Total points: 100 points

Name: ____________________________

Number: ____________________________

AMS 161–F20, HW Assignment 1

Due Date: Friday 09/11/2020
You MUST show all your work in order to get full credits

Question:	1	2	3	4	5	Total
Points:	30	10	18	30	12	100
Score:						

In this assignment, you will **modify the provided codes** in this file on the Jupyter notebook. You may download the file and work on it using `Anaconda Distribution` (local host) or `CoCalc`.

In order to be accepted, you **MUST** submit the two following files:

a. A single PDF file showing all your supporting work with the name:

Number-FirstName-LastName-AMS161-HW1.pdf

b. A file on Jupyter notebook including all coding assignments with the name:

Number-FirstName-LastName-AMS161-HW1.ipynb

1. As we know that the area and distance problems lead to the concept of a definite integral which is defined as

$$A = \int_b^a f(x)\, dx = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i^*)\Delta x \tag{1}$$

where $x_i^* \in [x_i, x_{i+1}]$, where $a = x_0 < x_1 < x_2 < \ldots x_{n-1} < x_n = b$ is a partition of the given interval $[a, b]$, and where $\Delta x = \dfrac{b-a}{n}$. Suppose that f is **integrable**, i.e. the limit in (1) exists and does not depend on the sample point. Let L_n, R_n, and M_n be the approximations to A using left endpoint, right endpoint, and midpoint respectively.

(a) Write a formula for R_n and L_n. (4)

(b) Suppose that $f(x) = 1 + x^2$. Estimate the area A under the graph of f from $x = -1$ to $x = 2$ using three rectangles and right endpoints (that is using R_3). Then improve your estimate by using six rectangles, i.e. using R_6. Sketch the curve and the approximating rectangles. (4)

(c) Repeat part (b) using left endpoints (i.e. L_3 and L_6). (4)

(d) Repeat part (b) using midpoints (i.e. M_3 and M_6). (4)

(e) From your sketches in parts (b) – (d), which appears to be the best estimate? (2)

(f) Do part (b), (c), and (d) in `Jupyter Notebook`. (6)

(g) Make the following table of values in `Jupyter Notebook`. (Just run the provided code there and do nothing then.) (2)

n	R_n	L_n	M_n
3			
6			
10			
20			
30			
50			
100			
200			
300			
400			

(h) Plot n, R_n, L_n, and M_n by running the provided code in `Jupyter Notebook`. Then you will get figure 1. Write any relevant comment about the behavior of R_n, L_n, and R_n in the long run. (4)

(i) **Extra credits (+ 4 pts): Write your own code to get figure 2.**

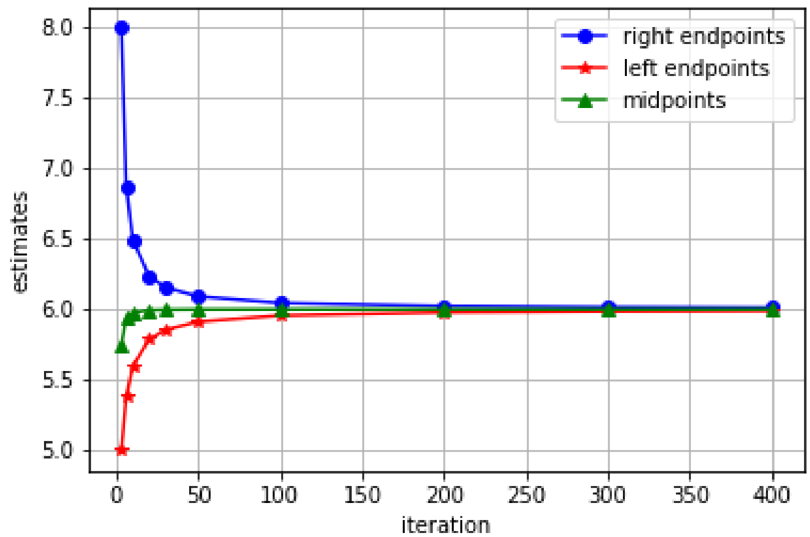


Figure 1: Behavior of R_n, L_n, and M_n.

2. **Distance Problem**

   (a) Oil leaked from a tank at a rate of $r(t)$ liters per hour. The rate decreased as time passed and values of the rate at two-hour time intervals are shown in the table. Find lower and upper estimates for the total amount of oil that leaked out. (4)

t (h)	0	2	4	6	8	10
$r(t)$ (L/h)	8.7	7.6	6.8	6.2	5.7	5.3

   (b) When we estimate distances from velocity data, it is sometimes necessary to use times $t_0, t_1, t_2, t_3, \ldots$ that are equally spaced. We can still estimate distances using the time periods $\Delta t_i = t_i - t_{i-1}$. For example, on May 7, 1992, the space shuttle *Endeavour* was launched on mission STS-49, the purpose of which was to install a new perigee kick motor in an Intelsat communications satellite. The table, provided by NASA, gives the velocity data for the shuttle between liftoff and the jettisoning of the solid rocket boosters. Use these data to estimate the height above the earth's surface of the *Endeavour*, 62 seconds after liftoff. (6)

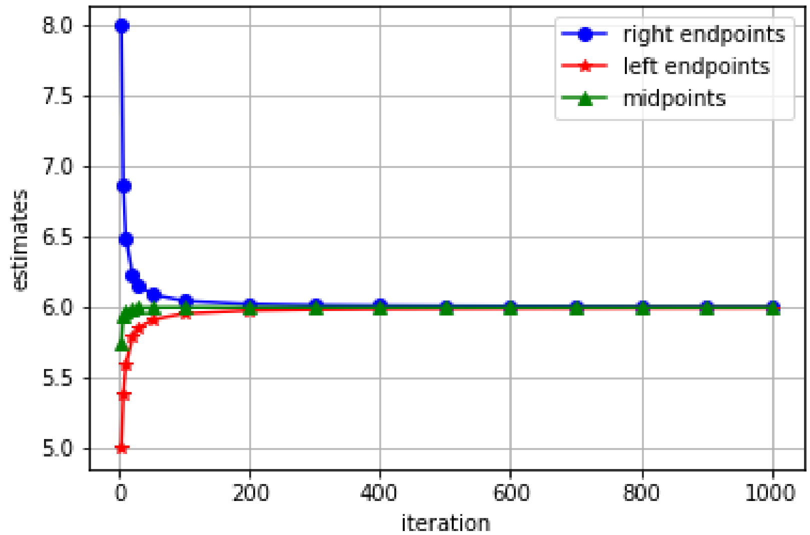


Figure 2: Behavior of $R_n, L_n,$ and M_n in the long run.

Event	Time (s)	Velocity (ft/s)
Launch	0	0
Begin roll maneuver	10	185
End roll maneuver	15	319
Throttle to 89%	20	447
Throttle to 67%	32	742
Throttle to 104%	59	1325
Maximum dynamic pressure	62	1445
Solid rocket booster separation	125	4151

Hint: You may estimate the distances using right endpoints.

(c) **Extra credits (+ 4 pts):** Explain the provided code in Jupyter Notebook.

3. **Understanding the concept of Definite integrals**

(a) Use the definition in question 1 to find an expression for the area under the graph of (4)

$$f(x) = \frac{2x}{x^2+1}, \quad 1 \le x \le 3$$

as a limit. Do not evaluate the limit.

(b) Determine a region whose area is equal to the limit (4)

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan \frac{i\pi}{4n}.$$

Do not evaluate the limit.

(c) Let A be the area under the graph of an increasing continuous function f from a to b, and let L_n and R_n be the approximations to A with n subintervals using left and right endpoints, respectively.

i. How are A, L_n, and R_n related? (4)

ii. Show that $R_n - L_n = \dfrac{b-a}{n}[f(b) - f(a)]$. (4)

iii. Deduce that $R_n - A < \dfrac{b-a}{n}[f(b) - f(a)]$. (2)

4. **Understanding the definition of Definite Integrals**

(a) Express the limit as a definite integral on the given interval.

i. $\lim_{n\to\infty} \sum_{i=1}^{n} x_i \ln(1 + x_i^2)\Delta x, \quad [2, 6]$ (4)

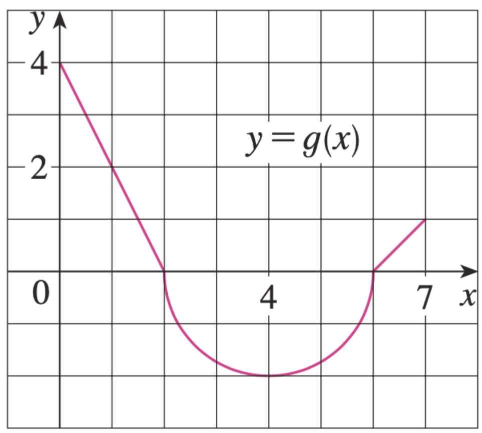
ii. $\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{2x_i^* + (x_i^*)^2}\Delta x, \quad [1, 8]$ (4)

(b) The graph of g consists of two straight lines and a semicircle. Use it to estimate each integral.

i. $\int_0^2 g(x)\, dx$ (2)

ii. $\int_2^6 g(x)\, dx$ (2)

iii. $\int_0^7 g(x)\, dx$ (2)



(c) Evaluate the integral by interpreting it in terms of areas.

i. $\int_{-3}^{0} (1 + \sqrt{9 - x^2})\, dx$ (4)

ii. $\int_{-1}^{2} |x|\, dx$ (4)

(d) Evaluate $\int_{\pi}^{\pi} \sin^2 x \cos^4 x\, dx$ (2)

(e) Given that $\int_0^1 3x\sqrt{x^2 + 4}\, dx = 5\sqrt{5} - 8$, what is $\int_0^1 3u\sqrt{u^2 + 4}\, du$? (2)

(f) Find $\int_0^5 f(x)\, dx$ if (4)

$$f(x) = \begin{cases} 3 & \text{for} \quad x < 3 \\ x & \text{for} \quad x \geq 3 \end{cases}$$

5. **Understanding the limit language** Suppose f is a continuous function on the interval $[a, b]$. Then f is integrable on $[a, b]$, that is the limit in (1) exists and does not depend on the sample point. The definition of the limit tells us that the Riemann sum (12)

$$A_n = \sum_{i=0}^{n-1} f(x_i^*)\Delta x$$

is approaching the exact value $A = \int_a^b f(x)\, dx$ as $n \to \infty$. This means if we increase the values of n we will get better approximations. In practice, we do not know whether we obtain a good approximation A_n or not since the exact value A is not known. To resolve this issue, we need to use the precise definition of the limit in (1) as follows:

Given $\epsilon > 0$ arbitrarily small, there exists N sufficiently large such that

$$|A - A_n| < \epsilon \quad \text{for all} \quad n \geq N.$$

Nevertheless, this definition is still ambiguous in practice for the following reasons:

- the exact value A is unknown,
- how to know if N is sufficiently large,
- how to find such an N.
- how to know if A_n is a good approximation of A.

For simplicity, suppose we want to estimate the integral $A = \int_0^1 (e^x + 1)\, dx$ using A_n, where A_n could be R_n, or L_n, or M_n. To proceed, we need something to 'measure' how good our estimate is. That is why ϵ, which is called a **degree of accuracy**, comes into the play.

We say that the estimate A_n is good (relative to ϵ) in the sense that $|A_n - A| < \epsilon$. However, since A is not known in advance, we should compare A_n with A_{n+1}. If $|A_n - A_{n+1}| < \epsilon$, then we are confident to say that A_n is a good approximation.

In summary, given a degree of accuracy $\epsilon > 0$, we calculate A_n for $n = 1, 2, 3, \ldots$ and check if

$$|A_n - A_{n+1}| < \epsilon.$$

There is some N satisfying the above inequality, we stop and say that A_N is a good approximation to within the given ϵ.

Go to the `Jupyter Notebook` to work on this problem.