Module 1. Laplace Transforms

INTRODUCTION: The knowledge of Laplace transforms has in recent years become an essential part of mathematical background required for engineers and scientists. This is because the transform methods provide an easy and effective means for the solution of many problems arising in engineering.

The method of Laplace transforms has the advantage of directly giving the solution of differential equations with given boundary values without the necessity of first finding the general solution and then evaluating from it the arbitrary constants. Moreover, the ready tables of Laplace transforms reduce the problem of solving differential equations to mere algebraic manipulations.

Definition: Let $f(t)$ be a function of t (defined for all positive values of t). The Laplace transform of $f(t)$, denoted by $L[f(t)]$, is defined by the equation

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

provided the integral converges. In the definition, s is a parameter which may be a real or complex number. $L[f(t)]$ is denoted by $\bar{f}(s)$ or $F(s)$. The symbol L is called the Laplace transform operator.

Remarks

1. The Laplace transform (an integral transform) converts a function $f(t)$ into a function $\bar{f}(s)$.

2. The Laplace transform technique is very useful in solving linear differential equations with initial conditions. It is a powerful tool for solving electrical circuit and systems problems.

Linearity Property

If a, b and c are constants and f, g and h are functions of t, then

$$L[a\,f(t)+b\,g(t)-c\,h(t)] = a\,L[f(t)]+b\,L[g(t)]-c\,L[h(t)]$$

1.1 Laplace Transform of Elementary Functions

1. $L[1]=\dfrac{1}{s}$ & $L[k]=\dfrac{k}{s}$ **(where k is any constant)**

Examples: $L[2]=\dfrac{2}{s}$ $\qquad L[-5]=\dfrac{-5}{s}$

2. $L[e^{at}]=\dfrac{1}{s-a}$ & $L[e^{-at}]=\dfrac{1}{s+a}$

Examples: $L[e^{2t}]=\dfrac{1}{s-2}$ $\qquad L[e^{-3t}]=\dfrac{1}{s+3}$

3. $L[\sin at]=\dfrac{a}{s^2+a^2}$ & $L[\cos at]=\dfrac{s}{s^2+a^2}$

Examples: $L[\sin 2t]=\dfrac{2}{s^2+2^2}=\dfrac{2}{s^2+4}$ $\qquad L[\cos t]=\dfrac{s}{s^2+1^2}=\dfrac{s}{s^2+1}$

4. $L[\sinh at]=\dfrac{a}{s^2-a^2}$ & $L[\cosh at]=\dfrac{s}{s^2-a^2}$

Examples: $L[\sinh t]=\dfrac{1}{s^2-1^2}=\dfrac{1}{s^2-1}$ $\qquad L[\cosh 2t]=\dfrac{s}{s^2-2^2}=\dfrac{s}{s^2-4}$

5. $L[t^n]=\dfrac{n!}{s^{n+1}}$, **if n is a positive integer**

Examples: $L[t]=\dfrac{1!}{s^{1+1}}=\dfrac{1}{s^2}$ $\qquad L[t^3]=\dfrac{3!}{s^{3+1}}=\dfrac{6}{s^4}$

6. $L[t^n]=\dfrac{\Gamma(n+1)}{s^{n+1}}$, **if n is real number**

Examples: $L\left[t^{-\frac{1}{2}}\right]=\dfrac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-\frac{1}{2}+1}}=\dfrac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}}=\dfrac{\sqrt{\pi}}{s^{\frac{1}{2}}}$

$$L\left[t^{1/2}\right]=\frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}}=\frac{\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}}=\frac{\frac{1}{2}\sqrt{\pi}}{s^{\frac{3}{2}}}=\frac{\sqrt{\pi}/2}{s^{\frac{3}{2}}}$$

$$L\left[t^{3/2}\right]=\frac{\Gamma\left(\frac{3}{2}+1\right)}{s^{\frac{3}{2}+1}}=\frac{\Gamma\left(\frac{5}{2}\right)}{s^{\frac{5}{2}}}=\frac{\frac{3}{2}\frac{1}{2}\sqrt{\pi}}{s^{\frac{5}{2}}}=\frac{3\sqrt{\pi}/4}{s^{\frac{5}{2}}}$$

SOLVED PROBLEMS

1. Find the Laplace transform of $e^{2t} + 4t^3 - 2sin3t + 3cos3t$

Solution: Let $f(t) = e^{2t} + 4t^3 - 2sin3t + 3cos3t$

Take the Laplace transform of both sides

$$L[f(t)] = L[e^{2t} + 4t^3 - 2sin3t + 3cos3t]$$

$$= L[e^{2t}] + 4L[t^3] - 2L[sin3t] + 3L[cos3t] \qquad \text{(by linearity)}$$

$$We\ have\ L[e^{at}] = \frac{1}{s-a}, L[t^n] = \frac{n!}{s^{n+1}}, L[\sin at] = \frac{a}{s^2+a^2}\ \&\ L[\cos at] = \frac{s}{s^2+a^2}$$

$$\therefore L[f(t)] = \frac{1}{s-2} + 4\left(\frac{3!}{s^4}\right) - 2\left(\frac{3}{s^2+9}\right) + 3\left(\frac{s}{s^2+9}\right)$$

$$= \frac{1}{s-2} + \frac{24}{s^4} - \frac{6}{s^2+9} + \frac{3s}{s^2+9}$$

Thus $\boldsymbol{L[f(t)] = \frac{1}{s-2} + \frac{24}{s^4} + \frac{3s-6}{s^2+9}}$.

2. Find the Laplace transform of $3\ cosh5t - 4\ sinh5t$

Solution: Let $f(t) = 3\ cosh5t - 4\ sinh5t$

Take the Laplace transform of both sides

$$L[f(t)] = L[3\ cosh5t - 4\ sinh5t]$$

$$= 3L[cosh5t] - 4L[sinh5t] \qquad \text{(by linearity)}$$

$$We\ have\ \ L[\cosh at] = \frac{s}{s^2-a^2} \quad \& \quad L[\sinh at] = \frac{a}{s^2-a^2}$$

$$\therefore L[f(t)] = 3\left(\frac{s}{s^2-5^2}\right) - 4\left(\frac{5}{s^2-5^2}\right)$$

$$= \frac{3s}{s^2-25} - \frac{20}{s^2-25}$$

Thus $\boldsymbol{L[f(t)] = \frac{3s-20}{s^2-25}}$.

3. Find the Laplace transform of $\sin 2t\ sin3t$

Solution: Let $f(t) = \sin 2t\ sin3t$

$$\text{We know that } \sin A\ sinB = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$\therefore f(t) = sin2t\ sin3t = \frac{1}{2}[\cos(-t) - cos5t]$$

$$f(t) = \frac{1}{2}[\cos t - \cos 5t] \qquad [\cos(-\theta) = \cos\theta]$$

Take the Laplace transform of both sides

$$L[f(t)] = L\left[\frac{1}{2}[\cos t - \cos 5t]\right]$$

$$L[f(t)] = \frac{1}{2}[L(\cos t) - L(\cos 5t)] \qquad \text{(by linearity)}$$

$$We\ have\ \ L[\cos at] = \frac{s}{s^2+a^2}$$

$$\text{Thus } \boldsymbol{L[f(t)] = \frac{1}{2}\left[\frac{s}{s^2+1} - \frac{s}{s^2+25}\right]}.$$

4. Find the Laplace transform of $cos^2 2t$

Solution: Let $f(t) = cos^2 2t$

$$\text{We know that } cos^2\theta = \frac{1}{2}(1 + cos2\theta)$$

$$\therefore f(t) = \frac{1}{2}(1 + \cos 4t)$$

Take the Laplace transform of both sides

$$L[f(t)] = L\left[\frac{1}{2}(1 + \cos 4t)\right]$$

$$L[f(t)] = \frac{1}{2}[L(1) + L(\cos 4t)] \qquad \text{(by linearity)}$$

$$We\ have\ \ L[1] = \frac{1}{s} \quad \& \quad L[\cos at] = \frac{s}{s^2+a^2}$$

$$\text{Thus } \boldsymbol{L[f(t)] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+16}\right]}.$$

5. Find the Laplace transform of $(sint - cost)^2$

Solution: Let $f(t) = (sint - cost)^2$

$$= \sin^2 t + \cos^2 t - 2\sin t \cos t$$

We know that $\sin^2\theta + \cos^2\theta = 1$ and $2\sin\theta \, cos\,\theta = \sin 2\theta$

$$\therefore f(t) = 1 - \sin 2t$$

Take the Laplace transform of both sides

$$L[f(t)] = L[1 - \sin 2t]$$

$$L[f(t)] = L(1) - L(\sin 2t) \qquad \text{(by linearity)}$$

$We\ have\ L[1] = \frac{1}{s}$ & $L[\sin at] = \frac{a}{s^2 + a^2}$

Thus $\boldsymbol{L[f(t)] = \frac{1}{s} - \frac{2}{s^2 + 4}}$.

6. Find the Laplace transform of $\cos(at + b)$

Solution: Let $f(t) = \cos(at + b)$

We know that $\cos(A + B) = cosA\, cosB - sinA \sin B$

$$\therefore f(t) = cosat\, cosb - sinat \sin b$$

Take the Laplace transform of both sides

$$L[f(t)] = L[cosat\, cosb - sinat \sin b]$$

$$L[f(t)] = \cos b\, L(\cos at) - \sin b\, L(\sin at) \qquad \text{(by linearity)}$$

$We\ have\ L[\cos at] = \frac{s}{s^2 + a^2}$ & $L[\sin at] = \frac{a}{s^2 + a^2}$

$$\therefore L[f(t)] = cosb\left(\frac{s}{s^2 + a^2}\right) - sinb\left(\frac{a}{s^2 + a^2}\right)$$

$$= \frac{s\cos b}{s^2 + a^2} - \frac{a\sin b}{s^2 + a^2}$$

Thus $\boldsymbol{L[f(t)] = \frac{s\, cosb - a\, sinb}{s^2 + a^2}}$.

7. Find the Laplace transform of $1 + 2\sqrt{t} + \frac{3}{\sqrt{t}}$

Solution: Let $f(t) = 1 + 2\sqrt{t} + \frac{3}{\sqrt{t}}$

$$f(t) = 1 + 2\,t^{\frac{1}{2}} + 3\,t^{-\frac{1}{2}}$$

Take the Laplace transform of both sides

$$L[f(t)] = L\left[1 + 2\,t^{\frac{1}{2}} + 3\,t^{-\frac{1}{2}}\right]$$

$$= L[1] + 2L\left[t^{\frac{1}{2}}\right] + 3L\left[t^{-\frac{1}{2}}\right] \qquad \text{(by linearity)}$$

$$We\ have\ \ L[1] = \frac{1}{s} \quad \& \quad L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$$

$$\therefore L[f(t)] = \frac{1}{s} + 2\frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}} + 3\frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-\frac{1}{2}+1}}$$

$$= \frac{1}{s} + 2\frac{\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}} + 3\frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}}$$

$$= \frac{1}{s} + 2\frac{\frac{1}{2}\sqrt{\pi}}{s^{\frac{3}{2}}} + 3\frac{\sqrt{\pi}}{s^{\frac{1}{2}}}$$

Thus $\boldsymbol{L[f(t)] = \frac{1}{s} + \frac{\sqrt{\pi}}{s^{\frac{3}{2}}} + \frac{3\sqrt{\pi}}{s^{\frac{1}{2}}}}$.

EXERCISE PROBLEMS

Find the Laplace transform of the following functions:

1. $\sin(a+bt)$ **Answer:** $\dfrac{s\sin a + b\cos a}{s^2+b^2}$

2. $\sin 3t \cos 2t$ **Answer:** $\dfrac{1}{2}\left\{\dfrac{5}{s^2+25}+\dfrac{1}{s^2+1}\right\}$

3. $\cos 3t \cos 2t$ **Answer:** $\dfrac{1}{2}\left\{\dfrac{s}{s^2+25}+\dfrac{s}{s^2+1}\right\}$

4. $\sin^2 2t$ **Answer:** $\dfrac{1}{2}\left\{\dfrac{1}{s}-\dfrac{s}{s^2+16}\right\}$

5. $\sin 2t \cos 2t$ **Answer:** $\dfrac{2}{s^2+16}$

6. $4t-3$ **Answer:** $\dfrac{4}{s^2}-\dfrac{3}{s}$

7. $3e^{-2t}$ **Answer:** $\dfrac{3}{s+2}$

8. $3\sinh 2t$ **Answer:** $\dfrac{6}{s^2-4}$

1.2 Properties of Laplace Transforms

Property 1 [First Shifting Property]

If $L[f(t)] = \bar{f}(s),$ then $L[e^{at}f(t)] = \bar{f}(s-a).$

Or

If $L[f(t)] = \bar{f}(s),$ then $L[e^{at}f(t)] = \left[\bar{f}(s)\right]_{s\to s-a}$ (s is replaced by $s-a$)

Or

$$L[e^{at}f(t)] = [L\{f(t)\}]_{s\to s-a}$$

SOLVED PROBLEMS

1. Find the Laplace transform of $e^{-3t}(2cos5t - 3sin5t)$

Solution: Let $f(t) = 2cos5t - 3sin5t$

Take the Laplace transform of both sides

$$L[f(t)] = L[2\cos 5t - 3\sin 5t]$$

$$= 2L[\cos 5t] - 3L[\sin 5t] \qquad \text{(by linearity)}$$

$We\ have\ \ L[\cos at] = \dfrac{s}{s^2+a^2} \quad \& \quad L[\sin at] = \dfrac{a}{s^2+a^2}$

$$\therefore L[f(t)] = 2\cdot\frac{s}{s^2+5^2} - 3\cdot\frac{5}{s^2+5^2}$$

$$= \frac{2s}{s^2+25} - \frac{15}{s^2+25}$$

$$\therefore L[f(t)] = \frac{2s-15}{s^2+25} = \bar{f}(s)$$

By shifting property, we have

If $L[f(t)] = \bar{f}(s),$ then $L[e^{at}f(t)] = \left[\bar{f}(s)\right]_{s\to s-a}$

$$\therefore L[e^{-3t}f(t)] = \left[\bar{f}(s)\right]_{s\to s+3}$$

$$= \left[\frac{2s-15}{s^2+25}\right]_{s\to s+3}$$

$$= \frac{2(s+3)-15}{(s+3)^2+25}$$

Thus $\boldsymbol{L[e^{-3t}(2\cos 5t - 3\sin 5t)] = \dfrac{2s-9}{s^2+6s+34}}$

2. Find the Laplace transform of $e^{2t}\,cos^2t$

Solution: Let $f(t) = cos^2t$

$$f(t) = \frac{1}{2}(1 + \cos 2t)$$

Take the Laplace transform of both sides

$$L[f(t)] = L\left[\frac{1}{2}(1 + \cos 2t)\right]$$

$$= \frac{1}{2}[L[1] + L[cos2t]] \qquad \text{(by linearity)}$$

$$We\ have\ \ L[1] = \frac{1}{s} \quad \& \quad L[\cos at] = \frac{s}{s^2 + a^2}$$

$$\therefore\ L[f(t)] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2 + 4}\right] = \bar{f}(s)$$

By shifting property, we have

If $L[f(t)] = \bar{f}(s)$, then $L[e^{at}f(t)] = \left[\bar{f}(s)\right]_{s\to s-a}$

$$\therefore L[e^{2t}f(t)] = \left[\bar{f}(s)\right]_{s\to s-2}$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2 + 4}\right]_{s\to s-2}$$

$$\text{Thus}\ \ \boldsymbol{L[e^{2t}cos^2t] = \frac{1}{2}\left[\frac{1}{s-2} + \frac{s-2}{(s-2)^2 + 4}\right]}$$

3. Find the Laplace transform of $\sqrt{t}\,e^{3t}$

Solution: Let $f(t) = \sqrt{t}$

$$f(t) = t^{\frac{1}{2}}$$

Take the Laplace transform of both sides

$$L[f(t)] = L\left[t^{\frac{1}{2}}\right]$$

$$We\ have\ \ L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$$

$$\therefore\ L[f(t)] = \frac{\Gamma\left(\frac{1}{2} + 1\right)}{s^{\frac{1}{2}+1}} = \frac{\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}} = \frac{\frac{1}{2}\sqrt{\pi}}{s^{\frac{3}{2}}}$$

$$L[f(t)] = \frac{1}{2}\frac{\sqrt{\pi}}{s^{\frac{3}{2}}} = \bar{f}(s)$$

By shifting property, we have

If $L[f(t)] = \bar{f}(s)$, then $L[e^{at}f(t)] = [\bar{f}(s)]_{s\to s-a}$

$$\therefore L[e^{3t}f(t)] = [\bar{f}(s)]_{s\to s-3}$$

$$= \frac{\sqrt{\pi}}{2}\left[\frac{1}{s^{\frac{3}{2}}}\right]_{s\to s-3}$$

$$\text{Thus } \boldsymbol{L[e^{3t}\sqrt{t}] = \frac{\sqrt{\pi}}{2(s-3)^{\frac{3}{2}}}}$$

4. Find the Laplace transform of $e^{-at}\, sinh\, bt$

Solution: Let $f(t) = sinh\, bt$

Take the Laplace transform of both sides

$$L[f(t)] = L[sinh\, bt]$$

We have $\quad L[\sinh at] = \dfrac{a}{s^2 - a^2}$

$$\therefore \; L[f(t)] = \frac{b}{s^2 - b^2} = \bar{f}(s)$$

By shifting property, we have

If $L[f(t)] = \bar{f}(s)$, then $L[e^{at}f(t)] = [\bar{f}(s)]_{s\to s-a}$

$$\therefore L[e^{-at}f(t)] = [\bar{f}(s)]_{s\to s+a}$$

$$= \left[\frac{b}{s^2 - b^2}\right]_{s\to s+a}$$

$$\text{Thus } \boldsymbol{L[e^{-at}\, sinh\, bt] = \frac{b}{(s+a)^2 - b^2}}$$

5. Find the Laplace transform of $t^2 e^{-2t}$

Solution: Let $f(t) = t^2$

Take the Laplace transform of both sides

$$L[f(t)] = L[t^2]$$

$We\ have\ \ L[t^n] = \dfrac{n!}{s^{n+1}}$

$$\therefore\ L[f(t)] = \frac{2!}{s^{2+1}} = \frac{2}{s^3} = \bar{f}(s)$$

By shifting property, we have

If $L[f(t)] = \bar{f}(s)$, then $L[e^{at}f(t)] = [\bar{f}(s)]_{s\to s-a}$

$$\therefore L[e^{-2t}f(t)] = [\bar{f}(s)]_{s\to s+2}$$

$$= \left[\frac{2}{s^3}\right]_{s\to s+2}$$

Thus $\boldsymbol{L[e^{-2t}\,t^2] = \dfrac{2}{(s+2)^3}}$

6. Find the Laplace transform of $\ cosh at\ sin at$

Solution: Given $cosh at \sin at = \left(\frac{e^{at}+e^{-at}}{2}\right)\sin at = \frac{1}{2}[e^{at}\sin at + e^{-at}\sin at]$

Take the Laplace transform of both sides

$$\therefore\ L[cosh at \sin at] = L\left[\frac{1}{2}[e^{at}\sin at + e^{-at}\sin at]\right]$$

$$= \frac{1}{2}[L[e^{at}\sin at] + L[e^{-at}\sin at]] \qquad \text{(by linearity)}$$

By shifting property, we have $L[e^{at}f(t)] = [L\{f(t)\}]_{s\to s-a}$

$$\therefore\ L[cosh at \sin at] = \frac{1}{2}[[L(\sin at)]_{s\to s-a} + [L(\sin at)]_{s\to s+a}]$$

$$= \frac{1}{2}\left[\left[\frac{a}{s^2+a^2}\right]_{s\to s-a} + \left[\frac{a}{s^2+a^2}\right]_{s\to s+a}\right]$$

$$= \frac{1}{2}\left[\frac{a}{(s-a)^2+a^2} + \frac{a}{(s+a)^2+a^2}\right]$$

Thus $\quad \boldsymbol{L[cosh\ at\ sin\ at] = \dfrac{a}{2}\left[\dfrac{1}{(s-a)^2+a^2} + \dfrac{1}{(s+a)^2+a^2}\right]}$

EXERCISE PROBLEMS

Find the Laplace transform of the following functions:

1. $e^{2t}(3t^2 - cos4t)$ **Answer**: $\dfrac{6}{(s-2)^3} - \dfrac{(s-2)}{s^2-4s+20}$

2. $e^{-t}sin^2 3t$ **Answer**: $\dfrac{1}{2}\left[\dfrac{1}{s+1} - \dfrac{s+1}{(s+1)^2+36}\right]$

3. $sinh3t\ cos^2 t$ **Answer**: $\dfrac{3}{2}\left[\dfrac{1}{s^3-9} + \dfrac{s^3-13}{s^4-10s^2+169}\right]$

4. $t^3\ cosht$ **Answer**: $\dfrac{1}{2}\left[\dfrac{6}{(s-1)^4} + \dfrac{6}{(s+1)^4}\right]$

5. $t^5 e^{4t} cosh3t$ **Answer**: $\dfrac{1}{2}\left[\dfrac{5!}{(s-7)^6} + \dfrac{5!}{(s-1)^6}\right]$

6. $cosht\ cost$ **Answer**: $\dfrac{1}{2}\left[\dfrac{s-1}{(s-1)^2+1} + \dfrac{s+1}{(s+1)^2+1}\right]$

7. $sinht\ cos2t$ **Answer**: $\dfrac{1}{2}\left[\dfrac{(s-1)}{(s-1)^2+4} - \dfrac{(s+1)}{(s+1)^2+4}\right]$

8. $e^{3t}\ (2t+5)^2$ **Answer**: $\dfrac{8}{(s-3)^3} + \dfrac{25}{(s-3)} + \dfrac{20}{(s-3)^2}$

<u>Property 2</u> [Multiplication by t^n property]

If $L[f(t)] = \bar{f}(s)$, then $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\bar{f}(s)]$, where $n = 1, 2, 3 \ldots$

Or

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[L\{f(t)\}], \quad \text{where } n = 1, 2, 3 \ldots$$

SOLVED PROBLEMS

1. Find the Laplace transform of $t\, cosat$

Solution: Let $f(t) = \cos at$

Take the Laplace transform of both sides

$$L[f(t)] = L[\cos at] = \frac{s}{s^2 + a^2} = \bar{f}(s)$$

By multiplication by t^n property, we have

If $L[f(t)] = \bar{f}(s)$, then $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\bar{f}(s)]$, where $n = 1, 2, 3 \ldots$

$$\therefore L[tf(t)] = (-1)^1 \frac{d^1}{ds^1}[\bar{f}(s)]$$

$$= -\frac{d}{ds}\left[\frac{s}{s^2 + a^2}\right]$$

$$= -\left[\frac{(s^2 + a^2)(1) - s(2s)}{(s^2 + a^2)^2}\right] \qquad \text{(by quotient rule)}$$

$$= -\left[\frac{s^2 + a^2 - 2s^2}{(s^2 + a^2)^2}\right] = -\left[\frac{a^2 - s^2}{(s^2 + a^2)^2}\right]$$

Thus $\boldsymbol{L[t\cos at] = \frac{s^2 - a^2}{(s^2 + a^2)^2}}$

2. Find the Laplace transform of $t^2\, sinat$

Solution: Let $f(t) = \sin at$

Take the Laplace transform of both sides

$$L[f(t)] = L[\sin at] = \frac{a}{s^2 + a^2} = \bar{f}(s)$$

By multiplication by t^n property, we have

If $L[f(t)] = \bar{f}(s)$, then $L[t^n f(t)] = (-1)^n \dfrac{d^n}{ds^n}[\bar{f}(s)]$, where $n = 1, 2, 3 \ldots$

$$\therefore L[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2}[\bar{f}(s)]$$

$$= \frac{d^2}{ds^2}\left[\frac{a}{s^2 + a^2}\right]$$

$$= \frac{d}{ds}\left[-\frac{a}{(s^2 + a^2)^2}(2s)\right] \qquad by \ \frac{d}{dx}\left[\frac{1}{x^n}\right] = -\frac{n}{x^{n+1}}$$

$$= -2a \frac{d}{ds}\left[\frac{s}{(s^2 + a^2)^2}\right]$$

$$= -2a\left[\frac{(s^2 + a^2)^2(1) - s[2(s^2 + a^2)(2s)]}{(s^2 + a^2)^4}\right]$$

$$= -2a\left[\frac{(s^2 + a^2)\{(s^2 + a^2) - 4s^2\}}{(s^2 + a^2)^4}\right]$$

$$= -2a\left[\frac{(a^2 - 3s^2)}{(s^2 + a^2)^3}\right]$$

Thus $\boldsymbol{L[t^2 \sin at] = \dfrac{2a(3s^2 - a^2)}{(s^2 + a^2)^3}}$

3. Find the Laplace transform of $t\, e^{-t} sin3t$

Solution: Rewrite the given as $e^{-t}\, t\, sin3t$

First let us find $L[t\, sin3t]$

By multiplication by t^n property, we have $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[L[f(t)]]$

$$\therefore L[t\, sin3t] = (-1)^1 \frac{d^1}{ds^1}[L[\sin 3t]] = -\frac{d}{ds}\left[\frac{3}{s^2 + 9}\right]$$

$$= -\left[-\frac{3}{(s^2 + 9)^2}(2s)\right]$$

$$L[t\, sin3t] = \frac{6s}{(s^2 + 9)^2}$$

By shifting property, we have $L[e^{at} f(t)] = [L\{f(t)\}]_{s \to s-a}$

$$\therefore L[e^{-t}\, (t\, sin3t)] = [L(t \sin 3t)]_{s \to s+1}$$

$$= \left[\frac{6s}{(s^2 + 9)^2}\right]_{s \to s+1}$$

$$= \left[\frac{6(s+1)}{[(s+1)^2+9]^2}\right]$$

$$\text{Thus } \boldsymbol{L[e^{-t}\, t\, sin3t]} = \frac{\boldsymbol{6(s+1)}}{\boldsymbol{(s^2+2s+10)^2}}$$

4. Find the Laplace transform of $t^4 e^{-3t}$

Solution: Let $f(t) = e^{-3t}$

Take the Laplace transform of both sides

$$L[f(t)] = L[e^{-3t}] = \frac{1}{s+3} = \bar{f}(s)$$

By multiplication by t^n property, we have

If $L[f(t)] = \bar{f}(s)$, then $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\bar{f}(s)]$, where $n = 1, 2, 3 \ldots$

$$\therefore L[t^4\, f(t)] = (-1)^4 \frac{d^4}{ds^4}[\bar{f}(s)]$$

$$= \frac{d^4}{ds^4}\left[\frac{1}{s+3}\right] \qquad use\ \frac{d}{dx}\left[\frac{1}{x^n}\right] = -\frac{n}{x^{n+1}}$$

$$= \frac{d^3}{ds^3}\left[-\frac{1}{(s+3)^2}\right] = -\frac{d^3}{ds^3}\left[\frac{1}{(s+3)^2}\right]$$

$$= -\frac{d^2}{ds^2}\left[-\frac{2}{(s+3)^3}\right] = 2\frac{d^2}{ds^2}\left[\frac{1}{(s+3)^3}\right]$$

$$= 2\frac{d}{ds}\left[-\frac{3}{(s+3)^4}\right] = -6\frac{d}{ds}\left[\frac{1}{(s+3)^4}\right]$$

$$= -6\left[-\frac{4}{(s+3)^5}\right]$$

$$\text{Thus } \boldsymbol{L[t^4\, e^{-3t}]} = \frac{\boldsymbol{24}}{\boldsymbol{(s+3)^5}}$$

Alternative Method: Rewrite the given as $e^{-3t}\, t^4$

By shifting property, we have $L[e^{at} f(t)] = [L\{f(t)\}]_{s \to s-a}$

$$\therefore L[e^{-3t}\, t^4] = [L(t^4)]_{s \to s+3}$$

$$= \left[\frac{4!}{s^{4+1}}\right]_{s \to s+3} = \left[\frac{24}{s^5}\right]_{s \to s+3}$$

$$\text{Thus } \boldsymbol{L[e^{-3t}\, t^4]} = \frac{\boldsymbol{24}}{\boldsymbol{(s+3)^5}}$$

EXERCISE PROBLEMS

Find the Laplace transform of the following functions:

1. $t\ sin^2t$ **Answer**: $\dfrac{2(3s^2+4)}{s^2(s^2+4)^2}$

2. $t^2\ cost$ **Answer**: $\dfrac{2s^3-6a^2s}{(s^2+a^2)^3}$

3. $t\ e^{-2t}sin4t$ **Answer**: $\dfrac{8(s+2)}{s^2+4s+20}$

<u>Property 3</u> [Division by t property]

If $L[f(t)] = \bar{f}(s)$, then $L\left[\frac{1}{t}f(t)\right] = \int_s^{\infty} \bar{f}(s)ds$ provided the integral exists

Or

$$L\left[\frac{1}{t}f(t)\right] = \int_s^{\infty} [L\{f(t)\}]ds$$

SOLVED PROBLEMS

1. Find the Laplace transform of $\frac{1-e^t}{t}$

Solution: Let $f(t) = 1 - e^t$

Take the Laplace transform of both sides

$$L[f(t)] = L[1 - e^t] = \frac{1}{s} - \frac{1}{s-1} = \bar{f}(s)$$

By division by t property, we have

If $L[f(t)] = \bar{f}(s)$, then $L\left[\frac{1}{t}f(t)\right] = \int_s^{\infty} \bar{f}(s)ds$ provided the integral exists

$$\therefore L\left[\frac{1}{t}f(t)\right] = \int_s^{\infty} \left(\frac{1}{s} - \frac{1}{s-1}\right) ds = [\log s - \log(s-1)]_s^{\infty}$$

$$= \left[\log\left(\frac{s}{s-1}\right)\right]_s^{\infty} = \left[\log\left\{\frac{s}{s\left(1-\frac{1}{s}\right)}\right\}\right]_s^{\infty}$$

$$= \left[\log\left(\frac{1}{1-\frac{1}{s}}\right)\right]_s^{\infty} = \log\left(\frac{1}{1-\frac{1}{\infty}}\right) - \log\left(\frac{1}{1-\frac{1}{s}}\right)$$

$$= \log\left(\frac{1}{1-0}\right) - \log\left(\frac{s}{s-1}\right) \qquad \left[\frac{1}{\infty} = 0\right]$$

$$= \log 1 - \log\left(\frac{s}{s-1}\right)$$

$$= 0 - \log\left(\frac{s}{s-1}\right) \qquad [\log 1 = 0]$$

$$= \log\left(\frac{s-1}{s}\right) \qquad \left[\log\left(\frac{a}{b}\right) = -\log\left(\frac{b}{a}\right)\right]$$

Thus $\boldsymbol{L\left[\frac{1-e^t}{t}\right] = \log\left(\frac{s-1}{s}\right)}$ ***or*** $\boldsymbol{L\left[\frac{1-e^t}{t}\right] = \log\left(1-\frac{1}{s}\right)}$

2. Find the Laplace transform of $\frac{cosat - cosbt}{t}$

Solution: Let $f(t) = \cos at - \cos bt$

Take the Laplace transform of both sides

$$L[f(t)] = L[\cos at - \cos bt] = \frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2} = \bar{f}(s)$$

By division by t property, we have

If $L[f(t)] = \bar{f}(s)$, then $L\left[\frac{1}{t} f(t)\right] = \int_s^\infty \bar{f}(s) ds$ provided the integral exists

$$\therefore L\left[\frac{1}{t} f(t)\right] = \int_s^\infty \left(\frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2}\right) ds$$

$$= \left[\frac{1}{2}\log(s^2 + a^2) - \frac{1}{2}\log(s^2 + b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2 + a^2}{s^2 + b^2}\right)\right]_s^\infty = \frac{1}{2}\left[\log\left\{\frac{s^2\left(1 + \frac{a^2}{s^2}\right)}{s^2\left(1 + \frac{b^2}{s^2}\right)}\right\}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{1 + \frac{a^2}{s^2}}{1 + \frac{b^2}{s^2}}\right)\right]_s^\infty = \frac{1}{2}\left[\log\left(\frac{1 + \frac{a^2}{\infty}}{1 + \frac{b^2}{\infty}}\right) - \log\left(\frac{1 + \frac{a^2}{s^2}}{1 + \frac{b^2}{s^2}}\right)\right]$$

$$= \frac{1}{2}\left[\log\left(\frac{1 + 0}{1 + 0}\right) - \log\left(\frac{s^2 + a^2}{s^2 + b^2}\right)\right] \qquad \left[\frac{1}{\infty} = 0\right]$$

$$= \frac{1}{2}\left[\log 1 - \log\left(\frac{s^2 + a^2}{s^2 + b^2}\right)\right]$$

$$= \frac{1}{2}\left[0 - \log\left(\frac{s^2 + a^2}{s^2 + b^2}\right)\right] \qquad [\log 1 = 0]$$

$$= \frac{1}{2}\log\left(\frac{s^2 + b^2}{s^2 + a^2}\right) \qquad \left[\log\left(\frac{a}{b}\right) = -\log\left(\frac{b}{a}\right)\right]$$

$$= \log\left(\frac{s^2 + b^2}{s^2 + a^2}\right)^{\frac{1}{2}} \qquad [\log m^n = n \log m]$$

Thus $\boldsymbol{L}\left[\frac{\boldsymbol{cosat - cosbt}}{\boldsymbol{t}}\right] = \boldsymbol{log}\left(\sqrt{\frac{s^2 + b^2}{s^2 + a^2}}\right)$

3. Find the Laplace transform of $\frac{e^{-3t}-e^{-4t}}{t}$

Solution: Let $f(t) = e^{-3t} - e^{-4t}$

Take the Laplace transform of both sides

$$L[f(t)] = L[e^{-3t} - e^{-4t}] = \frac{1}{s+3} - \frac{1}{s+4} = \bar{f}(s)$$

By division by t property, we have

If $L[f(t)] = \bar{f}(s)$, then $L\left[\frac{1}{t}f(t)\right] = \int_s^\infty \bar{f}(s)ds$ provided the integral exists

$$\therefore L\left[\frac{1}{t}f(t)\right] = \int_s^\infty \left(\frac{1}{s+3} - \frac{1}{s+4}\right) ds$$

$$= [\log(s+3) - \log(s+4)]_s^\infty$$

$$= \left[\log\left(\frac{s+3}{s+4}\right)\right]_s^\infty = \left[\log\left\{\frac{s(1+\frac{3}{s})}{s\left(1+\frac{4}{s}\right)}\right\}\right]_s^\infty$$

$$= \left[\log\left(\frac{1+\frac{3}{s}}{1+\frac{4}{s}}\right)\right]_s^\infty = \log\left(\frac{1+\frac{3}{\infty}}{1+\frac{4}{\infty}}\right) - \log\left(\frac{1+\frac{3}{s}}{1+\frac{4}{s}}\right)$$

$$= \log\left(\frac{1+0}{1+0}\right) - \log\left(\frac{s+3}{s+4}\right) \qquad \left[\frac{1}{\infty} = 0\right]$$

$$= \log 1 - \log\left(\frac{s+3}{s+4}\right)$$

$$= 0 - \log\left(\frac{s+3}{s+4}\right) \qquad [\log 1 = 0]$$

$$= \log\left(\frac{s+4}{s+3}\right) \qquad \left[\log\left(\frac{a}{b}\right) = -\log\left(\frac{b}{a}\right)\right]$$

Thus $\boldsymbol{L\left[\frac{e^{-3t}-e^{-4t}}{t}\right] = \log\left(\frac{s+4}{s+3}\right)}$

4. Find the Laplace transform of $\frac{1-\cos 3t}{t}$

Solution: Let $f(t) = 1 - \cos 3t$

Take the Laplace transform of both sides

$$L[f(t)] = L[1 - \cos 3t] = \frac{1}{s} - \frac{s}{s^2+9} = \bar{f}(s)$$

By division by t property, we have

If $L[f(t)] = \bar{f}(s)$, then $L\left[\frac{1}{t}f(t)\right] = \int_s^\infty \bar{f}(s)ds$ provided the integral exists

$$\therefore L\left[\frac{1}{t}f(t)\right] = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+9}\right) ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+9)\right]_s^\infty = \left[\frac{2}{2}\log s - \frac{1}{2}\log(s^2+9)\right]_s^\infty$$

$$= \frac{1}{2}\ [2\log s - \log(s^2+9)]_s^\infty = \frac{1}{2}\ [\log s^2 - \log(s^2+9)]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2}{s^2+9}\right)\right]_s^\infty = \frac{1}{2}\left[\log\left\{\frac{s^2}{s^2\left(1+\frac{9}{s^2}\right)}\right\}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{1}{1+\frac{9}{s^2}}\right)\right]_s^\infty = \frac{1}{2}\left[\log\left(\frac{1}{1+\frac{9}{\infty}}\right) - \log\left(\frac{1}{1+\frac{9}{s^2}}\right)\right]$$

$$= \frac{1}{2}\left[\log\left(\frac{1}{1+0}\right) - \log\left(\frac{s^2}{s^2+9}\right)\right] \qquad \left[\frac{1}{\infty} = 0\right]$$

$$= \frac{1}{2}\left[\log 1 - \log\left(\frac{s^2}{s^2+9}\right)\right]$$

$$= \frac{1}{2}\left[0 - \log\left(\frac{s^2}{s^2+9}\right)\right] \qquad [\log 1 = 0]$$

$$= \frac{1}{2}\log\left(\frac{s^2+9}{s^2}\right) \qquad \left[\log\left(\frac{a}{b}\right) = -\log\left(\frac{b}{a}\right)\right]$$

$$= \log\left(\frac{s^2+9}{s^2}\right)^{\frac{1}{2}} \qquad [\log m^n = n\log m]$$

$$= \log\left(\sqrt{\frac{s^2+9}{s^2}}\right)$$

Thus $\boldsymbol{L\left[\frac{1-cos3t}{t}\right] = log\left(\sqrt{\frac{s^2+9}{s^2}}\right)}$ ***or*** $\boldsymbol{L\left[\frac{1-cos3t}{t}\right] = log\left(\frac{\sqrt{s^2+9}}{s}\right)}$

5. Find the Laplace transform of $2^t + \dfrac{cos2t - cos3t}{t} + t\, sint$

Solution:

$$L\left[2^t + \frac{cos2t - cos3t}{t} + t\, sint\right] = L[2^t] + L\left[\frac{cos2t - cos3t}{t}\right] + L[t\, sint] - - - (1)$$

$i)$ $L[2^t] = L\left[e^{\log 2^t}\right] = L\left[e^{t\log 2}\right] = L\left[e^{(\log 2)t}\right]$

$$\text{We have } L[e^{at}] = \frac{1}{s-a}$$

$$\therefore L[2^t] = \frac{1}{s - \log 2} - - - -(2)$$

$ii)$ By division by t property, we have $L\left[\frac{1}{t}f(t)\right] = \int_s^\infty [L\{f(t)\}]ds$

$$\therefore\ L\left[\frac{cos2t - cos3t}{t}\right] = \int_s^\infty [L\{cos2t - cos3t\}]ds$$

$$= \int_s^\infty [L(cos2t) - L(cos3t)]ds$$

$$= \int_s^\infty \left(\frac{s}{s^2+4} - \frac{s}{s^2+9}\right) ds$$

$$= \left[\frac{1}{2}\log(s^2+4) - \frac{1}{2}\log(s^2+9)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+4}{s^2+9}\right)\right]_s^\infty = \frac{1}{2}\left[\log\left\{\frac{s^2\left(1+\frac{4}{s^2}\right)}{s^2\left(1+\frac{9}{s^2}\right)}\right\}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{1+\frac{4}{s^2}}{1+\frac{9}{s^2}}\right)\right]_s^\infty = \frac{1}{2}\left[\log\left(\frac{1+\frac{4}{\infty}}{1+\frac{9}{\infty}}\right) - \log\left(\frac{1+\frac{4}{s^2}}{1+\frac{9}{s^2}}\right)\right]$$

$$= \frac{1}{2}\left[\log\left(\frac{1+0}{1+0}\right) - \log\left(\frac{s^2+4}{s^2+9}\right)\right] \qquad \left[\frac{1}{\infty} = 0\right]$$

$$= \frac{1}{2}\left[\log 1 - \log\left(\frac{s^2+4}{s^2+9}\right)\right]$$

$$= \frac{1}{2}\left[0 - \log\left(\frac{s^2+4}{s^2+9}\right)\right] \qquad [\log 1 = 0]$$

$$= \frac{1}{2}\log\left(\frac{s^2+9}{s^2+4}\right) \qquad \left[\log\left(\frac{a}{b}\right) = -\log\left(\frac{b}{a}\right)\right]$$

$$= \log\left(\frac{s^2+9}{s^2+4}\right)^{\frac{1}{2}} \qquad [\log m^n = n\log m]$$

$$= \log\left(\sqrt{\frac{s^2+9}{s^2+4}}\right)$$

$$\therefore\ L\left[\frac{cos2t - cos3t}{t}\right] = log\left(\sqrt{\frac{s^2+9}{s^2+4}}\right) - - - - -(3)$$

$iii)$ By multiplication by t^n property, we have $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}\left[L[f(t)]\right]$

$$\therefore L[t\ sint] = (-1)^1 \frac{d^1}{ds^1}[L(\sin t)] = -\frac{d}{ds}\left[\frac{1}{s^2+1}\right]$$

$$= -\left[-\frac{1}{(s^2+1)^2}(2s)\right]$$

$$\therefore L[t\ sint] = \frac{2s}{(s^2+1)^2} - - - - (4)$$

Substituting (2), (3) & (4) in (1), we get

$$\boldsymbol{L\left[2^t + \frac{cos2t - cos3t}{t} + t\ sint\right] = \frac{1}{s - \log 2} + log\left(\sqrt{\frac{s^2+9}{s^2+4}}\right) + \frac{2s}{(s^2+1)^2}}$$

EXERCISE PROBLEMS

Find the Laplace transform of the following functions:

1. $\frac{sint}{t}$ **Answer**: $cot^{-1}s$

2. $\frac{sin^2t}{t}$ **Answer**: $\frac{1}{2}log\left(\frac{\sqrt{s^2+4}}{s}\right)$

3. $\frac{sin3t\ sint}{t}$ **Answer**: $\frac{1}{2}log\left(\sqrt{\frac{s^2+16}{s^2+4}}\right)$

4. $\frac{e^{at}-cosbt}{t}$ **Answer**: $log\left(\frac{\sqrt{s^2+b^2}}{(s-a)}\right)$

5. $\frac{e^{-at}-e^{-bt}}{t}$ **Answer**: $log\left(\frac{s+b}{s+a}\right)$

1.3 Laplace Transform of Periodic Functions

Definition: A function $f(t)$ is said to be a periodic function of period $T > 0$ if $f(t+T) = f(t)$.

For example, $\sin t$ and $\cos t$ are periodic functions of period 2π since $sin(t+2\pi) = sin\, t$ & $cos\,(t+2\pi) = cost$.

Theorem: If $f(t)$ is a periodic function with period T, then

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)dt$$

SOLVED PROBLEMS

1) Find the Laplace transform of the function

$$f(t) = \begin{cases} E\sin\omega t, & 0 < t < \frac{\pi}{\omega} \\ 0\,, & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases} \quad \text{with period } \frac{2\pi}{\omega}.$$

Solution: The Laplace transform of a periodic function is given by

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)dt. \quad \text{Here } T = \frac{2\pi}{\omega}.$$

$$\therefore L[f(t)] = \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\int_0^{\frac{2\pi}{\omega}} e^{-st} f(t)dt = \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left[\int_0^{\frac{\pi}{\omega}} e^{-st} f(t)dt + \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} e^{-st} f(t)dt\right]$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left[\int_0^{\frac{\pi}{\omega}} e^{-st} E\sin\omega t\, dt + \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} e^{-st}\,(0)\,dt\right]$$

$$= \frac{E}{1-e^{-\frac{2\pi s}{\omega}}}\int_0^{\frac{\pi}{\omega}} e^{-st}\sin\omega t\, dt$$

We know that $\int e^{at} sinbt\, dt = \frac{e^{at}}{a^2+b^2}(a\, sinbt - b\, cosbt)$

$$\therefore L[f(t)] = \frac{E}{1-e^{-\frac{2\pi s}{\omega}}}\left[\frac{e^{-st}}{(-s)^2+\omega^2}(-s\sin\omega t - \omega\cos\omega t)\right]_0^{\frac{\pi}{\omega}}$$

$$= \frac{-E}{1-e^{-\frac{2\pi s}{\omega}}}\left[\frac{e^{-st}}{s^2+\omega^2}(s\sin\omega t+\omega\cos\omega t)\right]_0^{\frac{\pi}{\omega}}$$

$$= \frac{-E}{\left(1-e^{-\frac{2\pi s}{\omega}}\right)(s^2+\omega^2)}\left[e^{-st}(s\sin\omega t+\omega\cos\omega t)\right]_0^{\pi/\omega}$$

$$= \frac{-E}{\left(1-e^{-\frac{2\pi s}{\omega}}\right)(s^2+\omega^2)}\left[e^{-s\frac{\pi}{\omega}}\left\{s\sin\left(\omega\frac{\pi}{\omega}\right)+\omega\cos\left(\omega\frac{\pi}{\omega}\right)\right\}\right.$$
$$\left. - e^{-s(0)}\{s\sin\big(\omega(0)\big)+\omega\cos(\omega(0))\}\right]$$

$$= \frac{-E}{\left(1-e^{-\frac{2\pi s}{\omega}}\right)(s^2+\omega^2)}\left[e^{-\frac{\pi s}{\omega}}\{s\sin\pi+\omega\cos\pi\}-e^0\{s\sin 0+\omega\cos 0\}\right]$$

$$= \frac{-E}{\left(1-e^{-\frac{2\pi s}{\omega}}\right)(s^2+\omega^2)}\left[e^{-\frac{\pi s}{\omega}}\{s\,(0)+\omega\,(-1)\}-1\{s\,(0)+\omega\,(1)\}\right]$$

$$= \frac{-E}{\left(1-e^{-\frac{2\pi s}{\omega}}\right)(s^2+\omega^2)}\left[e^{-\frac{\pi s}{\omega}}\{0-\omega\}-1\{0+\omega\}\right]$$

$$= \frac{-E}{\left(1-e^{-\frac{2\pi s}{\omega}}\right)(s^2+\omega^2)}\left[-\omega\, e^{-\frac{\pi s}{\omega}}-\omega\}\right]$$

$$= \frac{E\omega}{\left(1-e^{-\frac{2\pi s}{\omega}}\right)(s^2+\omega^2)}\left[e^{-\frac{\pi s}{\omega}}+1\right]$$

$$= \frac{E\omega\,\left(1+e^{-\frac{\pi s}{\omega}}\right)}{(s^2+\omega^2)\left(1-e^{-\frac{2\pi s}{\omega}}\right)} = \frac{E\omega\,\left(1+e^{-\frac{\pi s}{\omega}}\right)}{(s^2+\omega^2)\left[(1)^2-\left(e^{-\frac{\pi s}{\omega}}\right)^2\right]}$$

$$= \frac{E\omega\,\left(1+e^{-\frac{\pi s}{\omega}}\right)}{(s^2+\omega^2)\left[\left(1+e^{-\frac{\pi s}{\omega}}\right)\left(1-e^{-\frac{\pi s}{\omega}}\right)\right]}$$

Thus $\boldsymbol{L[f(t)]} = \dfrac{\boldsymbol{E\omega}}{\boldsymbol{(s^2+\omega^2)\left(1-e^{-\frac{\pi s}{\omega}}\right)}}.$

2) Find the Laplace transform of the function

$$f(t) = \begin{cases} t\,, & 0 < t < \pi \\ \pi - t\,, & \pi < t < 2\pi \end{cases} \quad \text{with period } 2\pi.$$

Solution: The Laplace transform of a periodic function is given by

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)dt\,. \qquad \text{Here } T = 2\pi.$$

$$\therefore L[f(t)] = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t)dt = \frac{1}{1-e^{-2\pi s}} \left[\int_0^{\pi} e^{-st} f(t)dt + \int_{\pi}^{2\pi} e^{-st} f(t)dt \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\int_0^{\pi} e^{-st}\, t\, dt + \int_{\pi}^{2\pi} e^{-st}\, (\pi - t)\, dt \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\int_0^{\pi} t\, e^{-st}\, dt + \int_{\pi}^{2\pi} (\pi - t)\, e^{-st}\, dt \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\left\{ (t)\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right) \right\}_0^{\pi} + \left\{ (\pi - t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{s^2}\right) \right\}_{\pi}^{2\pi} \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[-\frac{1}{s}(t\, e^{-st})_0^{\pi} - \frac{1}{s^2}(e^{-st})_0^{\pi} - \frac{1}{s}\{(\pi - t)e^{-st}\}_{\pi}^{2\pi} + \frac{1}{s^2}(e^{-st})_{\pi}^{2\pi} \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[-\frac{1}{s}\left\{\pi\, e^{-s(\pi)} - 0\, e^{-s(0)}\right\} - \frac{1}{s^2}\left\{e^{-s(\pi)} - e^{-s(0)}\right\} \right.$$
$$\left. -\frac{1}{s}\{(\pi - 2\pi)e^{-s(2\pi)} - (\pi - \pi)e^{-s(\pi)}\} + \frac{1}{s^2}\{e^{-s(2\pi)} - e^{-s(\pi)}\} \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[-\frac{1}{s}\{\pi\, e^{-\pi s} - 0\} - \frac{1}{s^2}\{e^{-\pi s} - e^{0}\} - \frac{1}{s}\{(-\pi)e^{-2\pi s} - 0\} \right.$$
$$\left. + \frac{1}{s^2}\{e^{-2\pi s} - e^{-\pi s}\} \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[-\frac{1}{s}\{\pi\, e^{-\pi s}\} - \frac{1}{s^2}\{e^{-\pi s} - 1\} - \frac{1}{s}\{(-\pi)e^{-2\pi s}\} + \frac{1}{s^2}\{e^{-2\pi s} - e^{-\pi s}\} \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{1}{s^2}\{e^{-2\pi s} - e^{-\pi s} - e^{-\pi s} + 1\} - \frac{1}{s}\{\pi\, e^{-\pi s} - \pi e^{-2\pi s}\} \right]$$

Thus $\boldsymbol{L[f(t)] = \frac{1}{1-e^{-2\pi s}} \left[\frac{1}{s^2}\{e^{-2\pi s} - 2e^{-\pi s} + 1\} - \frac{\pi}{s}\{e^{-\pi s} - e^{-2\pi s}\} \right]}$

3) Find the Laplace transform of the function

$$f(t) = \begin{cases} 1\,, & if\ \ 0 < t < a \\ -1\,, & if\ \ a < t < 2a \end{cases} \quad \text{with period } 2a.$$

Solution: The Laplace transform of a periodic function is given by

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)dt\,. \qquad \text{Here } T = 2a.$$

$$\therefore L[f(t)] = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)dt = \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st} f(t)dt + \int_a^{2a} e^{-st} f(t)dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st}\ (1)\ dt + \int_a^{2a} e^{-st}\ (-1)\ dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st}\ dt - \int_a^{2a} e^{-st}\ dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\left(\frac{e^{-st}}{-s} \right)_0^a - \left(\frac{e^{-st}}{-s} \right)_a^{2a} \right]$$

$$= \frac{1}{1-e^{-2as}} \left[-\frac{1}{s} (e^{-st})_0^a + \frac{1}{s} (e^{-st})_a^{2a} \right]$$

$$= \frac{1}{1-e^{-2as}} \left[-\frac{1}{s} \left\{ e^{-s(a)} - e^{-s(0)} \right\} + \frac{1}{s} \left\{ e^{-s(2a)} - e^{-s(a)} \right\} \right]$$

$$= \frac{1}{1-e^{-2as}} \left[-\frac{1}{s} \{ e^{-as} - e^{0} \} + \frac{1}{s} \{ e^{-2as} - e^{-as} \} \right]$$

$$= \frac{1}{s(1-e^{-2as})} [-e^{-as} + 1 + e^{-2as} - e^{-as}]$$

$$= \frac{(1 + e^{-2as} - 2e^{-as})}{s(1-e^{-2as})} = \frac{[(1)^2 + (e^{-as})^2 - 2\,(1)(e^{-as})]}{s\,[(1)^2 - (e^{-as})^2]}$$

$$= \frac{(1-e^{-as})^2}{s\,(1+e^{-as})(1-e^{-as})}$$

Thus $\boldsymbol{L[f(t)] = \dfrac{(1-e^{-as})}{s\,(1+e^{-as})}}$

EXERCISE PROBLEMS

1) Find the Laplace transform of the function

$$f(t) = \begin{cases} sint\,, & if\;\; 0 < t < \pi \\ 0\,, & if\;\; \pi < t < 2\pi \end{cases} \quad \text{with period } 2\pi.$$

Answer: $L[f(t)] = \dfrac{1}{(1 - e^{-\pi s})(s^2 + 1)}$

2) Find the Laplace transform of the function $f(t) = t$ for $0 < t < 1$ with period 1.

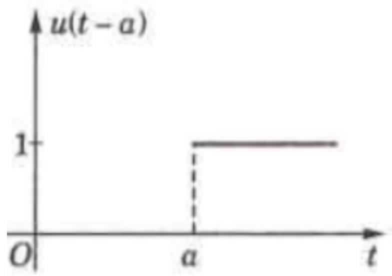
Answer: $L[f(t)] = \dfrac{1}{s^2(1 - e^{-s})}(-se^{-s} - e^{-s} + 1)$

1.4 Laplace Transform of Unit Step Functions

Definition: The unit step function $u(t-a)$ is defined as follows

$$u(t-a) = \begin{cases} 0 & for\ t < a \\ 1 & for\ t \geq a \end{cases}$$

where a is always positive. Unit step function is also called as Heaviside's unit function & denoted by $H(t-a)$.



Laplace transform of unit function: $L[u(t-a)] = \dfrac{e^{-as}}{s}$.

Examples: $L[u(t-2)] = \dfrac{e^{-2s}}{s}$ $\quad L[u(t-1)] = \dfrac{e^{-s}}{s}$ $\quad L[u(t-\pi)] = \dfrac{e^{-\pi s}}{s}$

Second shifting property:

If $L[f(t)] = \bar{f}(s)$, then $\boldsymbol{L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)}$.

NOTE:

1) $If\ f(t) = \begin{cases} f_1(t), & t \leq a \\ f_2(t), & t > a \end{cases},\quad then\ f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a)$

2) $If\ f(t) = \begin{cases} f_1(t), & t \leq a \\ f_2(t), & a < t \leq b, \\ f_3(t), & t > b \end{cases}$

$then\ f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a) + [f_3(t) - f_2(t)]u(t-b)$

SOLVED PROBLEMS

I. Find the Laplace transform of the following functions

1) $e^{t-1}\,u(t-1)$

Solution: Here $u(t-a) = u(t-1) \Longrightarrow \boldsymbol{a} = \mathbf{1}.$

$$\text{Let } f(t-1) = e^{t-1}$$

Replace t by $t+1$, we get

$$f(t+1-1) = e^{(t+1)-1}$$

$$f(t) = e^{t} \Longrightarrow L[f(t)] = L[e^{t}]$$

$$\therefore L[f(t)] = \frac{\mathbf{1}}{\boldsymbol{s}-\mathbf{1}} = \bar{\boldsymbol{f}}(\boldsymbol{s})$$

By second shifting property, we have $L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$.

$$\therefore L[e^{t-1}\,u(t-1)] = e^{-(1)s}\frac{1}{s-1}$$

$$\text{Thus } \boldsymbol{L}[\boldsymbol{e}^{\boldsymbol{t}-\mathbf{1}}\,\boldsymbol{u}(\boldsymbol{t}-\mathbf{1})] = \frac{\boldsymbol{e}^{-\boldsymbol{s}}}{\boldsymbol{s}-\mathbf{1}}.$$

2) $t^2\,u(t-3)$

Solution: Here $u(t-a) = u(t-3) \Longrightarrow \boldsymbol{a} = \mathbf{3}.$

$$\text{Let } f(t-3) = t^2$$

Replace t by $t+3$, we get

$$f(t+3-3) = (t+3)^2$$

$$f(t) = t^2 + 6t + 9 \Longrightarrow L[f(t)] = L[t^2 + 6t + 9] = L[t^2] + 6L[t] + L[9]$$

$$= \frac{2!}{s^{2+1}} + 6\frac{1!}{s^{1+1}} + \frac{9}{s}$$

$$\therefore L[f(t)] = \frac{\mathbf{2}}{\boldsymbol{s}^{\mathbf{3}}} + \frac{\mathbf{6}}{\boldsymbol{s}^{\mathbf{2}}} + \frac{\mathbf{9}}{\boldsymbol{s}} = \bar{\boldsymbol{f}}(\boldsymbol{s})$$

By second shifting property, we have $L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$.

$$\therefore L[t^2\,u(t-3)] = e^{-(3)s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right)$$

$$\text{Thus } \boldsymbol{L}[\boldsymbol{t}^{\mathbf{2}}\,\boldsymbol{u}(\boldsymbol{t}-\mathbf{3})] = \boldsymbol{e}^{-\mathbf{3}\boldsymbol{s}}\left(\frac{\mathbf{2}}{\boldsymbol{s}^{\mathbf{3}}} + \frac{\mathbf{6}}{\boldsymbol{s}^{\mathbf{2}}} + \frac{\mathbf{9}}{\boldsymbol{s}}\right).$$

3) $(t-1)^2\,u(t-1)$

Solution: Here $u(t-a) = u(t-1) \Longrightarrow \boldsymbol{a} = \mathbf{1}.$

$$\text{Let } f(t-1) = (t-1)^2$$

Replace t by $t+1$, we get

$$f(t+1-1) = [(t+1)-1]^2$$

$$f(t) = t^2 \Longrightarrow L[f(t)] = L[t^2] = \frac{2!}{s^{2+1}}$$

$$\therefore L[f(t)] = \frac{\mathbf{2}}{\boldsymbol{s}^3} = \bar{\boldsymbol{f}}(\boldsymbol{s})$$

By second shifting property, we have $L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$.

$$\therefore L[(t-1)^2\, u(t-1)] = e^{-(1)s}\frac{2}{s^3}$$

$$\text{Thus } \boldsymbol{L[(t-1)^2\, u(t-1)] = \frac{2e^{-s}}{s^3}}.$$

4) $\boldsymbol{sin\,[\pi(t-1)]\,u(t-1)}$

Solution: Here $u(t-a) = u(t-1) \Longrightarrow \boldsymbol{a = 1}$.

$$\text{Let } f(t-1) = \sin\{\pi(t-1)\}$$

Replace t by $t+1$, we get

$$f(t+1-1) = \sin\{\pi(t+1-1)\}$$

$$f(t) = \sin \pi t \Longrightarrow L[f(t)] = L[\sin \pi t]$$

$$\therefore L[f(t)] = \frac{\boldsymbol{\pi}}{\boldsymbol{s^2+\pi^2}} = \bar{\boldsymbol{f}}(\boldsymbol{s})$$

By second shifting property, we have $L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$.

$$\therefore L[sin\,\{\pi(t-1)\}\,u(t-1)] = e^{-(1)s}\frac{\pi}{s^2+\pi^2}$$

$$\text{Thus } \boldsymbol{L[sin\,\{\pi(t-1)\}\,u(t-1)] = \frac{\pi e^{-s}}{s^2+\pi^2}}.$$

II. Express the following functions in terms of unit step function and hence find their Laplace transform.

1) $\boldsymbol{f(t) = \begin{cases} t^2 & for \;\; 0 < t \le 1 \\ 4t & for \;\; t > 1 \end{cases}}$

Solution: Here $f_1(t) = t^2$, $\quad f_2(t) = 4t$ and $a = 1$

We have $f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a)$

$$\therefore f(t) = t^2 + [4t - t^2]u(t-1)$$

$$L[f(t)] = L[t^2] + L[(4t - t^2)u(t-1)]$$

$$= \frac{2!}{s^{2+1}} + L[(4t - t^2)u(t-1)]$$

$$L[f(t)] = \frac{2}{s^3} + L[(4t - t^2)u(t-1)] ------(\mathbf{1})$$

Consider $(4t - t^2)u(t-1)$

Here $u(t-a) = u(t-1) \Longrightarrow \boldsymbol{a} = \mathbf{1}.$

Let $f(t-1) = 4t - t^2$

Replace t by $t+1$, we get

$$f(t+1-1) = 4(t+1) - (t+1)^2 = 4t + 4 - (t^2 + 1 + 2t)$$

$$f(t) = 4t + 4 - t^2 - 1 - 2t \Longrightarrow f(t) = -t^2 + 2t + 3$$

$$L[f(t)] = L[-t^2 + 2t + 3] = -L[t^2] + 2L[t] + L[3]$$

$$= -\frac{2!}{s^{2+1}} + 2\frac{1!}{s^{1+1}} + \frac{3}{s}$$

$$\therefore L[f(t)] = -\frac{\mathbf{2}}{\boldsymbol{s}^{\mathbf{3}}} + \frac{\mathbf{2}}{\boldsymbol{s}^{\mathbf{2}}} + \frac{\mathbf{3}}{\boldsymbol{s}} = \bar{\boldsymbol{f}}(\boldsymbol{s})$$

By second shifting property, we have $L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$.

$$\therefore L[(4t - t^2)u(t-1)\,] = e^{-(1)s}\left(-\frac{2}{s^3} + \frac{2}{s^2} + \frac{3}{s}\right)$$

$$L[(4t - t^2)u(t-1)\,] = e^{-s}\left(-\frac{2}{s^3} + \frac{2}{s^2} + \frac{3}{s}\right).$$

Using above result in equation (1), we get

$$\boldsymbol{L}[\boldsymbol{f}(\boldsymbol{t})] = \frac{\mathbf{2}}{\boldsymbol{s}^{\mathbf{3}}} + \boldsymbol{e}^{-\boldsymbol{s}}\left(-\frac{\mathbf{2}}{\boldsymbol{s}^{\mathbf{3}}} + \frac{\mathbf{2}}{\boldsymbol{s}^{\mathbf{2}}} + \frac{\mathbf{3}}{\boldsymbol{s}}\right).$$

2) $\boldsymbol{f}(\boldsymbol{t}) = \begin{cases} \mathbf{0}, & \mathbf{0} < \boldsymbol{t} < \mathbf{1} \\ \boldsymbol{t} - \mathbf{1}, & \mathbf{1} < \boldsymbol{t} < \mathbf{2} \\ \mathbf{1}, & \boldsymbol{t} > \mathbf{2} \end{cases}$

Solution: Here $f_1(t) = 0,\; f_2(t) = t - 1,\; f_3(t) = 1$ and $a = 1,\; b = 2.$

We have $f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a) + [f_3(t) - f_2(t)]u(t-b)$

$$\therefore f(t) = 0 + [(t-1) - 0]u(t-1) + [1 - (t-1)]u(t-2)$$

$$f(t) = (t-1)u(t-1) + (-t+2)u(t-2)$$

$$L[f(t)] = L[(t-1)u(t-1)] + L[(-t+2)u(t-2)] ----(\mathbf{1})$$

Consider $(t-1)u(t-1)$

Here $u(t-a)=u(t-1) \Longrightarrow \boldsymbol{a=1}$

Let $f(t-1)=t-1$

Replace t by $t+1$, we get

$$f(t+1-1)=t+1-1$$

$$f(t)=t$$

$$L[f(t)]=L[t]=\frac{1!}{s^{1+1}}$$

$$\therefore L[f(t)]=\frac{\mathbf{1}}{\boldsymbol{s^2}}=\boldsymbol{\bar{f}(s)}$$

Consider $(-t+2)u(t-2)$

Here $u(t-a)=u(t-2) \Longrightarrow \boldsymbol{a=2}$

Let $f(t-2)=-t+2$

Replace t by $t+2$, we get

$$f(t+2-2)=-(t+2)+2$$

$$f(t)=-t$$

$$L[f(t)]=L[-t]=-\frac{1!}{s^{1+1}}$$

$$\therefore L[f(t)]=-\frac{\mathbf{1}}{\boldsymbol{s^2}}=\boldsymbol{\bar{f}(s)}$$

By second shifting property, we have $L[f(t-a)u(t-a)]=e^{-as}\bar{f}(s)$.

$$\therefore L[(t-1)\,u(t-1)]=e^{-(1)s}\frac{1}{s^2}$$

$$\boldsymbol{L[(t-1)\,u(t-1)]=\frac{e^{-s}}{s^2}}$$

$$\therefore L[(-t+2)\,u(t-2)]=e^{-(2)s}\left(-\frac{1}{s^2}\right)$$

$$\boldsymbol{L[(-t+2)u(t-2)]=-\frac{e^{-2s}}{s^2}}$$

Using above results in the equation (1), we get

$$L[f(t)]=\frac{e^{-s}}{s^2}-\frac{e^{-2s}}{s^2}$$

$$\text{Thus}\quad \boldsymbol{L[f(t)]=\frac{e^{-s}-e^{-2s}}{s^2}}.$$

3) $\boldsymbol{f(t)=\begin{cases} t^2, & 0<t<2 \\ 4t, & 2<t\le 4 \\ 8, & t>4 \end{cases}}$

Solution: Here $f_1(t)=t^2,\ \ f_2(t)=4t,\ \ f_3(t)=8$ and $a=2,\ \ b=4.$

We have $f(t)=f_1(t)+[f_2(t)-f_1(t)]u(t-a)+[f_3(t)-f_2(t)]u(t-b)$

$$\therefore f(t)=t^2+[4t-t^2]u(t-2)+[8-4t]u(t-4)$$

$$f(t)=t^2+(4t-t^2)u(t-2)+(8-4t)u(t-4)$$

$$L[f(t)]=L[t^2]+L[(4t-t^2)u(t-2)]+L[(8-4t)u(t-4)]$$

$$L[f(t)]=\frac{2!}{s^{2+1}}+L[(4t-t^2)u(t-2)]+L[(8-4t)u(t-4)]$$

$$L[f(t)]=\frac{2}{s^3}+L[(4t-t^2)u(t-2)]+L[(8-4t)u(t-4)]----(\mathbf{1})$$

Consider $(4t-t^2)u(t-2)$

Here $u(t-a)=u(t-2) \Longrightarrow \boldsymbol{a=2}$

Let $f(t-2)=4t-t^2$

Replace t by $t+2$, we get

$f(t+2-2)=4(t+2)-(t+2)^2$

$f(t)=4t+8-(t^2+4+4t)$

$f(t)=4t+8-t^2-4-4t$

$f(t)=-t^2+4$

$L[f(t)]=-L[t^2]+L[4]$

$$L[f(t)]=-\frac{2!}{s^{2+1}}+\frac{4}{s}$$

$$\therefore L[f(t)]=-\frac{\mathbf{2}}{\boldsymbol{s^3}}+\frac{\mathbf{4}}{\boldsymbol{s}}=\boldsymbol{\bar{f}(s)}$$

Consider $(8-4t)u(t-4)$

Here $u(t-a)=u(t-4) \Longrightarrow \boldsymbol{a=4}$

Let $f(t-4)=8-4t$

Replace t by $t+4$, we get

$f(t+4-4)=8-4(t+4)$

$f(t)=8-4t-16$

$f(t)=-4t-8$

$f(t)=-4t-8$

$L[f(t)]=-4L[t]-L[8]$

$$L[f(t)]=-4\frac{1!}{s^{1+1}}-\frac{8}{s}$$

$$\therefore L[f(t)]=-\frac{\mathbf{4}}{\boldsymbol{s^2}}-\frac{\mathbf{8}}{\boldsymbol{s}}=\boldsymbol{\bar{f}(s)}$$

By second shifting property, we have $L[f(t-a)u(t-a)]=e^{-as}\bar{f}(s)$.

$$\therefore L[(4t-t^2)u(t-2)]=e^{-(2)s}\left(\frac{4}{s}-\frac{2}{s^3}\right)$$

$$\boldsymbol{L[(4t-t^2)u(t-2)]=e^{-2s}\left(\frac{4}{s}-\frac{2}{s^3}\right)}$$

$$\therefore L[(8-4t)u(t-4)]=e^{-(4)s}\left(-\frac{4}{s^2}-\frac{8}{s}\right)$$

$$\boldsymbol{L[(8-4t)u(t-4)]=e^{-4s}\left(-\frac{4}{s^2}-\frac{8}{s}\right)}$$

Using above results in equation (1), we get

$$\boldsymbol{L[f(t)]=\frac{2}{s^3}+e^{-2s}\left(\frac{4}{s}-\frac{2}{s^3}\right)+e^{-4s}\left(-\frac{4}{s^2}-\frac{8}{s}\right)}$$

$$\mathbf{4)}\ \boldsymbol{f(t)=\begin{cases}\sin t, & 0\le t<\pi\\ \sin 2t, & \pi\le t<2\pi\\ \sin 3t, & t\ge 2\pi\end{cases}}$$

Solution: Here $f_1(t)=\sin t \quad f_2(t)=\sin 2t \quad f_3(t)=\sin 3t \ \& \ a=\pi \quad b=2\pi$.

We have $f(t)=f_1(t)+[f_2(t)-f_1(t)]u(t-a)+[f_3(t)-f_2(t)]u(t-b)$

$$\therefore f(t)=\sin t+[\sin 2t-\sin t]u(t-\pi)+[\sin 3t-\sin 2t]u(t-2\pi)$$

$$L[f(t)]=L[\sin t]+L[(\sin 2t-\sin t)u(t-\pi)]+L[(\sin 3t-\sin 2t)\,u(t-2\pi)]$$

$$L[f(t)]=\frac{1}{s^2+1}+L[(\sin 2t-\sin t)u(t-\pi)]$$

$$+L[(\sin 3t-\sin 2t)\,u(t-2\pi)]-------(\mathbf{1})$$

Consider $(\boldsymbol{sin\,2t - sin\,t})\boldsymbol{u(t-\pi)}$

$$\text{Here } u(t-a) = u(t-\pi) \Rightarrow \boldsymbol{a = \pi.}$$

$$\text{Let } f(t-\pi) = \sin 2t - \sin t$$

$$\text{Replace } t \text{ by } t+\pi, \text{ we get}$$

$$f(t+\pi-\pi) = \sin 2(t+\pi) - \sin(t+\pi)$$

$$f(t) = \sin(2\pi+2t) - \sin(\pi+t) = \sin 2t + \sin t$$

$$L[f(t)] = L[\sin 2t] + L[\sin t]$$

$$\therefore L[f(t)] = \frac{\mathbf{2}}{\boldsymbol{s^2+4}} + \frac{\mathbf{1}}{\boldsymbol{s^2+1}} = \boldsymbol{\bar{f}(s)}$$

By second shifting property, we have $L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$.

$$\therefore \boldsymbol{L[(\sin 2t - \sin t)u(t-\pi)] = e^{-\pi s}\left(\frac{2}{s^2+4} + \frac{1}{s^2+1}\right)} ------ (\mathbf{2})$$

Consider $(\boldsymbol{sin\,3t - sin\,2t})\boldsymbol{u(t-2\pi)}$

$$\text{Here } u(t-a) = u(t-2\pi) \Rightarrow \boldsymbol{a = 2\pi.}$$

$$\text{Let } f(t-2\pi) = \sin 3t - \sin 2t$$

$$\text{Replace } t \text{ by } t+2\pi, \text{ we get}$$

$$f(t+2\pi-2\pi) = \sin 3(t+2\pi) - \sin 2(t+2\pi)$$

$$f(t) = \sin(6\pi+3t) - \sin(4\pi+2t) = \sin 3t - \sin 2t$$

$$L[f(t)] = L[\sin 3t] - L[\sin 2t]$$

$$\therefore L[f(t)] = \frac{\mathbf{3}}{\boldsymbol{s^2+9}} - \frac{\mathbf{2}}{\boldsymbol{s^2+4}} = \boldsymbol{\bar{f}(s)}$$

By second shifting property, we have $L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$.

$$\therefore \boldsymbol{L[(\sin 3t - \sin 2t)u(t-2\pi)] = e^{-2\pi s}\left(\frac{3}{s^2+9} - \frac{2}{s^2+4}\right)} ----(\mathbf{3})$$

Using equations (2) and (3) in equation (1), we get

$$\boldsymbol{L[f(t)] = \frac{1}{s^2+1} + e^{-\pi s}\left(\frac{2}{s^2+4} + \frac{1}{s^2+1}\right) + e^{-2\pi s}\left(\frac{3}{s^2+9} - \frac{2}{s^2+4}\right)}$$

EXERCISE PROBLEMS

I. Find the Laplace transform of the following functions

1) $(t-3)u(t-3)$ **Answer**: $\dfrac{e^{-3s}}{s^2}$

2) $(1+2t-3t^2)u(t-2)$ **Answer**: $e^{-2s}\left(-\dfrac{6}{s^3}-\dfrac{10}{s^2}-\dfrac{7}{s}\right)$

II. Express the following functions in terms of unit step function and hence find their Laplace transform.

1) $f(t)=\begin{cases} t-1 & for\ \ 1<t<2 \\ 3-t & for\ \ 2<t<3 \end{cases}$

Answer: $\dfrac{e^{-s}(1-e^{-s})^2}{s^2}$

2) $f(t)=\begin{cases} cos\ t, & 0<t<\pi \\ 1, & \pi<t<2\pi \\ sin\ t, & t>2\pi \end{cases}$

Answer: $\dfrac{s}{s^2+1}+e^{-\pi s}\left(\dfrac{1}{s}+\dfrac{s}{s^2+1}\right)-e^{-2\pi s}\left(\dfrac{1}{s}-\dfrac{1}{s^2+1}\right)$

3) $f(t)=\begin{cases} \cos t, & 0<t<\pi \\ \cos 2t, & \pi<t<2\pi \\ \cos 3t, & t>2\pi \end{cases}$

Answer: $\dfrac{s}{s^2+1}+e^{-\pi s}\left(\dfrac{s}{s^2+4}-\dfrac{s}{s^2+1}\right)+e^{-2\pi s}\left(\dfrac{s}{s^2+9}-\dfrac{s}{s^2+4}\right)$