Phase 2 - Lecture 5

Black Scholes Equation

$$V(S, t; \sigma, \mu; E, T; r).$$

Notice that the semi-colons separate different types of variables and parameters:

- S and t are variables;
- σ and μ are parameters associated with the asset price;
- E and T are parameters associated with the details of the particular contract;
- r is a parameter associated with the currency in which the asset is quoted.

$$\Pi = V(S, t) - \Delta S.$$

$$dS = \mu S\, dt + \sigma S\, dX.$$

$$d\Pi = dV - \Delta\, dS.$$

$$dV = \frac{\partial V}{\partial t}\,dt + \frac{\partial V}{\partial S}\,dS + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\,dt.$$

Thus the portfolio changes by

$$d\Pi = \frac{\partial V}{\partial t}\,dt + \frac{\partial V}{\partial S}\,dS + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\,dt - \Delta\,dS.$$

Delta Hedging

$$\left(\frac{\partial V}{\partial S} - \Delta\right) dS.$$

$$\Delta = \frac{\partial V}{\partial S}$$

No Arbitrage

$$d\Pi = \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt.$$

$$d\Pi = r\Pi \, dt.$$

The Black Scholes Equation

$$\left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt = r\left(V - S\frac{\partial V}{\partial S}\right) dt.$$

On dividing by dt and rearranging we get

$$\boxed{\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.}$$

Assumptions of BSE

- Read from the book

Final conditions

$$V(S,T) = \max(S - E, 0).$$

For a put we have

$$V(S,T) = \max(E - S, 0),$$

for a binary call

$$V(S,T) = \mathcal{H}(S - E)$$

and for a binary put

$$V(S,T) = \mathcal{H}(E - S),$$

Different type of options

Dividend paying asset

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S\frac{\partial V}{\partial S} - rV = 0.$$

Different type of options

Currency Options

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - r_f) S \frac{\partial V}{\partial S} - rV = 0.$$

Forwards

$$V(S, T) = S - \overline{S}.$$

The solution of the equation with this final condition is

$$V(S, t) = S - \overline{S}e^{-r(T-t)}.$$

Futures

Remember that the value of the futures contract during its life is always zero because the change in value is settled daily. This cashflow must be taken into account in our analysis.

Set up a portfolio of one long futures contract and short Δ of the underlying:

$$\Pi = -\Delta S.$$

Where is the value of the futures contract? Is this a mistake? No, because it has no value it doesn't appear in the portfolio valuation equation. How does the portfolio change in value?

$$d\Pi = dF - \Delta\, dS.$$

The dF represents the cashflow due to the continual settlement. Applying Itô's lemma,

$$d\Pi = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S}dS + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2}dt - \Delta\, dS.$$

Choose

$$\Delta = \frac{\partial F}{\partial S}$$

to eliminate risk. Set

$$d\Pi = r\Pi\, dt$$

to get

$$\frac{\partial F}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + rS\frac{\partial F}{\partial S} = 0.$$

Final Black Scholes Formula

Call option value

$$Se^{-D(T-t)}N(d_1) - Ee^{-r(T-t)}N(d_2)$$

$$d_1 = \frac{\log(S/E) + \left(r - D + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\log(S/E) + \left(r - D - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$= d_1 - \sigma\sqrt{T-t}$$

Put option value

$$-Se^{-D(T-t)}N(-d_1) + Ee^{-r(T-t)}N(-d_2)$$

Next class

Greeks