Module: 1

Mathematical Logic

Propositional Logic

Propositions

A proposition or statement is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.

EXAMPLE

All the following declarative sentences are propositions.

1. Washington, D.C., is the capital of the United States of America.
2. Toronto is the capital of Canada.
3. $1 + 1 = 2$.
4. $2 + 2 = 3$.

Propositions 1 and 3 are true, whereas 2 and 4 are false.

Note:

Some sentences that are not propositions that can be simultaneously True or False. Consider the following sentences.

1. What time is it?
2. Read this carefully.
3. $x + 1 = 2$.
4. $x + y = z$.

Sentences 1 and 2 are not propositions because they are not declarative sentences. Sentences 3 and 4 are not propositions because they are neither true nor false. If sentences 3 and 4 can be turned into a proposition if we assign values to the variables.

CONNECTIVES

Declarative sentences which cannot be further split into simpler sentences are called atomic statements. Many mathematical statements are constructed by combining one or more propositions, new propositions called molecular or compound or compound propositions, are formed from existing propositions using logical operators.

The following symbols are used to represent connectives,

S.No	Symbol used	Connective word	Logical connectives	Symbolic form
1	$\neg$	Not	Negation	$\neg p$
2	$\wedge$	And	Conjunction	$p \wedge q$
3	$\vee$	Or	Disjunction	$p \vee q$
4	$\rightarrow$	If…then	Implication (or)conditional	$p \rightarrow q$
5	$\leftrightarrow$	If and only if	Equivalence (or) bi-conditional	$p \leftrightarrow q$

NOTE:

1. Propositional variables (or statement variables), that is, variables that represent propositions, just as letters are used to denote numerical variables. The conventional letters used for propositional variables are p, q, r, s, ...
2. The truth value of a proposition is true, denoted by T, if it is a true proposition, and the truth value of a proposition is false, denoted by F, if it is a false proposition.

NEGATION

The negation of a statement is generally formed by introducing a word 'not' at a proper place in a statement.

If p be a proposition. The negation of p, denoted by ¬p (also denoted by p) and read as "not p." The truth value of the negation of p, ¬p, is the opposite of the truth value of p.

The Truth table for the negation of a proposition:

p	¬p
T	F
F	T

EXAMPLE

1. Find the negation of each of the following proposition
   a) "Michael's PC runs Linux"
   b) "Vandana's smartphone has at least 32GB of memory" English.

Solution:
The negation is
   a) "Michael's PC does not run Linux."
   b) "Vandana's smartphone does not have at least 32GB of memory"

CONJUNCTION

Let p and q be propositions. The conjunction of p and q, denoted by p ∧ q, is the proposition "p and q." The conjunction p ∧ q is true when both p and q are true and is false otherwise.

The Truth table for the conjunction of two proposition:

p	q	p ∧ q
T	T	T
T	F	F
F	T	F
F	F	F

Example

1. The conjunction of
   p: It is raining today.
   q: There are 20 tables in this room.
   Solution:

$p \wedge q$: It is raining today and there are 20 tables in this room.

2. If p: It is snowing, q: I am cold.

   Solution:

   $p \wedge q$: It is snowing and I am cold.

DISJUNCTION

Let p and q be propositions. The disjunction of p and q, denoted by $p \vee q$, is the proposition "p or q." The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

The Truth table for the disjunction of two proposition:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

EXAMPLE

1.The disjunction of

p: I shall watch the game on T.V
q: Go to the stadium.

Solution:

$p \vee q$: I shall watch the game on T.V or go to the stadium.

2.Find the conjunction and disjunction of the propositions p and q where p is the proposition "Rebecca's PC has more than 16 GB free hard disk space" and q is the proposition "The processor in Rebecca's PC runs faster than 1 GHz."

Solution:

The conjunction of these propositions, $p \wedge q$, is the proposition "Rebecca's PC has more than 16 GB free hard disk space, and the processor in Rebecca's PC runs faster than 1 GHz."

The disjunction of p and q, $p \vee q$, is the proposition "Rebecca's PC has at least 16 GB free hard disk space, or the processor in Rebecca's PC runs faster than 1 GHz."

CONDITIONAL STATEMENT

Let p and q be propositions. The conditional statement $p \rightarrow q$ is the proposition "if p, then q." The conditional statement $p \rightarrow q$ is false when p is true and q is false, and true otherwise. In the conditional statement $p \rightarrow q$, p is called the hypothesis (or antecedent or premise) and q is called the conclusion (or consequence)

The Truth table for the conditional statement of two proposition:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

NOTE

The conditional statements play such an essential role in mathematical reasoning, a variety of terminology is used to express $p \to q$. You will encounter most if not all of the following ways to express this conditional statement:

"if p, then q"; "p implies q" ; "if p, q"; "p only if q" ; "p is sufficient for q" ; "a sufficient condition for q is p"; "q if p"; "q whenever p" ; "q when p" ; "q is necessary for p"; "a necessary condition for p is q" ; "q follows from p"; "q unless $\neg$p".

EXAMPLE:

Let p be the statement "Maria learns discrete mathematics" and q the statement "Maria will find a good job." Express the statement $p \to q$ as a statement in English.

Solution:

$p \to q$: If Maria learns discrete mathematics, then she will find a good job.

BICONDITIONAL STATEMENT

Let p and q be propositions. The biconditional statement $p \leftrightarrow q$ is the proposition "p if and only if q." The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called bi-implications.

The Truth table for the biconditional statement of two proposition:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

EXAMPLE

Let p be the statement "You can take the flight," and let q be the statement "You buy a ticket." Then $p \leftrightarrow q$ is the statement

Solution:

$p \leftrightarrow q$: You can take the flight if and only if you buy a ticket.

PROBLEMS

1. Which of these are propositions? What are the truth values of those that are propositions?
   a. Do not pass go.
   b. What time is it?
   c. $4 + x = 5$.
   d. The moon is made of green cheese.
   e. $2n \geq 100$.

Solution:

a) This is not a proposition; it's a command.
b) This is not a proposition; it's a question.
c) This is not a proposition; it's truth value depends on the value of x.
d) This is a proposition that is false.
e) This is not a proposition; it's truth value depends on the value of n.

2. Let p and q be the propositions "Swimming at the New Jersey shore is allowed" and "Sharks have been spotted near the shore," respectively. Express each of these compound propositions as an English sentence.

a) $\neg q$	e) $\neg q \rightarrow p$
b) $p \land q$	f) $\neg p \rightarrow \neg q$
c) $\neg p \lor q$	g) $p \leftrightarrow \neg q$
d) $p \rightarrow \neg q$	h) $\neg p \land (p \lor \neg q)$

Solution:

a) Sharks have not been spotted near the shore.
b) Swimming at the New Jersey shore is allowed and sharks have not been spotted near the shore.
c) Swimming at the New Jersey shore is not allowed or sharks have been spotted near the shore.
d) If Swimming at the New Jersey shore is allowed, then sharks have not been spotted near the shore.
e) If sharks have not been spotted near the shore, then swimming at the New Jersey shore is allowed.
f) If Swimming at the New Jersey shore is not allowed, then sharks have not been spotted near the shore.
g) Swimming at the New Jersey shore is allowed if and only if sharks have not been spotted near the shore.
h) Swimming at the New Jersey shore is not allowed and, either swimming at the New Jersey shore is allowed or sharks have not been spotted near the shore.

3. Let p and q be the propositions p : You drive over 65 miles per hour. q : You get a speeding ticket. Write these propositions using p and q and logical connectives (including negations).
   a. You do not drive over 65 miles per hour.
   b. You drive over 65 miles per hour, but you do not get a speeding ticket.
   c. You will get a speeding ticket if you drive over 65 miles per hour.
   d. If you do not drive over 65 miles per hour, then you will not get a speeding ticket.
   e. Driving over 65 miles per hour is sufficient for getting a speeding ticket.
   f. You get a speeding ticket, but you do not drive over 65 miles per hour.

Solution:

a. This is just the negation of p, so we write $\neg p$.
b. This is a conjunction ("but" means "and"): $p \land \neg q$.
c. The position of the word "if" tells us which is the antecedent and which is the consequence:

$$q \rightarrow p.$$

d. $\neg p \rightarrow \neg q$

e. The sufficient condition is the antecedent: $p \rightarrow q$.

f. $q \wedge \neg p$.

PRACTICE PROBLEMS

1. Let p and q denote the statements: p : You drive over 70 km per hour. q : You get a speeding ticket. Write the following statements into symbolic forms.
   i. You will get a speeding ticket if you drive over 70 km per hour.
   ii. Driving over 70 km per hour is sufficient for getting a speeding ticket.
   iii. If you do not drive over 70 km per hour then you will not get a speeding ticket.
   iv. Whenever you get a speeding ticket, you drive over 70 km per hour.
2. Write the negation of the following statements.
   a) Jan will take a job in industry or go to graduate school.
   b) If the processor is fast then the printer is slow.

WELL FORMED FORMULAS

A statement formula is an expression denoted by a string consisting of variables, parentheses and connective symbols.

EXAMPLE

1. $\neg (p \wedge q)$

2. $\neg (p \vee q)$

3. $(p \rightarrow (p \vee q))$

4. $(p \rightarrow (q \rightarrow r))$

TRUTH TABLE

The truth value of a proposition is either true (denoted as T) or false (denoted as F). A truth table is a table that shows the truth value of a compound proposition for all possible cases.

PROBLEMS

1. Construct the truth table for $\neg p \wedge q$.
   Solution:

p	q	¬p	¬p ∧ q
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

2. Construct the truth table for $(p \vee q) \vee \neg p$.

Solution:

p	q	p ∨q	¬p	(p∨ q) ∨¬p
T	T	T	F	T
T	F	T	F	T
F	T	T	T	T
F	F	F	T	T

3. Construct the truth table of the compound proposition (p ∨ ¬q) → (p ∧ q)
   Solution:

p	q	¬q	p ∨ ¬q	p ∧ q	(p ∨ ¬q) → (p ∧ q)
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

4. Construct a truth table for (p ↔ q) ↔ (r ↔ s).

   Solution:

p	q	r	s	$p \leftrightarrow q$	$r \leftrightarrow s$	$(p \leftrightarrow q) \leftrightarrow (r \leftrightarrow s)$
T	T	T	T	T	T	T
T	T	T	F	T	F	F
T	T	F	T	T	F	F
T	T	F	F	T	T	T
T	F	T	T	F	T	F
T	F	T	F	F	F	T
T	F	F	T	F	F	T
T	F	F	F	F	T	F
F	T	T	T	F	T	F
F	T	T	F	F	F	T
F	T	F	T	F	F	T
F	T	F	F	F	T	F
F	F	T	T	T	T	T
F	F	T	F	T	F	F
F	F	F	T	T	F	F
F	F	F	F	T	T	T

Practice Problems:

1. Construct the truth table for the following formulas

(i) ¬ (¬p ∨ ¬q)   (iii) p → (¬q ∨ r)

(ii) ¬ (¬p ∧ ¬q)

Duality Law

Two formulas A and A* are said to be duals of each other if either one can be obtained from the other by replacing ∧ by ∨ and ∨ by ∧. The connectives ∨ and ∧ are called duals of each other. If the formula A contains the special variable T or F , then A* , its dual is obtained by replacing T by F and F by T in addition to the above mentioned interchanges.

Example: Write the dual of the following formulas:

(i). (p ∨ q) ∧ r   (ii). (p ∧ q) ∨ T   (iii). (p ∧ q) ∨ (p ∨ ¬(q ∧ ¬ s))

Solution: The duals of the formulas may be written as

(i). (p ∧ q) ∨ r

(ii). (p ∨ q) ∧ F

(iii). (p ∨ q) ∧ (p ∧ ¬(q ∨ ¬s))

NOTE :

The negation of the formula is equivalent to its dual in which every variable is replaced by its negation. We can prove ¬A $(p_1, p_2, \ldots, p_n) \Leftrightarrow$ A $(\neg p_1, \neg p_2, \ldots, \neg p_n)$

CONVERSE, CONTRAPOSITIVE, AND INVERSE

If p → q is a conditional statement, then

(1). q → p is called its converse

(2). ¬p → ¬q is called its inverse

(3). ¬q → ¬p is called its contrapositive.

EXAMPLE

What are the contrapositive, the converse, and the inverse of the conditional statement "The home team wins whenever it is raining?"

Solution:

Because "q whenever p" is one of the ways to express the conditional statement p → q, the original statement can be rewritten as "If it is raining, then the home team wins."

Consequently, the contrapositive of this conditional statement is "If the home team does not win, then it is not raining."

The converse is "If the home team wins, then it is raining."

The inverse is "If it is not raining, then the home team does not win.

LOGIC AND BIT OPERATIONS

Computers represent information using bits. A bit has two possible values, namely 0 and 1. We will use a 1 bit to represent true and a 0 bit to represent false.

Truth value	Bit
T	1
F	0

EXAMPLE

Find the bitwise OR and bitwise AND of the bit strings 01 1011 0110 and 11 0001 1101. (Here, bit strings will be split into blocks of four bits to make them easier to read.)

Solution: The bitwise OR and bitwise AND of these strings are obtained by taking the OR and AND, of the corresponding bits, respectively. This gives us

01 1011 0110

11 0001 1101

11 1011 1111 bitwise OR

01 0001 0100 bitwise AND

Propositional Logic Equivalences

TAUTOLOGY

A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a tautology.

EXAMPLE: $p \vee \neg p$

CONTRADICTION

A compound proposition that is always false is called a contradiction.

EXAMPLE: $p \wedge \neg p$

CONTINGENCY

A compound proposition that is neither a tautology nor a contradiction is called a contingency.

LOGICALLY EQUIVALENT

The compound propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ or $p \Leftrightarrow q$ denotes that p and q are logically equivalent.

Method I. Truth Table Method: One method to determine whether any two statement formulas are equivalent is to construct their truth tables.

PROBLEMS

1. Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent by using truth table.

   Solution:

   Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

$\therefore \neg (p \vee q) \equiv \neg p \wedge \neg q$

2. Show that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent by using truth table.

Solution:

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

$\therefore p \rightarrow q \equiv \neg p \vee q$

3. Show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent.

Solution:

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

$\therefore p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.

4. Show that the proposition $(p \vee q) \leftrightarrow (q \vee p)$ is tautology.
Solution:

p	q	$p \vee q$	$q \vee p$	$(p \vee q) \leftrightarrow (q \vee p)$
T	T	T	T	T
T	F	T	T	T
F	T	T	T	T
F	F	F	F	T

The last column entries are T.
$\therefore$ $(p \vee q) \leftrightarrow (q \vee p)$ is tautology.

5. Show the truth table for $((\neg q \wedge p) \wedge q)$.
Solution:

p	q	$\neg q$	$\neg q \wedge p$	$((\neg q \wedge p) \wedge q)$
T	T	F	F	F

T	F	T	T	F
F	T	F	F	F
F	F	T	F	F

The last column entries are F.

∴ $((\neg q \wedge p) \wedge q)$ is contradiction.

Table Logic equivalences

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Table logical equivalences invoving conditionals

$$p \rightarrow q \equiv \neg p \vee q$$
$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$
$$p \vee q \equiv \neg p \rightarrow q$$
$$p \wedge q \equiv \neg(p \rightarrow \neg q)$$
$$\neg(p \rightarrow q) \equiv p \wedge \neg q$$
$$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$$
$$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$$
$$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$$
$$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$$

Table logical equivalences involving biconditionals.

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$
$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$
$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$
$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

Method II. Replacement Process: Consider a formula A : $p \to (q \to r)$. The formula $q \to r$ is a part of the formula A. If we replace $q \to r$ by an equivalent formula $\neg q \vee r$ in A, we get another formula B : $p \to (\neg q \vee r)$. One can easily verify that the formulas A and B are equivalent to each other. This process of obtaining B from A as the replacement process.

PROBLEMS

1. Show that $\neg(p \to q)$ and $p \wedge \neg q$ are logically equivalent.
   Solution:

$\neg(p \to q) \equiv \neg(\neg p \vee q)$	(by logical equivalences invoving condition)
$\equiv \neg(\neg p) \wedge \neg q$	(by De Morgan law)
$\equiv p \wedge \neg q$	(by double negation law)

2. Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences
   Solution:

$\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg(\neg p \wedge q)$	(by De Morgan law)
$\equiv \neg p \wedge [\neg(\neg p) \vee \neg q]$	(by De Morgan law)
$\equiv \neg p \wedge (p \vee \neg q)$	(by double negation law)
$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$	(by distributive law)
$\equiv F \vee (\neg p \wedge \neg q)$	(because $\neg p \wedge p \equiv F$)
$\equiv (\neg p \wedge \neg q) \vee F$	(by the commutative law for disjunction)
$\equiv \neg p \wedge \neg q$	(by the identity law for F)

3. Show that $(p \wedge q) \to (p \vee q)$ is a tautology.
   Solution:

$(p \wedge q) \to (p \vee q) \equiv \neg(p \wedge q) \vee (p \vee q)$	(by logical equivalences invoving condition)
$\equiv (\neg p \vee \neg q) \vee (p \vee q)$	(by De Morgan law)
$\equiv (\neg p \vee p) \vee (\neg q \vee q)$	(by the associative and commutative laws for disjunction)
$\equiv T \vee T$	(by Negation law and the commutative law for disjunction)
$\equiv T$	(by the domination law)

4. Show that $(p \to r) \vee (q \to r)$ and $(p \wedge q) \to r$ are logically equivalent.
   Solution:

$(p \to r) \vee (q \to r) \equiv (\neg p \vee r) \vee (\neg q \vee r)$	(by logical equivalences invoving condition
$\equiv (\neg p \vee \neg q) \vee (r \vee r)$	(by the associative and commutative laws for disjunction)
$\equiv \neg(p \wedge q) \vee r$	(by De Morgan law and idempotent law)
$\equiv (p \wedge q) \to r$	(by logical equivalences invoving condition)

PRACTICE PROBLEMS

1. Show $((p \vee q) \wedge \neg(\neg p \wedge (\neg q \vee \neg r))) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$ is tautology, by using truth table and replacement process.
2. Prove that $(p \to q) \wedge (r \to q) \Leftrightarrow (p \vee r) \to q$, by using truth table and replacement process.

3. Prove that $(\neg p \wedge (\neg q \wedge r)) \vee (q \wedge r) \vee (p \wedge r) \Leftrightarrow r$, by using truth table and replacement process.

4. Prove that conditional and contrapositive are equivalent.

Normal forms

ELEMENTARY PRODUCT

A conjunction of the variables and their negations is called an elementary product.

EXAMPLE:

Let p, q be any two atomic variables. Then $\neg p \wedge q$, $\neg q \wedge p$, $\neg q \wedge p \wedge \neg p$, $\neg p \wedge p$, $\neg p \wedge \neg q$ are some examples of elementary products.

ELEMENTARY SUM

A disjunction of the variables and their negations is called an elementary sum.

EXAMPLE

Let p, q be any two atomic variables. Then $\neg p \vee q$, $\neg q \vee p$, $\neg q \vee p \vee \neg p$, $\neg p \vee p$, $\neg p \vee \neg q$ are some examples of elementary sum.

FACTOR

Any part of an elementary product or elementary sum, which is itself an elementary product or sum is a factor of the product or sum.

EXAMPLE

$q \vee p$ is a factor of $\neg q \vee q \vee p$

DISJUNCTIVE NORMAL FORM (DNF)

A formula which is equivalent to a given formula and which consists of a sum of elementary products is called a disjunctive normal form of the given formula.

EXAMPLE

1. Obtain DNF of $p \wedge (p \rightarrow q)$

   Solution:

   $p \wedge (p \rightarrow q)) \equiv p \wedge (\neg p \vee q)$ (by logical equivalences invoving condition)

   $\equiv (p \wedge \neg p) \vee (p \wedge q)$ (by Distributive)

   Which is the required DNF.

2. Obtain the DNF $p \rightarrow [(p \rightarrow q) \wedge \neg(\neg q \vee \neg p)]$

   Solution:

   $\equiv \neg p \vee [(\neg p \vee q) \wedge \neg(\neg q \vee \neg p)]$ (by logical equivalences invoving condition)

   $\equiv \neg p \vee [(\neg p \vee q) \wedge (q \wedge p)]$ (by double negation law)

   $\equiv \neg p \vee [(\neg p \wedge q \wedge p) \vee (q \wedge q \wedge p)]$ (distributive law)

   $\equiv \neg p \vee [(F \wedge q) \vee (q \wedge p)]$ (negation law, idempotent law)

   $\equiv \neg p \vee [(F \vee (q \wedge p)]$ (domination law)

$\equiv \neg p \vee (q \wedge p)$ (identity law)

Which is the required DNF.

3. Obtain DNF of $\neg (p \rightarrow (q \wedge r))$.

Solution:

$\neg (p \rightarrow (q \wedge r))$

$\equiv \neg[\neg p \vee (q \wedge r)]$ (by logical equivalences invoving condition)

$\equiv p \wedge (\neg q \vee \neg r)$ (De Morgan law)

$\equiv (p \wedge \neg q) \vee (p \wedge \neg r)$ (Distributive law)

Conjunctive Normal Forms(CNF)

A formula which is equivalent to a given formula and which consists of a product of elementary sums is called a conjunctive normal form of a given formula.

Example

1. Obtain the conjunctive normal form for the following

(a) $p \wedge (p \rightarrow q)$

(b) $\neg (p \vee q)$

(c) $\neg (p \leftrightarrow q)$

Solution:

(a) $p \wedge (p\rightarrow q) \Leftrightarrow p \wedge (\neg p \vee q)$. (by logical equivalences invoving condition)

(b) $\neg (p \vee q) \Leftrightarrow \neg p \wedge \neg q$. (De Morgan's law)

(c) $\neg (p \leftrightarrow q) \Leftrightarrow \neg ((p\rightarrow q) \wedge (q\rightarrow p))$ (by logical equivalences invoving Bi-condition)

$\Leftrightarrow \neg ((\neg p \vee q) \wedge (\neg q \vee p))$ (by logical equivalences invoving condition)

$\Leftrightarrow \neg(\neg p \vee q) \vee \neg (\neg q \vee p)$ (De Morgan's law)

$\Leftrightarrow (p \wedge \neg q) \vee (q \wedge \neg p)$ (De Morgan's law)

$\Leftrightarrow (p \vee q) \wedge (q \vee \neg q) \wedge (p \vee \neg p) \wedge (\neg p \vee \neg q)$ (Distributive law)

Principal Disjunctive Normal Form(PDNF)

Minterm

For a given number of variables, the minterm consists of conjunctions in which each statement variable or its negation, but not both, appears only once.

Let p and q be the two statement variables. Then there are 2^2 minterms given by $p \wedge q$, $p \wedge \neg q$, $\neg p \wedge q$, and $\neg p \wedge \neg q$.

Minterms for three variables p , q and r are $p \wedge q \wedge r$, $p \wedge q \wedge \neg r$, $p \wedge \neg q \wedge r$, $p \wedge \neg q \wedge \neg r$, $\neg p \wedge q \wedge r$, $\neg p \wedge q \wedge \neg r$, $\neg p \wedge \neg q \wedge r$ and $\neg p \wedge \neg q \wedge \neg r$.

Definition

For a given formula, an equivalent formula consisting of disjunctions of minterms only is called the Principal disjunctive normal form of the formula. The principle disjunctive normal formula is also called the sum-of-products canonical form.

Methods to obtain PDNF of a given formula

(a). Without constructing the truth table:

In order to obtain the principal disjunctive normal form of a given formula is constructed as follows:

(1). First replace $\rightarrow$, by their equivalent formula containing only $\wedge$, $\vee$ and $\neg$.

(2). Next, negations are applied to the variables by De Morgan's laws followed by the application of distributive laws.

(3). Any elementarily product which is a contradiction is dropped. Minterms are ob-tained in the disjunctions by introducing the missing factors. Identical minterms appearing in the disjunctions are deleted.

Example

1. Obtain the principal disjunctive normal form of
   $(p \wedge q) \vee (\neg p \wedge r) \vee (q \wedge r)$.

Solution:

(i) $$\begin{aligned}(p \wedge q) \vee (\neg p \wedge r) \vee (q \wedge r) &\Leftrightarrow (p \wedge q \wedge T) \vee (\neg p \wedge r \wedge T) \vee (q \wedge r \wedge T)\\ &\Leftrightarrow (p \wedge q \wedge (r \vee \neg r)) \vee (\neg p \wedge r \wedge (q \vee \neg q))\\ &\qquad \vee (q \wedge r \wedge (p \vee \neg p))\\ &\Leftrightarrow (p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (\neg p \wedge r \wedge q) \vee\\ &\qquad (\neg p \wedge r \wedge \neg q) \vee (q \wedge r \wedge p) \vee (q \wedge r \wedge \neg p)\\ &\Leftrightarrow (p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r)\\ &\qquad \vee (\neg p \wedge \neg q \wedge r)\end{aligned}$$

which is the required PDNF.

2. Prove the following equivalences by showing that both are having same PDNF.
   (i) $p \vee (p \wedge q) \Leftrightarrow p$
   (ii) $p \vee (\neg p \wedge q) \Leftrightarrow p \vee q$

Solution:

(i) LHS $$\begin{aligned}p \vee (p \wedge q) &\Leftrightarrow (p \wedge T) \vee (p \wedge q) && [\because P \wedge Q \Leftrightarrow P]\\ &\Leftrightarrow (p \wedge (q \vee \neg q)) \vee (p \wedge q) && [\because P \vee \neg P \Leftrightarrow T]\\ &\Leftrightarrow ((p \wedge q) \vee (p \wedge \neg q)) \vee (p \wedge q) && [\text{by distributive laws}]\\ &\Leftrightarrow (p \wedge q) \vee (p \wedge \neg q) && [\because P \vee P \Leftrightarrow P]\end{aligned}$$

which is its PDNF.

Now, RHS $p \Leftrightarrow p \wedge T$

$\Leftrightarrow p \wedge (q \vee \neg q)$

$\Leftrightarrow (p \wedge q) \vee (q \wedge \neg q)$

which is its PDNF.

Clearly both are having same PDNF.

Hence, $p \vee (p \wedge q) \Leftrightarrow p$

(ii) LHS $p \vee (\neg p \wedge q) \Leftrightarrow (p \wedge T) \vee (\neg p \wedge q)$

$\Leftrightarrow (p \wedge (q \vee \neg q)) \vee (\neg p \wedge q)$

$\Leftrightarrow (p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge q)$

which is its PDNF.

Now RHS, $p \vee q \Leftrightarrow (p \wedge T) \vee (q \wedge T)$

$\Leftrightarrow (p \wedge (q \vee \neg q)) \vee (q \wedge (p \vee \neg p))$

$\Leftrightarrow (p \wedge q) \vee (p \wedge \neg q) \vee (q \wedge p) \vee (q \wedge \neg p)$

$\Leftrightarrow (p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge q)$

which is its PDNF.

Clearly, both are having same PDNF

Hence, $p \vee (\neg p \wedge q) \Leftrightarrow p \vee q$.

(b). By Truth table:

(i). Construct a truth table of the given formula.

(ii). For every truth value T in the truth table of the given formula, select the minterm which also has the value T for the same combination of the truth values of p and q.

(iii). The disjunction of these minterms will then be equivalent to the given formula.

Example:

1. Obtain the PDNF of $p \rightarrow q$.

Solution:

From the truth table of $p \rightarrow q$

p	q	$p \rightarrow q$	Minterm
T	T	T	$p \wedge q$
T	F	F	$p \wedge \neg q$
F	T	T	$\neg p \wedge q$
F	F	T	$\neg p \wedge \neg q$

The PDNF of $p \rightarrow q$ is $(p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q)$.

$\therefore p \rightarrow q \Leftrightarrow (p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q)$.

2. Obtain the PDNF for $(p \wedge q) \vee (\neg p \wedge r) \vee (q \wedge r)$.

Solution:

p	Q	r	Minterm	$p \wedge q$	$\neg p \wedge r$	$q \wedge r$	$(p \wedge q) \vee (\neg p \wedge r) \vee (q \wedge r)$
T	T	T	$p \wedge q \wedge r$	T	F	T	T
T	T	F	$p \wedge q \wedge \neg r$	T	F	F	T
T	F	T	$p \wedge \neg q \wedge r$	F	F	F	F
T	F	F	$p \wedge \neg q \wedge \neg r$	F	F	F	F
F	T	T	$\neg p \wedge q \wedge r$	F	T	T	T
F	T	F	$\neg p \wedge q \wedge \neg r$	F	F	F	F
F	F	T	$\neg p \wedge \neg q \wedge r$	F	T	F	T
F	F	F	$\neg p \wedge \neg q \wedge \neg r$	F	F	F	F

The PDNF of $(p \wedge q) \vee (\neg p \wedge r) \vee (q \wedge r)$is

$$(p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge r).$$

Principal Conjunctive Normal Form (PCNF)

The dual of a minterm is called a Maxterm. For a given number of variables, the maxterm consists of disjunctions in which each variable or its negation, but not both, appears only once. Each of the maxterm has the truth value F for exactly one combination of the truth values of the variables. Now we define the principal conjunctive normal form.

For a given formula, an equivalent formula consisting of conjunctions of the max-terms only is known as its principle conjunctive normal form. This normal form is also called the product-of-sums canonical form. The method for obtaining the PCNF for a given formula is similar to the one described previously for PDNF.

Example:

Obtain the principal conjunctive normal form of the formula $(\neg p \to r) \wedge (q \leftrightarrow p)$

Solution:

$(\neg p \to r) \wedge (q \leftrightarrow p)$

$\Leftrightarrow [\neg(\neg p) \vee r] \wedge [(q \to p) \wedge (p \to q)]$

$\Leftrightarrow (p \vee r) \wedge [(\neg q \vee p) \wedge (\neg p \vee q)]$

$\Leftrightarrow (p \vee r \vee F) \wedge [(\neg q \vee p \vee F) \wedge (\neg p \vee q \vee F)]$

$\Leftrightarrow [(p \vee r) \vee (q \wedge \neg q)] \wedge [\neg q \vee p) \vee (r \wedge \neg r)] \wedge [(\neg p \vee q) \vee (r \wedge \neg r)]$

$\Leftrightarrow (p \vee r \vee q) \wedge (p \vee r \vee \neg q) \wedge (p \vee \neg q \vee r) \wedge (p \vee \neg q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (\neg p \vee q \vee \neg r)$

$\Leftrightarrow (p \vee q \vee r) \wedge (p \vee \neg q \vee r) \wedge (p \vee \neg q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (\neg p \vee q \vee \neg r)$

which is required principal conjunctive normal form

Note:

1. A PDNF is sum of min-terms.

2. A minterm alone is also a PDNF.

3. If the given formula is a tautology, then only its PDNF exists (PCNF doesn't exist).

4. A PCNF is product of max-terms.

5. A max-term alone is also a PCNF.

6. If the given formula is a contradiction, then only its PCNF exists (PDNF doesn't exist).

PRACTICE PROBLEM

1. Find the PDNF form PCNF of S : $p \lor (\neg p \to (q \lor (\neg q \to r)))$.
2. Obtain CNF for $((p \to q) \land \neg q) \to \neg p$.
3. Obtain DNF for $(p \land r) \leftrightarrow (\sim q \lor r)$

ANSWER:

1. $p \lor q \lor r$ which is the PCNF.

PDNF of S: $(\neg p \land \neg q \land r) \lor (\neg p \land q \land \neg r) \lor (\neg p \land q \land r) \lor (p \land \neg q \land \neg r) \lor (p \land \neg q \land r) \lor (p \land q \land \neg r) \lor (p \land q \land r)$.

PCNF of S: $(p \lor q \lor \neg r) \land (p \lor \neg q \lor r) \land (p \lor \neg q \lor \neg r) \land (\neg p \lor q \lor r) \land (\neg p \lor q \lor \neg r) \land (\neg p \lor \neg q \lor r) \land (\neg p \lor \neg q \lor \neg r)$

2. $(p \lor \neg q) \land (\neg q \lor p)$

3. $(p \land q \land r) \lor (p \land q \land \neg r) \lor (p \land \neg q \land r) \lor (\neg p \land q \land \neg r)$

Definition: A statement formula 'A' is said to be logically implies another formula 'B' whenever $A \to B$ is a tautology.

We denote A logically implies B as $A \Rightarrow B$.

Example:

$(P \to Q) \land (Q \to R)$ logically implies $(P \to R)$.

Rules of Inference

ARGUMENT

A set of propositions $p_1, p_2, \ldots, p_n$ and q is said to be an argument if $(p_1 \land p_2 \land \ldots \land p_n) \Rightarrow q$. In other words $(p_1 \land p_2 \land \ldots \land p_n) \to q$ should be a tautology. Here $p_1, p_2, \ldots, p_n$ are called Premises of the argument and q is called a conclusion of the argument.

$$\begin{array}{l} p_1 \\ p_2 \\ . \\ . \\ . \\ p_n \\ \hline \therefore q \end{array}$$

An argument with premises $p_1, p_2, \ldots, p_n$ and conclusion q is said to valid if whenever each of premises $p_1, p_2, \ldots, p_n$ is true, then conclusion q is also true. Otherwise the argument is invalid.

RULES OF INFERENCE

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \land (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \land (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \land (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore q$	$((p \lor q) \land \neg p) \to q$	Disjunctive syllogism
p $\therefore p \lor q$	$p \to (p \lor q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \to p$	Simplification
p q $\therefore p \land q$	$((p) \land (q)) \to (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$	Resolution

PROBLEMS

1.State which rule of inference is the basis of the following argument: “It is below freezing now. Therefore, it is either below freezing or raining now.”
Solution:
Let p be the proposition “It is below freezing now” and q the proposition “It is raining now.” Then this argument is of the form

$$\begin{array}{l} \quad p \\ \hline \therefore p \lor q \end{array}$$

This is an argument that uses the addition rule.

3. State which rule of inference is the basis of the following argument: “It is below freezing and raining now. Therefore, it is below freezing now.”
Solution:
Let p be the proposition “It is below freezing now,” and let q be the proposition “It is raining now.” This argument is of the form

$$\begin{array}{l} \quad p \land q \\ \hline \therefore p \end{array}$$

This argument uses the simplification rule.

4. State which rule of inference is used in the argument: If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have a barbecue tomorrow.
   Solution:
   Let p be the proposition “It is raining today,” let q be the proposition “We will not have a barbecue today,” and let r be the proposition “We will have a barbecue tomorrow.” Then this argument is of the form

$$\begin{array}{rl} & p \to q \\ & q \to r \\ \hline \therefore & p \to r \end{array}$$

   Hence, this argument is a hypothetical syllogism.

5. Show that the premises “It is not sunny this afternoon and it is colder than yesterday,” “We will go swimming only if it is sunny,” “If we do not go swimming, then we will take a canoe trip,” and “If we take a canoe trip, then we will be home by sunset” lead to the conclusion “We will be home by sunset.”
   Solution:
   Let p : “It is sunny this afternoon,”
   q : “It is colder than yesterday,”
   r : “We will go swimming,”

   s: “We will take a canoe trip,” and

   t : “We will be home by sunset.”
   Then the premises become $\neg p \wedge q$, $r \to p$, $\neg r \to s$, and $s \to t$. The conclusion is simply t.
   We construct an argument to show that our premises lead to the desired conclusion as follows.

Step	**Reason**
1. $\neg p \wedge q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \to p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \to s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \to t$	Premise
8. t	Modus ponens using (6) and (7)

   Hence the result.

6. Show that the premises “If you send me an e-mail message, then I will finish writing the program,” “If you do not send me an e-mail message, then I will go to sleep early,” and “If I go to sleep early, then I will wake up feeling refreshed” lead to the conclusion “If I do not finish writing the program, then I will wake up feeling refreshed.”
   Solution:
   Let p : “You send me an e-mail message,”
   q : “I will finish writing the program,”
   r : “I will go to sleep early,” and
   s : “I will wake up feeling refreshed.”
   Then the premises are $p \to q$, $\neg p \to r$, and $r \to s$. The desired conclusion is $\neg q \to s$. This argument form shows that the premises lead to the desired conclusion.

Step	Reason
1. $p \to q$	Premise
2. $\neg q \to \neg p$	Contrapositive of (1)
3. $\neg p \to r$	Premise
4. $\neg q \to r$	Hypothetical syllogism using (2) and (3)
5. $r \to s$	Premise
6. $\neg q \to s$	Hypothetical syllogism using (4) and (5)

Hence the result.

7. Show that r ∨ s follows logically from the premises c ∨d, (c ∨d) → ¬h, ¬h → (a ∧ ¬b), and (a ∧ ¬b) → (r ∨ s).
Solution:

Step	Reason
(1) (c ∨ d) → ¬h	Premise
(2) ¬h → (a ∧ ¬b)	Premise
(3) (c ∨ d) → (a ∧ ¬b)	Hypothetical syllogism using (1) and (2)
(4) (a ∧ ¬b) → (r ∨ s)	Premise
(5) (c ∨ d) → (r ∨ s)	Hypothetical syllogism using (3) and (4)
(6) c ∨ d	Premise
(7) r ∨ s	Modus ponens using (5) and (6)

Hence the result.

8. Show that s ∨ r is tautologically implied by (p ∨ q) ∧ (p → r) ∧ (q → s).
Solution:

Step	Reason
(1) p ∨ q	Premise
(2) ¬p → q	p → q ⇔ ¬ p ∨ q (1)
(3) q → s	Premise
(4) ¬p → s	Hypothetical syllogism using (2) and (3)
(5) ¬s → p	p → q ⇔ ¬ q → ¬ p (4)
(6) p → r	Premise
(7) ¬s → r	Hypothetical syllogism using (5) and (6)
(8) s ∨ r	p → q ⇔ ¬ p ∨ q (7)

Hence the result.

Consistency of Premises

A set of formulas $H_1, H_2, ..., H_m$ is said to be consistent if their conjunction has the truth value T for some assignment of the truth values to the atomic variables appearing in $H_1, H_2, ..., H_m$. If, for every assignment of the truth values to the atomic variables, at least one of the formulas $H_1, H_2, ..., H_m$ is false, so that their conjunction is identically false, then the formulas $H_1, H_2, ..., H_m$ are called inconsistent.

Alternatively, a set of formulas $H_1, H_2, ..., H_m$ is inconsistent if their conjunction implies a contradiction, that is,

$$H_1, H_2, ..., H_m \Rightarrow r \wedge \neg r$$

where r is any formula.

PROBLEMS

1. Show that the following premises are inconsistent:

(1). If Jack misses many classes through illness, then he fails high school.
(2). If Jack fails high school, then he is uneducated.
(3). If Jack reads a lot of books, then he is not uneducated.
(4). Jack misses many classes through illness and reads a lot of books.

Solution: Let us indicate the statements as follows:
e: Jack misses many classes through illness.
s: Jack fails high school.
a: Jack reads a lot of books.
h: Jack is uneducated.
The premises are $e \to s$, $s \to h$, $a \to \neg h$, and $e \land a$.

Step	**Reason**
(1) $e \to s$	Premise
(2) $s \to h$	Premise
(3) $e \to h$	Hypothetical syllogism using (1) and (2)
(4) $a \to \neg h$	Premise
(5) $h \to \neg a$	$p \to q \Leftrightarrow \neg q \to \neg p$ (4)
(6) $e \to \neg a$	Hypothetical syllogism using (3) and (5)
(7) $\neg e \lor \neg a$	$p \to q \Leftrightarrow \neg p \lor q$ (6)
(8) $\neg(e \land a)$	$\neg(p \land q) \Leftrightarrow \neg p \lor \neg q$ (7)
(9) $e \land a$	Premise
(10) $\neg(e \land a) \land (e \land a)$	Conjunction (8), (9)

Thus, the given set of premises leads to a contradiction and hence it is inconsistent.

2. Show that the following set of premises is inconsistent: "If the contract is valid, then John is liable for penalty. If John is liable for penalty, he will go bankrupt. If the bank will loan him money, he will not go bankrupt. As a matter of fact, the contract is valid, and the bank will loan him money."

Solution: Let us indicate the statements as follows:
v : The contract is valid.
l: John is liable for penalty.
m: Bank will loan him money.
b: John will go bankrupt.

Step	**Reason**
(1) $v \to l$	Premise
(2) $l \to b$	Premise
(3) $v \to b$	Hypothetical syllogism using (1) and (2)
(4) $m \to \neg b$	Premise
(5) $b \to \neg m$	$p \to q \Leftrightarrow \neg q \to \neg p$ (4)
(6) $v \to \neg m$	Hypothetical syllogism using (3) and (5)
(7) $\neg v \lor \neg m$	$p \to q \Leftrightarrow \neg p \lor q$ (6)
(8) $\neg(v \land m)$	$\neg(p \land q) \Leftrightarrow \neg p \lor \neg q$ (7)
(9) $v \land m$	Premise

(10) ¬(v ∧ m) ∧ (v ∧ m) Conjunction (8), (9)

Thus, the given set of premises leads to a contradiction and hence it is inconsistent.

Indirect Method of Proof

The method of using the rule of conditional proof and the notion of an inconsistent set of premises is called the indirect method of proof or proof by contradiction.

In order to show that a conclusion C follows logically from the premises $H_1, H_2, ..., H_m$. Consider ¬C as an additional premise. If the new set of premises is inconsistent, so that they imply a contradiction. Therefore, $H_1, H_2, ..., H_m$ conclude C.

Hence, C is true whenever $H_1, H_2, ..., H_m$ are true. Thus, C follows logically from the premises . $H_1, H_2, ..., H_m$.

PROBLEMS

1.Show that ¬(p ∧ q) follows from ¬p ∧ ¬q.

Solution: We introduce ¬ ¬(p ∧q) as additional premise and show that this additional premise leads to a contradiction.

Step	Reason
(1) ¬ ¬ (p ∧ Q)	Premise (assumed)
(2) p ∧ q	¬ ¬P ⇔ P (1)
(3) p	Simplification, (2),
(4) ¬p ∧ ¬q	Premise
(5) ¬p	Simplification, (4)
(6) p ∧ ¬p	Conjunction, (3), (5)

Hence, our assumption is wrong.
Thus, ¬(p ∧ q) follows from ¬p ∧ ¬q.

2.Using the indirect method of proof, show that p → q, q → r, p ∨ r ⇒ r.

Solution: We include ¬R as an additional premise. Then we show that this leads to a contradiction.

Step	Reason
(1) p → q	Premise
(2) q → r	Premise
(3) p → r	Hypothetical syllogism using (1), (2)
(4) ¬r	Premise (assumed)
(5) ¬p	Modus Tollens, (4)
(6) p ∨ r	Premise

(7) r Disjunctive syllogism, (5), (6)

(8) $r \land \neg r$ Conjunction, (4), (7)

Hence, our assumption is wrong

PRACTICE PROBLEMS

1.Show that $r \land (p \lor q)$ is a valid conclusion from the premises $p \lor q$, $q \to r$, $p \to m$, and $\neg m$.

2.Show that $p \to s$ can be derived from the premises $\neg p \lor q$, $\neg q \lor r$, and $r \to s$.

3."If there was a ball game, then traveling was difficult. If they arrived on time, then traveling was not difficult. They arrived on time. Therefore, there was no ball game'. Show that these statements constitute a valid argument.

[Hint: $p \to q$, $r \to \neg q$, and r. The conclusion is $\neg p$]

4.By using the method of derivation, show that following statements constitute a valid argument: "If A works hard, then either B or C will enjoy. If B enjoys, then A will not work hard. If D enjoys, then C will not. Therefore, if A works hard, D will not enjoy.

[Hint: $a \to (b \lor c)$, $b \to \neg a$, and $d \to \neg c$. The conclusion is $a \to \neg d$].