Module 2: Inverse Laplace Transforms

Definition: If $L[f(t)] = \bar{f}(s)$, then $f(t)$ is called the **inverse Laplace transform** of $\bar{f}(s)$ and we write symbolically $L^{-1}[\bar{f}(s)] = f(t)$ where L^{-1} is called the inverse Laplace transformation operator.

Linearity Property

If $\bar{f}_1(s)$ and $\bar{f}_2(s)$ are the Laplace transforms of $f_1(t)$ and $f_2(t)$ respectively, then

$L^{-1}[c_1\bar{f}_1(s) + c_2\bar{f}_2(s)] = c_1L^{-1}[\bar{f}_1(s)] + c_2L^{-1}[\bar{f}_2(s)] = c_1f_1(t) + c_2f_2(t)$

where c_1 and c_2 are any constants.

Inverse Laplace Transform of Standard Functions

1. $L^{-1}\left[\frac{1}{s}\right] = 1$ & $L^{-1}\left[\frac{k}{s}\right] = k$ **(where k is any constant)**

Examples: $L^{-1}\left[\frac{3}{s}\right] = 3 \qquad L^{-1}\left[\frac{-7}{s}\right] = -7$

2. $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$ & $L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$

Examples: $L^{-1}\left[\frac{1}{s-2}\right] = e^{2t} \qquad L^{-1}\left[\frac{1}{s+5}\right] = e^{-5t}$

3. $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}sinat$ & $L^{-1}\left[\frac{s}{s^2+a^2}\right] = cosat$

Examples:

$$L^{-1}\left[\frac{1}{s^2+9}\right] = L^{-1}\left[\frac{1}{s^2+3^2}\right] = \frac{1}{3}\sin 3t$$

$$L^{-1}\left[\frac{1}{s^2+5}\right] = L^{-1}\left[\frac{1}{s^2+(\sqrt{5})^2}\right] = \frac{1}{\sqrt{5}}\sin\sqrt{5}t$$

$$L^{-1}\left[\frac{s}{s^2+4}\right] = L^{-1}\left[\frac{s}{s^2+2^2}\right] = \cos 2t$$

$$L^{-1}\left[\frac{s}{s^2+3}\right] = L^{-1}\left[\frac{s}{s^2+(\sqrt{3})^2}\right] = \cos\sqrt{3}t$$

4. $\mathbf{L^{-1}\left[\frac{1}{s^2-a^2}\right]=\frac{1}{a}sinhat}$ & $\mathbf{L^{-1}\left[\frac{s}{s^2-a^2}\right]=coshat}$

Examples: $\mathrm{L}^{-1}\left[\frac{1}{s^2-4}\right]=\mathrm{L}^{-1}\left[\frac{1}{s^2-2^2}\right]=\frac{1}{2}\sinh 2t$

$$\mathrm{L}^{-1}\left[\frac{1}{s^2-3}\right]=\mathrm{L}^{-1}\left[\frac{1}{s^2-(\sqrt{3})^2}\right]=\frac{1}{\sqrt{3}}\sinh\sqrt{3}t$$

$$\mathrm{L}^{-1}\left[\frac{s}{s^2-9}\right]=\mathrm{L}^{-1}\left[\frac{s}{s^2-3^2}\right]=\cosh 3t$$

$$\mathrm{L}^{-1}\left[\frac{s}{s^2-5}\right]=\mathrm{L}^{-1}\left[\frac{s}{s^2-(\sqrt{5})^2}\right]=\cosh\sqrt{5}t$$

5. $\mathbf{L^{-1}\left[\frac{1}{s^n}\right]=\frac{t^{n-1}}{(n-1)!}}$, **if n is a positive integer**

Examples: $\mathrm{L}^{-1}\left[\frac{1}{s^2}\right]=\frac{t^{2-1}}{(2-1)!}=\frac{t}{1}=t.$ $\quad\mathrm{L}^{-1}\left[\frac{1}{s^4}\right]=\frac{t^{4-1}}{(4-1)!}=\frac{t^3}{3!}=\frac{t^3}{6}$

1.1 Methods of Finding Inverse Laplace Transforms

METHOD 1: Direct Method

SOLVED PROBLEMS

1. Find the inverse Laplace transform of $\mathbf{\frac{2}{s+3}+\frac{5s}{s^2+9}}$

Solution: Let $\bar{f}(s)=\frac{2}{s+3}+\frac{5s}{s^2+9}$

Take the inverse Laplace transform of both sides

$$L^{-1}[\bar{f}(s)]=L^{-1}\left[\frac{2}{s+3}+\frac{5s}{s^2+9}\right]$$

$$=2L^{-1}\left[\frac{1}{s+3}\right]+5L^{-1}\left[\frac{s}{s^2+3^2}\right]\quad\text{(by linearity)}$$

We have $\mathbf{L^{-1}\left[\frac{1}{s+a}\right]=e^{-at}}$ & $\mathbf{L^{-1}\left[\frac{s}{s^2+a^2}\right]=\cos at}$

$$\therefore L^{-1}[\bar{f}(s)]=2e^{-3t}+5\cos 3t$$

Thus $\boldsymbol{L^{-1}[\bar{f}(s)]=f(t)=2e^{-3t}+5\cos 3t.}$

2. Find the inverse Laplace transform of $\dfrac{s^2 - 3s + 4}{s^3}$

Solution: Let $\bar{f}(s) = \dfrac{s^2 - 3s + 4}{s^3}$

The above can be written as

$$\bar{f}(s) = \frac{s^2}{s^3} - \frac{3s}{s^3} + \frac{4}{s^3} = \frac{1}{s} - \frac{3}{s^2} + \frac{4}{s^3}$$

Take the inverse Laplace transform of both sides

$$L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{1}{s} - \frac{3}{s^2} + \frac{4}{s^3}\right]$$

$$= L^{-1}\left[\frac{1}{s}\right] - 3L^{-1}\left[\frac{1}{s^2}\right] + 4L^{-1}\left[\frac{1}{s^3}\right] \quad \text{(by linearity)}$$

We have $\mathbf{L^{-1}}\left[\dfrac{1}{s}\right] = 1$ & $\mathbf{L^{-1}}\left[\dfrac{1}{s^n}\right] = \dfrac{t^{n-1}}{(n-1)!}$

$$\therefore L^{-1}[\bar{f}(s)] = 1 - 3\left(\frac{t^{2-1}}{(2-1)!}\right) + 4\left(\frac{t^{3-1}}{(3-1)!}\right)$$

$$= 1 - 3\left(\frac{t}{1!}\right) + 4\left(\frac{t^2}{2!}\right)$$

$$= 1 - 3\frac{t}{1} + 4\frac{t^2}{2}$$

Thus $\boldsymbol{L^{-1}[\bar{f}(s)] = f(t) = 1 - 3t + 2t^2}$.

3. Find the inverse Laplace transform of $\dfrac{2s + 3}{s^2 - 8}$

Solution: Let $\bar{f}(s) = \dfrac{2s + 3}{s^2 - 8}$

The above can be written as

$$\bar{f}(s) = \frac{2s}{s^2 - 8} + \frac{3}{s^2 - 8} = \frac{2s}{s^2 - (\sqrt{8})^2} + \frac{3}{s^2 - (\sqrt{8})^2}$$

Take the inverse Laplace transform of both sides

$$L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{2s}{s^2 - (\sqrt{8})^2} + \frac{3}{s^2 - (\sqrt{8})^2}\right]$$

$$= 2L^{-1}\left[\frac{s}{s^2 - (\sqrt{8})^2}\right] + 3L^{-1}\left[\frac{1}{s^2 - (\sqrt{8})^2}\right] \quad \text{(by linearity)}$$

We have $\mathbf{L^{-1}\left[\frac{s}{s^2-a^2}\right]=\cosh at}$ **&** $\mathbf{L^{-1}\left[\frac{1}{s^2-a^2}\right]=\frac{1}{a}\sinh at}$

$$\therefore L^{-1}[\bar{f}(s)] = 2\cosh\sqrt{8}t + 3\frac{1}{\sqrt{8}}\sinh\sqrt{8}t$$

Thus $\boldsymbol{L^{-1}[\bar{f}(s)] = f(t) = 2\cosh\sqrt{8}t + \frac{3}{\sqrt{8}}\sinh\sqrt{8}t}$

4. Find the inverse Laplace transform of $\mathbf{\frac{5}{2s-3}+\frac{4s}{9-s^2}}$

Solution: Let $\bar{f}(s) = \frac{5}{2s-3}+\frac{4s}{9-s^2}$

The above can be written as

$$\bar{f}(s) = \frac{5}{2(s-3/2)} - \frac{4s}{s^2-9}$$

Take the inverse Laplace transform of both sides

$$L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{5}{2(s-3/2)} - \frac{4s}{s^2-9}\right]$$

$$= \frac{5}{2}L^{-1}\left[\frac{1}{s-3/2}\right] - 4L^{-1}\left[\frac{s}{s^2-3^2}\right] \quad \text{(by linearity)}$$

We have $\boldsymbol{L^{-1}\left[\frac{1}{s-a}\right] = e^{at}}$ **&** $\boldsymbol{L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at}$

$$\therefore L^{-1}[\bar{f}(s)] = \frac{5}{2}e^{\frac{3}{2}t} - 4\cosh 3t$$

Thus $\boldsymbol{L^{-1}[\bar{f}(s)] = f(t) = \frac{5}{2}e^{\frac{3}{2}t} - 4\cosh 3t}$.

5. Find the inverse Laplace transform of $\mathbf{\frac{2s-5}{4s^2+25}}$

Solution: Let $\bar{f}(s) = \frac{2s-5}{4s^2+25}$

The above can be written as

$$\bar{f}(s) = \frac{2s-5}{4\left(s^2+\frac{25}{4}\right)} = \frac{2s}{4[s^2+(5/2)^2]} - \frac{5}{4[s^2+(5/2)^2]}$$

Take the inverse Laplace transform of both sides

$$L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{s}{2\{s^2+(5/2)^2\}} - \frac{5}{4\{s^2+(5/2)^2\}}\right]$$

$$= \frac{1}{2}L^{-1}\left[\frac{s}{s^2+(5/2)^2}\right] - \frac{5}{4}L^{-1}\left[\frac{1}{s^2+(5/2)^2}\right] \quad \text{(by linearity)}$$

We have $\mathbf{L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at}$ & $\mathbf{L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at}$

$$\therefore L^{-1}[\bar{f}(s)] = \frac{1}{2}\cos\left(\frac{5}{2}t\right) - \frac{5}{4}\cdot\frac{1}{5/2}\sin\left(\frac{5}{2}t\right)$$

$$= \frac{1}{2}\cos\left(\frac{5}{2}t\right) - \frac{5}{4}\cdot\frac{2}{5}\sin\left(\frac{5}{2}t\right)$$

$$= \frac{1}{2}\cos\left(\frac{5}{2}t\right) - \frac{1}{2}\sin\left(\frac{5}{2}t\right)$$

Thus $\boldsymbol{L^{-1}[\bar{f}(s)] = f(t) = \frac{1}{2}\left[\cos\left(\frac{5}{2}t\right) - \sin\left(\frac{5}{2}t\right)\right]}$.

EXERCISE PROBLEMS

Find the inverse Laplace transform of the following functions:

1. $\frac{2s+5}{s^2+4}$ **Answer**: $2\cos 2t + \frac{5}{2}\sin 2t$
2. $\frac{3s^2+4}{s^5}$ **Answer**: $\frac{3t^2}{2} + \frac{t^4}{6}$
3. $\frac{1}{3s^2+16}$ **Answer**: $\frac{1}{4\sqrt{3}}\sin\left(\frac{4}{\sqrt{3}}t\right)$
4. $\frac{3s-4}{16-s^2}$ **Answer**: $\sinh 4t - 3\cosh 4t$
5. $\frac{1}{s+2} + \frac{3}{2s+5} - \frac{4}{3s-2}$ **Answer**: $e^{-2t} + \frac{3}{2}e^{-\frac{5}{2}t} - \frac{4}{3}e^{\frac{2}{3}t}$

METHOD 2: Shifting Property

If $L^{-1}\left[\bar{f}(s)\right] = f(t)$, then $L^{-1}\left[\bar{f}(s-a)\right] = e^{at}L^{-1}\left[\bar{f}(s)\right] = e^{at}f(t)$

Or

If $L^{-1}\left[\bar{f}(s)\right] = f(t)$, then $L^{-1}\left[\bar{f}(s+a)\right] = e^{-at}L^{-1}\left[\bar{f}(s)\right] = e^{-at}f(t)$

SOLVED PROBLEMS

1. Find the inverse Laplace transform of $\frac{1}{(s+2)^2}$

Solution: By shifting property, we have

$$L^{-1}\left[\bar{f}(s+a)\right] = e^{-at}L^{-1}\left[\bar{f}(s)\right]$$

$$\therefore \mathrm{L}^{-1}\left[\frac{1}{(s+2)^2}\right] = e^{-2t}L^{-1}\left[\frac{1}{s^2}\right]$$

$$\textbf{We have } \mathbf{L^{-1}}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$$

$$\therefore \mathrm{L}^{-1}\left[\frac{1}{(s+2)^2}\right] = e^{-2t}\left(\frac{t^{2-1}}{(2-1)!}\right) = e^{-2t}\left(\frac{t}{1}\right)$$

$$\text{Thus } \mathbf{L^{-1}}\left[\frac{1}{(s+2)^2}\right] = e^{-at}f(t) = e^{-2t}t$$

2. Find the Inverse Laplace transform of $\frac{s}{(s-3)^5}$

Solution: $\frac{s}{(s-3)^5} = \frac{s+3-3}{(s-3)^5} = \frac{(s-3)+3}{(s-3)^5}$

By shifting property, we have

$$L^{-1}\left[\bar{f}(s-a)\right] = e^{at}L^{-1}\left[\bar{f}(s)\right]$$

$$\therefore \mathrm{L}^{-1}\left[\frac{s}{(s-3)^5}\right] = \mathrm{L}^{-1}\left[\frac{(s-3)+3}{(s-3)^5}\right] = e^{3t}L^{-1}\left[\frac{s+3}{s^5}\right]$$

$$= e^{3t}L^{-1}\left[\frac{s}{s^5}+\frac{3}{s^5}\right] = e^{3t}L^{-1}\left[\frac{1}{s^4}+\frac{3}{s^5}\right]$$

$$= e^{3t}\left[L^{-1}\left\{\frac{1}{s^4}\right\} + 3L^{-1}\left\{\frac{1}{s^5}\right\}\right] \qquad \text{by linearity}$$

$$\textbf{We have } \mathbf{L^{-1}}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$$

$$\therefore \mathrm{L}^{-1}\left[\frac{s}{(s-3)^5}\right] = e^{3t}\left[\left\{\frac{t^{4-1}}{(4-1)!}\right\} + 3\left\{\frac{t^{5-1}}{(5-1)!}\right\}\right]$$

$$= e^{3t}\left[\left\{\frac{t^3}{3!}\right\} + 3\left\{\frac{t^4}{4!}\right\}\right]$$

$$= e^{3t}\left[\left\{\frac{t^3}{6}\right\} + 3\left\{\frac{t^4}{24}\right\}\right]$$

Thus $\boldsymbol{L^{-1}\left[\frac{s}{(s-3)^5}\right] = e^{at}f(t) = e^{3t}\left(\frac{t^3}{6} + \frac{t^4}{8}\right)}$

3. Find the Inverse Laplace transform of $\frac{3s+1}{(s+1)^4}$

Solution: $\frac{3s+1}{(s+1)^4} = \frac{3s+1+3-3}{(s+1)^4} = \frac{3s+3-2}{(s+1)^4} = \frac{3(s+1)-2}{(s+1)^4}$

By shifting property, we have

$$L^{-1}[\bar{f}(s+a)] = e^{-at}L^{-1}[\bar{f}(s)]$$

$$\therefore \mathrm{L}^{-1}\left[\frac{3s+1}{(s+1)^4}\right] = \mathrm{L}^{-1}\left[\frac{3(s+1)-2}{(s+1)^4}\right] = e^{-(1)t}L^{-1}\left[\frac{3s-2}{s^4}\right]$$

$$= e^{-t}L^{-1}\left[\frac{3s}{s^4} - \frac{2}{s^4}\right] = e^{-t}L^{-1}\left[\frac{3}{s^3} - \frac{2}{s^4}\right]$$

$$= e^{-t}\left[3L^{-1}\left\{\frac{1}{s^3}\right\} - 2L^{-1}\left\{\frac{1}{s^4}\right\}\right] \qquad \text{by linearity}$$

We have $\boldsymbol{\mathrm{L}^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}}$

$$\therefore \mathrm{L}^{-1}\left[\frac{3s+1}{(s+1)^4}\right] = e^{-t}\left[3\left\{\frac{t^{3-1}}{(3-1)!}\right\} - 2\left\{\frac{t^{4-1}}{(4-1)!}\right\}\right]$$

$$= e^{-t}\left[3\left\{\frac{t^2}{2!}\right\} - 2\left\{\frac{t^3}{3!}\right\}\right]$$

$$= e^{-t}\left[3\left\{\frac{t^2}{2}\right\} - 2\left\{\frac{t^3}{6}\right\}\right]$$

Thus $\boldsymbol{L^{-1}\left[\frac{3s+1}{(s+1)^4}\right] = e^{-at}f(t) = e^{-t}\left(\frac{3t^2}{2} - \frac{t^3}{3}\right)}$

4. Find the inverse Laplace transform of $\dfrac{s+3}{s^2-4s+13}$

Solution: $s^2-4s+13 = s^2-4s+13+2^2-2^2$

$$= [s^2-(2)(2)s+2^2]+13-2^2$$

$$= (s-2)^2+9$$

Now $\dfrac{s+3}{s^2-4s+13} = \dfrac{s+3}{(s-2)^2+9} = \dfrac{s+3+2-2}{(s-2)^2+9} = \dfrac{(s-2)+5}{(s-2)^2+9}$

By shifting property, we have

$$L^{-1}[\bar{f}(s-a)] = e^{at}L^{-1}[\bar{f}(s)]$$

$$\therefore \mathrm{L}^{-1}\left[\frac{s+3}{s^2-4s+13}\right] = \mathrm{L}^{-1}\left[\frac{(s-2)+5}{(s-2)^2+9}\right] = e^{(2)t}L^{-1}\left[\frac{s+5}{s^2+9}\right]$$

$$= e^{2t}\left[L^{-1}\left\{\frac{s}{s^2+3^2}+\frac{5}{s^2+3^2}\right\}\right]$$

$$= e^{2t}\left[L^{-1}\left\{\frac{s}{s^2+3^2}\right\}+5L^{-1}\left\{\frac{1}{s^2+3^2}\right\}\right] \quad \text{by linearity}$$

We have $\mathbf{L}^{-1}\left[\dfrac{s}{s^2+a^2}\right] = \cos at$ & $\mathbf{L}^{-1}\left[\dfrac{1}{s^2+a^2}\right] = \dfrac{1}{a}\sin at$

$$\therefore \mathrm{L}^{-1}\left[\frac{s+3}{s^2-4s+13}\right] = e^{2t}\left[\cos 3t + 5\frac{1}{3}\sin 3t\right]$$

Thus $\boldsymbol{L^{-1}\left[\dfrac{s+3}{s^2-4s+13}\right] = e^{at}f(t)} = e^{2t}\left(\cos 3t+\dfrac{5}{3}\sin 3t\right)$

EXERCISE PROBLEMS

Find the inverse Laplace transform of the following functions:

1. $\dfrac{s}{(s+2)^2}$ **Answer**: $e^{-2t}(1-2t)$

2. $\dfrac{s}{s^2+4s+5}$ **Answer**: $e^{-2t}(\cos t - 2\sin t)$

3. $\dfrac{s+2}{s^2-2s+5}$ **Answer**: $e^{t}(\cos 2t + \dfrac{3}{2}\sin 2t)$

4. $\dfrac{2s}{s^2+2s+5}$ **Answer**: $e^{-t}(2\cos 2t - \sin 2t)$

METHOD 3: Inverse Laplace Transform Using Derivatives

If $L^{-1}[\bar{f}(s)] = f(t)$, then $L^{-1}\left[-\frac{d}{ds}\{\bar{f}(s)\}\right] = tf(t)$

Or

If $L^{-1}[\bar{f}(s)] = f(t)$, then $L^{-1}[-\bar{f}'(s)] = tf(t)$

Note: This method is used to find inverse Laplace transform of logarithmic functions and inverse functions.

SOLVED PROBLEMS

1. Find the inverse Laplace transform of $log\left(\frac{s+1}{s-1}\right)$

Solution: Let $\bar{f}(s) = log\left(\frac{s+1}{s-1}\right)$

$$\bar{f}(s) = \log(s+1) - \log(s-1) \qquad by \log\left(\frac{m}{n}\right) = \log m - \log n$$

Differentiate the above with respect to s, we get

$$\frac{d}{ds}\{\bar{f}(s)\} = \frac{d}{ds}[\log(s+1) - \log(s-1)]$$

$$\bar{f}'(s) = \frac{d}{ds}\{\log(s+1)\} - \frac{d}{ds}\{\log(s-1)\}$$

$$\bar{f}'(s) = \frac{1}{s+1} - \frac{1}{s-1}$$

$$\therefore\ -\bar{f}'(s) = \frac{1}{s-1} - \frac{1}{s+1}$$

Take the inverse Laplace transform of both sides

$$L^{-1}[-\bar{f}'(s)] = L^{-1}\left[\frac{1}{s-1} - \frac{1}{s+1}\right]$$

Using $L^{-1}[-\bar{f}'(s)] = tf(t)$ and linear property in the above, we get

$$tf(t) = L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{1}{s+1}\right]$$

We have, $\mathbf{L^{-1}\left[\frac{1}{s-a}\right] = e^{at}\ \&\ L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}}$

$$\therefore tf(t) = e^{(1)t} - e^{(-1)t}$$

$$tf(t) = e^{t} - e^{-t}$$

$$f(t)=\frac{e^{t}-e^{-t}}{t}$$

$$\text{Thus } \mathbf{L^{-1}}\left[\boldsymbol{log}\left(\frac{\boldsymbol{s+1}}{\boldsymbol{s-1}}\right)\right]=\boldsymbol{f(t)}=\frac{\boldsymbol{e^{t}-e^{-t}}}{\boldsymbol{t}}$$

2. Find the inverse Laplace transform of $\boldsymbol{log\left(\frac{s^2+1}{s^2+4}\right)}$

Solution: Let $\bar{f}(s)=log\left(\frac{s^2+1}{s^2+4}\right)$

$$\bar{f}(s)=\log(s^2+1)-\log(s^2+4) \qquad by \log\left(\frac{m}{n}\right)=\log m-\log n$$

Differentiate the above with respect to s, we get

$$\frac{d}{ds}\{\bar{f}(s)\}=\frac{d}{ds}[\log(s^2+1)-\log(s^2+4)]$$

$$\bar{f}'(s)=\frac{d}{ds}\{\log(s^2+1)\}-\frac{d}{ds}\{\log(s^2+4)\}$$

$$\bar{f}'(s)=\frac{1}{s^2+1}(2s)-\frac{1}{s^2+4}(2s)$$

$$\bar{f}'(s)=\frac{2s}{s^2+1}-\frac{2s}{s^2+4}$$

$$\therefore\ -\bar{f}'(s)=\frac{2s}{s^2+4}-\frac{2s}{s^2+1}$$

Take the inverse Laplace transform of both sides

$$L^{-1}[-\bar{f}'(s)]=L^{-1}\left[\frac{2s}{s^2+4}-\frac{2s}{s^2+1}\right]$$

Using $L^{-1}[-\bar{f}'(s)]=tf(t)$ and linear property in the above, we get

$$tf(t)=2L^{-1}\left[\frac{s}{s^2+2^2}\right]-2L^{-1}\left[\frac{s}{s^2+1^2}\right]$$

We have, $\mathbf{L^{-1}}\left[\frac{\boldsymbol{s}}{\boldsymbol{s^2+a^2}}\right]=\boldsymbol{cos\ at}$

$$\therefore tf(t)=2\cos 2t-2\cos t$$

$$tf(t)=2(\cos 2t-\cos t)$$

$$f(t)=\frac{2(\cos 2t-\cos t)}{t}$$

$$\text{Thus } \mathbf{L^{-1}}\left[\boldsymbol{log}\left(\frac{\boldsymbol{s^2+1}}{\boldsymbol{s^2+4}}\right)\right]=\boldsymbol{f(t)}=\frac{\boldsymbol{2(\cos 2t-\cos t)}}{\boldsymbol{t}}$$

3. Find the inverse Laplace transform of $\boldsymbol{log\left(\frac{s^2+1}{s(s+1)}\right)}$

Solution: Let $\bar{f}(s) = log\left(\frac{s^2+1}{s(s+1)}\right)$

$$\bar{f}(s) = \log(s^2+1) - \log\{s(s+1)\} \qquad by \log\left(\frac{m}{n}\right) = \log m - \log n$$

$$\bar{f}(s) = \log(s^2+1) - [\log s + \log(s+1)] \qquad by \log(mn) = \log m + \log n$$

$$\bar{f}(s) = \log(s^2+1) - \log s - \log(s+1)$$

Differentiate the above with respect to s, we get

$$\frac{d}{ds}\{\bar{f}(s)\} = \frac{d}{ds}[\log(s^2+1) - \log s - \log(s+1)]$$

$$\bar{f}'(s) = \frac{d}{ds}\{\log(s^2+1)\} - \frac{d}{ds}\{\log s\} - \frac{d}{ds}\{\log(s+1)\}$$

$$\bar{f}'(s) = \frac{1}{s^2+1}(2s) - \frac{1}{s} - \frac{1}{s+1}$$

$$\bar{f}'(s) = \frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}$$

$$\therefore\ -\bar{f}'(s) = \frac{1}{s} + \frac{1}{s+1} - \frac{2s}{s^2+1}$$

Take the inverse Laplace transform of both sides

$$L^{-1}[-\bar{f}'(s)] = L^{-1}\left[\frac{1}{s} + \frac{1}{s+1} - \frac{2s}{s^2+1}\right]$$

Using $L^{-1}[-\bar{f}'(s)] = tf(t)$ and linear property in the above, we get

$$tf(t) = L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s+1}\right] - 2L^{-1}\left[\frac{s}{s^2+1^2}\right]$$

We have, $\mathbf{L}^{-1}\left[\frac{1}{s}\right] = 1,\quad \mathbf{L}^{-1}\left[\frac{1}{s+a}\right] = e^{-at} \quad \& \quad \mathbf{L}^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

$$\therefore tf(t) = 1 + e^{(-1)t} - 2\cos t$$

$$tf(t) = 1 + e^{-t} - 2\cos t$$

$$f(t) = \frac{1 + e^{-t} - 2\cos t}{t}$$

Thus $\mathbf{L}^{-1}\left[\boldsymbol{log\left(\frac{s^2+1}{s(s+1)}\right)}\right] = \boldsymbol{f(t)} = \boldsymbol{\frac{1+e^{-t}-2\cos t}{t}}$

4. Find the inverse Laplace transform of $\boldsymbol{tan^{-1}\left(\frac{1}{s}\right)}$

Solution: Let $\bar{f}(s) = tan^{-1}\left(\frac{1}{s}\right)$

Differentiate the above with respect to s, we get

$$\frac{d}{ds}\{\bar{f}(s)\} = \frac{d}{ds}\left[tan^{-1}\left(\frac{1}{s}\right)\right]$$

$$\bar{f}'(s) = \frac{1}{1+\left(\frac{1}{s}\right)^2}\left(\frac{-1}{s^2}\right) \qquad by\frac{d}{dx}[\tan^{-1}x] = \frac{1}{1+x^2}$$

$$\bar{f}'(s) = \frac{-1}{1+\frac{1}{s^2}}\left(\frac{1}{s^2}\right) = \frac{-1}{\frac{s^2+1}{s^2}}\left(\frac{1}{s^2}\right) = \frac{-1}{s^2+1}$$

$$\therefore\ -\bar{f}'(s) = \frac{1}{s^2+1}$$

Take the inverse Laplace transform of both sides

$$L^{-1}[-\bar{f}'(s)] = L^{-1}\left[\frac{1}{s^2+1}\right]$$

Using $L^{-1}[-\bar{f}'(s)] = tf(t)$ in the above, we get

$$tf(t) = L^{-1}\left[\frac{1}{s^2+1^2}\right]$$

We have, $\boldsymbol{L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\, sin\, at}$

$$\therefore tf(t) = \frac{1}{1}\sin 1t$$

$$tf(t) = \sin t$$

$$f(t) = \frac{\sin t}{t}$$

$$\text{Thus } \mathbf{L^{-1}}\left[\boldsymbol{tan^{-1}\left(\frac{1}{s}\right)}\right] = \boldsymbol{f(t) = \frac{\sin t}{t}}$$

EXERCISE PROBLEMS

Find the inverse Laplace transform of the following functions:

1. $log\left(\frac{s+6}{s+2}\right)$ **Answer**: $\frac{e^{-2t}-e^{-6t}}{t}$

2. $log\left(\frac{s+1}{s^2}\right)$ **Answer**: $\frac{2-e^{-t}}{t}$

3. $log\left(\frac{s^2+1}{(s-1)^2}\right)$ **Answer**: $\frac{2(e^t-\cos t)}{t}$

4. $\tan^{-1}\left(\frac{2}{s}\right)$ **Answer**: $\frac{\sin 2t}{t}$

5. $\cot^{-1}\left(\frac{s}{a}\right)$ **Answer**: $\frac{\sin at}{t}$

METHOD 4: Partial Fraction Method

By partial fraction, we have

1. $\frac{1}{(s+a)(s+b)} = \frac{A}{(s+a)} + \frac{B}{(s+b)}$

2. $\frac{1}{(s+a)(s+b)^2} = \frac{A}{(s+a)} + \frac{B}{(s+b)} + \frac{C}{(s+b)^2}$

3. $\frac{1}{(s+a)(s^2+bs+c)} = \frac{A}{(s+a)} + \frac{Bs+C}{(s^2+bs+c)}$

SOLVED PROBLEMS

1. Find the Inverse Laplace transform of $\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}$

Solution: By partial fraction we can write the given as

$$\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)} = \frac{A}{(s-1)} + \frac{B}{(s-2)} + \frac{C}{(s-3)}$$

$$2s^2-6s+5 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2) - - - - (1)$$

Put s = 1 in (1) $2-6+5 = A(1-2)(1-3)$ $1 = A(-1)(-2)$ $1 = 2A$ $\therefore \boldsymbol{A = \frac{1}{2}}$	Put s = 2 in (1) $2(4)-6(2)+5 = B(2-1)(2-3)$ $8-12+5 = B(1)(-1)$ $1 = -B$ $\therefore \boldsymbol{B = -1}$

Put s = 3 in (1) $2(9)-6(3)+5 = C(3-1)(3-2)$ $18-18+5 = C(1)(2)$ $5 = 2C \quad \therefore \boldsymbol{C = \frac{5}{2}}$

$$\therefore \frac{2s^2-6s+5}{(s-1)(s-2)(s-3)} = \frac{1/2}{s-1} + \frac{(-1)}{s-2} + \frac{5/2}{s-3}$$

Take the inverse Laplace transform of both sides

$$L^{-1}\left[\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}\right] = L^{-1}\left[\frac{1/2}{s-1} + \frac{(-1)}{s-2} + \frac{5/2}{s-3}\right]$$

$$= \frac{1}{2}L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{1}{s-2}\right] + \frac{5}{2}L^{-1}\left[\frac{1}{s-3}\right] \quad \text{(by linearity)}$$

We have $\mathbf{L^{-1}\left[\frac{1}{s-a}\right] = e^{at}}$

$$\therefore\ L^{-1}\left[\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}\right] = \frac{1}{2}e^{t} - e^{2t} + \frac{5}{2}e^{3t}$$

$$\text{Thus } \boldsymbol{L^{-1}\left[\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}\right] = f(t) = \frac{1}{2}e^{t} - e^{2t} + \frac{5}{2}e^{3t}}.$$

2. Find the Inverse Laplace transform of $\frac{s-1}{s^2+3s+2}$

Solution: Here $\frac{s-1}{s^2+3s+2} = \frac{s-1}{(s+1)(s+2)}$ by Factorization

By partial fraction we can write the above as

$$\frac{s-1}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

$$s-1 = A(s+2) + B(s+1) ----(1)$$

Put $s = -2$ in (1) $-2-1 = A(0) + B(-2+1)$ $-3 = -B$ $\therefore\ \boldsymbol{B = 3}$	Put $s = -1$ in (1) $-1-1 = A(-1+2) + B(0)$ $-2 = A$ $\therefore\ \boldsymbol{A = -2}$

$$\therefore\ \frac{s-1}{(s+1)(s+2)} = \frac{-2}{(s+1)} + \frac{3}{(s+2)}$$

Take the inverse Laplace transform of both sides

$$L^{-1}\left[\frac{s-1}{(s+1)(s+2)}\right] = L^{-1}\left[\frac{-2}{(s+1)} + \frac{3}{(s+2)}\right]$$

$$= -2L^{-1}\left[\frac{1}{s+1}\right] + 3L^{-1}\left[\frac{1}{s+2}\right] \quad \text{(by linearity)}$$

We have $\mathbf{L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}}$

$$\therefore\ L^{-1}\left[\frac{s-1}{(s+1)(s+2)}\right] = -2e^{-t} + 3e^{-2t}$$

$$\text{Thus } \boldsymbol{L^{-1}\left[\frac{s-1}{s^2+3s+2}\right] = f(t) = 3e^{-2t} - 2e^{-t}}.$$

3. Find the Inverse Laplace transform of $\frac{4s+5}{(s+2)(s-1)^2}$

Solution: By partial fraction we can write the given as

$$\frac{4s+5}{(s+2)(s-1)^2} = \frac{A}{(s+2)} + \frac{B}{(s-1)} + \frac{C}{(s-1)^2}$$

$$4s+5 = A(s-1)^2 + B(s+2)(s-1) + C(s+2) ----(1)$$

Put $\mathrm{s} = -2$ in (1) $-8+5 = A(-2-1)^2 + B(0) + C(0)$ $-3 = A(9)$ $\therefore \boldsymbol{A} = \frac{-\mathbf{1}}{\mathbf{3}}$	Put $\mathrm{s} = 1$ in (1) $4+5 = A(0) + B(0) + C(1+2)$ $9 = C(3)$ $\therefore \boldsymbol{C} = \mathbf{3}$

Put $\mathrm{s} = 0$ in (1) $5 = A - 2B + 2C$ $5 = -\frac{1}{3} - 2B + 2(3) = \frac{17}{3} - 2B$ $2B = \frac{17}{3} - 5$ $2B = \frac{2}{3} \quad \therefore \boldsymbol{B} = \frac{\mathbf{1}}{\mathbf{3}}$

$$\therefore \frac{4s+5}{(s+2)(s-1)^2} = \frac{-\frac{1}{3}}{(s+2)} + \frac{\frac{1}{3}}{(s-1)} + \frac{3}{(s-1)^2}$$

Take the inverse Laplace transform of both sides

$$L^{-1}\left[\frac{4s+5}{(s+2)(s-1)^2}\right] = L^{-1}\left[\frac{-\frac{1}{3}}{(s+2)} + \frac{\frac{1}{3}}{(s-1)} + \frac{3}{(s-1)^2}\right]$$

$$= -\frac{1}{3}L^{-1}\left[\frac{1}{s+2}\right] + \frac{1}{3}L^{-1}\left[\frac{1}{s-1}\right] + 3L^{-1}\left[\frac{1}{(s-1)^2}\right] \qquad \text{(by linearity)}$$

We have $\mathbf{L}^{-1}\left[\frac{\mathbf{1}}{\boldsymbol{s}+\boldsymbol{a}}\right] = \boldsymbol{e}^{-\boldsymbol{at}}, \mathbf{L}^{-1}\left[\frac{\mathbf{1}}{\boldsymbol{s}-\boldsymbol{a}}\right] = \boldsymbol{e}^{\boldsymbol{at}}$ & $\boldsymbol{L}^{-1}[\bar{\boldsymbol{f}}(\boldsymbol{s}-\boldsymbol{a})] = \boldsymbol{e}^{\boldsymbol{at}}\boldsymbol{L}^{-1}[\bar{\boldsymbol{f}}(\boldsymbol{s})]$

$$\therefore L^{-1}\left[\frac{4s+5}{(s+2)(s-1)^2}\right] = -\frac{1}{3}e^{-2t} + \frac{1}{3}e^{t} + 3\,e^{(1)t}L^{-1}\left[\frac{1}{s^2}\right]$$

We have $\mathbf{L}^{-1}\left[\frac{\mathbf{1}}{\boldsymbol{s}^{\boldsymbol{n}}}\right] = \frac{\boldsymbol{t}^{\boldsymbol{n}-\mathbf{1}}}{(\boldsymbol{n}-\mathbf{1})!}$

$$\therefore\ L^{-1}\left[\frac{4s+5}{(s+2)(s-1)^2}\right] = -\frac{1}{3}e^{-2t} + \frac{1}{3}e^{t} + 3\,e^{t}\,\frac{t^{2-1}}{(2-1)!}$$

$$= -\frac{1}{3}e^{-2t} + \frac{1}{3}e^{t} + 3\,e^{t}\,\frac{t}{1!}$$

Thus $\boldsymbol{L^{-1}\left[\frac{4s+5}{(s+2)(s-1)^2}\right] = f(t) = \left(\frac{1}{3}+3t\right)e^{t} - \frac{1}{3}e^{-2t}}$.

4. Find the Inverse Laplace transform of $\frac{s}{(s-3)(s^2+4)}$

Solution: By partial fraction we can write the given as

$$\frac{s}{(s-3)(s^2+4)} = \frac{A}{(s-3)} + \frac{Bs+C}{(s^2+4)}$$

$$s = A(s^2+4) + (Bs+C)(s-3) - - - - (1)$$

Put $s = 3$ in (1) $3 = A(9+4) + (Bs+C)(0)$ $3 = A(13)$ $\therefore \boldsymbol{A = \frac{3}{13}}$	Put $s = 0$ in (1) $0 = 4A - 3C$ $0 = 4\left(\frac{3}{13}\right) - 3C =$ $3C = \frac{12}{13}$ $\quad\therefore \boldsymbol{C = \frac{4}{13}}$

$$(1) \Longrightarrow s = As^2 + 4A + Bs^2 - 3Bs + Cs - 3C - - - - (2)$$

Comparing the coefficients of s^2 in (2), we get

$$0 = A + B$$

$$B = -A = -\left(\frac{3}{13}\right)$$

$$\therefore \boldsymbol{B = -\frac{3}{13}}$$

$$\therefore \frac{s}{(s-3)(s^2+4)} = \frac{\frac{3}{13}}{(s-3)} + \frac{\left(\frac{-3}{13}\right)s + \frac{4}{13}}{(s^2+4)}$$

Take the inverse Laplace transform of both sides

$$L^{-1}\left[\frac{s}{(s-3)(s^2+4)}\right] = L^{-1}\left[\frac{\frac{3}{13}}{(s-3)} + \frac{\left(\frac{-3}{13}\right)s + \frac{4}{13}}{(s^2+4)}\right]$$

$$= L^{-1}\left[\frac{\frac{3}{13}}{(s-3)} + \frac{\left(\frac{-3}{13}\right)s}{(s^2+4)} + \frac{\frac{4}{13}}{(s^2+4)}\right]$$

$$= \frac{3}{13}L^{-1}\left[\frac{1}{s-3}\right] - \frac{3}{13}L^{-1}\left[\frac{s}{s^2+2^2}\right] + \frac{4}{13}L^{-1}\left[\frac{1}{s^2+2^2}\right] \quad \text{(by linearity)}$$

We have $\mathbf{L^{-1}\left[\frac{1}{s-a}\right] = e^{at}, L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at}$ **&** $\mathbf{L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at}$

$$\therefore\ L^{-1}\left[\frac{s}{(s-3)(s^2+4)}\right] = \frac{3}{13}e^{3t} - \frac{3}{13}\cos 2t + \frac{4}{13}\left(\frac{1}{2}\sin 2t\right)$$

$$= \frac{3}{13}e^{3t} - \frac{3}{13}\cos 2t + \frac{2}{13}\sin 2t$$

Thus $\boldsymbol{L^{-1}\left[\frac{s}{(s-3)(s^2+4)}\right] = f(t) = \frac{1}{13}[3e^{3t} - 3\cos 2t + 2\sin 2t]}$

EXERCISE PROBLEMS

Find the inverse Laplace transform of the following functions:

1. $\frac{s}{(s+2)(s+3)}$ **Answer**: $-2e^{-2t} + 3e^{-3t}$

2. $\frac{1-7s}{(s-3)(s-1)(s+2)}$ **Answer**: $-2e^{-2t} + 3e^{-3t}$

3. $\frac{s^2+s-2}{s(s+3)(s-2)}$ **Answer**: $\frac{7}{3} + \frac{4}{15}e^{-3t} + \frac{2}{5}e^{2t}$

4. $\frac{1}{(s+2)(s+1)^2}$ **Answer**: $e^{-2t} - e^{-t} + te^{-t}$

5. $\frac{1}{(s-1)(s^2+1)}$ **Answer**: $\frac{1}{2}[e^t - \cos t - \sin t]$

1.2 Convolution

Definition: The convolution of two functions $f(t)$ & $g(t)$ usually denoted by $f(t) * g(t)$ is defined in the form of an integral as follows

$$f(t) * g(t) = \int_0^t f(u)g(t-u)du$$

Note: Convolution operation '$*$' is commutative i.e., $f(t) * g(t) = g(t) * f(t)$.

Convolution Theorem: If $L^{-1}\left[\bar{f}(s)\right] = f(t)$ & $L^{-1}[\bar{g}(s)] = g(t)$, then

$$L^{-1}\left[\bar{f}(s)\,\bar{g}(s)\right] = \int_0^t f(u)g(t-u)du = f(t) * g(t)$$

Or

$$L^{-1}\left[\bar{f}(s)\,\bar{g}(s)\right] = \int_0^t f(t-u)g(u)du = f(t) * g(t)$$

SOLVED PROBLEMS

1) Apply convolution theorem to find inverse Laplace transform of $\frac{1}{s(s^2+4)}$.

Solution: First we express the given function as a product of two functions

$$\frac{1}{s(s^2+4)} = \frac{1}{s}\frac{1}{(s^2+4)} = \bar{f}(s)\,\bar{g}(s)$$

$$\text{Let } \bar{f}(s) = \frac{1}{s} \quad \& \quad \bar{g}(s) = \frac{1}{(s^2+4)}$$

$$f(t) = L^{-1}\left[\bar{f}(s)\right] = L^{-1}\left[\frac{1}{s}\right] = 1$$

$$g(t) = L^{-1}[\bar{g}(s)] = L^{-1}\left[\frac{1}{s^2+4}\right] = L^{-1}\left[\frac{1}{s^2+2^2}\right] = \frac{1}{2}\sin 2t$$

$$\therefore\ \boldsymbol{f(t) = 1 \qquad \& \qquad g(t) = \frac{1}{2}\sin 2t}$$

By Convolution theorem, we have

$$L^{-1}\left[\bar{f}(s)\,\bar{g}(s)\right] = \int_0^t f(u)g(t-u)du = f(t) * g(t)$$

$$\therefore\ L^{-1}\left[\frac{1}{s(s^2+4)}\right] = \int_0^t (1)\left[\frac{1}{2}\sin 2(t-u)\right] du$$

$$= \frac{1}{2}\int_0^t \sin(2t-2u)\, du$$

$$= \frac{1}{2}\left[-\frac{\cos(2t-2u)}{(-2)}\right]_0^t$$

$$= \frac{1}{4}[\cos(2t-2u)]_0^t$$

$$= \frac{1}{4}[\cos(2t-2t)-\cos(2t-0)]$$

$$= \frac{1}{4}[\cos 0-\cos 2t] = \frac{1}{4}[1-\cos 2t]$$

Thus $\boldsymbol{L^{-1}\left[\frac{1}{s(s^2+4)}\right] = \frac{1}{4}[1-\cos 2t]}$

2) Use convolution theorem to find inverse Laplace transform of $\boldsymbol{\frac{s}{(s^2+a^2)^2}}$.

Solution: First we express the given function as a product of two functions

$$\frac{s}{(s^2+a^2)^2} = \frac{1}{(s^2+a^2)}\frac{s}{(s^2+a^2)} = \bar{f}(s)\,\bar{g}(s)$$

$$\text{Let } \bar{f}(s) = \frac{1}{(s^2+a^2)} \quad \& \quad \bar{g}(s) = \frac{s}{(s^2+a^2)}$$

$$f(t) = L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$$

$$g(t) = L^{-1}[\bar{g}(s)] = L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$$

$$\therefore\ \boldsymbol{f(t) = \frac{1}{a}\sin at \qquad \& \qquad g(t) = \cos at}$$

By Convolution theorem, we have

$$L^{-1}[\bar{f}(s)\,\bar{g}(s)] = \int_0^t f(u)g(t-u)du = f(t)*g(t)$$

$$\therefore\ L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \int_0^t \left[\frac{1}{a}\sin au\right][\cos a(t-u)]du$$

$$= \frac{1}{a}\int_0^t \sin au \cos(at - au)\, du$$

We know that $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

$$\therefore L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{a}\int_0^t \frac{1}{2}[\sin\{au + (at - au)\} + \sin\{au - (at - au)\}]du$$

$$= \frac{1}{2a}\int_0^t [\sin\{au + at - au\} + \sin\{au - at + au\}]du$$

$$= \frac{1}{2a}\int_0^t [\sin at + \sin(2au - at)]du$$

$$= \frac{1}{2a}\left[\int_0^t \sin at\, du + \int_0^t \sin(2au - at)\, du\right]$$

$$= \frac{1}{2a}\left[\sin at \int_0^t 1\, du + \int_0^t \sin(2au - at)\, du\right]$$

$$= \frac{1}{2a}\left[\sin at\, \{u\}_0^t + \left\{-\frac{\cos(2au - at)}{2a}\right\}_0^t\right]$$

$$= \frac{1}{2a}\left[\sin at\, \{u\}_0^t - \frac{1}{2a}\{\cos(2au - at)\}_0^t\right]$$

$$= \frac{1}{2a}\left[\sin at\, \{t - 0\} - \frac{1}{2a}\{\cos(2at - at) - \cos(0 - at)\}\right]$$

$$= \frac{1}{2a}\left[t \sin at - \frac{1}{2a}\{\cos at - \cos(-at)\}\right]$$

$$= \frac{1}{2a}\left[t \sin at - \frac{1}{2a}\{\cos at - \cos at\}\right]$$

$$= \frac{1}{2a}\left[t \sin at - \frac{1}{2a}(0)\right] = \frac{1}{2a} t \sin at$$

Thus $\boldsymbol{L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{2a} t \sin at}$

3) Apply convolution theorem to evaluate $\boldsymbol{L^{-1}\left[\frac{1}{(s^2+1)(s^2+9)}\right]}$.

Solution: First we express the given function as a product of two functions

$$\frac{1}{(s^2+1)(s^2+9)} = \frac{1}{(s^2+1)}\frac{1}{(s^2+9)} = \bar{f}(s)\,\bar{g}(s)$$

$$\text{Let } \bar{f}(s) = \frac{1}{(s^2+1)} \quad \& \quad \bar{g}(s) = \frac{1}{(s^2+9)}$$

$$f(t) = L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{1}{s^2+1}\right] = L^{-1}\left[\frac{1}{s^2+1^2}\right] = \frac{1}{1}\sin 1t = \sin t$$

$$g(t) = L^{-1}[\bar{g}(s)] = L^{-1}\left[\frac{1}{s^2+9}\right] = L^{-1}\left[\frac{1}{s^2+3^2}\right] = \frac{1}{3}\sin 3t$$

$$\therefore\ \boldsymbol{f(t) = \sin t} \qquad \& \qquad \boldsymbol{g(t) = \frac{1}{3}\sin 3t}$$

By Convolution theorem, we have

$$L^{-1}[\bar{f}(s)\,\bar{g}(s)] = \int_0^t f(u)g(t-u)du = f(t) * g(t)$$

$$\therefore L^{-1}\left[\frac{1}{(s^2+1)(s^2+9)}\right] = \int_0^t [\sin u]\left[\frac{1}{3}\sin 3(t-u)\right] du$$

$$= \frac{1}{3}\int_0^t \sin u \sin(3t-3u)\, du$$

We know that $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$$\therefore L^{-1}\left[\frac{1}{(s^2+1)(s^2+9)}\right] = \frac{1}{3}\int_0^t \frac{1}{2}[\cos\{u-(3t-3u)\} - \cos\{u+(3t-3u)\}]du$$

$$= \frac{1}{6}\int_0^t [\cos\{u-3t+3u\} - \cos\{u+3t-3u\}]du$$

$$= \frac{1}{6}\int_0^t [\cos(4u-3t) - \cos(3t-2u)]du$$

$$= \frac{1}{6}\left[\int_0^t \cos(4u-3t)\, du - \int_0^t \cos(3t-2u)\, du\right]$$

$$= \frac{1}{6}\left[\left\{\frac{\sin(4u-3t)}{4}\right\}_0^t - \left\{\frac{\sin(3t-2u)}{(-2)}\right\}_0^t\right]$$

$$= \frac{1}{6}\left[\frac{1}{4}\{\sin(4u-3t)\}_0^t + \frac{1}{2}\{\sin(3t-2u)\}_0^t\right]$$

$$= \frac{1}{6}\left[\frac{1}{4}\{\sin(4t-3t) - \sin(0-3t)\} + \frac{1}{2}\{\sin(3t-2t) - \sin(3t-0)\}\right]$$

$$= \frac{1}{6}\left[\frac{1}{4}\{\sin t - \sin(-3t)\} + \frac{1}{2}\{\sin t - \sin 3t\}\right]$$

$$= \frac{1}{6}\left[\frac{1}{4}\{\sin t + \sin 3t\} + \frac{1}{2}\{\sin t - \sin 3t\}\right]$$

$$= \frac{1}{6}\left[\frac{1}{4}\sin t + \frac{1}{4}\sin 3t + \frac{1}{2}\sin t - \frac{1}{2}\sin 3t\right]$$

$$= \frac{1}{6}\left[\frac{3}{4}\sin t - \frac{1}{4}\sin 3t\right]$$

Thus $\boldsymbol{L^{-1}\left[\frac{1}{(s^2+1)(s^2+9)}\right] = \frac{1}{24}(3\sin t - \sin 3t)}$

4) Using convolution theorem find inverse Laplace transform of $\boldsymbol{\frac{s}{(s+2)(s^2+9)}}$.

Solution: First we express the given function as a product of two functions

$$\frac{s}{(s+2)(s^2+9)} = \frac{1}{(s+2)}\frac{s}{(s^2+9)} = \bar{f}(s)\,\bar{g}(s)$$

$$\text{Let } \bar{f}(s) = \frac{1}{(s+2)} \quad \& \quad \bar{g}(s) = \frac{s}{(s^2+9)}$$

$$f(t) = L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{1}{s+2}\right] = e^{-2t}$$

$$g(t) = L^{-1}[\bar{g}(s)] = L^{-1}\left[\frac{s}{s^2+9}\right] = L^{-1}\left[\frac{s}{s^2+3^2}\right] = \cos 3t$$

$$\therefore\ \boldsymbol{f(t) = e^{-2t}} \qquad \& \qquad \boldsymbol{g(t) = \cos 3t}$$

By Convolution theorem, we have

$$L^{-1}[\bar{f}(s)\,\bar{g}(s)] = \int_0^t f(t-u)g(u)du = f(t) * g(t)$$

$$\therefore\ L^{-1}\left[\frac{1}{(s^2+1)(s^2+9)}\right] = \int_0^t [e^{-2(t-u)}][\cos 3u)]du$$

$$= \int_0^t e^{-2t+2u}\cos 3u\,du = \int_0^t e^{-2t}e^{2u}\cos 3u\,du$$

$$= e^{-2t}\int_0^t e^{2u}\cos 3u\,du$$

We know that $\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2+b^2}(a\cos bu + b\sin bu)$

$$\therefore\ L^{-1}\left[\frac{1}{(s^2+1)(s^2+9)}\right] = e^{-2t}\left[\frac{e^{2u}}{2^2+3^2}(2\cos 3u + 3\sin 3u)\right]_0^t$$

$$= e^{-2t}\left[\frac{e^{2u}}{13}(2\cos 3u + 3\sin 3u)\right]_0^t$$

$$= \frac{e^{-2t}}{13}[e^{2u}(2\cos 3u + 3\sin 3u)]_0^t$$

$$= \frac{e^{-2t}}{13}[\{e^{2t}(2\cos 3t + 3\sin 3t)\} - \{e^0(2\cos 0 + 3\sin 0)\}]$$

$$= \frac{e^{-2t}}{13}[e^{2t}(2\cos 3t + 3\sin 3t) - (1)\{2(1) + 3(0)\}]$$

$$= \frac{e^{-2t}}{13}(2e^{2t}\cos 3t + 3e^{2t}\sin 3t - 2)$$

$$= \frac{1}{13}(2\cos 3t + 3\sin 3t - 2e^{-2t})$$

Thus $\boldsymbol{L^{-1}\left[\frac{s}{(s+2)(s^2+9)}\right] = \frac{1}{13}(2\cos 3t + 3\sin 3t - 2e^{-2t})}$

EXERCISE PROBLEMS

Find the inverse Laplace transform of the following functions using convolution theorem:

1. $\frac{1}{s(s^2+a^2)}$ **Answer**: $\frac{1}{a^2}(1-\cos at)$
2. $\frac{s}{(s^2+4)^2}$ **Answer**: $\frac{t\sin 2t}{4}$
3. $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$ **Answer**: $\frac{a\sin at - b\sin bt}{a^2-b^2}$
4. $\frac{1}{(s-1)(s^2+1)}$ **Answer**: $\frac{1}{2}(e^t - \sin t - \cos t)$
5. $\frac{2}{s^2(s^2+4)}$ **Answer**: $\frac{1}{4}(2t - \sin 2t)$

1.3 Application of Laplace Transform to Differential Equations

Laplace transform method of solving differential equations yields particular solution without the necessity of first finding the general solution and then evaluating the arbitrary constants. In general, this method is shorter than our earlier methods and is specially useful for solving linear differential equations with constant coefficients.

Working Procedure to solve linear differential equation with constant coefficient by Laplace transform method.

- Take the Laplace transform of both sides of the differential equation and use the following formulae then the given initial conditions

$$L[f'(t)] = s\bar{f}(s) - f(0) \quad \boldsymbol{or} \quad \boldsymbol{L[y'(t)] = s\bar{y}(s) - y(0)}$$

$$L[f''(t)] = s^2\bar{f}(s) - sf(0) - f'(0) \quad \boldsymbol{or} \quad \boldsymbol{L[y''(t)] = s^2\bar{y}(s) - sy(0) - y'(0)}$$

$$L[f'''(t)] = s^3\bar{f}(s) - s^2f(0) - sf'(0) - f''(0)$$

$$\boldsymbol{or}$$

$$\boldsymbol{L[y'''(t)] = s^3\bar{y}(s) - s^2y(0) - sy'(0) - y''(0)}$$

$$\boldsymbol{In\ general,\ \ L[f^n(t)] = s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{n-1}(0)}$$

$$\boldsymbol{L[y^n(t)] = s^n\bar{y}(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{n-1}(0)}$$

$$\boldsymbol{where\ \ \bar{y}(s) = L[y(t)]}$$

- Transpose the terms with minus sign to the RHS.
- Divide by the coefficient of $\bar{y}$, getting $\bar{y}$ as a known function of s.
- Take the inverse transform of both sides and use appropriate method to find inverse Laplace transform. This gives y as a function of t which is desired solution satisfying the given conditions.

SOLVED PROBLEMS

1) Use Laplace transform method to solve $\boldsymbol{\frac{dy}{dt} + y = te^{-t}}$ $\boldsymbol{with\ y(0) = 2}$.

Solution: Given $\quad y'(t) + y(t) = te^{-t}$

Taking the Laplace transform of both the sides

$$L[y'(t)] + L[y(t)] = L[te^{-t}]$$

$$\textbf{Using } L[y'(t)] = s\bar{y}(s) - y(0), \quad L[y(t)] = \bar{y}(s)$$

$$\& \ L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[L\{f(t)\}], \textbf{ we get}$$

$$[s\bar{y}(s) - y(0)] + \bar{y}(s) = -\frac{d}{ds}[L\{e^{-t}\}]$$

Using the given condition $y(0) = 2$, the above reduces to

$$s\bar{y}(s) - 2 + \bar{y}(s) = -\frac{d}{ds}\left[\frac{1}{s+1}\right]$$

$$(s+1)\bar{y}(s) - 2 = -\left[-\frac{1}{(s+1)^2}\right]$$

$$(s+1)\bar{y}(s) = \frac{1}{(s+1)^2} + 2$$

$$(s+1)\bar{y}(s) = \frac{1 + 2(s+1)^2}{(s+1)^2}$$

$$\therefore \bar{y}(s) = \frac{1 + 2(s+1)^2}{(s+1)^3}$$

Take the inverse Laplace transform of both sides

$$L^{-1}[\bar{y}(s)] = y(t) = L^{-1}\left[\frac{1 + 2(s+1)^2}{(s+1)^3}\right]$$

$$\textit{We have } L^{-1}[\bar{f}(s+a)] = e^{-at}L^{-1}[\bar{f}(s)]$$

$$\therefore y(t) = e^{-t}L^{-1}\left[\frac{1+2s^2}{s^3}\right] = e^{-t}L^{-1}\left[\frac{1}{s^3} + \frac{2s^2}{s^3}\right] = e^{-t}L^{-1}\left[\frac{1}{s^3} + \frac{2}{s}\right]$$

$$= e^{-t}\left[L^{-1}\left(\frac{1}{s^3}\right) + 2L^{-1}\left(\frac{1}{s}\right)\right] \qquad \text{by linearity}$$

$$\textbf{We have } \mathrm{L}^{-1}\left[\frac{1}{s}\right] = 1 \quad \& \quad \mathrm{L}^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$$

$$= e^{-t}\left[\frac{t^{3-1}}{(3-1)!} + 2(1)\right] = e^{-t}\left[\frac{t^2}{2!} + 2(1)\right]$$

$$\text{Thus } y(t) = e^{-t}\left(2 + \frac{t^2}{2}\right)$$

2) Use Laplace transform technique to solve $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = e^{-t}$ ***with*** $y(0) = y'(0) = 0$.

Solution: Given $y''(t) + 4y'(t) + 3y(t) = e^{-t}$

Taking the Laplace transform of both the sides

$$L[y''(t)] + 4L[y'(t)] + 3L[y(t)] = L[e^{-t}]$$

Using $\boldsymbol{L[y''(t)] = s^2\bar{y}(s) - sy(0) - y'(0)}$, $\boldsymbol{L[y'(t)] = s\bar{y}(s) - y(0)}$

$\boldsymbol{L[y(t)] = \bar{y}(s)}$ and $\boldsymbol{L[e^{-at}] = \dfrac{1}{s+a}}$ we get

$$[s^2\bar{y}(s) - sy(0) - y'(0)] + 4[s\bar{y}(s) - y(0)] + 3\bar{y}(s) = \frac{1}{s+1}$$

Using the initial conditions $y(0) = 0$ & $y'(0) = 0$, the above reduces to

$$s^2\bar{y}(s) + 4s\bar{y}(s) + 3\bar{y}(s) = \frac{1}{s+1}$$

$$(s^2 + 4s + 3)\bar{y}(s) = \frac{1}{s+1}$$

$$(s+1)(s+3)\bar{y}(s) = \frac{1}{(s+1)}$$

$$\therefore \bar{y}(s) = \frac{1}{(s+1)^2(s+3)}$$

Take the inverse Laplace transform of both sides

$$L^{-1}[\bar{y}(s)] = y(t) = L^{-1}\left[\frac{1}{(s+1)^2(s+3)}\right] - - - - - - (1)$$

Let $\dfrac{1}{(s+1)^2(s+3)} = \dfrac{A}{(s+1)} + \dfrac{B}{(s+1)^2} + \dfrac{C}{s+3}$ by partial fraction

$$\therefore 1 = A(s+1)(s+3) + B(s+3) + C(s+1)^2 - - - - (2)$$

Put s $= -1$ in (2)	Put s $= -3$ in (2)
$1 = A(0) + B(-1+3) + C(0)$	$1 = A(0) + B(0) + C(-3+1)^2$
$1 = 2B$	$1 = C(4)$
$\therefore \boldsymbol{B = \frac{1}{2}}$	$\therefore \boldsymbol{C = \frac{1}{4}}$

Put s $= 0$ in (2)

$$1 = 3A + 3B + C$$

$$1 = 3A + 3\left(\frac{1}{2}\right) + \frac{1}{4} = 3A + \frac{7}{4}$$

$$3A = 1 - \frac{7}{4} = -\frac{3}{4} \qquad \therefore \boldsymbol{A = -\frac{1}{4}}$$

$$\therefore \frac{1}{(s+1)^2(s+3)} = \frac{-1/4}{(s+1)} + \frac{1/2}{(s+1)^2} + \frac{1/4}{s+3}$$

Using the above in (1), we get

$$y(t) = L^{-1}\left[\frac{1}{(s+1)^2(s+3)}\right] = L^{-1}\left[\frac{-1/4}{(s+1)} + \frac{1/2}{(s+1)^2} + \frac{1/4}{s+3}\right]$$

$$y(t) = -\frac{1}{4}L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{(s+1)^2}\right] + \frac{1}{4}L^{-1}\left[\frac{1}{s+3}\right] \qquad \text{by linearity}$$

We have $\mathbf{L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}}$ **&** $\mathbf{L^{-1}[\bar{f}(s+a)] = e^{-at}L^{-1}[\bar{f}(s)]}$

$$\therefore y(t) = -\frac{1}{4}e^{-(1)t} + \frac{1}{2}e^{-(1)t}L^{-1}\left[\frac{1}{s^2}\right] + \frac{1}{4}e^{-3t}$$

We have $\mathbf{L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}}$

$$\therefore y(t) = -\frac{1}{4}e^{-t} + \frac{1}{2}e^{-t}\left[\frac{t^{2-1}}{(2-1)!}\right] + \frac{1}{4}e^{-3t}$$

$$= -\frac{1}{4}e^{-t} + \frac{1}{2}e^{-t}\left(\frac{t}{1}\right) + \frac{1}{4}e^{-3t}$$

Thus $\mathbf{y(t) = \left(\frac{t}{2} - \frac{1}{4}\right)e^{-t} + \frac{1}{4}e^{-3t}}$

3) Apply Laplace transform method to solve $\mathbf{\frac{d^2y}{dt^2} - y = t \ \ with\ y(0) = 0}$ **and** $\mathbf{y'(0) = 0}$**.**

Solution: Given $y''(t) - y(t) = t$

Taking the Laplace transform of both the sides

$$L[y''(t)] - L[y(t)] = L[t]$$

Using $\mathbf{L[y''(t)] = s^2\bar{y}(s) - sy(0) - y'(0)}$ **,** $\mathbf{L[y(t)] = \bar{y}(s)}$

and $\mathbf{L[t] = \frac{1}{s^2}}$ **we get**

$$[s^2\bar{y}(s) - sy(0) - y'(0)] - \bar{y}(s) = \frac{1}{s^2}$$

Using the initial conditions $y(0) = 0$ & $y'(0) = 0$, the above reduces to

$$s^2\bar{y}(s) - \bar{y}(s) = \frac{1}{s^2}$$

$$(s^2 - 1)\bar{y}(s) = \frac{1}{s^2}$$

$$\therefore \bar{y}(s) = \frac{1}{s^2(s^2-1)}$$

Take the inverse Laplace transform of both sides

$$L^{-1}[\bar{y}(s)] = y(t) = L^{-1}\left[\frac{1}{s^2(s^2-1)}\right] - - - - - (1)$$

$$\text{Let } \frac{1}{s^2(s^2-1)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2-1} \quad \text{by partial fraction}$$

$$1 = (As+B)(s^2-1) + (Cs+D)s^2$$

$$\therefore 1 = As^3 + Bs^2 - As - B + Cs^3 + Ds^2 - - - - (2)$$

Put $s = 0$ in (2), we get

$$1 = -B \quad \therefore \boldsymbol{B = -1}$$

Comparing the coefficients of s in (2), we get

$$0 = -A \quad \therefore \boldsymbol{A = 0}$$

Comparing the coefficients of s^2 in (2), we get

$$0 = B + D$$

$$D = -B = -(-1) = 1 \quad \therefore \boldsymbol{D = 1}$$

Comparing the coefficients of s^3 in (2), we get

$$0 = A + C$$

$$C = -A = -(0) = 0 \quad \therefore \boldsymbol{C = 0}$$

$$\therefore \frac{1}{s^2(s^2-1)} = \frac{(0)s+(-1)}{s^2} + \frac{(0)s+1}{s^2-1} = \frac{-1}{s^2} + \frac{1}{s^2-1}$$

Using the above in (1), we get

$$y(t) = L^{-1}\left[\frac{1}{s^2(s^2-1)}\right] = L^{-1}\left[\frac{-1}{s^2} + \frac{1}{s^2-1}\right]$$

$$y(t) = -L^{-1}\left[\frac{1}{s^2}\right] + L^{-1}\left[\frac{1}{s^2-1^2}\right] \quad \text{by linearity}$$

We have $\boldsymbol{L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}} \quad \boldsymbol{L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{1}{a} sinh\, at}$

$$\therefore y(t) = -\frac{t^{2-1}}{(2-1)!} + \frac{1}{1}\sinh 1t$$

$$\therefore y(t) = -\left(\frac{t}{1!}\right) + \sinh t = -t + \sinh t$$

$$\text{Thus } \boldsymbol{y(t) = sinh\, t - t}$$

4) Apply Laplace transform technique to solve $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t$

with $x = 2$, $\frac{dx}{dt} = -1$ at $t = 0$.

Solution: *Given* $x''(t) - 2\,x'(t) + x(t) = e^t$

Taking the Laplace transform of both the sides

$$L[x''(t)] - 2\,L[x'(t)] + L[x(t)] = L[e^t]$$

Using $L[x''(t)] = s^2\bar{x}(s) - sx(0) - x'(0)$,

$L[x'(t)] = s\bar{x}(s) - x(0)$ and $L[x(t)] = \bar{x}(s)$, we get

$$[s^2\bar{x}(s) - sx(0) - x'(0)] - 2[s\bar{x}(s) - x(0)] + \bar{x}(s) = \frac{1}{s-1}$$

Using the initial conditions $x(0) = 2$ & $x'(0) = -1$, the above reduces to

$$[s^2\bar{x}(s) - s(2) - (-1)] - 2[s\bar{x}(s) - 2] + \bar{x}(s) = \frac{1}{s-1}$$

$$(s^2 - 2s + 1)\bar{x}(s) - 2s + 5 = \frac{1}{s-1}$$

$$(s^2 - 2s + 1)\bar{x}(s) = \frac{1}{s-1} + 2s - 5 = \frac{1 + (2s-5)(s-1)}{s-1}$$

$$(s-1)^2\bar{x}(s) = \frac{1 + (2s^2 - 7s + 5)}{s-1} = \frac{2s^2 - 7s + 6}{s-1}$$

$$\therefore \bar{x}(s) = \frac{2s^2 - 7s + 6}{(s-1)^3}$$

Take the inverse Laplace transform of both sides

$$L^{-1}[\bar{x}(s)] = x(t) = L^{-1}\left[\frac{2s^2 - 7s + 6}{(s-1)^3}\right] ----(1)$$

$$\frac{2s^2 - 7s + 6}{(s-1)^3} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3} \qquad \text{by partial fraction}$$

$$\therefore 2s^2 - 7s + 6 = A(s-1)^2 + B(s-1) + C$$

$$2s^2 - 7s + 6 = A[s^2 - 2s + 1] + Bs - B + C$$

$$2s^2 - 7s + 6 = As^2 - 2As + A + Bs - B + C ----(2)$$

Comparing the coefficients of s^2 in (2), we get

$$2 = A \qquad \therefore \boldsymbol{A = 2}$$

Comparing the coefficients of s in (2), we get

$$-7 = -2A + B$$

$$-7 = -2(2) + B$$

$$B = -7 + 4$$

$$\therefore B = -3$$

Comparing the constants in (2), we get **or** Put $s = 0$ in (2), we get

$$6 = A - B + C$$

$$6 = 2 - (-3) + C$$

$$C = 6 - 5$$

$$\therefore C = 1$$

$$\therefore \frac{2s^2 - 7s + 6}{(s-1)^3} = \frac{2}{s-1} + \frac{-3}{(s-1)^2} + \frac{1}{(s-1)^3}$$

Using the above in (1), we get

$$x(t) = L^{-1}\left[\frac{2s^2 - 7s + 6}{(s-1)^3}\right] = L^{-1}\left[\frac{2}{s-1} + \frac{-3}{(s-1)^2} + \frac{1}{(s-1)^3}\right]$$

$$x(t) = 2L^{-1}\left[\frac{1}{s-1}\right] - 3L^{-1}\left[\frac{1}{(s-1)^2}\right] + L^{-1}\left[\frac{1}{(s-1)^3}\right] \qquad \text{by linearity}$$

We have $\mathbf{L^{-1}\left[\frac{1}{s-a}\right] = e^{at}}$ **&** $\mathbf{L^{-1}[\bar{f}(s-a)] = e^{at}L^{-1}[\bar{f}(s)]}$

$$\therefore x(t) = 2e^{(1)t} - 3e^{(1)t}L^{-1}\left[\frac{1}{s^2}\right] + e^{(1)t}L^{-1}\left[\frac{1}{s^3}\right]$$

We have $\mathbf{L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}}$

$$\therefore x(t) = 2e^t - 3e^t\left[\frac{t^{2-1}}{(2-1)!}\right] + e^t\left[\frac{t^{3-1}}{(3-1)!}\right]$$

$$= 2e^t - 3e^t\left(\frac{t}{1!}\right) + e^t\left(\frac{t^2}{2!}\right)$$

$$= 2e^t - 3te^t + \frac{t^2}{2}e^t$$

Thus $\boldsymbol{x(t) = \left(2 - 3t + \frac{t^2}{2}\right)e^t}$

5) An altenating e. m. f. $E\sin\omega t$ is applied to an inductance L and capacitance C in series, show that by using Laplace transform method that the current in the circuit is $\frac{E\omega}{L(p^2-\omega^2)}(\cos\omega t - \cos pt)$ where $p^2 = \frac{1}{LC}$.

Solution: Let i be a current and q be the charge in the circuit, then its differential equation is

$$\mathbf{L}\frac{di}{dt} + \frac{q}{\boldsymbol{C}} = \boldsymbol{E}\sin\omega t$$

$$\mathbf{L}\ \mathrm{i}'(\mathrm{t}) + \frac{q}{\boldsymbol{C}} = \boldsymbol{E}\sin\omega t$$

Taking the Laplace transform of both the sides

$$\mathbf{L}\ L[i'(t)] + \frac{1}{\boldsymbol{C}}L[q] = \boldsymbol{E}L[\sin\omega t]$$

Using $L[i'(t)] = s\,\bar{i}(s) - i(0)$ and $L[i(t)] = \bar{i}(s)$, we get

$$\mathbf{L}[s\,\bar{i}(s) - i(0)] + \frac{1}{\boldsymbol{C}}L[q] = \boldsymbol{E}\left[\frac{\omega}{s^2+\omega^2}\right]$$

Since $i = 0$ & $q = 0$ $at\ t = 0$ $[intially]$, the above reduces to

$$\mathbf{L}\ s\,\bar{i}(s) + \frac{1}{\boldsymbol{C}}L[q] = \frac{\boldsymbol{E}\omega}{s^2+\omega^2} - - - - (1)$$

Also, we have $i = \frac{dq}{dt} \Rightarrow L[i(t)] = L[q'(t)] \Rightarrow \bar{i}(s) = sL[q] - q(0)$

$$\Rightarrow\ \bar{i}(s) = sL[q] - 0 \Rightarrow \boldsymbol{L}[\boldsymbol{q}] = \frac{\bar{\boldsymbol{i}}(\boldsymbol{s})}{\boldsymbol{s}}$$

Now equation (1) become

$$\mathbf{L}\ s\,\bar{i}(s) + \frac{1}{\boldsymbol{C}}\frac{\bar{i}(s)}{s} = \frac{\boldsymbol{E}\omega}{s^2+\omega^2} \Rightarrow \left[\mathbf{L}\ s\ + \frac{1}{\boldsymbol{C}s}\right]\bar{i}(s) = \frac{\boldsymbol{E}\omega}{s^2+\omega^2}$$

$$\frac{\mathbf{L}}{\mathrm{s}}\left[s^2\ + \frac{1}{\mathbf{LC}}\right]\bar{i}(s) = \frac{\boldsymbol{E}\omega}{s^2+\omega^2}$$

$$\frac{\mathbf{L}}{\mathrm{s}}[s^2\ + p^2]\bar{i}(s) = \frac{\boldsymbol{E}\omega}{s^2+\omega^2} \qquad where\ p^2 = \frac{1}{\mathbf{LC}}$$

$$\therefore\ \bar{i}(s) = \frac{\boldsymbol{E}\omega s}{\mathbf{L}(\mathrm{s}^2+\mathrm{p}^2)(s^2+\omega^2)}$$

$$\bar{i}(s) = \frac{\boldsymbol{E}\omega}{\mathbf{L}} \frac{s}{(\mathrm{s}^2 + \mathrm{p}^2)(s^2 + \omega^2)}$$

Take the inverse Laplace transform of both sides

$$L^{-1}[\bar{i}(s)] = i(t) = L^{-1}\left[\frac{\boldsymbol{E}\omega}{\mathbf{L}} \frac{s}{(\mathrm{s}^2 + \mathrm{p}^2)(s^2 + \omega^2)}\right]$$

$$i(t) = \frac{\boldsymbol{E}\omega}{\mathbf{L}} L^{-1}\left[\frac{s}{(\mathrm{s}^2 + \mathrm{p}^2)(s^2 + \omega^2)}\right] ----(2)$$

$$\frac{s}{(\mathrm{s}^2 + \mathrm{p}^2)(s^2 + \omega^2)} = \frac{As + B}{s^2 + p^2} + \frac{Cs + D}{s^2 + \omega^2} \quad \text{by partial fraction}$$

$$\therefore s = (As + B)(s^2 + \omega^2) + (Cs + D)(s^2 + p^2)$$

$$s = As^3 + Bs^2 + \omega^2 As + B\omega^2 + Cs^3 + Ds^2 + p^2 Cs + Dp^2 ----(3)$$

Comparing the coefficients of s^3 in (3), we get

$$0 = A + C \quad \Rightarrow \quad C = -A$$

Comparing the coefficients of s^2 in (3), we get

$$0 = B + D \quad \Rightarrow \quad D = -B$$

Comparing the coefficients of s in (3), we get

$$1 = \omega^2 A + p^2 C \ \Rightarrow 1 = \omega^2 A - p^2 A$$

$$\Rightarrow 1 = (\omega^2 - p^2)A \Rightarrow \boldsymbol{A} = \frac{\mathbf{1}}{\boldsymbol{\omega^2 - p^2}}$$

$$\text{Since } \ C = -A \Rightarrow \boldsymbol{C} = -\frac{\mathbf{1}}{\boldsymbol{\omega^2 - p^2}}$$

Comparing the constants in (3), we get **or** Put $s = 0$ in (2), we get

$$0 = \ \omega^2 B + p^2 D \Rightarrow 0 = \omega^2 B - p^2 B$$

$$\Rightarrow 0 = (\omega^2 - p^2)B \Rightarrow B = \frac{0}{\omega^2 - p^2} \Rightarrow \boldsymbol{B} = \mathbf{0}$$

$$\text{Since } \ D = -B \Rightarrow \boldsymbol{D} = \mathbf{0}$$

$$\therefore \frac{s}{(\mathrm{s}^2 + \mathrm{p}^2)(s^2 + \omega^2)} = \frac{\left(\frac{1}{\omega^2 - p^2}\right)s + (0)}{s^2 + p^2} + \frac{\left(-\frac{1}{\omega^2 - p^2}\right)s + (0)}{s^2 + \omega^2}$$

$$\frac{s}{(\mathrm{s}^2 + \mathrm{p}^2)(s^2 + \omega^2)} = \left(\frac{1}{\omega^2 - p^2}\right)\frac{s}{s^2 + p^2} - \left(\frac{1}{\omega^2 - p^2}\right)\frac{s}{s^2 + \omega^2}$$

Take the inverse Laplace transform of both sides

$$\therefore L^{-1}\left[\frac{s}{(\mathrm{s}^2 + \mathrm{p}^2)(s^2 + \omega^2)}\right] = L^{-1}\left[\left(\frac{1}{\omega^2 - p^2}\right)\frac{s}{s^2 + p^2} - \left(\frac{1}{\omega^2 - p^2}\right)\frac{s}{s^2 + \omega^2}\right]$$

$$L^{-1}\left[\frac{s}{(\mathrm{s}^2+\mathrm{p}^2)(s^2+\omega^2)}\right] = \left(\frac{1}{\omega^2-p^2}\right)L^{-1}\left[\frac{s}{s^2+p^2}\right] - \left(\frac{1}{\omega^2-p^2}\right)L^{-1}\left[\frac{s}{s^2+\omega^2}\right]$$

We have $\mathbf{L^{-1}\left[\frac{s}{s^2+a^2}\right] = \boldsymbol{cos\ at}}$

$$\therefore L^{-1}\left[\frac{s}{(\mathrm{s}^2+\mathrm{p}^2)(s^2+\omega^2)}\right] = \left(\frac{1}{\omega^2-p^2}\right)\cos pt - \left(\frac{1}{\omega^2-p^2}\right)\cos \omega t$$

$$= \left(\frac{1}{\omega^2-p^2}\right)[\cos pt - \cos \omega t]$$

$$= \frac{-(\cos \omega t - \cos pt)}{-(p^2-\omega^2)}$$

$$\therefore L^{-1}\left[\frac{s}{(\mathrm{s}^2+\mathrm{p}^2)(s^2+\omega^2)}\right] = \frac{\cos \omega t - \cos pt}{p^2-\omega^2}$$

Using above result in (2), we get

$$i(t) = \frac{E\omega}{\mathrm{L}}\left[\frac{\cos \omega t - \cos pt}{p^2-\omega^2}\right]$$

$$\boldsymbol{Thus} \quad \boldsymbol{i(t)} = \frac{\boldsymbol{E\omega}}{\mathbf{L}(\boldsymbol{p^2-\omega^2})}(\mathbf{cos}\,\boldsymbol{\omega t} - \mathbf{cos}\,\boldsymbol{pt})$$

EXERCISE PROBLEMS

1) Apply Laplace transform method to solve $\frac{dy}{dt} + y = sin\ t\ \ with\ y(0) = 0.$

Answer: $y(t) = \frac{1}{2}[e^{-t} + \sin t - \cos t]$

2) Use Laplace transform method to solve $\frac{d^2y}{dt^2} + y = t\ \ with\ y(0) = y'(0) = 0$.

Answer: $y(t) = t - \sin t + \cos t$

3) Apply Laplace transform technique to solve $y'' - 3y' - 4y = 2\, e^{-t}\ \ with$

$y(0) = 1\ and\ y'(0) = 1$.

Answer: $y(t) = \frac{12}{25}e^{4t} + \frac{13}{25}e^{-t} - \frac{2}{5}te^{-t}$

4) Use Laplace transform technique to solve $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 10 \sin t\ \ e^{-t}\ \ with$

$y(0) = 0\ and\ y'(0) = 0$.

Answer: $y(t) = \sin t - 2\cos t + \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t}$

5) Use Laplace transform method to solve $y''' + 2y'' - y' - 2y = 0\ \ given$

$y(0) = y'(0) = 0\ and\ y''(0) = 6.$

Answer: $y(t) = e^{t} - 3e^{-t} + 2e^{-2t}$