FIXED - POINT ITERATION

Consider $x^2 - 2x + 1 = 0$

$$\therefore x = \underbrace{\frac{x^2+1}{2}}_{g(x)}$$

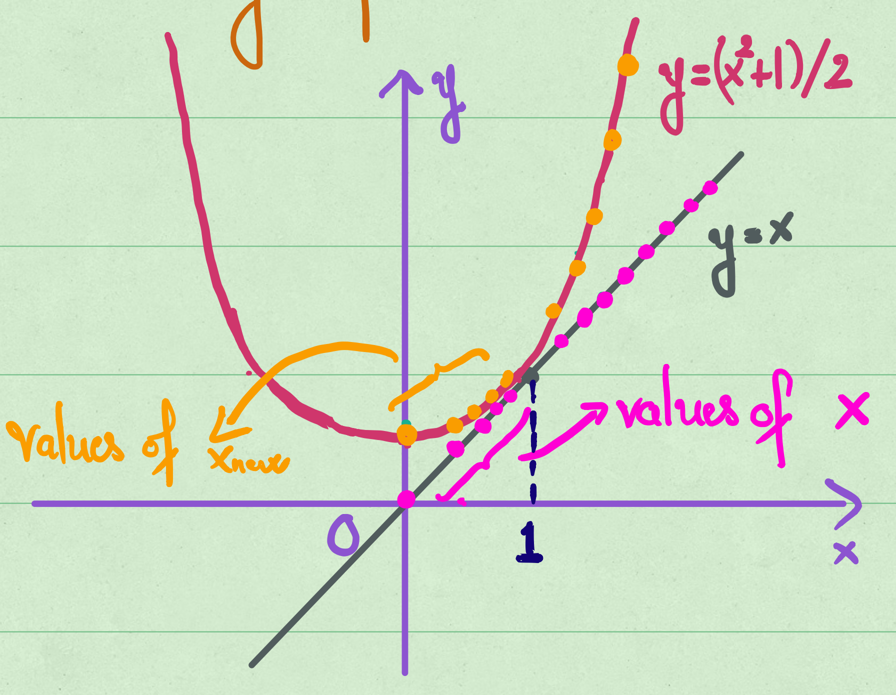
$$x_{new} = 0.5x^2 + 0.5$$

We use simple iteration method to solve this equation numerically

Key point: The values of x_{new} and x are getting close to each other after several iterations.

How to make it possible?

This may depend on the initial guess.



Choosing a good initial guess to start the first iteration is not easy.

It requires further analysis.

Mathematicians can help!

CS Math

Hey, we should work together

The existence of a solution of $g(x) = x$ depends on the behaviors of $g(x)$.

<u>Definition</u>: The # p is a fixed point for a given function g if $g(p) = p$.

We can reduce a problem of solving equation

$f(x) = 0$ to a problem of finding a fixed point of some function g.

Root-finding form	Fixed-point form
$f(x) = 0$	$g(p) = p$
$\underbrace{x + f(x)}_{g(x)} = x$	$\underbrace{g(p) - p}_{f(p)} = 0$
$g(x) = x$	$f(p) = 0$

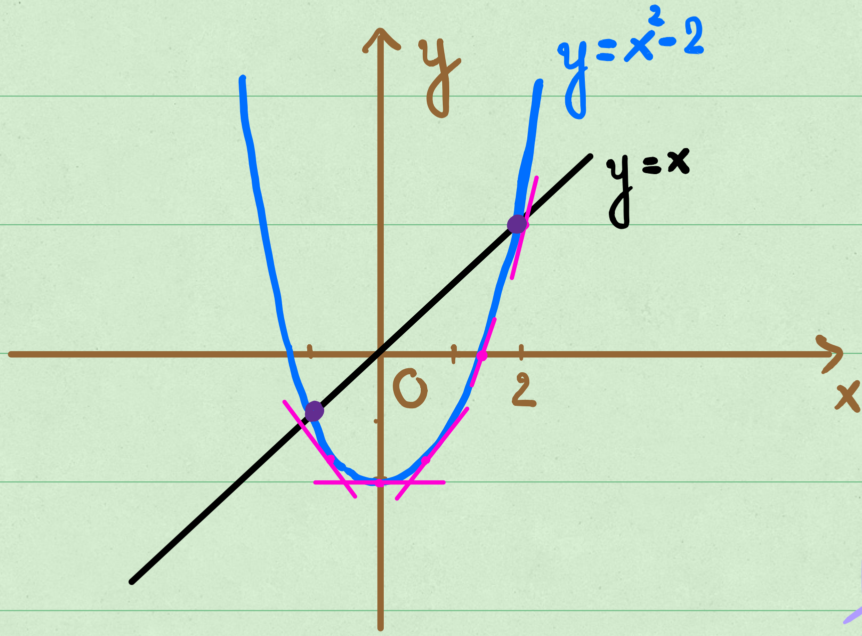
Question: Why fixed-point form?

- It is easy to analyze.
- Certain fixed-point choices lead to very powerful root-finding techniques.

Example: Determine any fixed points of

$$g(x) = x^2 - 2$$

Ans. $p = -1$, $p = 2$



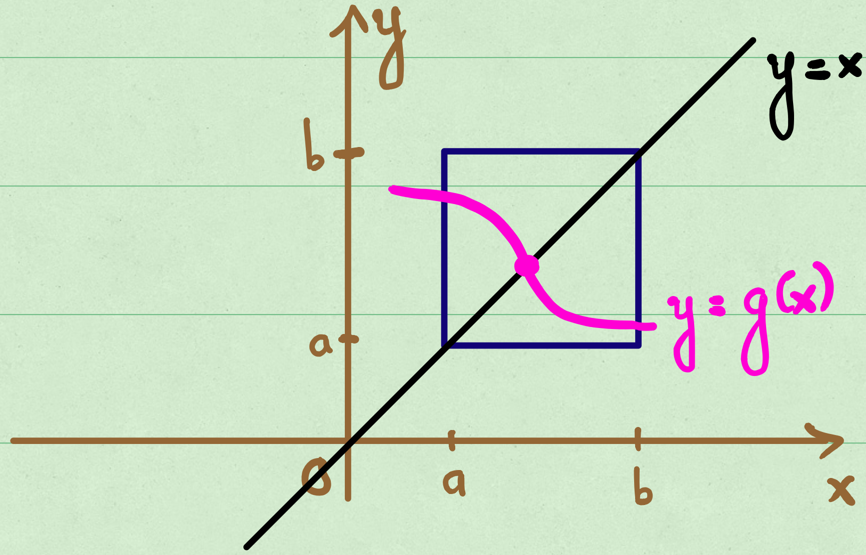
look like g doesn't change too much around fixed points.

Question: How do we know if a function a fixed point?

☺ Theorem:

a) $g \in C([a,b])$
$a \leq g(x) \leq b, \forall x \in [a,b]$ $\xRightarrow{\text{Existence}}$ g has at least one fixed point in $[a,b]$.

b) If, in addition, $g'(x)$ exists on (a,b) and $0<k<1$ exists s.t. $|g'(x)| \leq k, \forall x \in (a,b)$ $\xRightarrow{\text{Uniqueness}}$ g has exactly one fixed point in $[a,b]$



◎ Proof:

a) • If either $g(a) = a$ OR $g(b) = b$

$\Longrightarrow$ we are done.

• Suppose $g(a) > a$ and $g(b) < b$.

- Set $h(x) = g(x) - x$

- Then $\begin{cases} h(a) = g(a) - a > 0 \\ h(b) = g(b) - b < 0 \end{cases}$

- Using the Intermediate Theorem

love it

$\Rightarrow \exists\, p \in (a,b)$ s.t. $h(p) = 0$

$\therefore g(p) = p$. ✓ Great ☺

b) • Suppose, in addition, that $|g'(x)| < 1 \ \forall x \in (a,b)$ and that g has two fixed points p and q.

• If $p \neq q$, using the Mean Value Theorem

$\Rightarrow \exists \xi$ between p and q s.t.

$$g'(\xi) = \frac{g(p) - g(q)}{p - q} = \frac{p - q}{p - q} = 1.$$

• Then $|g'(\xi)| = 1$ Oh, a contradiction !!!

$|g'(x)| \leq k < 1, \forall x \in (a,b)$

• This ensures that $p = q$. ✓ Oh yeah ! 😊

<u>Question</u>: What happens if $|g'(x)| \leq 1 \ \forall x \in (a,b)$?

Fixed Point Iteration

<u>Question</u>: How to find fixed points of a function numerically ?

Fixed-Point Illustration

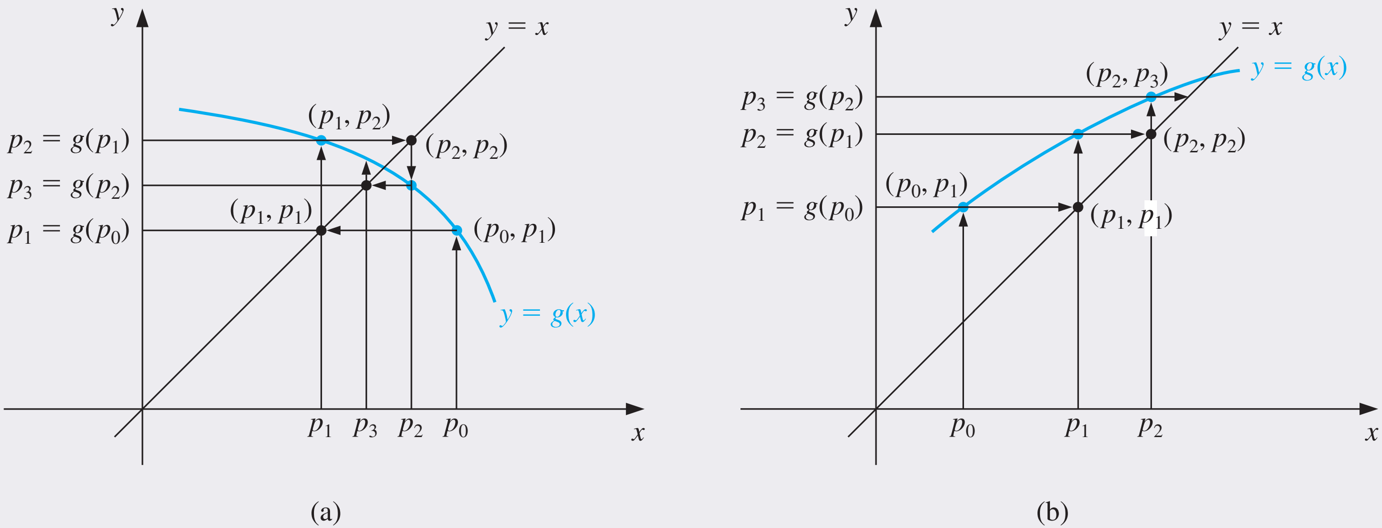


Figure: Figure 2.6

Goal: Approximate a fixed point of a function to any specified degree of accuracy.

Given g and its fixed point p.

- Choose an initial approximation p_0
- Construct a sequence $\{p_n\}_{n=0}^{\infty}$ by setting

$$\boxed{p_n = g(p_{n-1}) \quad \text{for each } n \geqslant 1.}$$

- Then $p = \lim_{n\to\infty} p_n = \lim g(p_{n-1}) = g(\lim p_{n-1}) = g(p)$.

So p_n is an approximation for p.

Theorem (Fixed-point Theorem)

Given

1) $g \in C([a,b])$, $a \leqslant g(x) \leqslant b \ \forall x \in [a,b]$
2) $g'(x)$ exists on (a,b) s.t.

$$|g'(x)| \leqslant k$$

Then, for any $p_0 \in [a,b]$,

the sequence $\{p_n\}_{n=0}^{\infty}$ defined by

$$p_n = g(p_{n-1}) \quad n \geqslant 1$$

converges to the unique fixed point p in $[a,b]$.

◎ **Proof**

- There exists a unique fixed point $p \in [a,b]$. ✓
- Since $g : [a,b] \longrightarrow [a,b]$,
  the sequence $\{p_n\}_{n=0}^{\infty}$ is defined for all $n \geqslant 0$
  and $p_n \in [a,b] \ \forall n \geqslant 0$.
- Consider

$$|p_n - p| = |g(p_{n-1}) - g(p)|$$

$$\overset{\text{Mean Value}}{\underset{\text{Theorem}}{=}} |g'(\underbrace{\xi_n}_{\in (a,b)})| \, |p_{n-1} - p| \leqslant k |p_{n-1} - p|.$$

- Applying this inequality inductively gives

$$|p_n - p| \leqslant k |p_{n-1} - p| \leqslant k^2 |p_{n-2} - p|$$
$$\leqslant \dots \leqslant k^n |p_0 - p|$$

• Since $0 < k < 1$, $\lim_{n\to\infty} k^n = 0$.

Thus $\lim_{n\to\infty} p_n = p$. Oh yeah ! 😊

The error bounds involved in using p_n to approximate p are given by

$$\boxed{|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\}}$$

and

$$\boxed{|p_n - p| \leq \frac{k^n}{1-k} \underbrace{|p_1 - p_0|}_{|g(p_0) - p_0|}}$$

Remark:

- The rate of convergence depends on k^n.
- The smaller k is, the faster the convergence is.
- The initial guess also affects the rate of convergence as well.

Question: How can we find a fixed-point problem that produces a sequence that reliably and rapidly converges to a root to $f(x) = 0$?

Ans. Manipulate the root-finding problem into a fixed-point problem that satisfies the conditions of Fixed-point Theorem and has a derivative that is as small as possible near the fixed point.

This YouTube video developed by Oscar Veliz can serve as a good illustration of the Fixed-Point Method for students:
https://www.youtube.com/embed/OLqdJMjzib8