Practical 4

Show that the image of the open disk

D1(-1-i)= {z: |z + 1 +i| < 1}
under the linear transformation

w = f(z) = (3 – 4i)z + 6 + 2i

is the open disk:

D5(–1 + 3i) = {w: |w + 1 – 3i| < 5}.

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(%i1) kill(all);
(%o0) *done*

(%i1) f(z):=block(
           [x, y],
           x:realpart(z),
           y:imagpart(z),
           w:rectform((3−4·%i)·(x+y·%i)+(6+2·%i))
       );
(%o1) $f(z):=\text{block}([x,y],x:\text{realpart}(z),y:\text{imagpart}(z),w:$
       $\text{rectform}((3-4\,\%i)\,(x+y\,\%i)+(6+2\,\%i)))$

(%i2) c:f(−1−%i);
(c) $3\,\%i-1$

(%i3) r(t, s):=−1−%i+s·(cos(t)+%i·sin(t));
(%o3) $r(t,s):=-1-\%i+s\,(\cos(t)+\%i\,\sin(t))$

(%i4) zdomain:makelist(parametric(realpart(r(t, s)), imagpart(r(t, s)), t,   0, 2·%pi),

(zdomain) $\left[\text{parametric}\left(\dfrac{\cos(t)}{5}-1,\dfrac{\sin(t)}{5}-1,t,0,2\,\pi\right),\right.$

$\text{parametric}\left(\dfrac{2\cos(t)}{5}-1,\dfrac{2\sin(t)}{5}-1,t,0,2\,\pi\right),$

$\text{parametric}\left(\dfrac{3\cos(t)}{5}-1,\dfrac{3\sin(t)}{5}-1,t,0,2\,\pi\right),$

$\text{parametric}\left(\dfrac{4\cos(t)}{5}-1,\dfrac{4\sin(t)}{5}-1,t,0,2\,\pi\right),$

$\left.\text{parametric}(\cos(t)-1,\sin(t)-1,t,0,2\,\pi)\right]$

(%i5) wxdraw2d(
    xaxis = true, xaxis_type = solid, xrange = [−3, 1],
    yaxis = true, yaxis_type = solid, yrange = [−3, 1],
    proportional_axes = xy,
    line_width = 2,
    nticks = 600,
    zdomain
);

(%t5)
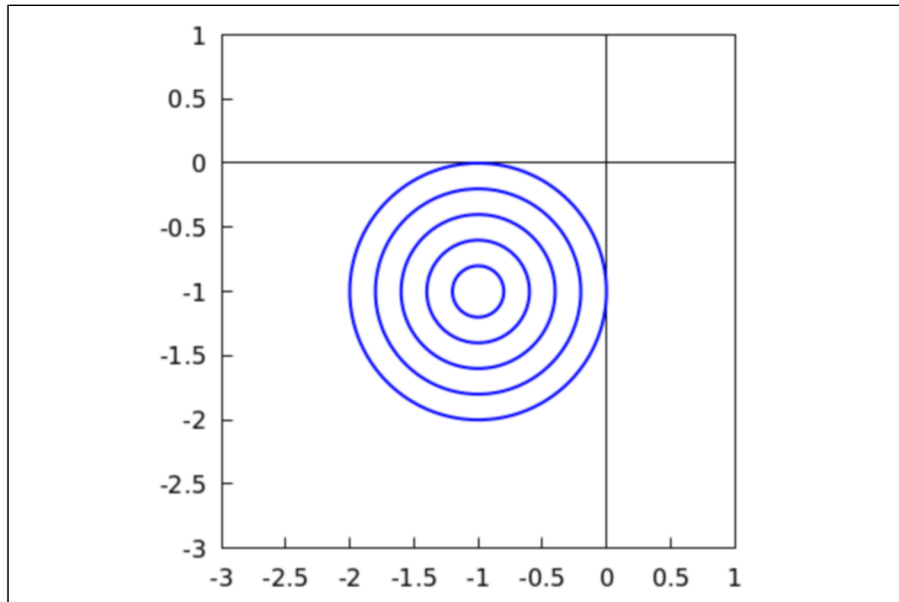
(%o5)

(%i6) w(t, s):=f(r(t, s));
(%o6) $w(t,s):=f(r(t,s))$

(%i7) wdomain:makelist(parametric(realpart(w(t, s)), imagpart(w(t, s)), t, 0, 2·%pi

(wdomain) $\left[\text{parametric}\left(4\left(\frac{\sin(t)}{5}-1 \right)+3\left(\frac{\cos(t)}{5}-1 \right)+6, 3 \right.\right.$

$\left(\frac{\sin(t)}{5}-1 \right)-4\left(\frac{\cos(t)}{5}-1 \right)+2, t, 0, 2\pi \bigg), \text{parametric}\bigg(4$

$\left(\frac{2\sin(t)}{5}-1 \right)+3\left(\frac{2\cos(t)}{5}-1 \right)+6, 3\left(\frac{2\sin(t)}{5}-1 \right)-4$

$\left(\frac{2\cos(t)}{5}-1 \right)+2, t, 0, 2\pi \bigg), \text{parametric}\bigg(4\left(\frac{3\sin(t)}{5}-1 \right)+3$

$\left(\frac{3\cos(t)}{5}-1 \right)+6, 3\left(\frac{3\sin(t)}{5}-1 \right)-4\left(\frac{3\cos(t)}{5}-1 \right)+2, t, 0, 2\pi \bigg),$

$\text{parametric}\bigg(4\left(\frac{4\sin(t)}{5}-1 \right)+3\left(\frac{4\cos(t)}{5}-1 \right)+6, 3\left(\frac{4\sin(t)}{5}-1 \right)$

$-4\left(\frac{4\cos(t)}{5}-1 \right)+2, t, 0, 2\pi \bigg), \text{parametric}\big(4\left(\sin(t)-1 \right)+3$

$\left(\cos(t)-1 \right)+6, 3\left(\sin(t)-1 \right)-4\left(\cos(t)-1 \right)+2, t, 0, 2\pi \big) \Big]$

(%i8) cabs(f(r(t, s)));

$$(\%o8)\quad \sqrt{\left(4\,s\,\sin\left(t\right)+3\,s\,\cos\left(t\right)-1\right)^{2}+\left(3\,s\,\sin\left(t\right)-4\,s\,\cos\left(t\right)+3\right)^{2}}$$

(%i9) wxdraw2d(
        xaxis = true, xaxis_type = solid, xrange = [−7, 10],
        yaxis = true, yaxis_type = solid, yrange = [−4, 10],
        proportional_axes = xy,
        nticks = 600,
        line_width = 2,
        color = red,
        wdomain
      );

(%t9)
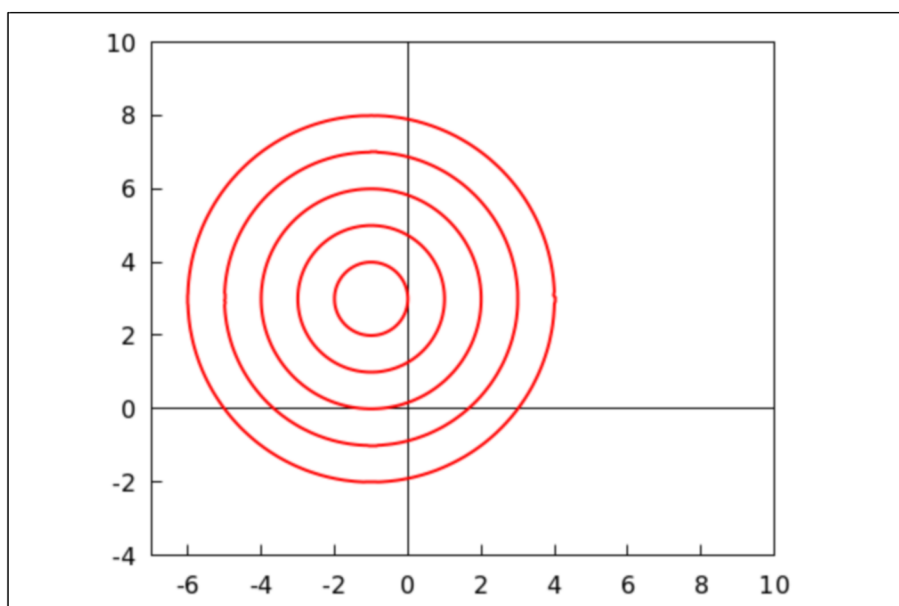
(%o9)

(%i10) cabs(c−f(r(t, s)));

$$(\%o10)\quad \sqrt{\left(4\,s\,\cos\left(t\right)-3\,s\,\sin\left(t\right)\right)^{2}+\left(-4\,s\,\sin\left(t\right)-3\,s\,\cos\left(t\right)\right)^{2}}$$

(%i11) trigsimp(%);

$$(\%o11)\quad 5\,\left|s\right|$$

(%i12) cabs(c−f(r(t, 1)));

$$(\%o12)\quad \sqrt{\left(4\,\cos\left(t\right)-3\,\sin\left(t\right)\right)^{2}+\left(-4\,\sin\left(t\right)-3\,\cos\left(t\right)\right)^{2}}$$

(%i13) trigsimp(%);

$$(\%o13)\quad 5$$

(%i14) makelist(cabs(c−f(r(t, s))), s, 1/5, 1, 1/5);

$$(\%o14) \quad \left[\sqrt{\left(\frac{4\cos(t)}{5} - \frac{3\sin(t)}{5}\right)^2 + \left(-\frac{4\sin(t)}{5} - \frac{3\cos(t)}{5}\right)^2},\right.$$

$$\sqrt{\left(\frac{8\cos(t)}{5} - \frac{6\sin(t)}{5}\right)^2 + \left(-\frac{8\sin(t)}{5} - \frac{6\cos(t)}{5}\right)^2},$$

$$\sqrt{\left(\frac{12\cos(t)}{5} - \frac{9\sin(t)}{5}\right)^2 + \left(-\frac{12\sin(t)}{5} - \frac{9\cos(t)}{5}\right)^2},$$

$$\sqrt{\left(\frac{16\cos(t)}{5} - \frac{12\sin(t)}{5}\right)^2 + \left(-\frac{16\sin(t)}{5} - \frac{12\cos(t)}{5}\right)^2},$$

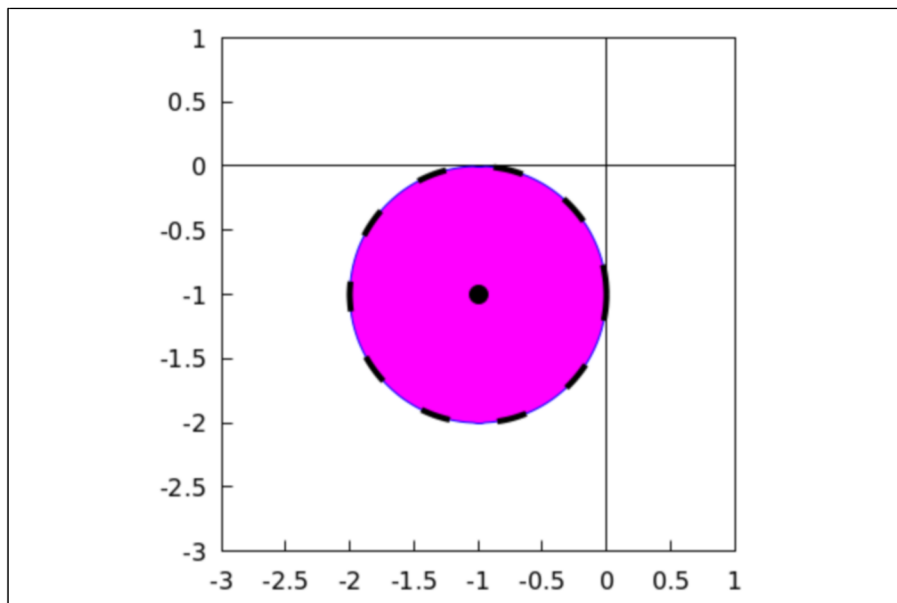$$\left.\sqrt{(4\cos(t) - 3\sin(t))^2 + (-4\sin(t) - 3\cos(t))^2}\right]$$

(%i15) trigsimp(%);

$$(\%o15) \quad [1, 2, 3, 4, 5]$$

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(%i16) wxdraw2d(
        xaxis = true, xaxis_type = solid, xrange = [−3, 1],
        yaxis = true, yaxis_type = solid, yrange = [−3, 1],
        proportional_axes = xy,
        nticks = 200,
        fill_color = magenta,
        ellipse(−1, −1, 1, 1, 0,   360),
        color = black,
        line_type = dashes,
        line_width = 4,
        parametric(−1+cos(t),   −1+ sin(t),   t,   0, 2·%pi),
        point_size = 2,
        point_type = 7,
        points([[−1, −1]])
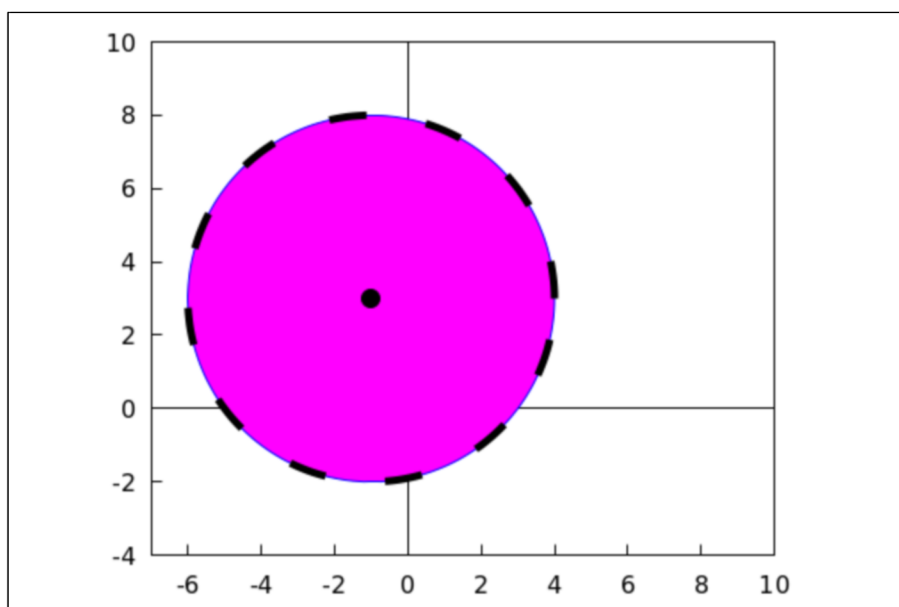
);

(%t16)

(%o16)

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(%i17) wxdraw2d(
        xaxis = true, xaxis_type = solid, xrange = [−7, 10],
        yaxis = true, yaxis_type = solid, yrange = [−4, 10],
        proportional_axes = xy,
        nticks = 200,
        fill_color = magenta,
        ellipse(−1, 3, 5, 5, 0, 360),
        color = black,
        line_type = dashes,
        line_width = 5,
        parametric(−1+5·cos(t), 3+5·sin(t), t, 0, 2·%pi),
        point_size = 2,
        point_type = 7,
        points([[−1, 3]])

);

(%t17)

(%o17)

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Exercise

Figure 1:

Let $w = f(z) = (3 + 4i)z - 2 + i$.

(a) Find the image of the disk $|z - 1| < 1$.

(b) Find the image of the line $x = t$, $y = 1 - 2t$ for $-\infty < t < \infty$.

(c) Find the image of the half-plane $\text{Im}(z) > 1$.

(d) For parts a and b, and c, sketch the mapping, identify the points $z_1 = 0$, $z_2 = 1 - i$, and $z_3 = 2$, and indicate their images.