Phase 2 - Lecture 5

Black Scholes Equation

$V(S, t; \sigma, \mu; E, T; r)$.

Notice that the semi-colons separate different types of variables and parameters:

- S and t are variables;
- σ and μ are parameters associated with the asset price;
- E and T are parameters associated with the details of the particular contract;
- r is a parameter associated with the currency in which the asset is quoted.

$$\Pi = V(S, t) - \Delta S.$$

 $dS = \mu S dt + \sigma S dX.$

$$d\Pi = dV - \Delta dS$$
.

Thus the portfolio changes by

 $dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt.$

 $d\Pi = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt - \Delta dS.$



Delta Hedging

$$\left(\frac{\partial V}{\partial S} - \Delta\right) dS.$$

$$\Delta = \frac{\partial V}{\partial S}$$

No Arbitrage

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt.$$

$$d\Pi = r\Pi dt$$
.

The Black Scholes Equation

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt = r \left(V - S \frac{\partial V}{\partial S}\right) dt.$$

On dividing by dt and rearranging we get

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0.$$

Assumptions of BSE

• Read from the book

Final conditions

$$V(S,T) = \max(S - E, 0).$$

For a put we have

$$V(S,T) = \max(E - S, 0),$$

for a binary call

$$V(S,T) = \mathcal{H}(S-E)$$

and for a binary put

$$V(S,T)=\mathcal{H}(E-S),$$

Different type of options

Dividend paying asset

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S \frac{\partial V}{\partial S} - rV = 0.$$

Different type of options

Currency Options

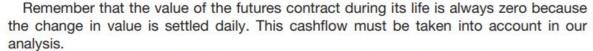
$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - r_f) S \frac{\partial V}{\partial S} - rV = 0.$$

Forwards

$$V(S,T)=S-\overline{S}.$$

The solution of the equation with this final condition is

$$V(S, t) = S - \overline{S}e^{-r(T-t)}$$
.



Futures

Set up a portfolio of one long futures contract and short Δ of the underlying:

$$\Pi = -\Delta S$$
.

Where is the value of the futures contract? Is this a mistake? No, because it has no value it doesn't appear in the portfolio valuation equation. How does the portfolio change in value?

$$d\Pi = dF - \Delta dS$$
.

The dF represents the cashflow due to the continual settlement. Applying Itô's lemma,

$$d\Pi = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2}dt - \Delta dS.$$

Choose

$$\Delta = \frac{\partial F}{\partial S}$$

to eliminate risk. Set

$$d\Pi = r\Pi dt$$

to get

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + rS \frac{\partial F}{\partial S} = 0.$$

Final Black Scholes Formula

Call option value

$$Se^{-D(T-t)}N(d_{1}) - Ee^{-r(T-t)}N(d_{2})$$

$$d_{1} = \frac{\log(S/E) + (r - D + \frac{1}{2}\sigma^{2})(T - t)}{\sigma\sqrt{T - t}}$$

$$d_{2} = \frac{\log(S/E) + (r - D - \frac{1}{2}\sigma^{2})(T - t)}{\sigma\sqrt{T - t}}$$

$$= d_{1} - \sigma\sqrt{T - t}$$

Put option value

$$-Se^{-D(T-t)}N(-d_1) + Ee^{-r(T-t)}N(-d_2)$$

Next class

Greeks