

Assignment 3

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Download the python codes from

<https://github.com/tanayyadav28/Assignments/blob/main/Assignment%203/code/assignment3.py>

and latex-tikz codes from

<https://github.com/tanayyadav28/Assignments/blob/main/Assignment%203/assignment3.tex>

1 PROBLEM

(GATE EC, Q. 25) A fair coin is tossed till a head appears for the first time. The probability that the number of required tosses is odd, is

- 1) $\frac{1}{3}$
- 2) $\frac{1}{2}$
- 3) $\frac{2}{3}$
- 4) $\frac{3}{4}$

2 SOLUTION

Given that the coin is tossed until a head appears on an odd toss.

$$p = \frac{1}{2}, q = \frac{1}{2} \quad (2.0.1)$$

Let's define a Markov chain $\{X_n, n = 0, 1, 2, \dots\}$, where $X_n \in S = \{1, 2, 3, 4\}$, such that:

TABLE 4: States and their notations

Notation	State
$S = 1$	Odd try
$S = 2$	Even try
$S = 3$	Loss
$S = 4$	Success

The state transition matrix for the Markov chain is:

$$P = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix} \quad (2.0.1)$$

The transient states are 1, 2 and the absorbing states are 3 and 4. The standard form of the matrix is;

$$P = \begin{matrix} & \begin{matrix} A & N \end{matrix} \\ \begin{matrix} A \\ N \end{matrix} & \begin{bmatrix} I & O \\ R & Q \end{bmatrix} \end{matrix} \quad (2.0.2)$$

where,

TABLE 4: Notations and their meanings

Notation	Meaning
A	Absorbing states
N	Non-absorbing states
I	Identity matrix
O	Zero matrix
R, Q	Other sub-matrices

Now, we convert the transition matrix to this standard form.

$$P = \begin{matrix} & \begin{matrix} 3 & 4 & 1 & 2 \end{matrix} \\ \begin{matrix} 3 \\ 4 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \end{bmatrix} \end{matrix} \quad (2.0.3)$$

From (2.0.3),

$$R = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}, Q = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix} \quad (2.0.4)$$

The limiting matrix for absorbing Markov chain is,

$$\bar{P} = \begin{bmatrix} I & O \\ FR & O \end{bmatrix} \quad (2.0.5)$$

where,

$$F = (I - Q)^{-1} \quad (2.0.6)$$

is called the fundamental matrix of P .

On solving we get,

$$\bar{P} = \begin{matrix} & \begin{matrix} 3 & 4 & 1 & 2 \end{matrix} \\ \begin{matrix} 3 \\ 4 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.333 & 0.667 & 0 & 0 \\ 0.667 & 0.333 & 0 & 0 \end{bmatrix} \end{matrix} \quad (2.0.7)$$

An element \bar{p}_{ij} of \bar{P} denotes the absorption probability to the state j , starting from the state i .

Let $\Pr(A)$ be the probability that the first head is obtained on an odd toss. Then,

$$\Pr(A) = p_{14} \quad (2.0.2)$$

$$= 0.667 \quad (2.0.3)$$

$$\therefore \Pr(A) = \frac{2}{3} \quad (2.0.4)$$

Hence, option (3) is the correct answer.

sim_plot.png

Plot: Theory vs Simulation.

Markov chain diagram

