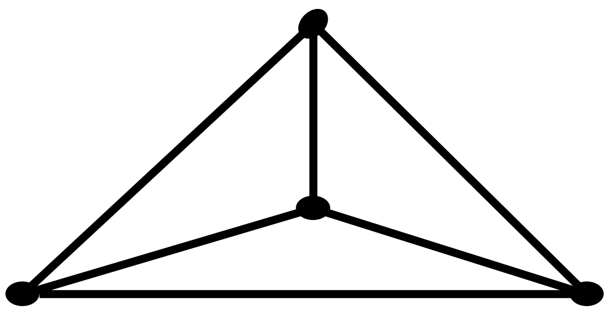
Cycles

The *cycle* C_n , $n \geq 3$, consists of n vertices v_1, v_2,…,v_n and edges $\{v_1,v_2\}$, $\{v_2,v_3\}$,…$\{v_{n-1},v_n\}$ and $\{v_n,v_1\}$.

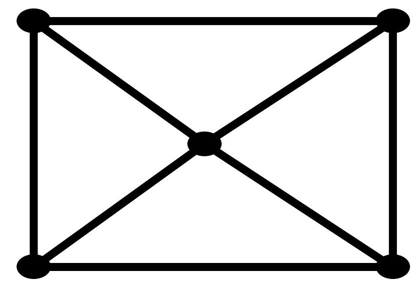


Wheels

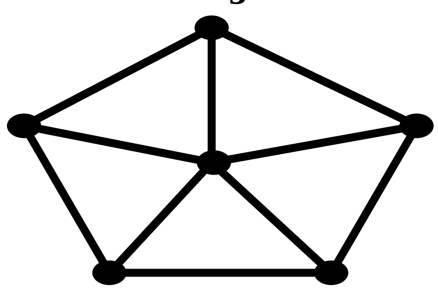
We obtain the *wheel W_n* when we add an additional vertex to the cycle C_n , for $n \geq 3$, and connect this new vertex to each of the n vertices in C_n , by new edges.”



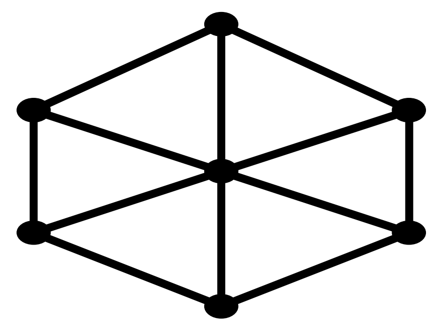
$\mathbf{W_3}$



$\mathbf{W_4}$



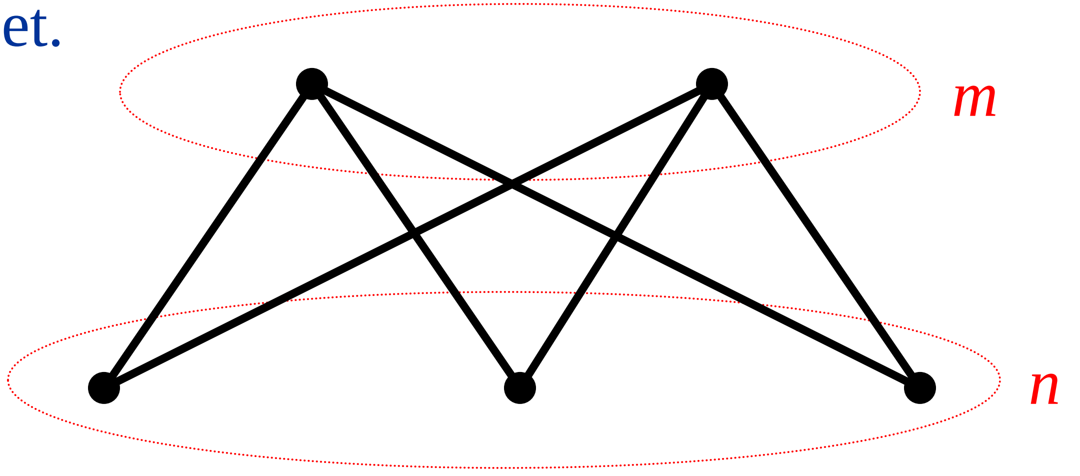
$\mathbf{W_5}$



$\mathbf{W_6}$

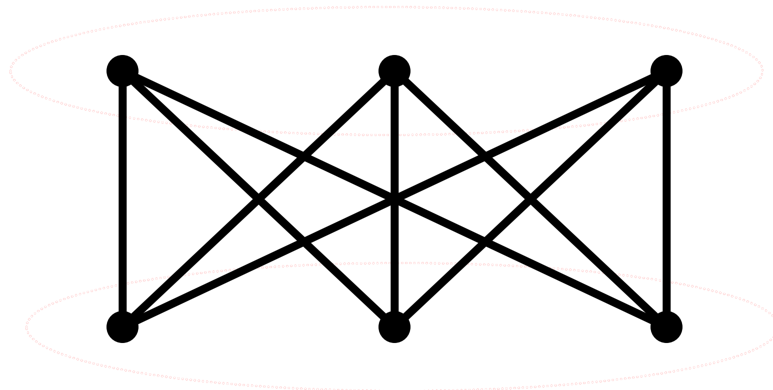
Complete Bipartite Graphs

The ***complete bipartite graph*** $K_{m,n}$, m, $n \geq 1$, is the graph with $m+n$ vertices that has its vertex set partitioned into two subsets m and n vertices, respectively **and** there is an edge between two vertices **if and only if** one vertex is in the first subset and the other vertex is in the second subset.

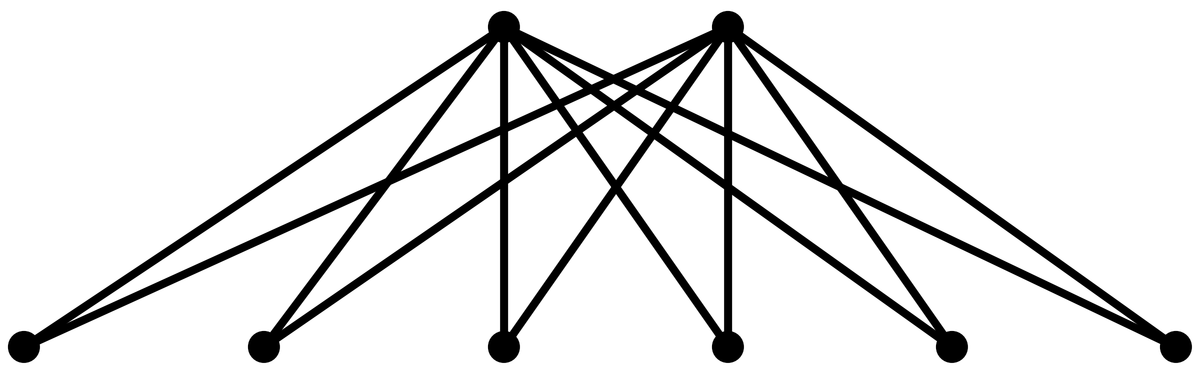


$K_{2,3}$

Complete Bipartite Graphs



$K_{3,3}$

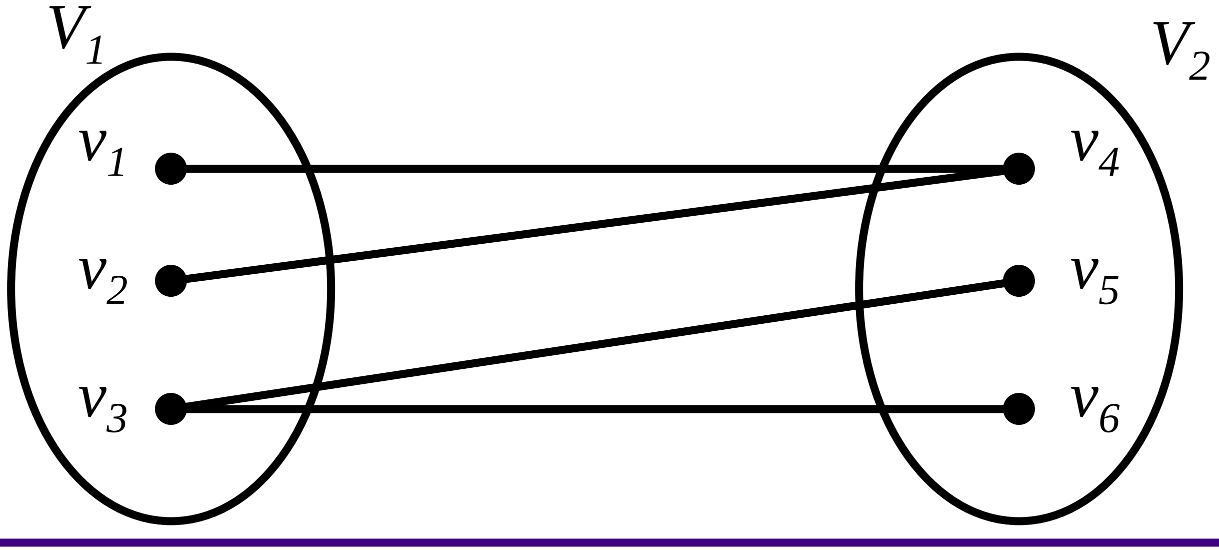


$K_{2,6}$

Bipartite Graphs

A simple graph G is called ***bipartite*** if its vertex V can be partitioned into **two disjoint nonempty sets V_1 and V_2** , i.e. $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$, such that every edge in the graph connects a vertex in V_1 and a vertex in V_2

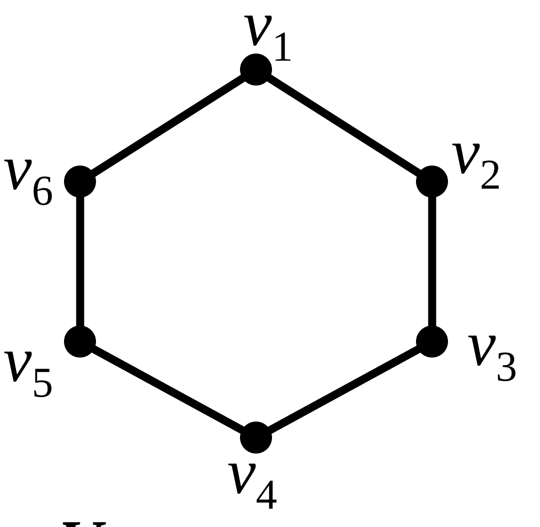
(so that no edge in G connects either two vertices in V_1 or two vertices in V_2)



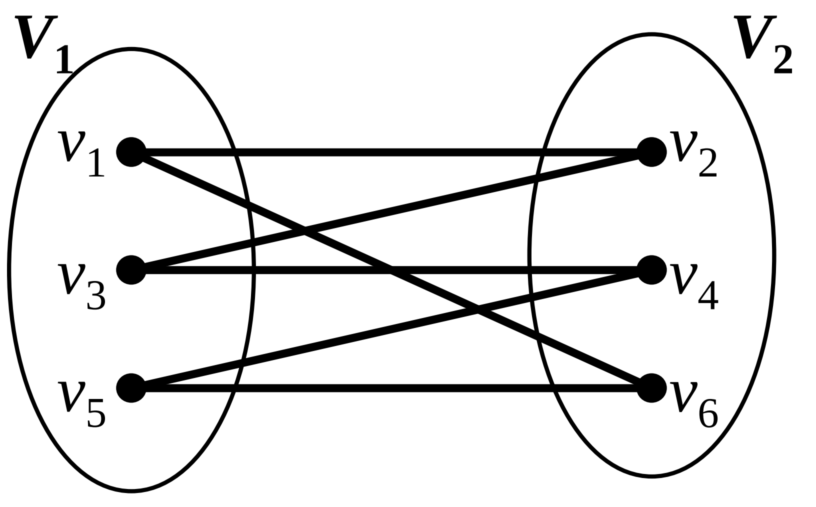
Bipartite Graphs

Is C_6 bipartite?

Can we partition $V=\{v_1, v_2, v_3, v_4, v_5, v_6\}$ into non-empty sets V_1 and V_2 such that every edge in C_6 connects a vertex in V_1 and a vertex V_2 ?

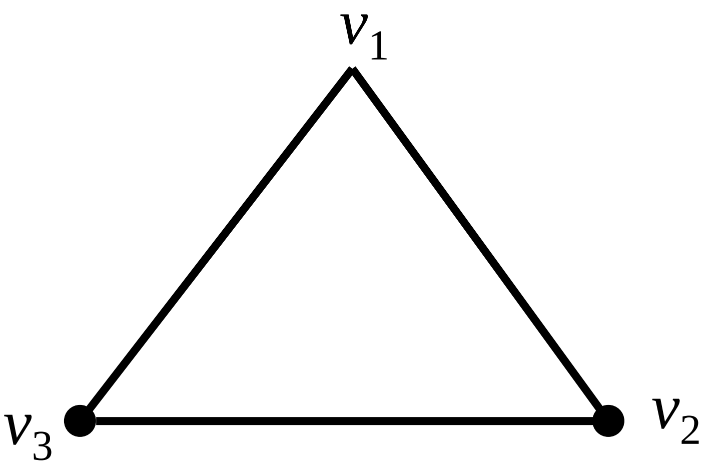


Yes!



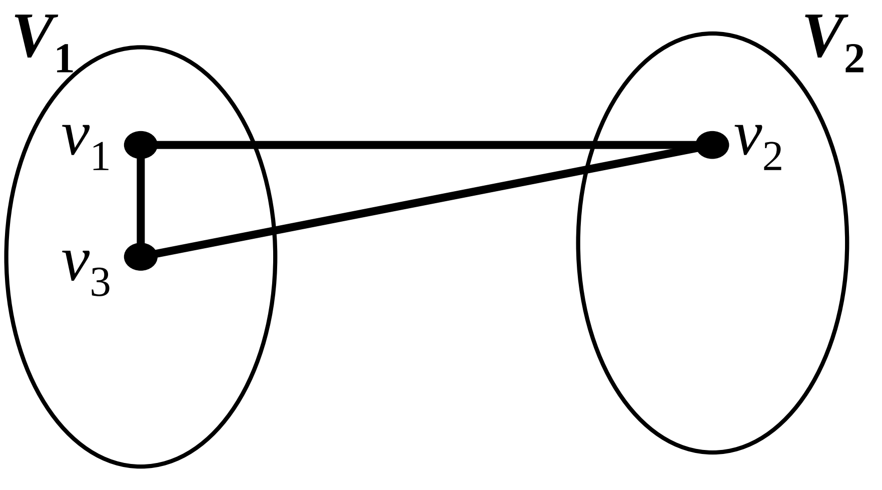
Bipartite Graphs

Is K_3 bipartite?



Can we partition $V=\{v_1, v_2, v_3\}$ into non-empty sets V_1 and V_2 such that every edge in K_3 connects a vertex in V_1 and a vertex V_2 ?

No!



Summary

- Simple Graphs
- Multigraphs
- Pseudographs
- Directed graphs
- Directed Multigraphs
- Concept of the Degree
- Adjacent Matrices
- Isomorphism

Summary

- **Special types of Graphs**
  - *Complete Graph*
  - *Cycle*
  - *Wheel*
  - *(Complete) Bipartite Graph*