Phase 2 - Lecture 4

Stochastic Calculus

Stock price follows:

$dS = uSdt + \sigma SdX$

Stochastic Differential Equations

$$W(t) = \int_0^t f(\tau)\, dX(\tau). \tag{5.1}$$

That shorthand comes from 'differentiating' (5.1) and is

$$dW = f(t)\, dX. \tag{5.2}$$

Think of dX as being an increment in X, i.e. a Normal random variable with mean zero and standard deviation $dt^{1/2}$.

Equations (5.1) and (5.2) are meant to be equivalent. One of the reasons for this shorthand is that the equation (5.2) looks a lot like an ordinary differential equation. We *do not* go the further step of dividing by dt to make it look exactly like an ordinary differential equation because then we would have the difficult task of defining $\frac{dX}{dt}$.

Pursuing this idea further, imagine what might be meant by

$$dW = g(t)\, dt + f(t)\, dX. \tag{5.3}$$

Ito's Lemma and its interpretation

$$\boxed{dF = \frac{dF}{dX}\,dX + \frac{1}{2}\frac{d^2F}{dX^2}\,dt.}$$

$$dS = \mu S\,dt + \sigma S\,dX,$$

$$dV = \underline{\qquad}\,dt + \underline{\qquad}\,dX.$$

Ito’s Lemma in Higher Dimension

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S_1} dS_1 + \frac{\partial V}{\partial S_2} dS_2 + \tfrac{1}{2} b_1^2 \frac{\partial^2 V}{\partial S_1^2} dt + \rho b_1 b_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} dt + \tfrac{1}{2} b_2^2 \frac{\partial^2 V}{\partial S_2^2} dt. \quad (5.9)$$

Some Examples

The first example is like the simple Brownian motion but with a drift: $dS = \mu\, dt + \sigma\, dX$.

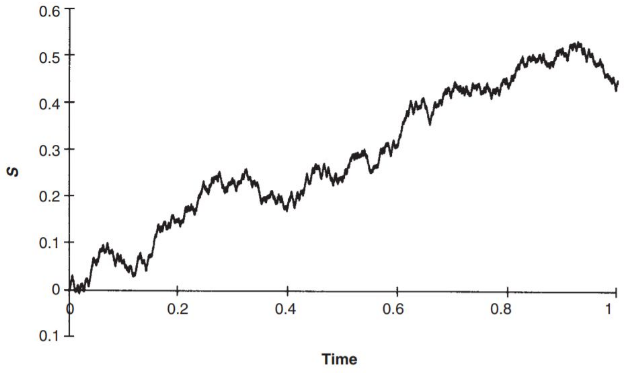

Figure 5.5 A realization of $dS = \mu\, dt + \sigma\, dX$.

Some Examples

The Lognormal Random Walk: F(S) = log S

$$dF = \frac{dF}{dS}\,dS + \tfrac{1}{2}\sigma^2 S^2 \frac{d^2F}{dS^2}\,dt = \frac{1}{S}\left(\mu S\,dt + \sigma S\,dX\right) - \tfrac{1}{2}\sigma^2\,dt$$

$$= \left(\mu - \tfrac{1}{2}\sigma^2\right) dt + \sigma\,dX.$$

Some Examples

Mean reverting random walk: $dS = (\nu - \mu S)\,dt + \sigma\,dX.$

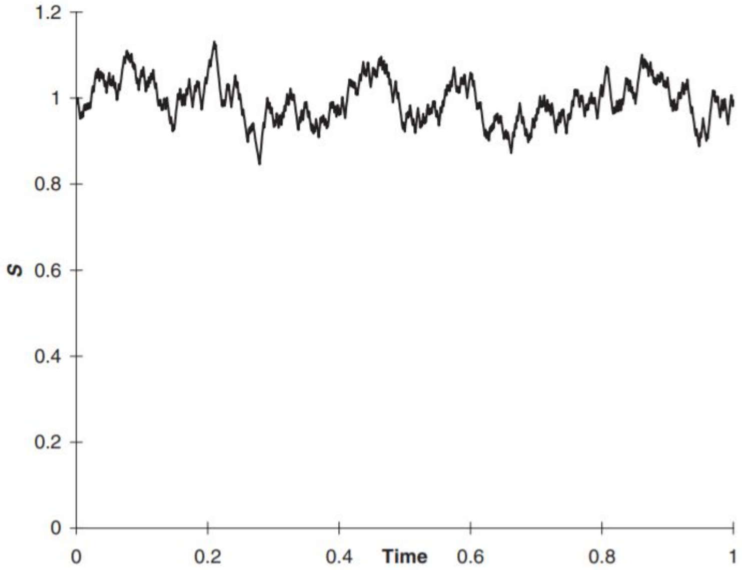

Figure 5.8 A realization of $dS = (\nu - \mu S)\,dt + \sigma\,dX$.

Further Reading

https://en.wikipedia.org/wiki/Euler%E2%80%93Maruyama_method#:~:text=In%20It%C3%B4%20calculus%2C%20the%20Euler,equations%20to%20stochastic%20differential%20equations.