Assignment 4

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Download the python codes from:

https://github.com/tanayyadav28/EE3900-Assignments/blob/main/Assignment_4/code/Assignment_4.py

Download the latex-tikz codes from:

https://github.com/tanayyadav28/EE3900-Assignments/blob/main/Assignment_4/Assignment_4.tex

1 Problem

[Linear Forms 2.23] Find the shortest distance between the lines:

$$L_1 : \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \tag{1.0.1}$$

$$L_2 : \mathbf{x} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \tag{1.0.2}$$

2 Solution

Given,

$$\mathbf{A_1} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \tag{2.0.1}$$

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{A_2} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{m_2} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \tag{2.0.4}$$

The lines L_1 and L_2 are intersecting if,

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \tag{2.0.5}$$

$$\lambda_1 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} - \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \tag{2.0.6}$$

$$\begin{pmatrix} 1 & 2 \\ -3 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \tag{2.0.7}$$

Row reducing the augmented matrix,

$$\begin{pmatrix} 1 & 2 & 3 \\ -3 & 3 & 3 \\ 2 & 1 & 3 \end{pmatrix} \xleftrightarrow{R_2=R_2+3R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 9 & 12 \\ 1 & 2 & 3 \end{pmatrix} \tag{2.0.8}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 9 & 12 \\ 2 & 1 & 3 \end{pmatrix} \xleftrightarrow{R_3=R_3-2R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 9 & 12 \\ 0 & -3 & -3 \end{pmatrix} \tag{2.0.9}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 9 & 12 \\ 0 & -3 & -3 \end{pmatrix} \xleftrightarrow{R_3=R_3+\frac{R_2}{3}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 9 & 12 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.0.10}$$

The above matrix has *rank* = 3. Hence, the given lines are skew-lines.
Let d be the shortest distance and $\mathbf{p_1}, \mathbf{p_2}$ be positional vectors of its end points. For d to be shortest, we know that,

$$\mathbf{m_1}^\top (\mathbf{p_2} - \mathbf{p_1}) = 0 \tag{2.0.11}$$

$$\mathbf{m_2}^\top (\mathbf{p_2} - \mathbf{p_1}) = 0 \tag{2.0.12}$$

$$\mathbf{m_1}^\top ((\mathbf{A}_2 - \mathbf{A}_1)) + \begin{pmatrix} \mathbf{m_2} & \mathbf{m}_1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0 \tag{2.0.13}$$

$$\mathbf{m_2}^\top ((\mathbf{A}_2 - \mathbf{A}_1)) + \begin{pmatrix} \mathbf{m_2} & \mathbf{m_1} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0 \tag{2.0.14}$$

Let

$$\mathbf{M} = \begin{pmatrix} \mathbf{m_2} & \mathbf{m}_1 \end{pmatrix} \qquad \mathbf{M}^\top = \begin{pmatrix} \mathbf{m_2}^\top \\ \mathbf{m_1}^\top \end{pmatrix} \tag{2.0.15}$$

By combining equations (2.0.13) and (2.0.14) and writing in terms of $\mathbf{M}$ and $\mathbf{M}^\top$ using (2.0.15), we get

$$\mathbf{M}^{\top}\mathbf{M}\begin{pmatrix}\lambda_2\\-\lambda_1\end{pmatrix} = \mathbf{M}^{\top}\left(\mathbf{A}_1 - \mathbf{A}_2\right) \tag{2.0.16}$$

By putting the values of $\mathbf{A_1}, \mathbf{A_2}, \mathbf{m_1}, \mathbf{m_2}$ in (2.0.16), we get

$$\begin{pmatrix}14 & -5\\-5 & 14\end{pmatrix}\begin{pmatrix}\lambda_2\\-\lambda_1\end{pmatrix} = \begin{pmatrix}-18\\0\end{pmatrix} \tag{2.0.17}$$

Solving (2.0.17), we get

$$\begin{pmatrix}\lambda_2\\-\lambda_1\end{pmatrix} = \begin{pmatrix}-1.4736\\-0.5263\end{pmatrix} \tag{2.0.18}$$

Substituting the value of λ_1 and λ_2 in (1.0.1) and (1.0.2), we get

$$\mathbf{p_1} = \begin{pmatrix}1.5263\\0.4210\\4.0526\end{pmatrix} \qquad \mathbf{p_2} = \begin{pmatrix}1.0526\\0.5789\\4.5263\end{pmatrix} \tag{2.0.19}$$

Hence, the shortest distance between these two skew lines is

$$d = \|\mathbf{p_2} - \mathbf{p_1}\| = 0.6882 \tag{2.0.20}$$

Hence, the shortest distance between the given lines is 0.6882.

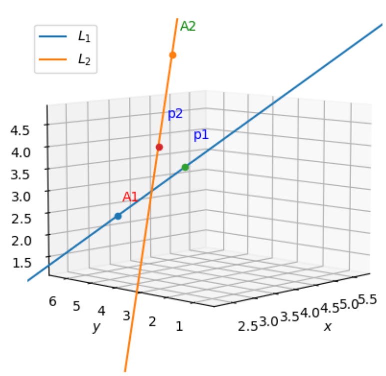

Fig. 0: Plot from Python Code.