Assignment 4

Tanay Yadav - AI20BTECH11026

Download the python codes from:

https://github.com/tanayyadav28/EE3900–Assignments/blob/main/Assignment_4/code/Assignment_4.py

Download the latex-tikz codes from:

https://github.com/tanayyadav28/EE3900–Assignments/blob/main/Assignment_4/Assignment_4.tex

1 Problem

[Linear Forms 2.23] Find the shortest distance between the lines:

$$L_1 : \mathbf{x} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda_1 \begin{pmatrix}1\\-3\\2\end{pmatrix} \tag{1.0.1}$$

$$L_2 : \mathbf{x} = \begin{pmatrix}4\\5\\6\end{pmatrix} + \lambda_2 \begin{pmatrix}2\\3\\1\end{pmatrix} \tag{1.0.2}$$

2 Solution

Given,

$$\mathbf{A_1} = \begin{pmatrix}1\\2\\3\end{pmatrix} \tag{2.0.1}$$

$$\mathbf{m_1} = \begin{pmatrix}1\\-3\\2\end{pmatrix} \tag{2.0.2}$$

$$\mathbf{A_2} = \begin{pmatrix}4\\5\\6\end{pmatrix} \tag{2.0.3}$$

$$\mathbf{m_2} = \begin{pmatrix}2\\3\\1\end{pmatrix} \tag{2.0.4}$$

The lines L_1 and L_2 are intersecting if,

$$\begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda_1 \begin{pmatrix}1\\-3\\2\end{pmatrix} = \begin{pmatrix}4\\5\\6\end{pmatrix} + \lambda_2 \begin{pmatrix}2\\3\\1\end{pmatrix} \tag{2.0.5}$$

$$\lambda_1 \begin{pmatrix}1\\-3\\2\end{pmatrix} - \lambda_2 \begin{pmatrix}2\\3\\1\end{pmatrix} = \begin{pmatrix}4\\5\\6\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix} \tag{2.0.6}$$

$$\begin{pmatrix}1 & 2\\-3 & 3\\2 & 1\end{pmatrix} \begin{pmatrix}\lambda_1\\-\lambda_2\end{pmatrix} = \begin{pmatrix}3\\3\\3\end{pmatrix} \tag{2.0.7}$$

Row reducing the augmented matrix,

$$\begin{pmatrix}1 & 2 & 3\\-3 & 3 & 3\\2 & 1 & 3\end{pmatrix} \xleftrightarrow{R_2=R_2+3R_1} \begin{pmatrix}1 & 2 & 3\\0 & 9 & 12\\1 & 2 & 3\end{pmatrix} \tag{2.0.8}$$

$$\begin{pmatrix}1 & 2 & 3\\0 & 9 & 12\\2 & 1 & 3\end{pmatrix} \xleftrightarrow{R_3=R_3-2R_1} \begin{pmatrix}1 & 2 & 3\\0 & 9 & 12\\0 & -3 & -3\end{pmatrix} \tag{2.0.9}$$

$$\begin{pmatrix}1 & 2 & 3\\0 & 9 & 12\\0 & -3 & -3\end{pmatrix} \xleftrightarrow{R_3=R_3+\frac{R_2}{3}} \begin{pmatrix}1 & 2 & 3\\0 & 9 & 12\\0 & 0 & 1\end{pmatrix} \tag{2.0.10}$$

The above matrix has *rank* = 3. Hence, the given lines are skew-lines.
The normal to both the lines is:

$$\mathbf{n} = \mathbf{m}_1 \times \mathbf{m}_2 \tag{2.0.11}$$

The equation of the plane is then obtained as:

$$\mathbf{n}^T\mathbf{x} = \mathbf{n}^T\mathbf{A}_2 \tag{2.0.12}$$

The distance of the above line from $\mathbf{A}_2$ is obtained as:

$$\frac{n^T(\mathbf{A}_2 - \mathbf{A}_1)}{\|\mathbf{n}\|} = \frac{(\mathbf{A}_2 - \mathbf{A}_1)^T(\mathbf{m}_1 \times \mathbf{m}_2)}{\|\mathbf{m}_1 \times \mathbf{m}_2\|} \tag{2.0.13}$$

$$= \frac{\begin{pmatrix}3\\3\\3\end{pmatrix}^T \begin{pmatrix}-9\\3\\9\end{pmatrix}}{\sqrt{\begin{pmatrix}-9\\3\\9\end{pmatrix}^T \begin{pmatrix}-9\\3\\9\end{pmatrix}}} \tag{2.0.14}$$

$$= \frac{9}{\sqrt{171}} \tag{2.0.15}$$

$$= \frac{3}{\sqrt{19}} \tag{2.0.16}$$

Let $\mathbf{P}$ and $\mathbf{Q}$ be the nearest points on the lines L_1 and L_2 respectively.

$$\mathbf{P} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \tag{2.0.17}$$

$$\mathbf{Q} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \tag{2.0.18}$$

$$\mathbf{PQ} = \mathbf{Q} - \mathbf{P} \tag{2.0.19}$$

$$\mathbf{PQ} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} + \begin{pmatrix} 2\lambda_2 - \lambda_1 \\ 3\lambda_2 + 3\lambda_1 \\ \lambda_2 - 2\lambda_1 \end{pmatrix} \tag{2.0.20}$$

$$\mathbf{PQ} = \begin{pmatrix} 3 + 2\lambda_2 - \lambda_1 \\ 3 + 3\lambda_2 + 3\lambda_1 \\ 3 + \lambda_2 - 2\lambda_1 \end{pmatrix} \tag{2.0.21}$$

As $\mathbf{PQ}$ is the shortest distance between the lines L_1 and L_2, it is perpendicular to both.

$$\mathbf{PQ}^T \mathbf{m_1} = 0 \tag{2.0.22}$$

$$\mathbf{PQ}^T \mathbf{m_2} = 0 \tag{2.0.23}$$

$$\therefore \mathbf{PQ}^T \mathbf{m_1} = 14\lambda_1 + 5\lambda_2 \tag{2.0.24}$$

$$\therefore \mathbf{PQ}^T \mathbf{m_2} = 5\lambda_1 + 14\lambda_2 + 18 \tag{2.0.25}$$

Now,

$$14\lambda_1 + 5\lambda_2 = 0 \tag{2.0.26}$$

$$5\lambda_1 + 14\lambda_2 + 18 = 0 \tag{2.0.27}$$

Hence,

$$\lambda_1 = \frac{10}{19} \tag{2.0.28}$$

$$\lambda_2 = \frac{-28}{19} \tag{2.0.29}$$

$$\therefore \mathbf{P} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \frac{10}{19} \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = \frac{1}{19} \begin{pmatrix} 29 \\ 8 \\ 77 \end{pmatrix} \tag{2.0.30}$$

$$\therefore \mathbf{Q} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \frac{-28}{19} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \frac{1}{19} \begin{pmatrix} 20 \\ 11 \\ 86 \end{pmatrix} \tag{2.0.31}$$

Hence, the shortest distance between the given lines is $\frac{3}{\sqrt{19}}$.

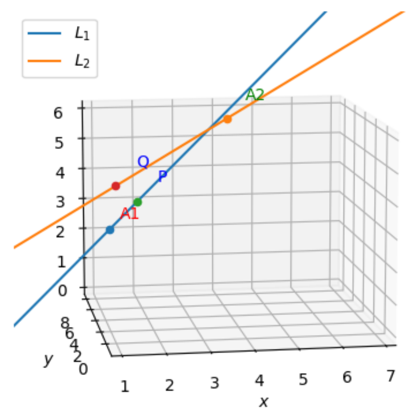

Fig. 0: Plot from Python Code.