Assignment 3

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Download the python codes from

https://github.com/tanayyadav28/Assignments/blob/main/Assignment%203/code/assignment3.py

and latex-tikz codes from

https://github.com/tanayyadav28/Assignments/blob/main/Assignment%203/assignment3.tex

1 Problem

(GATE EC, Q. 25) A fair coin is tossed till a head appears for the first time. The probability that the number of requried tosses is odd,is
(A) $\frac{1}{3}$ (B) $\frac{1}{2}$ (C) $\frac{2}{3}$ (D) $\frac{3}{4}$

2 Solution

Given that the coin is tossed until a head appears on an odd toss.

$$p = \frac{1}{2}, q = \frac{1}{2} \tag{2.0.1}$$

Let's define a Markov chain $\{X_n, n = 0, 1, 2, \dots\}$, where $X_n \in S = \{1, 2, 3, 4\}$, such that:

TABLE 0: States and their notations

Notation	State
$S = 1$	Odd try
$S = 2$	Even try
$S = 3$	Loss
$S = 4$	Success

The state transition matrix for the Markov chain is:

$$P = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \tag{2.0.1}$$

The transient states are 1,2 and the absorbing state is 3. The standard form of the matrix is;

$$P = \begin{array}{c c} & \begin{array}{cc} A & N \end{array} \\ \begin{array}{c} A \\ N \end{array} & \begin{bmatrix} I & O \\ R & Q \end{bmatrix} \end{array} \tag{2.0.2}$$

where,

TABLE 0: Notations and their meanings

Notation	Meaning
A	Absorbing states
N	Non-absorbing states
I	Identity matrix
O	Zero matrix
R, Q	Other sub-matrices

Now, we convert the transition matrix to this standard form.

$$P = \begin{array}{c c} & \begin{array}{cccc} 3 & 4 & 1 & 2 \end{array} \\ \begin{array}{c} 3 \\ 4 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \end{bmatrix} \end{array} \tag{2.0.3}$$

From (2.0.3),

$$R = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}, Q = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix} \tag{2.0.4}$$

The limiting matrix for absorbing Markov chain is,

$$\bar{P} = \begin{bmatrix} I & O \\ FR & O \end{bmatrix} \tag{2.0.5}$$

where,

$$F = (I - Q)^{-1} \tag{2.0.6}$$

is called the fundamental matrix of P.

On solving we get,

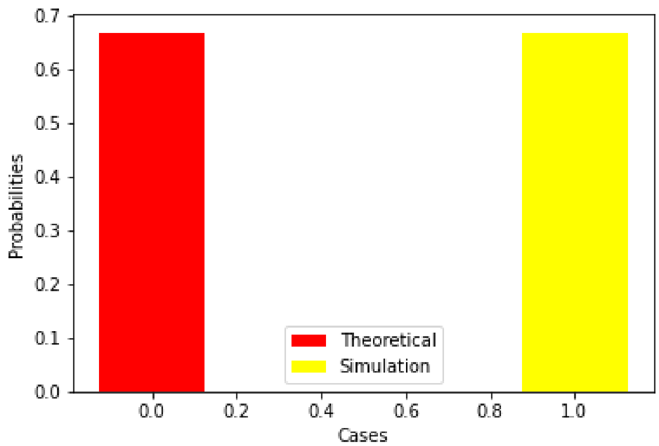
$$\bar{P} = \begin{array}{c} \\ 3 \\ 4 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.333 & 0.667 & 0 & 0 \\ 0.667 & 0.333 & 0 & 0 \end{bmatrix} \end{array} \tag{2.0.7}$$

An element $\bar{p}_{ij}$ of $\bar{P}$ denotes the absorption probability to the state j, starting from the state i.
Let $\Pr(A)$ be the probability that the first head is obtained on an odd toss. Then,

$$\Pr(A) = p_{14} \tag{2.0.2}$$
$$= 0.667 \tag{2.0.3}$$
$$\therefore \Pr(A) = \frac{2}{3} \tag{2.0.4}$$

Hence, option (C) is the correct answer.

Plot: Theory vs Simulation.



Markov chain diagram