CSC263: Problem Set 2

September 24, 2019

1 Problem 2

(a) The index of the parent is given by $2^j/2 = 2^{(j-1)}$.

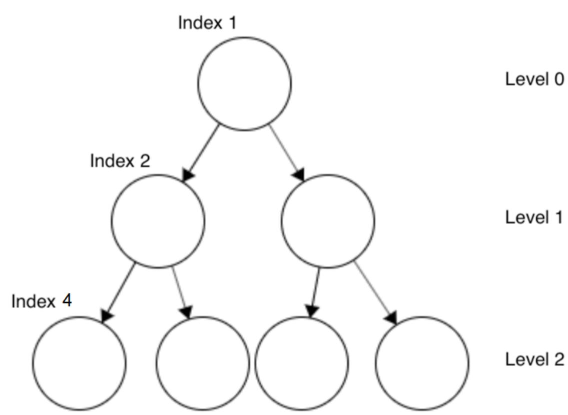

Figure 1: A binary heap

(b)

(c) $P(x < a_j) = 1 - P(a_j \leq x) = 1 - p^j$. So the probability of $P(x < a_j) = 1 - p^j$.

(d) $P(x < a_j \cap x < a_{j+1}) = 1 - p^j$. We note that the probability here is the same as in part c because $a_j \leq a_{j+1}$ as this is a binary min-heap. So if $x < a_j$ occurs then $x < a_{j+1}$ will also occur. So, $P(x < a_j \cap x < a_{j+1})$ is dependent on $P(x < a_j)$. Thus $P(x < a_j \cap x < a_{j+1}) = 1 - p^j$.

(e) $P(x < a_{j+1}|x < a_j) = 1$. For conditional probability we are given $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$. Here $P(A|B)$ is $P(x < a_{j+1}|x < a_j)$, $P(A \cap B)$ is $P(x < a_{j+1} \cap x < a_j)$, and $P(B)$ is $P(x < a_j)$. So, $P(x < a_{j+1}|x < a_j) = \dfrac{P(x < a_j \cap x < a_{j+1})}{P(x < a_j)} = \dfrac{1-p^j}{1-p^j} = 1$.

(f) $P(x < a_j|x < a_{j+1}) = \dfrac{1-p^j}{1-p^{j+1}}$. For conditional probability we are given $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$. Here $P(A|B)$ is $P(x < a_j|x < a_{j+1})$, $P(A \cap B)$ is $P(x < a_j \cap x < a_{j+1})$, and $P(B)$ is $P(x < a_{j+1})$. $P(x < a_{j+1}) = 1 - P(a_{j+1} \leq x) = 1 - p^{j+1}$. So, $P(x < a_j|x < a_{j+1}) = \dfrac{P(x < a_j \cap x < a_{j+1})}{P(x < a_{j+1})} = \dfrac{1-p^j}{1-p^{j+1}}$.

(g) $P(a_j \leq x|x < a_{j+1}) = \dfrac{p^j - p^{j+1}}{1-p^{j+1}}$. For conditional probability we are given $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$. Here $P(B)$ is $P(x < a_{j+1}) = 1 - p^{j+1}$. $P(A \cap B)$ is $P(a_j \leq x \cap x < a_{j+1}) = 1 - P(x < a_j) - P(a_{j+1} \leq x) = 1-(1-p^j)-p^{j+1} = p^j - p^{j+1}$. Note that we are looking for the probability of $a_j \leq x < a_{j+1}$, so to find this, we simply subtract from 1 the probability of $x < a_j$ and $a_{j+1} \leq x$ so that we are left with just the probability of the intersection. So, $P(a_j \leq x|x < a_{j+1}) = \dfrac{P(a_j \leq x \cap x < a_{j+1})}{P(x < a_{j+1})} = \dfrac{p^j - p^{j+1}}{1-p^{j+1}}$.

(h) Q.insert(x) would only perform one swap if x is less than its parent but not less than its parent's parent. In other words if x was inserted at the bottom of the tree say at index a_{k+1} then for only one swap to occur, a_k should be greater than x and a_{k-1} should be less than or equal to x.

(i) The probability that Q.insert(x) performs exactly one swap based on the condition given in part h.) is given by $p^{k-1} - p^k$. This is because the probability of only swap occurring is given by $P(a_{k-1} \leq x \cap x < a_k) = 1 - P(x < a_{k-1}) - P(a_k \leq x) = 1 - (1 - p^{k-1}) - p^k = p^{k-1} - p^k$.

(j) Q.insert(x) would only perform two swap if x is less than its parent and less than its parent's parent but greater than its parent's parents parent. In other words if x was inserted at the bottom of the tree say at index a_{k+1} then for the first swap to occur, a_k should be greater than x. For the second swap to occur then x should be less than a_{k-1} and it's other child but for another not to occur x should not be less than a_{k-2}.

(k) The probability of Q.insert(x) performing two swaps based on the condition given in part j.) is given by $p^{k-2} - p^{k-1}$. This is because the probability of only two swaps occurring is given by $P(a_{k-2} \leq x \cap x < a_{k-1}) = 1 - P(x < a_{k-2}) - P(a_{k-1} \leq x) = 1 - (1 - p^{k-2}) - p^{k-1} = p^{k-2} - p^{k-1}$.

(l) Q.insert(x) would perform exactly j swaps when x is less than j of its parents. That is to say, x's parent's parent's ... parent j times. To simplify this, let's denote x's parent with $parent_1$ and x's parent's parent with $parent_2$ and so on till we reach $parent_j$. If $x < parent_{j+1}$, then we would have performed $j+1$ swaps assuming that x is inserted at position a_{k+1}. So to ensure that only j swaps are performed we require $x < a_{k-j+1}$ and $x \geq a_{k-j}$.

(m) The probability of Q.insert(x) performing j swaps based on the condition given in part l.) is given by $p^{k-j} - p^{k-j+1}$. This is because the probability of only j swaps occurring is given by $P(a_{k-j} \leq x \cap x < a_{k-j+1}) = 1 - P(x < a_{k-j}) - P(a_{k-j+1} \leq x) = 1-(1-p^{k-j}) - p^{k-j+1} = p^{k-j} - p^{k-j+1}$.

(n) The expected number of swaps for Q.insert(x) is $E(m) = \sum_{j=0}^{m} j \cdot (p^{k-j} - p^{k-j+1})$. This is because the expected value of Y is the summation of y_i multiplied by $p(y_i)$, where $p(y_i)$ is the probability of event y_i occurring, and y_i for $i = 1, .., n$, is the number of finite outcomes.