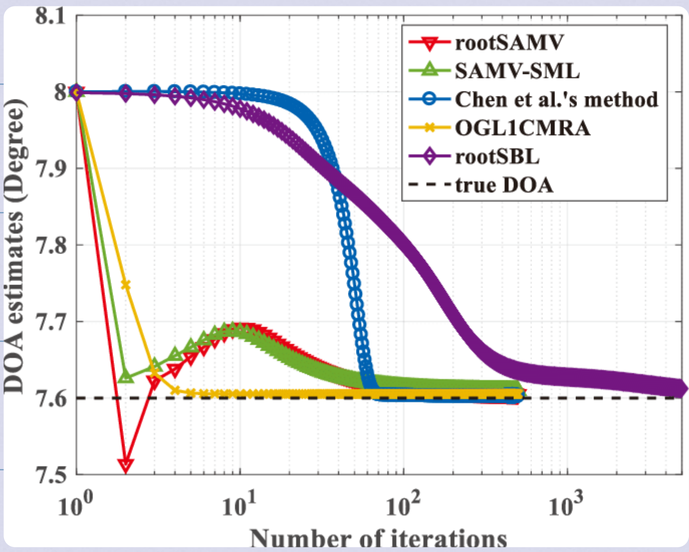
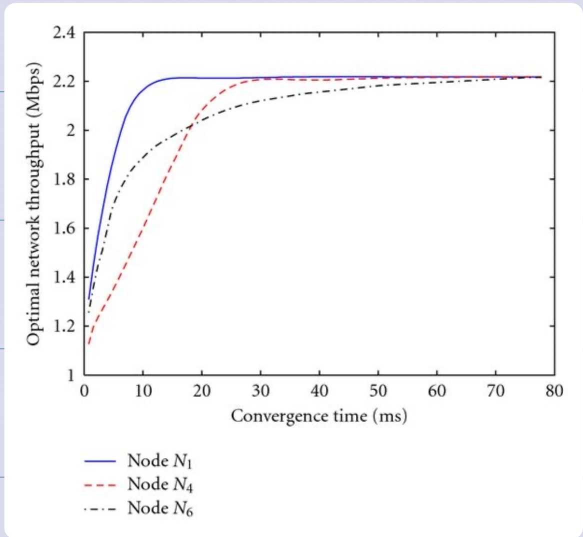
ERROR ANALYSIS FOR ITERATIVE METHODS

Goals:

- Investigate the order of convergence of the iteration schemes we have studied so far
- Improve convergence rate of Newton's method in special circumstances





- Fixed-point
- Newton-Raphson
- Bisection
- Secant, False position

$\Big\}$ $\xRightarrow{\text{generate}}$ $\{p_n\} \longrightarrow p$ a root of $f(x) = 0$

Question: How to compare the convergence rates of these methods?

→ measuring how rapidly a sequence converges. 😑

§ Order of Convergence

Definition:

Suppose $\{p_n\} \longrightarrow p$ as $n \longrightarrow \infty$

$p_n \neq p$ for all n.

If there exist constants λ and α for which

$$\boxed{\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda}$$

then $\{p_n\} \longrightarrow p$ of order α

with asymptotic error constant λ.

$|p_{n+1} - p| \approx \lambda |p_n - p|^{\alpha}$ for n sufficiently large

- An iterative method of the form $p_n = g(p_{n-1})$ is said to be of order α if $\{p_n\} \longrightarrow p$: solution of $p = g(p)$ of order α.

😄 The bigger α is, the faster the sequence converges!

- If $\alpha = 1$ (and $\lambda < 1$), the sequence is linearly convergent.
- If $\alpha = 2$, the sequence is quadratically convergent.

◎ Theorem:

- Let $g \in C([a,b])$ s.t. $a \leq g(x) \leq b$
- Suppose g' is continuous on (a, b) and there exists a constant $k < 1$ s.t.

$$|g'(x)| < k, \text{ for all } x \in (a,b)$$

If $g'(p) \neq 0$, then for any $p_0 \neq p \in [a,b]$, the sequence

$$p_n = g(p_{n-1}) \quad \text{for } n \geq 1$$

converges only linearly to the unique fixed point $p \in [a,b]$.

Proof

- It follows from the Fixed-point Theorem that $\{p_n\} \longrightarrow p$.
- Using the Mean Value Theorem gives us ξ_n between p_n and p s.t.

$$p_{n+1} - p = g(p_n) - g(p) = g'(\xi_n)(p_n - p)$$

- So

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha=1}} = \lim_{n\to\infty} |g'(\xi_n)| \overset{\text{continuity}}{\underset{\text{of } g'}{=\!=}} |g'(p)| \neq 0$$

and we complete the proof. ☺

This theorem implies that an arbitrary fixed-point technique that generates a convergent sequence does so only linearly.

Question: Can we get the quadratic convergence?

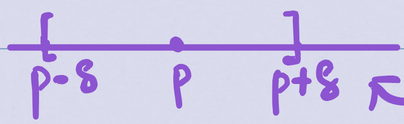
Getting a better result?

$\longrightarrow$ pay more!

we do need additional assumptions.

Theorem

- Let p be a solution of $g(x) = x$
- Suppose $\begin{cases} g'(p) = 0 \\ g'' \text{ is continuous with } |g''(x)| < M \text{ on} \end{cases}$



interval I $\qquad I \ni p$.

- Then $\exists\, \delta > 0$ such that $p_0 \in [p-\delta, p+\delta]$ the sequence $\{p_n\}$ defined by

$$p_n = g(p_{n-1})$$

converges at least quadratically to p.

- Moreover,

$$\boxed{|p_{n+1} - p| < \frac{M}{2}|p_n - p|^2}$$

Proof:

- Choose $k \in (0,1)$.

Since $\left.\begin{array}{l} g'(p) = 0 \\ g' \text{ is continuous on } I \ni p \end{array}\right\}$

$\Longrightarrow \exists\, \delta > 0$ such that

$$|g'(x)| < k < 1 \text{ for all } x \in [p-\delta, p+\delta]$$

- Moreover we can verify that

$$g : [p-\delta, p+\delta] \longrightarrow [p-\delta, p+\delta]$$

i.e. $g(x) \in [p-\delta, p+\delta]$, $\forall x \in [p-\delta, p+\delta]$

(by Mean Value Theorem)

• Expanding $g(x)$ in a linear Taylor polynomial for $x \in [p-\delta, p+\delta]$ gives us

$$\boxed{g(x) = \underbrace{g(p)}_{=\,p} + \underbrace{g'(p)}_{=\,0}(x-p) + \frac{g''(\xi)}{2}(x-p)^2}$$

where ξ lies between x and p.

So $g(x) = p + \frac{g''(\xi)}{2}(x-p)^2$

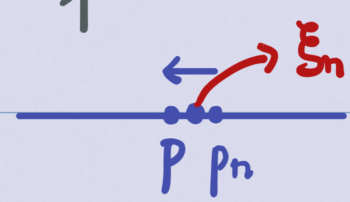
• Then $p_{n+1} = g(p_n) = p + \frac{g''(\xi_n)}{2}(p_n - p)^2$

where ξ_n lies between p_n and p.

So $p_{n+1} - p = \frac{g''(\xi_n)}{2}(p_n - p)^2$

• Since $\begin{cases} |g(x)| \leq k < 1 \quad \forall x \in [p-\delta, p+\delta] \\ g \text{ maps } [p-\delta, p+\delta] \text{ into itself} \end{cases}$

Fixed-point $\Longrightarrow$ Theorem $\{p_n\} \xrightarrow{n \to \infty} p$



which also implies $\lim_{n\to\infty} \xi_n = p$.

• Thus $\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^2} = \frac{|g''(p)|}{2} < \frac{M}{2}$

$$\Longrightarrow \{p_n\} \text{ is } \begin{cases} \text{quadratically convergent if } g''(p) \neq 0. \\ \text{of higher-order convergent if } g''(p) = 0. \end{cases}$$

• Moreover, it follows from the continuity of g'' and the boundedness of g'' on $[p-\delta, p+\delta]$ that

$$|p_{n+1}-p| < \frac{M}{2}|p_n-p|^2$$

for n sufficiently large. 😄

Remark:

For a fixed point method to converge quadratically, we need $\begin{cases} 1)\ g(p) = p \\ 2)\ g'(p) = 0 \end{cases}$

Consider the root-finding problem:

$$f(x) = 0$$

$$\Big\downarrow \text{ convert to}$$

a fixed-point problem: $g(x) = x$

How?
$$f(x) = 0$$

$$\underbrace{x - f(x)}_{g(x)\,?} = x \quad \text{(not efficient)}$$

→ modify a bit: $g(x) = x - \underbrace{\phi(x)}_{\text{will be chosen later}} f(x)$

ϕ: differentiable

Goal: Construct the iterative procedure from g that is quadratically convergent.

We need $g'(p) = 0$ when $f(p) = 0$.

- We have
$$g'(x) = 1 - \phi'(x)f(x) - \phi(x)f'(x)$$
$$g'(p) = 1 - \phi'(p)f(p) - \phi(p)f'(p)$$

- So $\quad 0 = 1 - 0 - \phi(p) f'(p)$

$$\therefore \phi(p) = \frac{1}{f'(p)}$$

Choose $\phi(x) = \dfrac{1}{f'(x)}$

- Then $\boxed{p_n = g(p_{n-1}) = p_{n-1} - \dfrac{f(p_{n-1})}{f'(p_{n-1})}}$

Newton - Raphson Method

So if $f(p) = 0$ & $f'(p) \neq 0$, then for starting values sufficiently close to p, Newton-Raphson method will converge at least quadratically.

Issue : What happens if $f'(p) = 0$ when $f(p) = 0$?

Multiple Roots

Definition:

A solution p of $f(x) = 0$ is a zero of multiplicity m of f if for $x \neq p$, we can write

$$\boxed{f(x) = (x-p)^m q(x), \text{ where } \lim_{x \to p} q(x) \neq 0.}$$

$$\lim_{x \to p} \frac{f(x)}{\underbrace{(x-p)^m}_{q(x)}} \neq 0$$

$q(x)$ represents a portion of $f(x)$ that does not contribute to the zero of f.

Example:

$$f(x) = x^2 (x-2)^3 (x+1)^4$$

$x = 0$ is a zero of multiplicity 2

$x = 2$ is a zero of multiplicity 3

$x = -1$ is a zero of multiplicity 4

Theorem:

The function $f \in C^1([a,b])$ has a simple zero at p in (a,b) iff $\begin{cases} f(p) = 0 \\ f'(p) \neq 0 \end{cases}$

Proof:

$(\Rightarrow)$ Suppose f has a simple zero at p in (a,b).

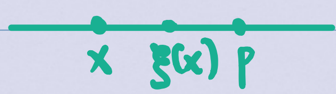
Then $f(x) = (x-p)\,q(x)$, where $\lim_{x\to p} q(x) \neq 0$.

- So $f'(x) = q(x) + (x-p)\,q'(x)$
- Thus $f'(p) = \lim_{x\to p} f'(x) = \lim_{x\to p} q(x) \neq 0.$

$(\Leftarrow)$ Suppose $f(p) = 0$ and $f'(p) \neq 0$.

- Expanding f in a Taylor polynomial about p gives us

$$f(x) = \underbrace{f(p)}_{}^{\,0} + f'(\xi(x))(x-p)$$



where $\xi(x)$ lies between x and p.

- So $f(x) = \underbrace{f'(\xi(x))}_{g(x)}(x-p)$

which implies that $\lim\limits_{x \to p} g(x) = \lim\limits_{x \to p} f'(\xi(x))$

$$= f'(p) \neq 0. \quad 😄$$

Theorem:

The function $f \in C^m([a,b])$ has a zero of multiplicity m at p in (a,b) iff

$$\begin{cases} 0 = f(p) = f''(p) = \dots = f^{(m-1)}(p) \\ f^{(m)}(p) \neq 0 \end{cases}$$

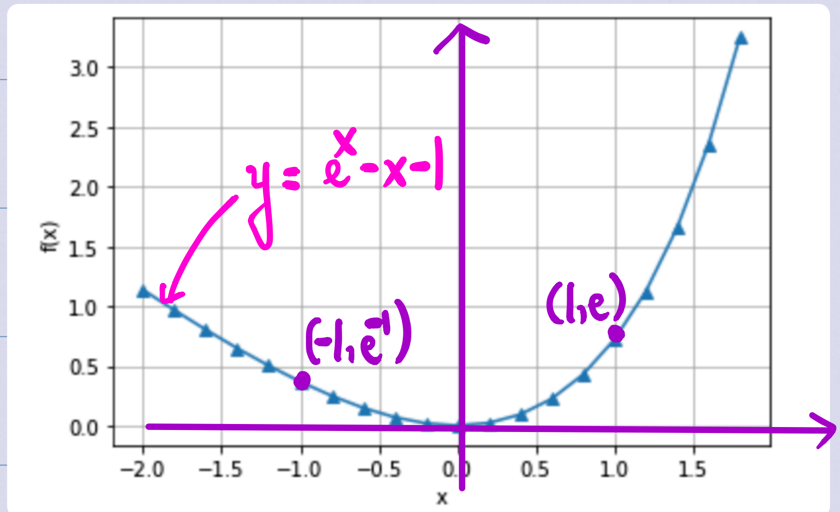
Example: Let $f(x) = e^x - x - 1$.

(a) f has a zero of multiplicity 2 at $x = 0$.

- We have $f'(x) = e^x - 1$ and $f''(x) = e^x$
- So $f(0) = 0$, $f'(0) = 0$, but $f''(0) = 1 \neq 0$.

(b) The Newton's method with $p_0 = 1$ converges to this zero but not quadratically.

```
n = 0 and pn = 1.000000e+00
n = 1 and pn = 5.819767e-01
n = 2 and pn = 3.190550e-01
n = 3 and pn = 1.679962e-01
n = 4 and pn = 8.634887e-02
n = 5 and pn = 4.379570e-02
n = 6 and pn = 2.205769e-02
n = 7 and pn = 1.106939e-02
n = 8 and pn = 5.544905e-03
n = 9 and pn = 2.775014e-03
n = 10 and pn = 1.388149e-03
n = 11 and pn = 6.942351e-04
n = 12 and pn = 3.471577e-04
n = 13 and pn = 1.735889e-04
n = 14 and pn = 8.679696e-05
n = 15 and pn = 4.339911e-05
n = 16 and pn = 2.169971e-05
```



$f'(p_n) \approx 0$ which slow down the rate of convergence.

Question: How to handle the problem of multiple roots of a function f ?

Introduce a new function μ as follows:

$$\boxed{\mu(x) = \frac{f(x)}{f'(x)}}$$

- Suppose p is a zero of f of multiplicity m with

$$f(x) = (x-p)^m q(x)$$

- Then
$$\mu(x) = \frac{(x-p)^m q(x)}{m(x-p)^{m-1} q(x) + (x-p)^m q'(x)}$$
$$= (x-p) \frac{q(x)}{\underbrace{m q(x) + (x-p) q'(x)}_{= Q(x)}}$$

- We have
$$Q(p) = \frac{\cancel{q(p)}}{m \cancel{q(p)}} = \frac{1}{m} \neq 0$$

so p is a simple zero of $\mu(x)$.

Yeah, no zero of multiplicity m any more !

- Apply Newton's method to $\mu(x)$, we get
$$g(x) = x - \frac{\mu(x)}{\mu'(x)} = x - \frac{f(x)/f'(x)}{\{[f'(x)]^2 - f(x)f''(x)\}/[f'(x)]^2}$$

which implies that

$$\boxed{g(x) = x - \frac{f(x)f'(x)}{[f'(x)]^2 - f(x)f''(x)}}$$

* Under appropriate assumptions, functional iteration applied to g will be quadratically convergent Nice regardless of the multiplicity of the zero of f.

* Drawbacks:

  - Compute $f''(x)$
  - Round-off problems: $[f'(x)]^2 - f(x)f''(x)$ may be 0.

The modification of Newton's method improves the rate of convergence.

```
n = 0 and pn = -2.342106e-01
n = 1 and pn = -8.458280e-03
n = 2 and pn = -1.189018e-05
n = 3 and pn = -4.218591e-11
n = 4 and pn = -4.218591e-11
n = 5 and pn = -4.218591e-11
```

Example: Find a zero of $f(x) = x^3 + 4x^2 - 10 = 0$ using Newton's method & its modification.

($p = 1.36523001$, $p_0 = 1.5$)