Assignment 4

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1 PROBLEM

[Linear Forms 2.23] Find the shortest distance between the lines:

$$L_1 : \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \tag{1.0.1}$$

$$L_2 : \mathbf{x} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \tag{1.0.2}$$

2 SOLUTION

Given,

$$\mathbf{A_1} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \tag{2.0.1}$$

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{A_2} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{m_2} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \tag{2.0.4}$$

The lines L_1 and L_2 are intersecting if,

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \tag{2.0.5}$$

$$\lambda_1 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} - \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \tag{2.0.6}$$

$$\begin{pmatrix} 1 & 2 \\ -3 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \tag{2.0.7}$$

Row reducing the augmented matrix,

$$\begin{pmatrix} 1 & 2 & 3 \\ -3 & 3 & 3 \\ 2 & 1 & 3 \end{pmatrix} \xleftarrow{R_2=R_2+3R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 9 & 12 \\ 1 & 2 & 3 \end{pmatrix} \tag{2.0.8}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 9 & 12 \\ 2 & 1 & 3 \end{pmatrix} \xleftarrow{R_3=R_3-2R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 9 & 12 \\ 0 & -3 & -3 \end{pmatrix} \tag{2.0.9}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 9 & 12 \\ 0 & -3 & -3 \end{pmatrix} \xleftarrow{R_3=R_3+\frac{R_2}{3}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 9 & 12 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.0.10}$$

The above matrix has *rank* = 3. Hence, the given lines are skew-lines.
The normal to both the lines is:

$$\mathbf{n} = \mathbf{m_1} \times \mathbf{m_2} \tag{2.0.11}$$

The equation of the plane is then obtained as:

$$\mathbf{n}^T \mathbf{x} = \mathbf{n}^T \mathbf{A_2} \tag{2.0.12}$$

The distance of the above line from $\mathbf{A_2}$ is obtained as:

$$\frac{n^T(\mathbf{A_2} - \mathbf{A_1})}{\|\mathbf{n}\|} = \frac{(\mathbf{A_2} - \mathbf{A_1})^T(\mathbf{m_1} \times \mathbf{m_2})}{\|\mathbf{m_1} \times \mathbf{m_2}\|} \tag{2.0.13}$$

$$= \frac{\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}^T \begin{pmatrix} -9 \\ 3 \\ 9 \end{pmatrix}}{\sqrt{\begin{pmatrix} -9 \\ 3 \\ 9 \end{pmatrix}^T \begin{pmatrix} -9 \\ 3 \\ 9 \end{pmatrix}}} \tag{2.0.14}$$

$$= \frac{9}{\sqrt{171}} \qquad (2.0.15)$$

$$= \frac{3}{\sqrt{19}} \qquad (2.0.16)$$
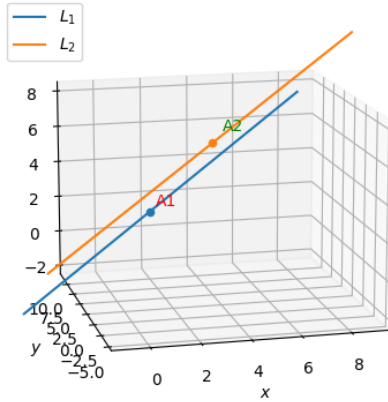
Hence, the shortest distance between the given lines is $\frac{3}{\sqrt{19}}$.

Fig. 0: Plot from Python Code.