Applied Calculus in Real-World Situations

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1 Introduction

1.1 Course Overview

The course provides students some fundamental knowledge and understanding about Calculus, one of the most important branches of mathematics: the study of continuous change. In this course, I aim to focus on the following topics:

1. The concept of functions
2. Elementary functions
3. Derivatives of functions and their applications (differential calculus)
4. Numerical methods of solving root-finding problems

The purpose of the course is to help students understand how calculus is used in a lot of ways and applications. Among the disciplines that utilize calculus include physics, engineering, economics, statistics, and medicine. It is used to create mathematical models in order to arrive into optimal solution. For example, in physics, calculus is used a lot such as motion, electricity, heat, light, harmonics, acoustics, astronomy, and dynamics. Even advanced physics concepts including electromagnetism and Einsteins theory of relativity use calculus. In the field of chemistry, calculus is used to predict functions such as reaction rates and radioactive decay. Meanwhile, in biology, it is utilized to formulate rates such as birth and death rates. In economics, calculus is used to compute marginal cost and marginal revenue, enabling economists to predict maximum profit in a specific setting. In addition, it is used to check answers for different mathematical disciplines such as statistics, analytical geometry, and algebra. So students can see that calculus has a huge role in the real world. During the course, students not only learn important concepts in

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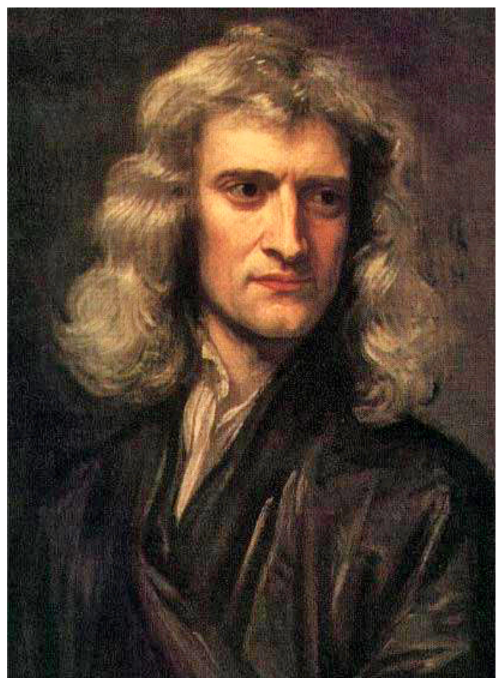
calculus but also learn how those concepts arise, the motivations and stories behind them. More importantly, students would develop their critical thinking and reasoning skills by working on the assignments and projects (some would require using Python), which is very crucial for their college life in the future.

Please click here for the coding materials used in the course.

1.2 Overview of Calculus

Calculus, or **infinitesimal calculus** or "the calculus of infinitesimals" is the study of continuous change, in the same way that geometry is the study of shape and algebra is the study of generalizations of arithmetic operations.
Calculus is used in every branch of the physical sciences, actuarial science, computer science, statistics, engineering, economics, business, medicine, demography, and in other fields wherever a problem can be mathematically modeled and an optimal solution is desired.

More information about Calculus can be found here.

1.3 Installing Python

We recommend beginners to install the Anaconda distribution of Python powered by Continuum Analytics because it comes as a convenient combo of many basic things you need to get started with Python. You can download the Anaconda distribution from here. You should choose the installer according to your operating system (Windows / Mac). This document assumes you are using Python 3.7 (Graphical Installer). You can then follow the instructions and install Anaconda with default settings.

2 Function

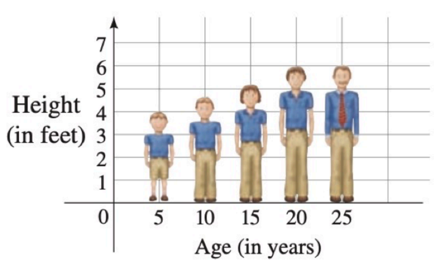
2.1 Functions All Around Us

In may real-world situations, many quantities do not exist independently but they depend on one another. Understanding a relationship among them would provide us useful information.

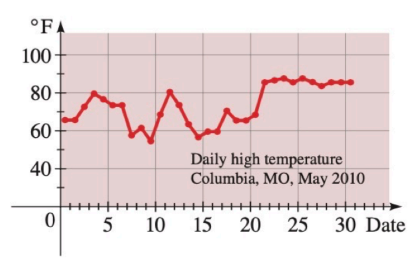
For example, knowing a relationship between a distance a **skydiver** falls and the time he/she has been falling allows him/her to determine when to open the parachute. Knowing a relationship between the temperature and the date may help us to predict the temperature on a particular date. Knowing a relationship between postage and the weight can help us to estimate the amount of money that we should pay for our particular package.

Example 2.1 The following examples illustrate a dependence of one quantity on another.

- The height **depends on** the age.
- The temperature **depends on** the date.
- The cost of mailing a package **depends on** its weight.



Height is a function of age.



Temperature is a function of date.

w (ounces)	2014 Postage (dollars)
$0 < w \le 1$	0.98
$1 < w \le 2$	1.19
$2 < w \le 3$	1.40
$3 < w \le 4$	1.61
$4 < w \le 5$	1.82
$5 < w \le 6$	2.03

Postage is a function of weight.

In Mathematics, we use the term "function" to describe this dependence. We can restate the above statements as follows:

- The height is **a function of** the age.
- The temperature is **a function of** the date.
- The cost of mailing a package is **a function of** its weight.

`Python Code`: In the Jupyter Notebook, replace "**depends on**" by "**is a function of**".

Example 2.2 There are many examples of functions in our real-world situations.

(i) The area of a circle is a function of its radius: $\boxed{A = \pi r^2}$.

(ii) The number of bacteria in a culture is a function of time.

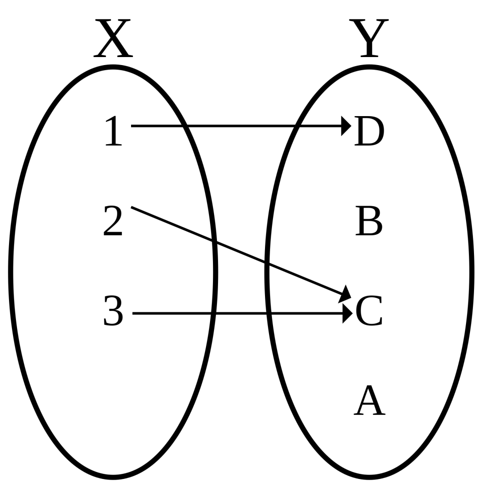
(iii) The weight of an astronaut is a function of his/her elevation.

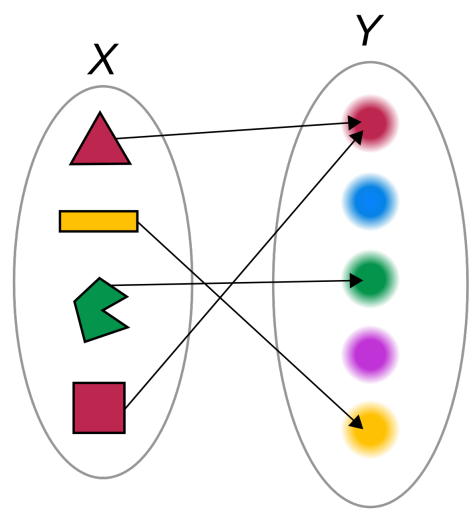
(iv) The price of a commodity is a function of the demand for that commodity.

(v) The temperature of water from the faucet is a function of time.

Your turn: Give at least two examples of functions.

Definition 2.3 *A function (of the single variable) is a rule that assigns to each element x in a set X exactly one element y in a set Y.*





To indicate that y and x are related by the rule f, we write $y = f(x)$ and read "f of x" or "f at x".

Remark 2.4 We call x an **independent variable** or an **input** and call y a **dependent variable** or **output**. The notation $f(x)$ should be understood that we apply a rule f at the input x to get the output y; or equivalently the rule f transform x into y.

Definition 2.5 *The domain of a function $f : X \to Y$, denoted by D_f, is the set of all possible inputs; that is*

$$D_f = \{x \in X |\ f(x) \text{ is defined}\}.$$

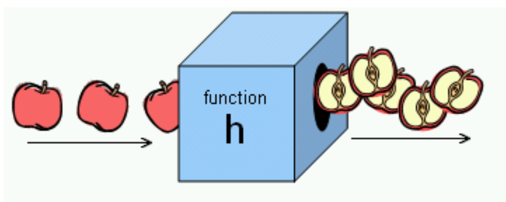
The range of f, denoted by R_f, is the set of all possible outputs; that is

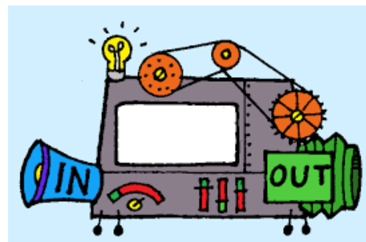
$$R_f = \{f(x) |\ x \in D_f\}.$$

The graph of f, denoted by $gph(f)$, is the set

$$gph(f) = \{(x, f(x)) \in \mathbb{R}^2 |\ x \in D_f\},$$

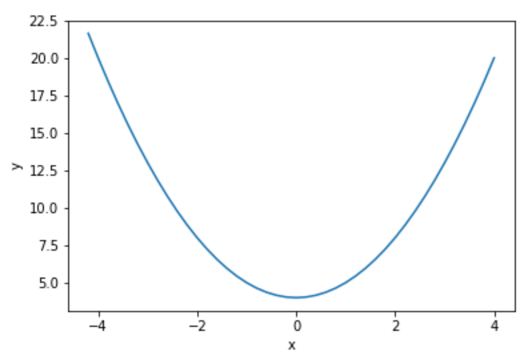
We can think of a function as a machine.





Example 2.6 Let f be a function defined by

$$f(x) = x^2 + 4$$



(i) Express in words how f acts on the input x to produce the output $f(x)$.

(ii) Evaluate $f(3)$, $f(-2)$, and $f(\sqrt{5})$.

(iii) Suppose the input changes from $x = 3$ to $x = 6$, compute $f(6) - f(3)$. This quantity measures the change of the output and is also called the net change.

(iv) Suppose the input changes in h units from $x = a$ to $x = a + h$, the corresponding net change is $f(a + h) - f(a)$. Simplify $\dfrac{f(a+h) - f(a)}{h}$, this difference quotient would tell us how fast the values of f change relative to the change of the input.

(v) Find the domain and the range of f.

(vi) Draw a machine diagram for f.

The function f in this example is described by an explicit formula.

`Python Code`: Solve parts (i), (ii), (iii), and (vi) in Jupyter Notebook.

`Python Code`: Write a function to tell us the sign of a number in Jupyter Notebook.

A Piecewise Defined Function: In some situations, using a single explicit formula to describe a function f is not enough. Instead, its formula may vary depending on the domains of the input.

Example 2.7 A cell phone plan costs \$39 a month. The plan includes 2 gigabytes (GB) of free data and charges \$15 per GB for any additional data used. The monthly charges are a function of the number of GBs of data used, given by

$$C(x) = \begin{cases} 39 & \text{if } 0 \leq x \leq 2 \\ 39 + 15(x-2) & \text{if } x > 2 \end{cases}$$

(i) How much do I have to pay if I use 0.5 GB, 2 GBs, 4 GBs?

(ii) To pay \$60, how much data should I spend?

Example 2.8 The weight of an Astronaut If an astronaut weighs 130 lb on the surface of the earth, then her weight when she is h miles above the earth is given by the function

$$w(h) = 130\left(\frac{3960}{3960+h}\right)^2$$

(i) What is her weight when she is 100 mi above the earth?

(ii) Construct a table of values for the function w that gives her weight at heights from 0 to 500 mi. What do you conclude from the table?

(iii) Fine the net change in the astronaut's weight from ground level to a height of 500 mi.

2.2 Four Ways to Represent a Function

We can represent a function by four different ways:

- Verbal: using words
- Algebra: using a formula
- Visual: using a graph

- Numerical: using a table of values

Assign more problems here.

3 Mathematical Models

Suppose we are given a real-world problem (P) and would like to solve it. To proceed, our first task is to formulate a mathematical model (P_m) that represents the real-world problem (P) quite well. This task, sometimes, is very challenging since the real-world problem (P) may be very complicated. The point is we need to understand the relationships among the relevant quantities in the problem and make some assumptions that **simplify the problem enough** to make it mathematically tractable. A mathematical model is basically a mathematical description (often by means of a function or equation) of a real-world phenomenon and its purpose is to understand the phenomenon and perhaps to make prediction about the future behavior.

The next task is to apply our mathematical knowledge to solve the formulated model (P_m) which hence enables us to draw some useful conclusions. Then, in the third task, we will use those mathematical conclusions and interpret them as valuable information about the original real-world problem (P). The final step is to test our predictions by checking against new real data. If the predictions do not compare well with the reality, that is (P_m) misrepresents the real-world model (P), we need to modify our mathematical model or reformulate it and start the cycle again.

A mathematical model (P_m) is never a completely accurate representation of a physical situation (the real-world problem (P)) – it is an *idealization*. A good model would simplify the reality enough to permit mathematical calculations but it should be accurate enough to provide valuable conclusions. It is important to realize the limitations of the model. In the end, Mother Nature has the final say.

There are many different types of functions that can be used to model relationships observed in the real world. We will study and explore them in this course.

3.1 Linear Models

A **linear function** is a function of the form

$$y = f(x) = mx + b$$

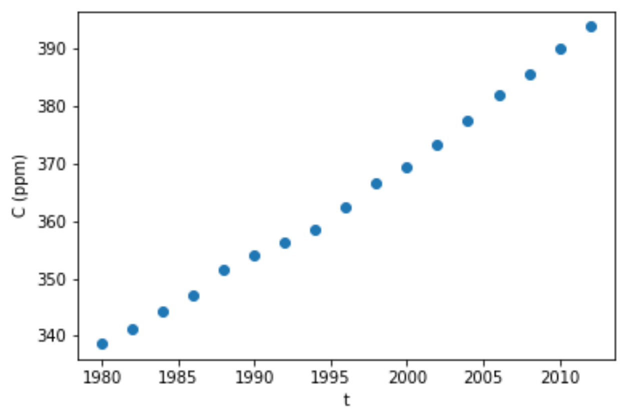
where m is the slope of the line and b is the y-intercept.

`Python Code`: Write a code to graph a linear function.

Example 3.1 The table below lists the average carbon dioxide level in the atmosphere, measured in parts per million at Mauna Loa Observatory from 1980 to 2012. Use the data in Table 1 to find a model for the carbon dioxide level.

Year	CO_2 level (in ppm)	Year	CO_2 level (in ppm)
1980	338.7	1998	366.5
1982	341.2	2000	369.4
1984	344.4	2002	373.2
1986	347.2	2004	377.5
1988	351.5	2006	381.9
1990	354.2	2008	385.6
1992	356.3	2010	389.9
1994	358.6	2012	393.8
1996	362.4		

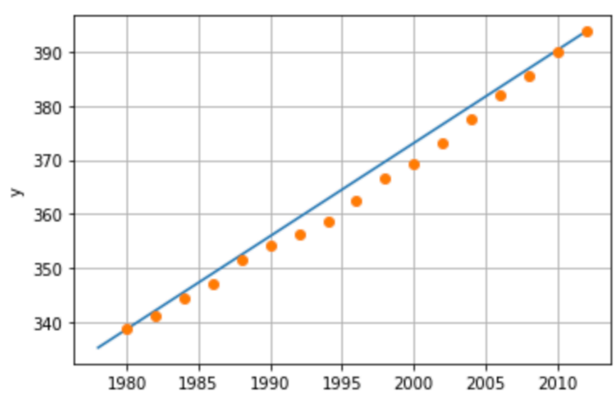
We can make a scatter plot showing a relationship between Year and CO_2 level in Python.



The scatter plot shows that our data points appear to lie close to a straight line (C and t have a linear relationship), so it we will use a linear model in this case. But there are many possible lines that approximate these data points, so which one should we use? On possibility is the line passing through the first and last data points. The equation of that line is given by

$$C = 1.722t - 3070.86$$

whose graph is given as follows.



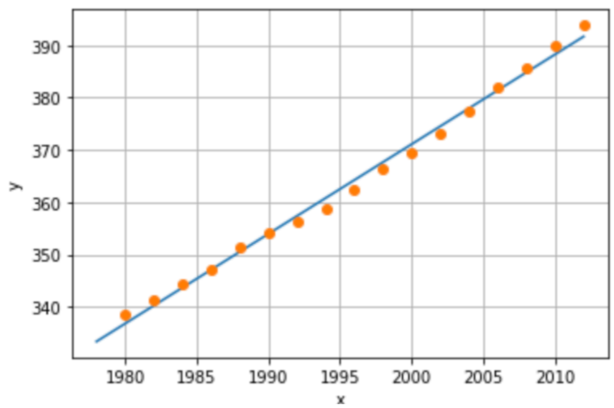
Note that our model gives values higher than most of the actual CO_2 levels. better linear model is obtained by a procedure from statistics called **linear regression**. Using a code in Python gives us the slope and y-intercept of the regression line as

$$m = 1.71262, \quad b = -3054.14$$

So our least squares model for the CO_2 level is

$$C = 1.71262t - 3054.14$$

The graph of this line is given below.



We can compare whether our best-fitting line gives a good approximation by examining the following table.

	Year	Exact value of C	Approximate value of C	Error
0	1980.0	338.7	336.850980	1.849020
1	1982.0	341.2	340.276225	0.923775
2	1984.0	344.4	343.701471	0.698529
3	1986.0	347.2	347.126716	0.073284
4	1988.0	351.5	350.551961	0.948039
5	1990.0	354.2	353.977206	0.222794
6	1992.0	356.3	357.402451	1.102451
7	1994.0	358.6	360.827696	2.227696
8	1996.0	362.4	364.252941	1.852941
9	1998.0	366.5	367.678186	1.178186
10	2000.0	369.4	371.103431	1.703431
11	2002.0	373.2	374.528676	1.328676
12	2004.0	377.5	377.953922	0.453922
13	2006.0	381.9	381.379167	0.520833
14	2008.0	385.6	384.804412	0.795588
15	2010.0	389.9	388.229657	1.670343
16	2012.0	393.8	391.654902	2.145098

It looks like our regression line gives good approximations. Using its equation, we can estimate the average CO_2 level for 1987 and predict the level for the year 2020. We even can predict when the CO_2 level will exceed 420 ppm.

3.2 Polynomials

A function P is called a polynomial if

$$\boxed{P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0}$$

where n is a nonnegative integer and the constants $a_0, a_1, a_2, \ldots, a_n$ are called the **coefficients** of the polynomial.

- If $a_n \neq 0$, then the **degree** of the polynomial is n.
- A polynomial of degree 1 is also a linear function.
- A polynomial of degree 2 is called a **quadratic function**.]item A polynomial of degree 3 is called a **cubic function**.

`Python Code`: graph the following functions in the Jupyter Notebook.

(i) $y = x^3 - x + 1$

(ii) $y = x^4 - 3x^2 + x$

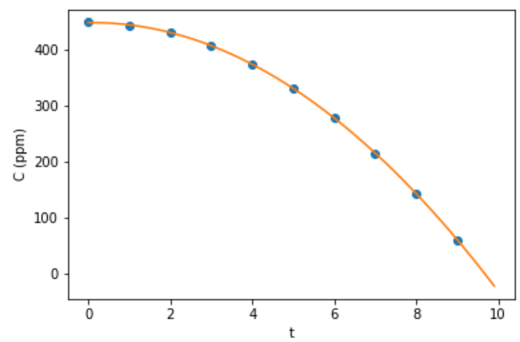
(iii) $y = 3x^5 - 25x^3 + 60x$

Example 3.2 A ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground, and its height h above the ground is recorded at 1-second intervals in Table 2. Find a model to fit the data and use the model to predict the time at which the ball hits the ground.

Time (seconds)	Height (meters)
0	450
1	445
2	431
3	408
4	375
5	332
6	279
7	216
8	143
9	61

Using a graphing calculator or computer algebra system (which uses the least squares method), we obtain the following quadratic model:

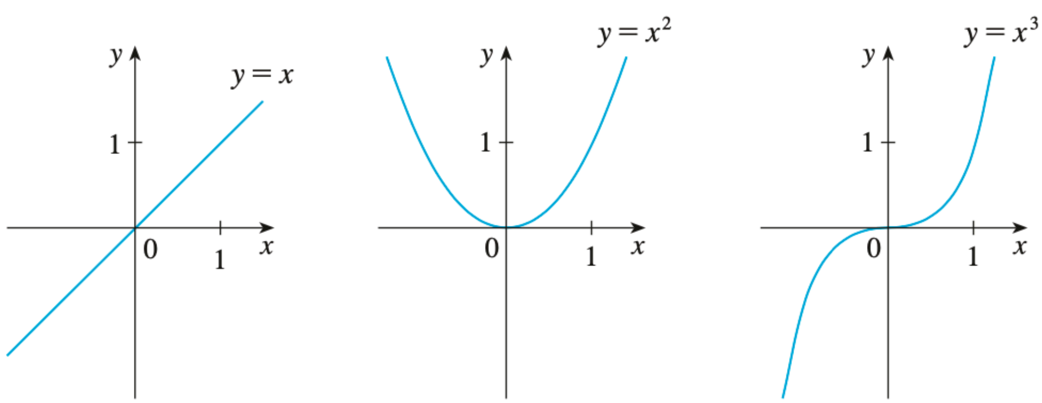
$$h = 449.36 + 0.96t - 4.90t^2$$

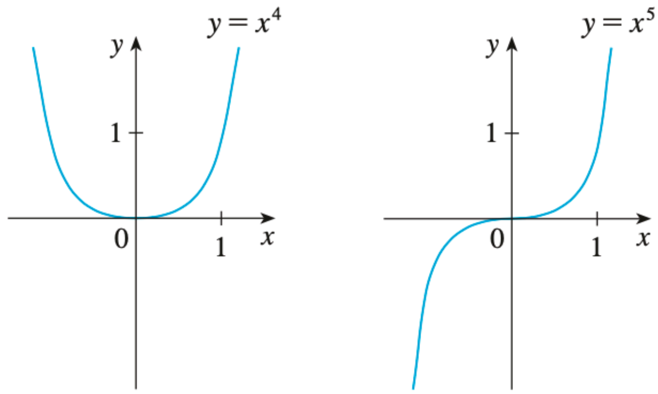


3.3 Power functions

A function of the form $\boxed{f(x) = x^a}$, where a is a constant, is called a power function. Consider the following cases.

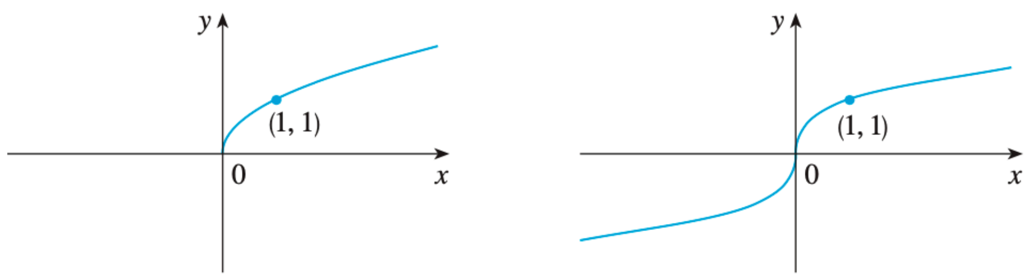
- $a = n$, where n is a positive integer. The graphs of several power functions are given as follows.





`Python Code`: graph the these functions in the Jupyter Notebook.

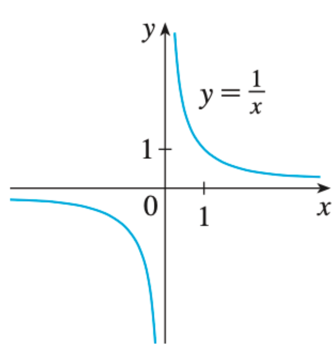
- $a = 1/n$, where n is a positive integer. The function $\boxed{f(x) = x^{1/n} = \sqrt[n]{x}}$ is a **root function**.



(a) $f(x) = \sqrt{x}$ (b) $f(x) = \sqrt[3]{x}$

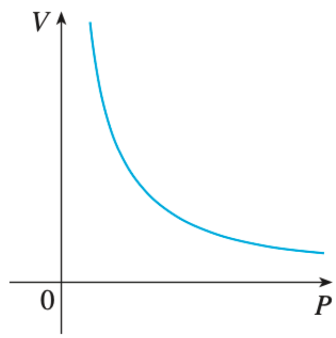
`Python Code`: graph the these functions in the Jupyter Notebook.

- $a = -1$. The graph of the reciprocal function $f(x) = x^{-1}$ is shown below.



- This kind of function arises in physics and chemistry in connection with Boyles Law.

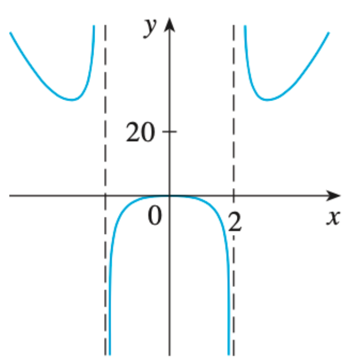
- Boyles Law: *when the temperature is constant, the volume V of a gas is inversely proportional to the pressure P:* $\boxed{V = \dfrac{C}{P}}$, where C is a constant.



- Power functions are also used to model **species-area relationships**, **illumination as a function of distance from a light source**, and **the period of revolution of a planet as a function of its distance from the sun.**

3.4 Rational Functions

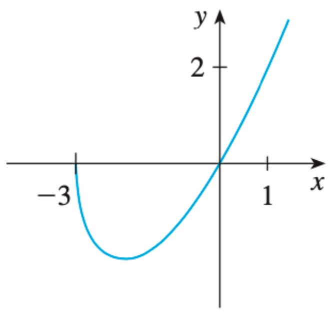
A **rational function** f is a ratio of two polynomials: $\boxed{f(x) = \dfrac{P(x)}{Q(x)}}$, where P and Q are polynomials. For example, the function $\boxed{f(x) = \dfrac{2x^4 - x^2 + 1}{x^2 - 4}}$ is a rational function. Its graph is shown below.



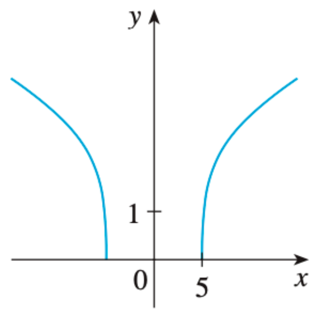
3.5 Algebraic Functions

- A function f is called an **algebraic function** if it can be constructed using algebraic operations *(such as addition, subtraction, multiplication, division, and taking roots)* starting with **polynomials**.

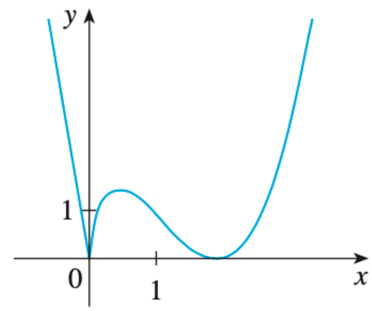
- The graphs of some algebraic functions are given below.



(a) $f(x) = x\sqrt{x+3}$



(b) $g(x) = \sqrt[4]{x^2 - 25}$

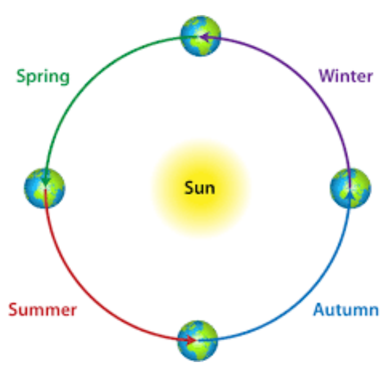


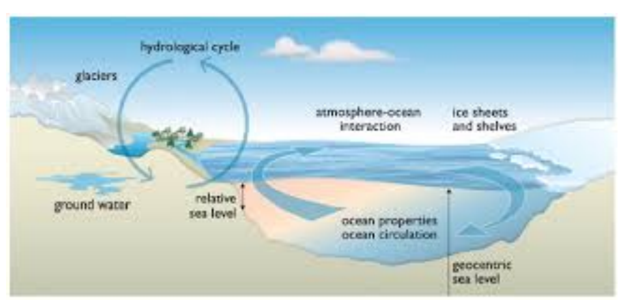
(c) $h(x) = x^{2/3}(x-2)^2$

- An example of an algebraic function occurs in the **theory of relativity**. The mass of a particle with velocity v is $\boxed{m = f(v) = \dfrac{m_0}{1 - v^2/c^2}}$, where m_0 is the rest mass of the particle and $c = 3.0 \times 10^5$ km/s is the speed of light in a vacuum.

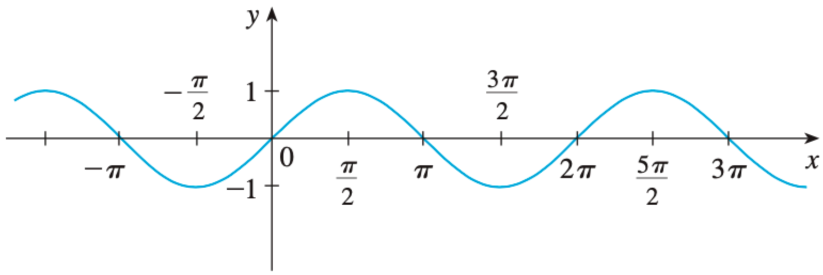
3.6 Trigonometric Functions

- One of the main applications of **trigonometry** is **periodic motion**.
- Periodic motion occurs often in nature, as in the daily rising and setting of the sun, the daily variation in tide levels, the vibrations of a leaf in the wind, and many more.

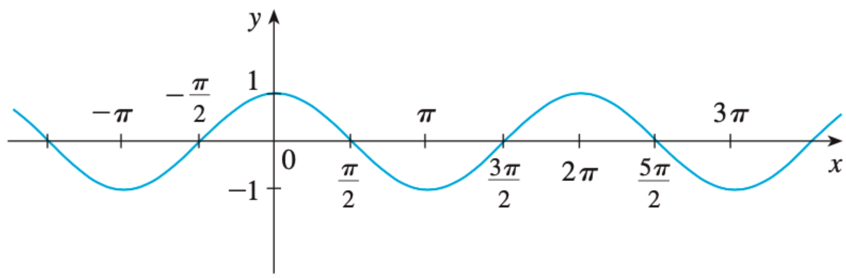




- Use the unit circle approach to define sine and cosine functions: $y = \sin x$ and $y = \cos x$.



(a) $f(x) = \sin x$



(b) $g(x) = \cos x$

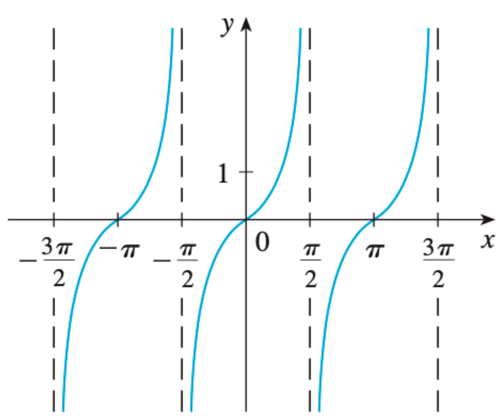
- The number of hours of daylight in Philadelphia t days after January 1 is given by the function

$$\boxed{L(t) = 12 + 2.8\sin\left[\frac{2\pi}{365}(t-80)\right]}$$

- The tangent function is given by

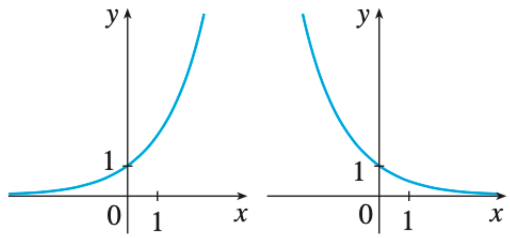
$$\boxed{\tan x = \frac{\sin x}{\cos x}}$$

and its graph is shown below.



3.7 Exponential Functions

- The exponential functions are the functions of the form $\boxed{f(x) = a^x}$, where the base a is a positive constant.

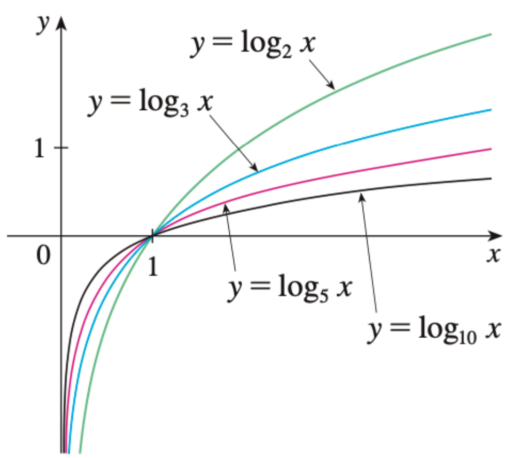


(a) $y = 2^x$ (b) $y = (0.5)^x$

- The **exponential functions** are useful for modeling many natural phenomena, such as **population growth** (if $a > 1$) and **radioactive decay** (if $a < 1$).

3.8 logarithmic functions

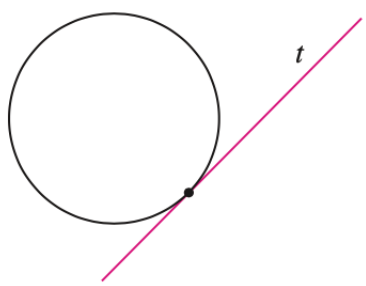
The **logarithmic functions** $\boxed{f(x) = \log_a x}$, where the base a is a positive constant, are the inverse functions of the exponential functions.

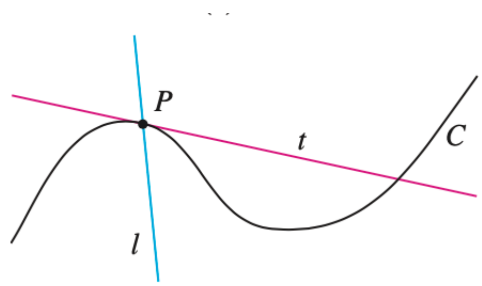


4 Limit Concepts

4.1 The Tangent Problems

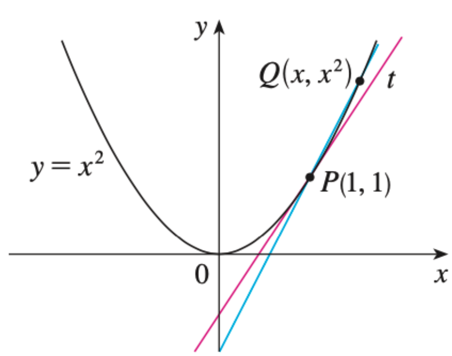
- The tangent and velocity problems are very important problems that lead to the concept of **limits** and **derivatives**.
- The word tangent is derived from the Latin word **tangens**, which means **"touching"**.
- Thus a tangent to a curve is a line that touches the curve. In other words, a tangent line should have the same direction as the curve at the point of contact.
- A tangent line does not necessarily intersect the curve only once.





Example 4.1 Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

- To find an equation of the line we should know one point that it passing through and its slope m.
- The difficulty is that we know only one point P, on t, whereas we need two points to compute the slope.



- We can approximate the slope m by choosing a nearby point $Q(x, x^2)$ on the parabola and computing the slope m_{PQ} of the **secant line**.

- We have $\boxed{m_{PQ} = \dfrac{x^2 - 1}{x - 1}}$.

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

- The table shows the values of m_{PQ} for several values of x close to 1. As Q is getting closer to P, x is getting closer to 1 and it appears from the tables that m_{PQ} is approaching 2.

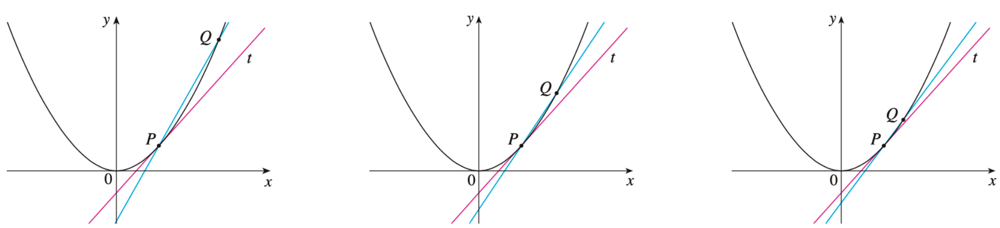
- We say that the slope of the tangent line is the limit of the slopes of the secant lines, and we express this symbolically by writing

$$\lim_{Q \to P} m_{PQ} = m \quad \text{and} \quad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$$

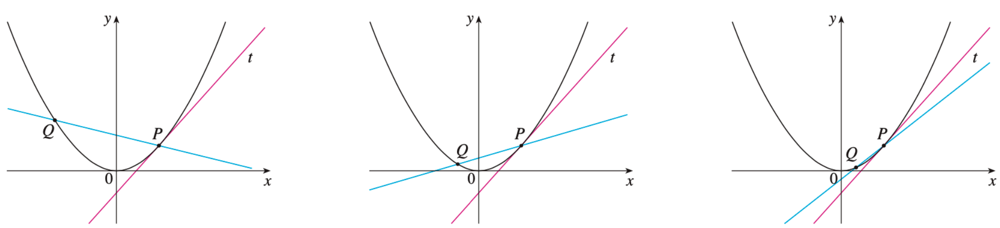
- Assume that the slope of the tangent line is indeed 2, then the point-slope form of the equation of the tangent line is

$$y - 1 = 2(x - 1) \quad \text{or} \quad y = 2x - 1.$$

- The figures below illustrate the limiting process in this example.



Q approaches *P* from the right



Q approaches *P* from the left

4.2 The Velocity Problem

- If you watch the speedometer of a car as you travel in city traffic, you see that the speed doesnt stay the same for very long; that is, the velocity of the car is not constant.
- We assume from watching the speedometer that the car has a definite velocity at each moment, but how is the **"instantaneous"** velocity defined?

Example 4.2 Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds.

The CN Tower in Toronto was the tallest freestanding building in the world for 32 years.

- Assume that this model for free fall neglects air resistance.
- Let $s(t)$ denote the distance fallen after t seconds, then it follows from Galileo's law that $\boxed{s(t) = 4.9t^2}$.
- Challenging issue: there is no time interval involved.

- We can approximate the instantaneous velocity after 5 seconds by computing the **average velocity** over the brief time interval from $t = 5$ to $t = 5.1$:

$$\boxed{\text{average velocity} = \frac{\text{change in position}}{\text{time elapsed}} = \frac{s(5.1) - s(5)}{0.1} = 49.49 \text{ m/s.}}$$

- Similarly, we calculate the average velocity over successively smaller time periods.

Time interval	Average velocity (m/s)
$5 \leqslant t \leqslant 6$	53.9
$5 \leqslant t \leqslant 5.1$	49.49
$5 \leqslant t \leqslant 5.05$	49.245
$5 \leqslant t \leqslant 5.01$	49.049
$5 \leqslant t \leqslant 5.001$	49.0049

- It appears that the average velocity is getting closer to 49 m/s as the time period is shorten.

- The instantaneous velocity when $t = 5$ is defined to be the limiting value of these average velocities over shorter and shorter time periods that start at $t = 5$.

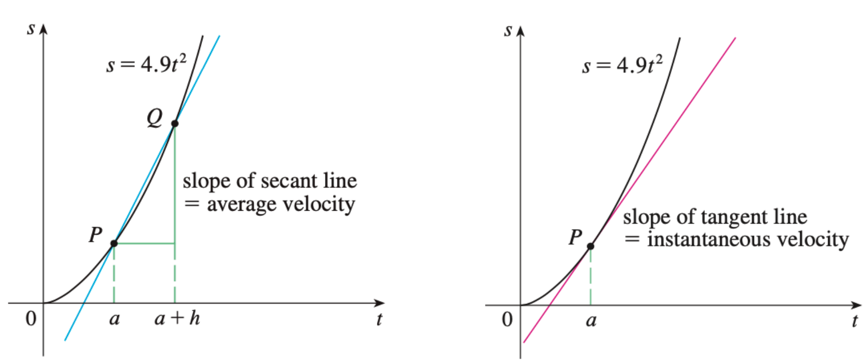
- Thus, the instantaneous velocity after 5 seconds is $\boxed{v = 49 \text{ m/s}}$.

4.3 Relationship between the Velocity Problem and the Tangent Problem

- There is a close connection between the tangent problem and the problem of the velocity problem.

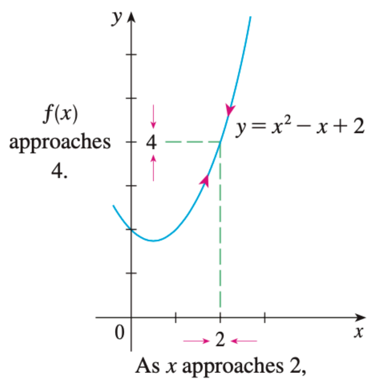
- If we draw the graph of the distance function of the ball and consider the points $P(a, 4.9a^2)$ and $Q(a+h, 4.9(a+h)^2)$ on the graph, then slope of the secant line PQ is

$$\boxed{m_{PQ} = \frac{4.9(a+h)^2 - 4.9a^2}{(a+h) - a}}$$



4.4 The Limit of a Function

- We now pay attention to the concept of limits in general.
- Let us investigate the behavior of the function $f(x) = x^2 - x + 2$ for values of x near 2.



x	$f(x)$	x	$f(x)$
1.0	2.000000	3.0	8.000000
1.5	2.750000	2.5	5.750000
1.8	3.440000	2.2	4.640000
1.9	3.710000	2.1	4.310000
1.95	3.852500	2.05	4.152500
1.99	3.970100	2.01	4.030100
1.995	3.985025	2.005	4.015025
1.999	3.997001	2.001	4.003001

- We can make the values of $f(x)$ as close as we like to 4 by taking x sufficiently close to 2.
- We write $\boxed{\lim_{x\to 2}(x^2 - x + 2) = 4}$.

Definition 4.3 *Suppose $f(x)$ is defined when x is near the number a. Then we write*

$$\boxed{\lim_{x\to a} f(x) = L}$$

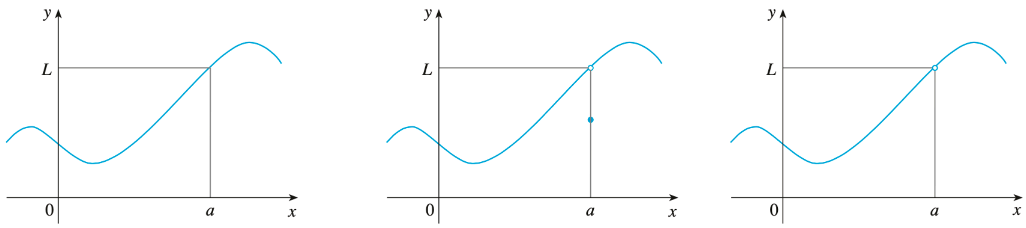
and say **“the limit of $f(x)$, as x approaches a, equals L”** *if we can make the values of $f(x)$ arbitrarily close to L by restricting x to be sufficiently close to a (on either side of a) but not equal to a.*

An Alternative notation is

$$\boxed{f(x) \to L \quad as \quad x \to a}$$

which is usually read “$f(x)$ approaches L as x approaches a.”

- When finding the limit of $f(x)$ as x approaches a, we never consider $x = a$.
- Sometimes, $f(x)$ need not even be defined when $x = a$.
- The only thing that matters is how f is defined near a.



(a) (b) (c)

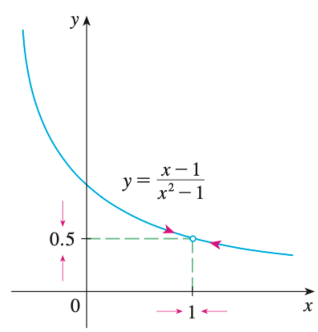
Example 4.4 Guess the value of $\lim_{x \to 1} \dfrac{x-1}{x^2-1}$.

– Make the tables of values

$x < 1$	$f(x)$
0.5	0.666667
0.9	0.526316
0.99	0.502513
0.999	0.500250
0.9999	0.500025

$x > 1$	$f(x)$
1.5	0.400000
1.1	0.476190
1.01	0.497512
1.001	0.499750
1.0001	0.499975

– Then $\lim_{x \to 1} \dfrac{x-1}{x^2-1} = 0.5$.



`Python Code`: Make the above tables of values in Python.

Example 4.5 Estimate the value of $\lim_{t \to 0} \dfrac{\sqrt{t^2+9}-3}{t^2}$.

`Python Code`: We will solve this problem in Python.

Example 4.6 Guess the value of $\lim_{x \to 0} \dfrac{\sin x}{x}$.

`Python Code`: We will solve this problem in Python.

Example 4.7 Guess the value of $\lim_{x \to 0} \sin \dfrac{\pi}{x}$.

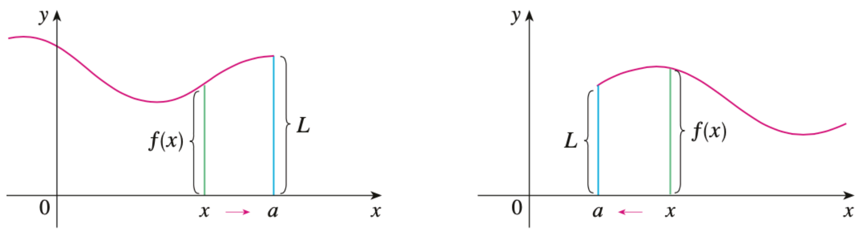
`Python Code`: We will solve this problem in Python.

Definition 4.8 One-Sided Limits *We write*

$$\boxed{\lim_{x \to a^-} f(x) = L}$$

and say the **left-hand limit** *of $f(x)$ as x approaches a is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a with x less than a.*

The **right-hand limit** *of $f(x)$ as x approaches a, written as $\boxed{\lim_{x \to a^+} f(x) = L}$, is defined similarly.*



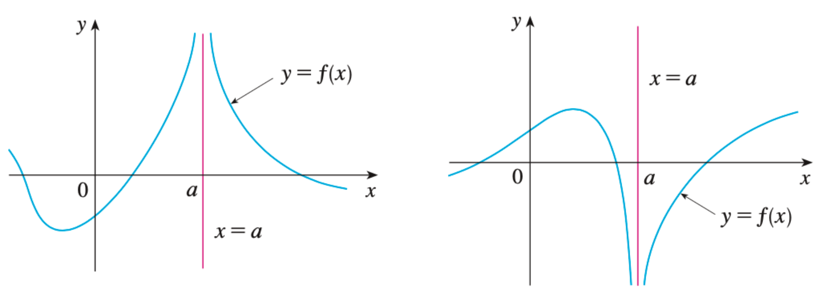
(a) $\lim_{x \to a^-} f(x) = L$ (b) $\lim_{x \to a^+} f(x) = L$

Definition 4.9 (Infinite Limits) *Let f be a function defined on both sides of a, except possibly at a itself. Then*

$$\boxed{\lim_{x \to a} f(x) = \infty}$$

means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a, but not equal to a.

The limit $\boxed{\lim_{x \to a} f(x) = -\infty}$ is defined similarly.



4.5 Calculating Limits Using the Limit Laws

Theorem 4.10 *Suppose that c is a constant and the limits $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist.*

(i) $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

(ii) $\lim_{x\to a}[cf(x)] = c \lim_{x\to a} f(x)$

(iii) $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

(iv) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ *if* $\lim_{x\to a} g(x) \neq 0$.

(v) $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$, *where n is a positive integer.*

(vi) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, *where n is a positive integer.*

Theorem 4.11 $\lim_{x\to a} f(x) = L$ *if and only if* $\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$.

4.6 Continuity

Definition 4.12 *A function f is* **continuous at a number** *a if*

$$\boxed{\lim_{x\to a} f(x) = f(a)}.$$

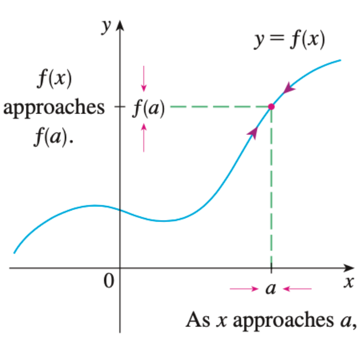
This definition requires three things:

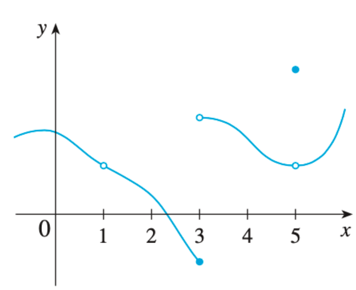
(i) *$f(a)$ is defined (that is, a is in the domain of f).*

(ii) *$\lim_{x\to a} f(x)$ exists.*

(iii) *$\lim_{x\to a} f(x) = f(a)$.*

- If f is defined near a, we say that f is **discontinuous** at a if f is not continuous at a.
- Physical phenomena are usually continuous. For instance, the displacement or velocity of a vehicle varies continuously with time, as does a persons height. But discontinuities do occur in such situations as electric currents.





Definition 4.13 *A function f is continuous from the right at a number a if*

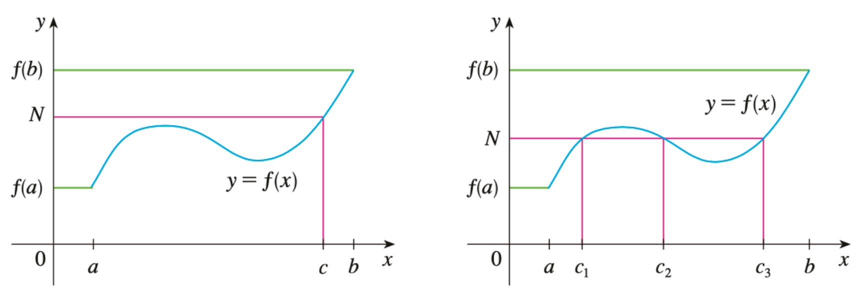
$$\boxed{\lim_{x\to a^+} f(x) = f(a)}$$

and f is continuous from the left at a if

$$\boxed{\lim_{x\to a^-} f(x) = f(a)}.$$

Definition 4.14 *A function f is continuous on an interval if it is continuous at every number in the interval.*

Theorem 4.15 (The Intermediate Value Theorem) *Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.*



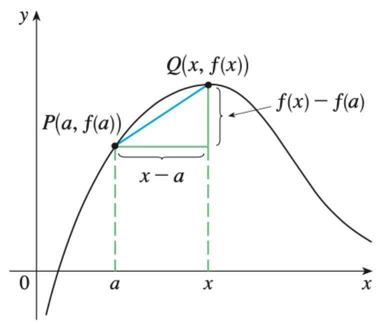
One use of the Intermediate Value Theorem is in locating roots of equations.

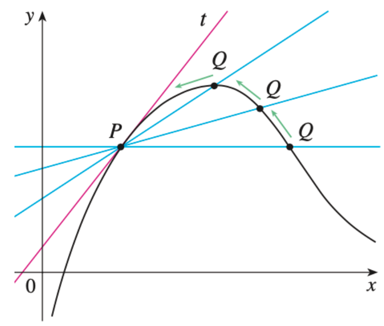
5 Derivatives and Rates of Changes

5.1 Tangents

- As we know that the problem of finding the tangent line to a curve and the problem of finding the velocity of an object both involve finding the same type of limit.
- This special kind of limit is called a **derivative** and it is interpreted as a rate of change in any of the natural or social sciences or engineering.
- Suppose a curve C has equation $y = f(x)$. We would like to find the tangent line to C at the point $P(a, f(a))$. Consider a nearby point $Q(x, f(a))$, where $x \neq a$.
- The slope of the secant line PQ is $\boxed{m_{PQ} = \dfrac{f(x) - f(a)}{x - a}}$.
- Let Q approach P along the curve C by letting x approach a.

- If m_{PQ} approaches a number m, then we define the **tangent** t to be the line through P with slope $\boxed{m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$.

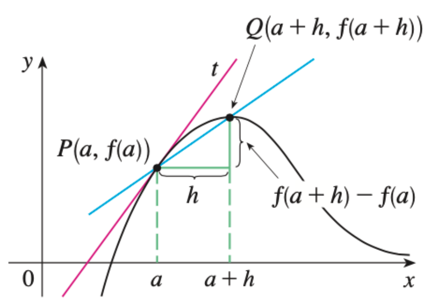




- If $h = x - a$, then $x = a + h$, and so the slope of the secant line PQ is

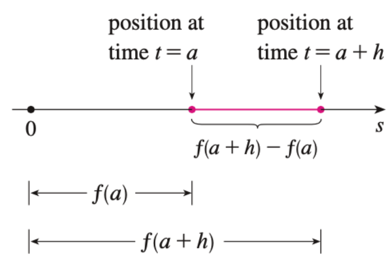
$$m_{PQ} = \frac{f(a+h) - f(a)}{h}$$

- The slope of the tangent line is $\boxed{m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$.



5.2 Velocities

- Suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the displacement (directed distance) of the object from the origin at time t.
- The function f that describes the motion is called the position function of the object.
- In the time interval from $t = a$ to $t = a + h$ the change in position is $f(a+h) - f(a)$.

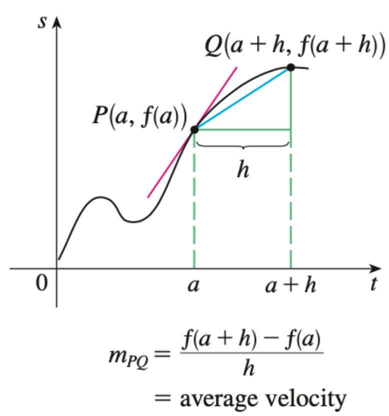


- The average velocity over this time interval is

$$\boxed{\text{average velocity } = \frac{f(a+h) - f(a)}{h}}.$$

- The instantaneous velocity $v(a)$ at time $t = a$ is defined to be

$$\boxed{v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$



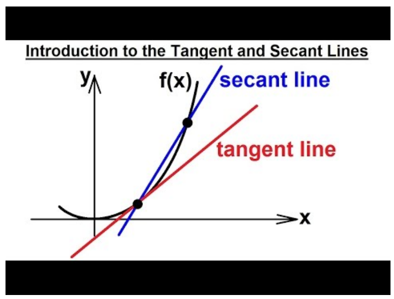
5.3 Derivatives: Rate of Change

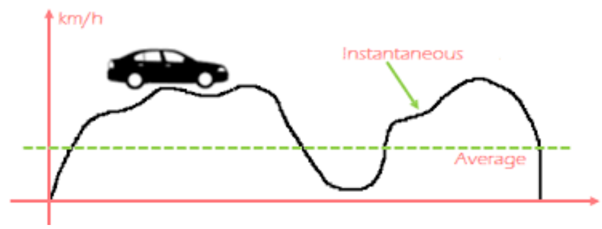
- The limit of the form $\lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ arises whenever we calculate a rate of change in any of the sciences or engineering, such as a rate of reaction in chemistry or a marginal cost in economics.
- Since this type of limit occurs so widely, it is given a special name and notation.
- The **derivative** of f at a, written $f'(a)$, is defined as

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

If the limit exists, then f is said to be **differentiable** at a.

- The computation of *rates of changes* is important in all of the **natural sciences**, in **engineering**, and even in the **social sciences**.

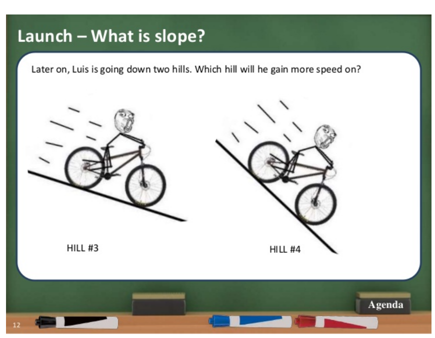


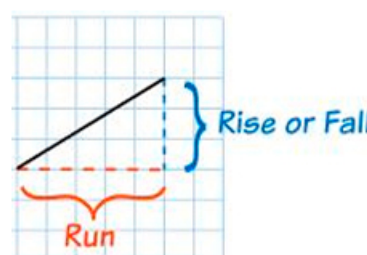


5.4 The Rate of Change of Linear (Affine) Functions

- Consider the *(affine)* function $f(x) = mx + b$ whose graph is the **straight line**.
- We find a way to measure the *"steepness"* of a line, which shows how **quickly** it rises /falls as we move from left to right.
- To proceed, we measure
  - **Run** to be the distance we move to the right.
  - **Rise** to be the corresponding distance that the line rises/falls.

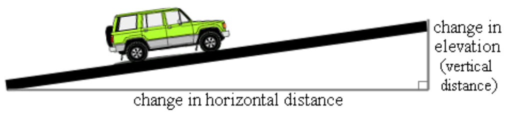
The **slope** of a line is then defined as $\boxed{\text{slope} = \frac{\text{rise}}{\text{run}}}$.

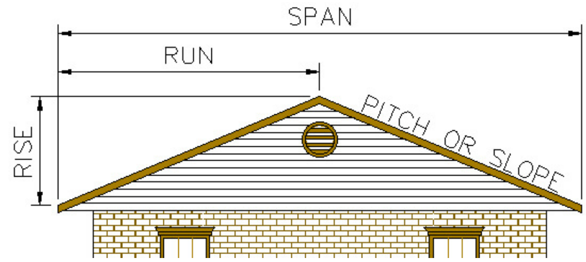




Source: mathsisfun.com

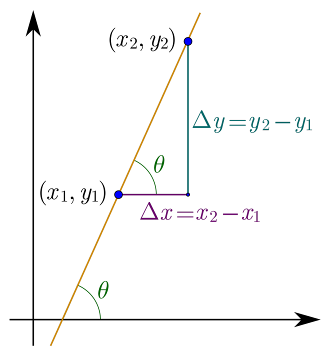
- The slope is very important and has many applications in the real world situations. For example, carpenters use the term **pitch** for the slope of a roof or a staircase; the term **grade** for the slope of a road.





- If a line is placed in a coordinate plane, then
  - the *run* is the change in the x-**coordinate** and
  - the *rise* is the change in the y-**coordinate**

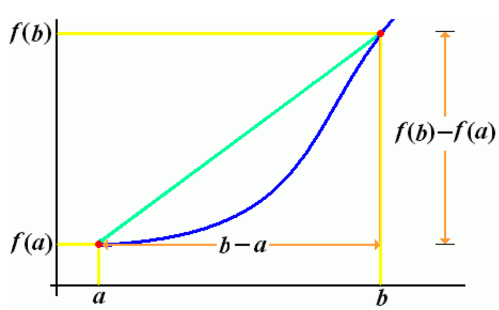
between two points on the line. The slope is $\boxed{m = \dfrac{y_2 - y_1}{x_2 - x_1}}$.



- The rate of change of this function is *constant*.

5.5 The Rate of Change of Nonlinear Functions

- How to measure the *rate of change* of general functions whose graphs are curves (not straight lines)?
- Average rate of change of a function f from $x = a$ to $x = b$ is $\boxed{\dfrac{f(b) - f(a)}{b - a}}$



- We can find the rate of change of f at a using the average rate of change over the interval from a to $a+h$: $\boxed{\dfrac{f(a+h)-f(a)}{h}}$, where h is sufficiently small.
- The **derivative** of f at a, written $f'(a)$, is defined as

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}}$$

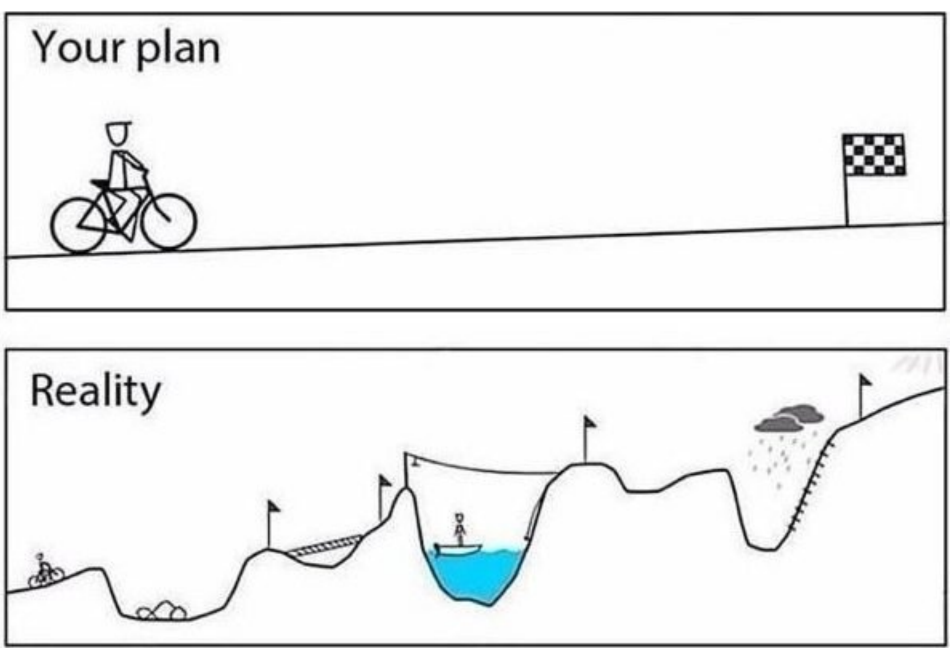
If the limit exists, then f is said to be **differentiable** at a.

5.6 Understanding the derivative

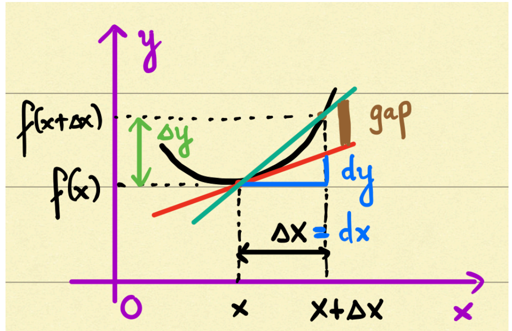
- Consider the function $y = f(x)$.

  - Small change in **inputs**: $x \longrightarrow x + \Delta x$.

  - Small change in **outputs**: $f(x) \longrightarrow f(x+\Delta x)$.
- How to relate $\Delta x = (x+\Delta x) - x$ and $\Delta y = f(x+\Delta x) - f(x)$?
- If $f(x) = mx + b$, then $\dfrac{\Delta y}{\Delta x} = m \iff \Delta y = m\Delta x$.
- That is $\boxed{\underbrace{f(x+\Delta x)}_{future} = \underbrace{f(x)}_{present} + m\Delta x.}$



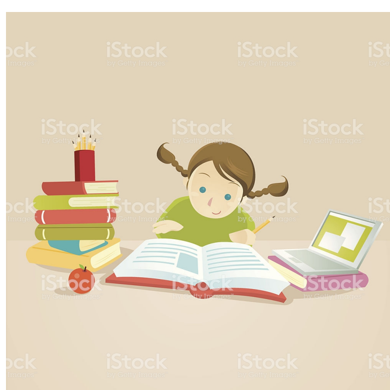
- If f is nonlinear, then $\dfrac{\Delta y}{\Delta x}$ is no longer constant.



- Suppose that f is **differentiable** at x. Then $\boxed{f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{dy}{dx}}$.
- This represents the quotient of two *infinitesimally* small numbers.
- dy is the *infinitesimally* small change in y caused by an *infinitesimally* small change dx applied to x.
- We have

$$\begin{aligned} dy &= f'(x)dx \\ \Delta y &= dy + \text{gap} \\ \Delta y &= f'(x)dx + \text{gap} \\ f(x+\Delta x) - f(x) &= f'(x)\Delta x + \text{gap} \\ f(x+\Delta x) &= f(x) + f'(x)\Delta x + \underbrace{\text{gap}}_{\approx 0}. \end{aligned}$$

- So $\boxed{\underbrace{f(x+\Delta x)}_{future} \approx \underbrace{f(x)}_{present} + \underbrace{f'(x)}_{effort}\,\underbrace{\Delta x}_{time}}$ for Δx sufficiently small.



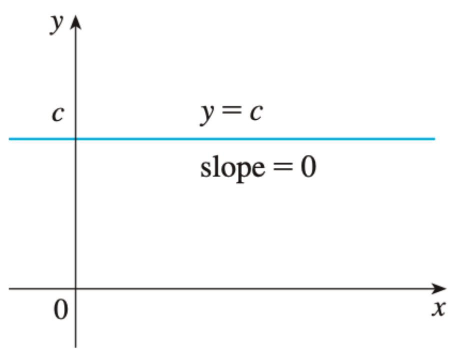
- Consider a uniform partition of the interval $[a, b]$ as follow $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ with $\Delta x = \dfrac{b-a}{n}$.

- We have $\begin{cases} f(x_1) - f(x_0) \approx f'(x_0)\Delta x \\ f(x_2) - f(x_1) \approx f'(x_1)\Delta x \\ \dots \\ f(x_n) - f(x_{n-1}) \approx f'(x_{n-1})\Delta x. \end{cases}$
- So $f(x_n) - f(x_0) \approx [f'(x_0) + f'(x_1) + \ldots + f'(x_{n-1})]\Delta x$.
- Or equivalently $\boxed{\underbrace{f(b)}_{far\ \ future} \approx \underbrace{f(a)}_{present} + \underbrace{[f'(x_0) + f'(x_1) + \ldots + f'(x_{n-1})]}_{accumulated\ \ efforts} \underbrace{\Delta x}_{time}}$.
- This reminds us a version of FTC:

$$\boxed{f(b) = f(a) + \int_a^b f'(x)dx}.$$

5.7 Derivatives of Polynomials and Exponential Functions

- The derivative of a constant function is 0: $\boxed{\frac{d}{dx}(c) = 0}$.



- If n is any real number, then $\boxed{\frac{d}{dx}(x^n) = nx^{n-1}}$.
- If c is a constant and f is a differentiable function, then $\boxed{\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)}$.
- **The Sum Rule:** If f and g are both differentiable, then

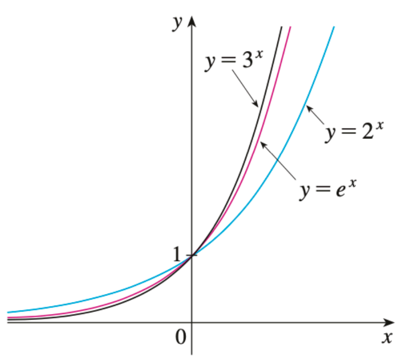
$$\boxed{\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)}$$

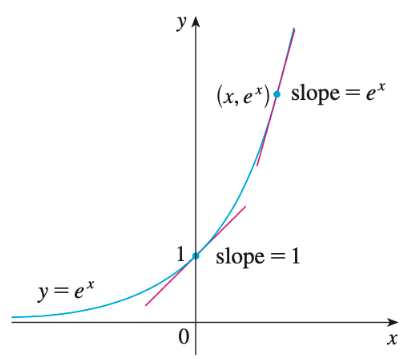
- For $f(x) = b^x$, we have $f'(x) = f'(0)b^x$.

- We would like to find the base b such that $f'(0) = 1$, for simplicity, that is $\boxed{\lim_{h\to 0} \frac{b^h - 1}{h} = 1}$.
- If $b = 2$, then $f'(0) \approx 0.693147$. If $b = 3$, then $f'(0) \approx 1.098612$.
- So b should be a number between 2 and 3. It is traditional to denote this value by the letter e.

Definition 5.1 *e is the number such that* $\boxed{\lim_{h\to 0} \frac{e^h - 1}{h} = 1}$.

- Derivative of the natural exponential function is $\boxed{\frac{d}{dx}(e^x) = e^x}$.
- The exponential function $f(x) = e^x$ has the nice property that it is its own derivative.
- The **geometrical significance** of this fact is that the slope of a tangent line to the curve $y = e^x$ is equal to the y-coordinate of the point.





5.8 The Product Rule

- If f is differentiable at x, then $\boxed{f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}}$.
- So $f(x + \Delta x) = f(x) + f'(x)\Delta x + \epsilon\Delta x$, where $\epsilon \to 0$ as $\Delta x \to 0$.

Theorem 5.2 *If f and g are both differentiable, then*

$$\boxed{\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]}$$

Proof.

- We have $\dfrac{d}{dx}[f(x)g(x)] = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}$.

- Use $\begin{cases} f(x+\Delta x) = f(x) + f'(x)\Delta x + \epsilon_f \Delta x \\ g(x+\Delta x) = g(x) + g'(x)\Delta x + \epsilon_g \Delta x. \end{cases}$ $\square$

5.9 The Quotient Rule

Theorem 5.3 *If f and g are both differentiable, then*

$$\boxed{\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}}$$

Proof.

- We have $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \lim\limits_{\Delta x \to 0} \left[\dfrac{\dfrac{f(x+\Delta x)}{g(x+\Delta x)} - \dfrac{f(x)}{g(x)}}{\Delta x}\right]$.

- Use $\begin{cases} f(x+\Delta x) = f(x) + f'(x)\Delta x + \epsilon_f \Delta x \\ g(x+\Delta x) = g(x) + g'(x)\Delta x + \epsilon_g \Delta x. \end{cases}$

$\square$

5.10 Derivatives of Trigonometric Functions

Theorem 5.4 *The derivatives of the trigonometric functions are given by*

(i) $\dfrac{d}{dx}(\sin x) = \cos x$

(ii) $\dfrac{d}{dx}(\cos x) = -\sin x$

(iii) $\dfrac{d}{dx}(\tan x) = \sec^2 x$

(iv) $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$

(v) $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

(vi) $\dfrac{d}{dx}(\cot x) = -\csc^2 x$

5.11 The Chain Rule

Theorem 5.5 *If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product* $\boxed{F'(x) = f'(g(x)) \cdot g'(x)}$

Proof.

$$\begin{aligned}
F'(x) &= \lim_{\Delta x \to 0} \frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(g(x) + g'(x)\Delta x + \epsilon_g \Delta x) - f(g(x))}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(g(x) + [g'(x) + \epsilon_g]\Delta x) - f(g(x))}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f'(g(x))[g'(x) + \epsilon_g]\Delta x + \epsilon_f [g'(x) + \epsilon_g]\Delta x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left(f'(g(x))[g'(x) + \epsilon_g] + \epsilon_f [g'(x) + \epsilon_g]\right)
\end{aligned}$$

□

5.12 Implicit Derivatives

- Some functions $y = f(x)$ are defined implicitly, for example $x^2 + y^2 = 25, \quad x^3 + [f(x)]^3 = 6xf(x)$. It may be hard to compute $\dfrac{dy}{dx}$ directly.
- Fortunately, we don't need to solve an equation for y in terms of x in order to find $\dfrac{dy}{dx}$. Instead we can use the method of **implicit differentiation**.
- This method consists of differentiation both sides of the equation with respect to x and then solving the resulting equation for y'.

Theorem 5.6 *The derivatives of the inverse trigonometric functions are given by*

(i) $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$

(ii) $\dfrac{d}{dx}(\cos^{-1} x) = -\dfrac{1}{\sqrt{1-x^2}}$

(iii) $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$

(iv) $\dfrac{d}{dx}(\csc^{-1} x) = -\dfrac{1}{x\sqrt{1-x^2}}$

(v) $\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{x\sqrt{1-x^2}}$

(vi) $\dfrac{d}{dx}(\cot^{-1} x) = -\dfrac{1}{1+x^2}$

5.13 Derivatives of Logarithmic Functions

Theorem 5.7 (Derivatives of Logarithmic Functions)

(i) $\dfrac{d}{dx}(\log_b x) = \dfrac{1}{x \ln b}$

(ii) $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$

(iii) $\dfrac{d}{dx}[\ln g(x)] = \dfrac{g'(x)}{g(x)}$

(iv) $\dfrac{d}{dx}(\ln |x|) = \dfrac{1}{x}$

Theorem 5.8 *The Power Rule if n is any real number and $f(x) = x^n$, then*

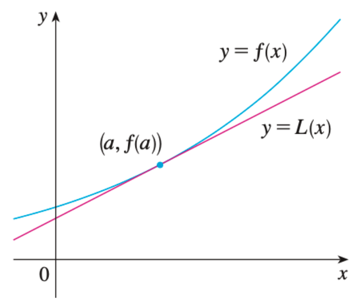
$$\boxed{f'(x) = nx^{n-1}}$$

Corollary 5.9 *We have the following results.*

- $\dfrac{d}{dx}(b^n) = 0$
- $\dfrac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} f'(x)$
- $\dfrac{d}{dx}[b^{g(x)}] = b^{g(x)}(\ln b) g'(x)$
- *To find $\dfrac{d}{dx}[f(x)]^{g(x)}$, logarithmic differentiation can be used.*

6 Linear Approximations and Differentials

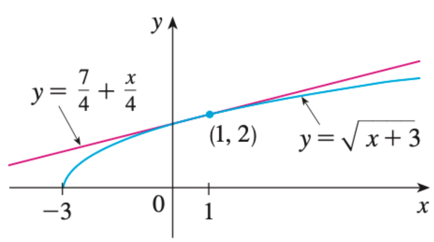
- Let $y = f(x)$ be a differentiable function. Suppose that it is easy to calculate the value $f(a)$, but difficult (or even impossible) to compute nearby values of f.
- The idea is we use the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$ to approximate $f(x)$ when x is near a.



- So $\boxed{f(x) \approx f(a) + f'(a)(x-a)}$.
- This approximation is called the **linear approximation** or **tangent line approximation** of f at a.
- The linear function $\boxed{L(x) = f(a) + f'(a)(x-a)}$ is called the **linearization** of f at a.

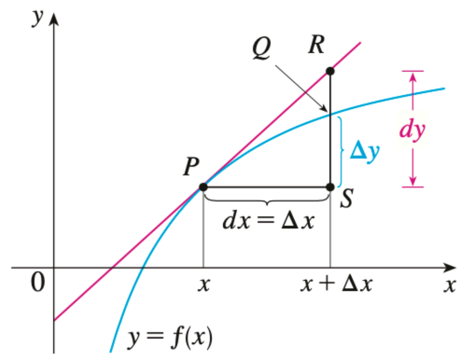
Example 6.1 (Differentials)

- Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimate?



	x	From $L(x)$	Actual value
$\sqrt{3.9}$	0.9	1.975	1.97484176...
$\sqrt{3.98}$	0.98	1.995	1.99499373...
$\sqrt{4}$	1	2	2.00000000...
$\sqrt{4.05}$	1.05	2.0125	2.01246117...
$\sqrt{4.1}$	1.1	2.025	2.02484567...
$\sqrt{5}$	2	2.25	2.23606797...
$\sqrt{6}$	3	2.5	2.44948974...

- The tangent line approximation gives good estimates when x is close to 1 but the accuracy of the approximation deteriorates when x is farther away from 1.
- If f is a differentiable function, then the **differential** dx is an independent variable.
- The **differential** dy is then defined in terms of dx by the equation $\boxed{dy = f'(x)dx}$.



- Let $P(x, f(x))$ and $Q(x + \Delta x, f(x + \Delta x))$ be points on the graph of f and let $dx = \Delta x$.The corresponding change in y is $\boxed{\Delta y = f(x + \Delta x) - f(x)}$.

- The quantity dy represents the amount that the tangent line rises or falls (the change in the linearization), whereas Δy represents the amount that the curve $y = f(x)$ rises or falls when x changes by an amount dx.

Example 6.2 Compare the values of Δy and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes

(i) from 2 to 2.05

(ii) from 2 to 2.01

7 Applications of Differentiation

7.1 Maximum and Minimum Values

- Some of the most important applications of differential calculus are **optimization problems**, in which we are required to find the optimal (best) way of doing something.

  (i) What is the shape of a can that minimizes manufacturing costs?
  (ii) What is the maximum acceleration of a space shuttle? (This is an important question to the astronauts who have to withstand the effects of acceleration.)
  (iii) What is the radius of a contracted windpipe that expels air most rapidly during a cough?
  (iv) At what angle should blood vessels branch so as to minimize the energy expended by the heart in pumping blood?

- These problems can be reduced to finding the maximum or minimum values of a function.

Definition 7.1 (Maximum and Minimum Values) *Let c be a number in the domain D of a function f. Then $f(c)$ is the*

- **absolute (global) maximum** *value of f on D if $f(c) \geq f(x)$ for all x in D.*
- **absolute (global) minimum** *value of f on D if $f(c) \leq f(x)$ for all x in D.*

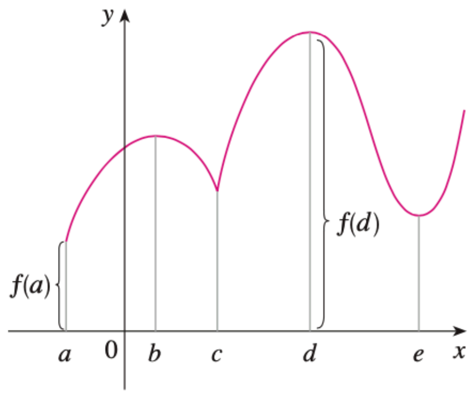
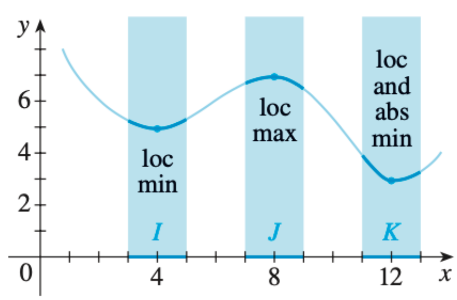


FIGURE 2
Abs min $f(a)$, abs max $f(d)$,
loc min $f(c), f(e)$, loc max $f(b), f(d)$

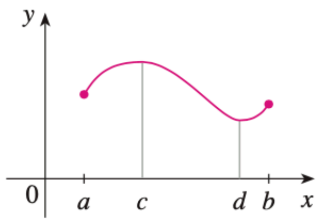
The maximum and minimum values of f are called **extreme values** of f.

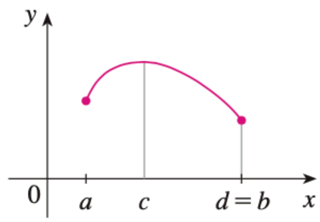
Definition 7.2 (Maximum and Minimum Values) *The number $f(c)$ is a*

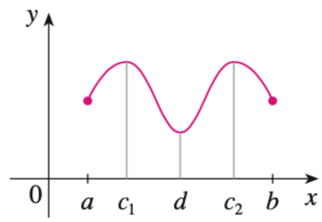
- **local maximum** *value if $f(c) \geq f(x)$ when x is near c.*
- **local minimum** *value if $f(c) \leq f(x)$ when x is near c.*



Theorem 7.3 (The Extreme Value Theorem) *If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.*







Theorem 7.4 (Fermat's Theorem) *If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.*

Definition 7.5 *A* **critical number** *of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.*

The Closed Interval Method To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

(i) Find the values of f at the critical numbers of f in $[a, b]$.

(ii) Find the values of f at the endpoints of the interval.

(iii) The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example 7.6 The Hubble Space Telescope was deployed on April 24, 1990, by the space shuttle *Discovery*. A model for the velocity of the shuttle during this mission, from liftoff at $t = 0$ until the solid rocket boosters were jettisoned at $t = 126$ seconds, is given by

$$v(t) = 0.001302t^3 - 0.09029t^2 + 23.61t - 3.083$$

(in feet per second). Using this model, estimate the absolute maximum and minimum values of the *acceleration* of the shuttle between liftoff and the jettisoning of the boosters.

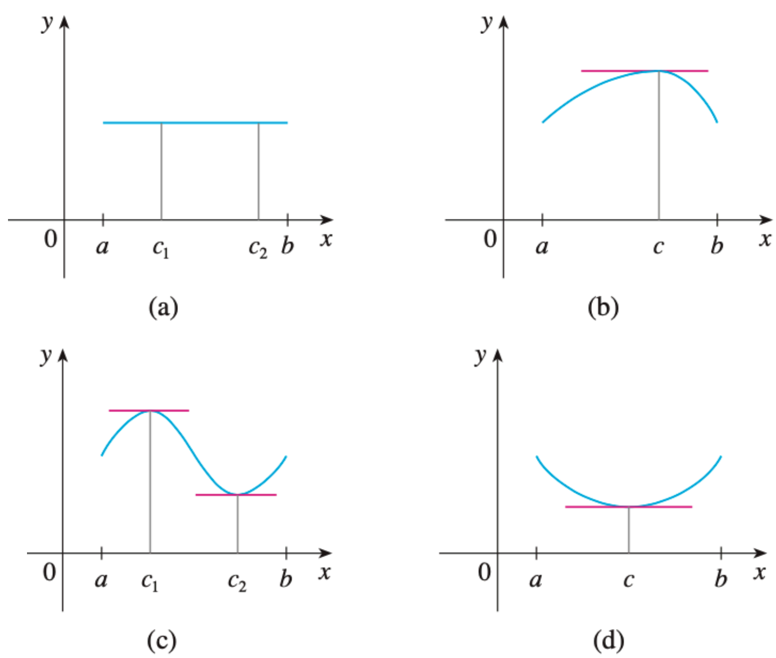
7.2 The Mean Value Theorem

Theorem 7.7 (Rolle's Theorem) *Let f be a function that satisfies the following three hypotheses:*

(i) *f is continuous on the closed interval $[a, b]$.*

(ii) *f is differentiable on the open interval (a, b).*

(iii) *$f(a) = f(b)$ Then there is a number c in (a, b) such that $f'(c) = 0$.*



Example 7.8 Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root.

Theorem 7.9 (The Mean Value Theorem) *Let f be a function that satisfies the following hypotheses:*

(i) *f is continuous on the closed interval $[a, b]$.*

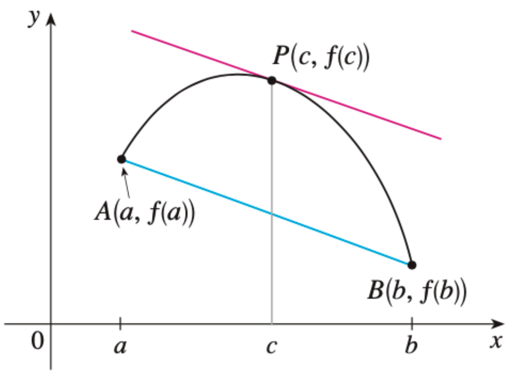
(ii) *f is differentiable on the open interval (a, b).*

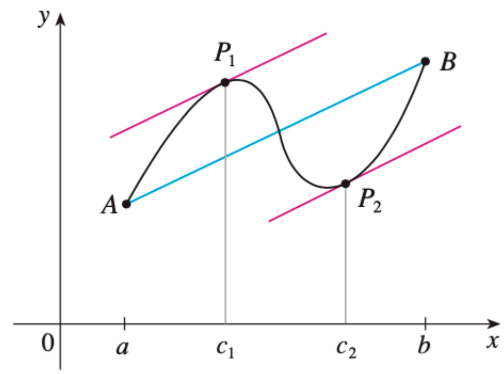
Then there exists a number c in (a, b) such that

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}$$

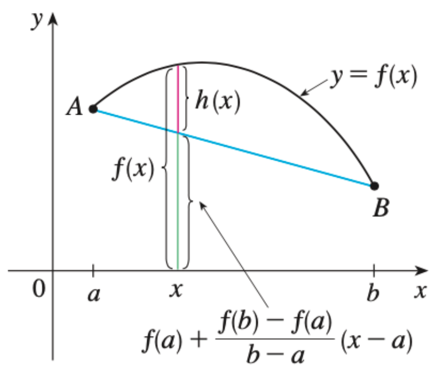
or equivalently

$$\boxed{f(b) - f(a) = f'(c)(b - a)}$$





Proof.



Apply the Rolle's Theorem for the function

$$\boxed{h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)}$$

$\square$

Theorem 7.10 (The Mean Value Theorem) *If $f'(x) = 0$ for all x in the interval (a, b), then f is constant on (a, b).*

Corollary 7.11 *If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is a constant.*

Example 7.12 Show that $\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}$.

7.3 Numerical Methods of Solving Root-Finding Problems

We refer this section to following lectures in AMS 326 (Numerical Analysis) that we taught in SUNY Korea.

1. Roots of High-Degree Equations
2. Fixed-Point Iterations
3. Netwon-Raphson's Method
4. Error analysis for iterative methods

8 Problem Set

1. The table shows (lifetime) peptic ulcer rates (per 100 popula- tion) for various family incomes as reported by the National Health Interview Survey.

Income	Ulcer rate (per 100 population)
\$4,000	14.1
\$6,000	13.0
\$8,000	13.4
\$12,000	12.5
\$16,000	12.0
\$20,000	12.4
\$30,000	10.5
\$45,000	9.4
\$60,000	8.2

   (a) Make a scatter plot of these data and decide whether a linear model is appropriate.
   (b) Find and graph a linear model using the first and last data points.
   (c) Find and graph the least squares regression line.
   (d) Use the linear model in part (c) to estimate the ulcer rate for an income of \$25,000.
   (e) According to the model, how likely is someone with an income of \$80,000 to suffer from peptic ulcers?
   (f) Do you think it would be reasonable to apply the model to someone with an income of \$200,000?

2. The table shows average US retail residential prices of electricity from 2000 to 2012, measured in cents per kilowatt hour.
   (a) Make a scatter plot. Is a linear model appropriate?
   (b) Find and graph the regression line.
   (c) Use your linear model from part (b) to estimate the average retail price of electricity in 2005 and 2013.

Years since 2000	Cents/kWh
0	8.24
2	8.44
4	8.95
6	10.40
8	11.26
10	11.54
12	11.58

3. The table shows the number N of species of reptiles and amphibians inhabiting Caribbean islands and the area A of the island in square miles.

   (a) Use a power function to model N as a function of A.

   (b) The Caribbean island of Dominica has area 291 mi^2 . How many species of reptiles and amphibians would you expect to find on Dominica?

Island	A	N
Saba	4	5
Monserrat	40	9
Puerto Rico	3,459	40
Jamaica	4,411	39
Hispaniola	29,418	84
Cuba	44,218	76

4. The table shows the mean (average) distances d of the planets from the sun (taking the unit of measurement to be the distance from the earth to the sun) and their periods T (time of revolution in years).

   (a) Fit a power model to the data.

   (b) Keplers Third Law of Planetary Motion states that "The square of the period of revolution of a planet is proportional to the cube of its mean distance from the sun." Does your model corroborate Keplers Third Law?

Planet	d	T
Mercury	0.387	0.241
Venus	0.723	0.615
Earth	1.000	1.000
Mars	1.523	1.881
Jupiter	5.203	11.861
Saturn	9.541	29.457
Uranus	19.190	84.008
Neptune	30.086	164.784

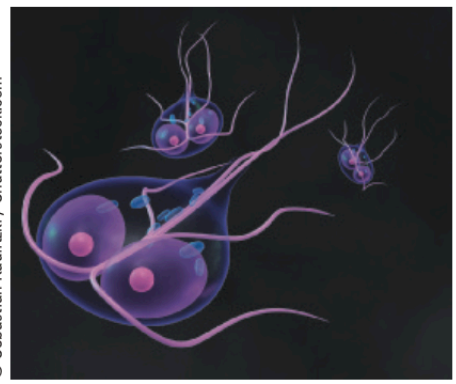
5. A researcher is trying to determine the doubling time for a population of the bacterium **Giardia lamblia**. He starts a culture in a nutrient solution and estimates the bacteria count every four hours. His data are shown in the table.

Time (hours)	0	4	8	12	16	20	24
Bacteria count (CFU/mL)	37	47	63	78	105	130	173

(a) Make a scatter plot of the data.

(b) Use a graphing calculator to find an exponential curve $f(t) = a \cdot b^t$? that models the bacteria population t hours later.

(c) Graph the model from part (b) together with the scatter plot in part (a). Use the TRACE feature to determine how long it takes for the bacteria count to double.



G. lamblia

6. A cardiac monitor is used to measure the heart rate of a patient after surgery. It compiles the number of heartbeats after t minutes. When the data in the table are graphed, the slope of the tangent line represents the heart rate in beats per minute.

t (min)	36	38	40	42	44
Heartbeats	2530	2661	2806	2948	3080

The monitor estimates this value by calculating the slope of a secant line. Use the data to estimate the patient's heart rate after 42 minutes using the secant line between the points with the given values of t.

(a) $t = 36$ and $t = 42$

(b) $t = 38$ and $t = 42$

(c) $t = 40$ and $t = 42$

(d) $t = 42$ and $t = 44$

7. If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height in meters t seconds later is given by $y = 10t - 1.86t^2$.

   (a) Find the average velocity over the given time intervals:

   $$[1, 2] \;\; [1, 1, 5] \;\; [1, 1.1] \;\; [1, 1.01] \;\; [1, 1.001]$$

   (b) Estimate the instantaneous velocity when $t = 1$.

8. A Tibetan monk leaves the monastery at 7:00 am and takes his usual path to the top of the mountain, arriving at 7:00 pm. The following morning, he starts at 7:00 am at the top and takes the same path back, arriving at the monastery at 7:00 pm. Use the Intermediate Value Theorem to show that there is a point on the path that the monk will cross at exactly the same time of day on both days.

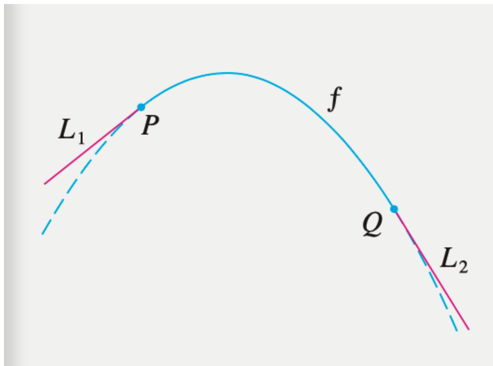
9 Applied Projects

This section includes a list of applied projects during the course.

Project 1: Suppose you are serving in the customer service and want to advertise the above cell phone plan to customers. This is a kind of boring job since you have to repeat the necessary information over and over. However, your schedule is very tight so you want to make a chatbot to do that job for you. You can make it by writing a selling program in Python.
Please refer to the "**Project 1**" in the Jupyter Notebook for more detail.

Project 2: This project is taken from section 3.1 in the textbook [1].
Suppose you are asked to design the first ascent and drop for a new roller coaster. By studying photographs of your favorite coasters, you decide to make the slope of the ascent 0.8 and the slope of the drop -1.6. You decide to connect these two straight stretches $y = L_1(x)$ and $y = L_2(x)$ with part of a parabola $y = f(x) = ax^2 + bx + c$, where x and $f(x)$ are measured in feet. For the track to be smooth there can't be abrupt changes in direction, so you want the linear segments L_1 and L_2 to be tangent to the parabola at

the transition points P and Q. (See the figure.) To simplify the equations, you decide to place the origin at P.



(i) (a) Suppose the horizontal distance between P and Q is 100 ft. Write equations in a, b, and c that will ensure that the track is smooth at the transition points.

(b) Solve the equations in part (a) for a, b, and c to find a formula for $f(x)$.

(c) Plot L_1, f, and L_2 to verify graphically that the transitions are smooth.

(d) Find the difference in elevation between P and Q.

(ii) The solution in Problem 1 might *look smooth*, but it might not *feel* smooth because the piecewise defined function [consisting of $L_1(x)$ for $x < 0$, $f(x)$ for $0 \le x \le 100$, and $L_2(x)$ for $x > 100$] doesn't have a continuous second derivative. So you decide to improve the design by using a quadratic function

$$q(x) = ax^2 + bx + c$$

only on the interval $10 \le x \le 90$ and connecting it to the linear functions by means of two cubic functions:

$$\begin{cases} g(x) = kx^3 + lx^2 + mx + n & \text{for } 0 \le x < 10 \\ h(x) = px^3 + qx^2 + rx + s & \text{for } 90 < x \le 100 \end{cases}$$

(a) Write a system of equations in 11 unknowns that ensure that the functions and their first two derivatives agree at the transition points.

(b) Solve the equations in part (a) with a computer algebra system to find formulas for $g(x), q(x)$, and $h(x)$.

(c) Plot L_1, g, q, h, and L_2, and compare with the plot in Problem 1(c).

Please refer to the "**Project 2**" in the Jupyter Notebook for more detail.

Project 3: This project is taken from section 3.10 in the textbook [1]. The tangent line approximation $L(x)$ is the best first-degree (linear) approximation to

$f(x)$ near $x = a$ because $f(x)$ and $L(x)$ have the same rate of change (derivative) at a. For a better approximation than a linear one, let's try a second-degree (quadratic) approximation $P(x)$. In other words, we approximate a curve by a parabola instead of by a straight line. To make sure that the approximation is a good one, we stipulate the following:

(i) $P(a) = f(a)$ (P and f should have the same value at a.)

(ii) $P'(a) = f'(a)$ (P and f should have the same rate of change at a.)

(iii) $P''(a) = f''(a)$ (The slope of P and f should change at the same rate at a.)

1. Find the quadratic approximation $P(x) = A+Bx+Cx^2$ to the function $f(x) = \cos x$ that satisfies condition (i), (ii), and (iii) with $a = 0$. Graph P, f, and the linear approximation $L(x) = 1$ on a common screen. Comment on how well the functions P and L approximate f.

2. Determine the values of x for which the quadratic approximation $f(x) \approx P(x)$ in Problem 1 is accurate to within 0.1 [*Hint:* Graph $y = P(x)$, $y = \cos x - 0.1$, and $y = \cos x + 0.1$ on a common screen.]

3. To approximate a function f by a quadratic function P near a number a, it is best to write P in the form

$$P(x) = A + B(x - a) + C(x - a)^2$$

Show that the quadratic function that satisfies condition (i), (ii), and (iii) is

$$P(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2$$

4. Find the quadratic approximation to $f(x) = \sqrt{x + 3}$ near $a = 1$. Graph f, the quadratic approximation, and the linear approximation on a common screen. What do you conclude?

5. Instead of being satisfied with a linear or quadratic approximation to $f(x)$ near $x = a$, let's try to find better approximations with higher-degree polynomials. We look for an nth-degree polynomial

$$T_n(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + \ldots + c_n(x - a)^n$$

such that T_n and its first n derivatives have the same values at $x = a$ as f and its first n derivatives. By differentiating repeatedly and setting $x = a$, show that these conditions are satisfied if

$$c_0 = f(a), \quad c_1 = f'(a), \quad c_2 = \frac{1}{2}f''(a)$$

and in general $c_k = \dfrac{f^{(k)}(a)}{k!}$, where $k! = 1 \cdot 2 \cdot 3 \dots k$. The resulting polynomial

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

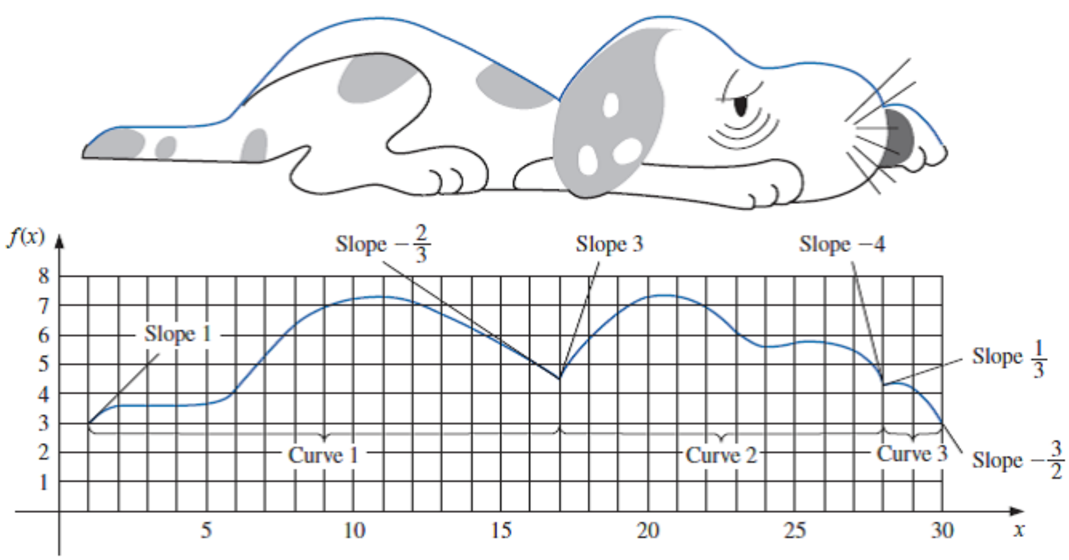
is called the **nth-degree Taylor polynomial of f centered at a.**

6. Find the 8th-degree Taylor polynomial centered at $a = 0$ for the function $f(x) = \cos x$. Graph f together with the Taylor polynomials T_2, T_4, T_6, T_8 in the viewing rectangle $[-5, 5]$ by $[-1.4, 1.4]$ and comment how well they approximate f.

Please refer to the "**Project 3**" in the Jupyter Notebook for more detail.

Project 4: The functions g and h in project 2 are called **cubic splines**. Please read this lecture to do the following project.

The upper portion of this noble beast is to be approximated using clamped cubic spline interpolants. The curve is drawn on a grid from which table is constructed. Use the provided code to construct three clamped cubic splines.



Curve 1				Curve 2				Curve 3			
i	x_i	$f(x_i)$	$f'(x_i)$	i	x_i	$f(x_i)$	$f'(x_i)$	i	x_i	$f(x_i)$	$f'(x_i)$
0	1	3.0	1.0	0	17	4.5	3.0	0	27.7	4.1	0.33
1	2	3.7		1	20	7.0		1	28	4.3	
2	5	3.9		2	23	6.1		2	29	4.1	
3	6	4.2		3	24	5.6		3	30	3.0	−1.5
4	7	5.7		4	25	5.8					
5	8	6.6		5	27	5.2					
6	10	7.1		6	27.7	4.1	−4.0				
7	13	6.7									
8	17	4.5	−0.67								

Modify the code in the Jupyter notebook to recover the desired graph above.

References

[1] James Stewart: *Calculus, Early Transcendentals,* eighth edition.

[2] Richard L. Burden, Douglas J. Faires, Annette M. Burden: *Numerical Analysis,* tenth edition.