泛函分析讨论班

第二次

定义. X. Y 赋范线.

$T: X \longmapsto Y$ 线性.

有界 iff. $\exists M>0$. $\forall x \in X$.

有 $\|Tx\|_Y \le M\|x\|_X$.

有界线性泛函 $Y = \mathbb{K}$ ($\mathbb{R}$ / $\mathbb{C}$)

引理: 线 $T: X \longmapsto Y$
赋范线

T 有界 iff T 连续.

"$\Rightarrow$" $\|Tx - Ty\| \le M\|x-y\|$

"$\Leftarrow$" 否: $n \in \mathbb{N}$. $\exists x_n \in X$.

$\|Tx_n\| > n\|x_n\|$.

$y_n = \frac{x_n}{n\|x_n\|}$. $\|y_n\| = \frac{1}{n} \to 0$.

$\|Ty_n\| = \frac{1}{n\|x_n\|}\|Tx_n\| > 1$. 矛盾

Remark: 线性算子 T 连续 $\Leftrightarrow$ T 在 0 处连续.

"$\Rightarrow$" ✓ "$\Leftarrow$" $\|Tx - Ty\| = \|T(x-y)\|$

$\mathbb{R}^n \to \mathbb{R}^m$ $\quad X \to Y$

$\|A\|_\infty$

$\mathcal{L}(X, Y)$ 有界线性映射

$T_1, T_2 \in \mathcal{L}(X, Y) \qquad T_1 + T_2 \overset{?}{\in} \mathcal{L}(X, Y)$

$\|(T_1+T_2)x\| \leq M_1\|x\| + M_2\|x\|$

$\|T_1\| = \sup \frac{\|T_1 x\|}{\|x\|} = \sup_{\|x\|=1} \|T_1 x\|$

$\mathcal{L}(X, \mathcal{L}(X, Y))$

Thm. $(\mathcal{L}(X, Y), \|\cdot\|)$ 为赋范线性空间

Y 为 Banach 空间. $(\mathcal{L}(X, Y), \|\cdot\|)$ 也为 Banach 空间

$\{T_n\}$. Cauchy 列

$\|T_n - T_m\| \to 0. \qquad n \geq m \to \infty$

$\{T_n x\} \cdot Y \quad \|T_n x - T_m x\| \leq \|T_n - T_m\| \, \|x\|$

$\downarrow$

$Tx. \qquad T_n x \to Tx$

$Tx + Ty \leftarrow T_n x + T_n y = T_n(x+y) \to T(x+y)$

$\sup_{\|x\|=1} \|T_n x - Tx\| \to 0.$

$$\sup_{\|x\|=1} \|T_n x - Tx\| \to 0.$$

$$\|T_n - T\| \to 0$$

反: $\exists\ \varepsilon>0$. $\{n_k\}$. $\exists \{x_{n_k}\}$. $\|x_{n_k}\|=1$

$$\|T_{n_k} x_{n_k} - T x_{n_k}\| \ge \varepsilon.$$

$\exists$ $x_{n_{k_l}} \to x$.

$$\|T_{n_{k_l}} \cancel{x_{n_{k_l}}} - Tx\| \ge \frac{\varepsilon}{2}.$$

$Y = \mathbb{K}(\mathbb{R}/\mathbb{C})$

$\mathcal{L}(X, \mathbb{K}) = X^*$.

Hahn-Banach定理.

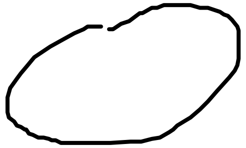
ZCC.

设X为线性空间. $V \le X$ f是定义在V上的线性泛函. $\forall x\in V$. $|f(x)| \le p(x)$. (p为半模). 则可将f延拓到整个X上.

但 $|F(x)| \le p(x)$.

半模 def $\forall x, y \in X$. $a \in K$.

$p(ax) = |a| p(x)$. $p(x+y) \le p(x) + p(y)$



$\forall y_0 \notin V$. $y_0 \in X$.

$\forall x \in V$.

$\underline{f(x) + a f(y_0)} = f(x + a y_0) \le p(a y_0 + x)$

① $a > 0$. $f(\frac{x}{a}) + f(y_0) \le p(y_0 + \frac{x}{a})$

$f(x) + f(y_0) \le p(y_0 + x)$ (1)

$x + a y_0$

$\mathrm{span}(V, y_0)$

② $a < 0$. $b = -a$.

$f(x) - b f(y_0) \le p(x - b y_0)$

$f(\frac{x}{b}) - f(y_0) \le p(\frac{x}{b} - y_0)$

$f(x) - f(y_0) \le p(x - y_0)$ $x \in V$. (2)

$\forall x, y \in V$

$f(x) - p(x - y_0) \le f(y_0) \le p(y_0 + y) - f(y)$

$f(x+y) \le p(x - y_0) + p(y_0 + y)$ $x + y \in V$.

V. $\ge p(x+y)$

$\ge f(x+y)$

$S = \{ (U, g) \mid$ $U \subseteq X$ $\underset{=}{g}$ f.

$g(x) \le p(x)$ $\forall x \in U$

$V \subset U$. $g|_V = f$.

S 非空. 偏序

$(\tilde{U}, \tilde{f})$

$\tilde{U} = X$

复值. $\cancel{f(x) = g(x)}$ $\frac{\mathrm{Re} f = g(x)}{\hat{f}}$ $a + bi \to (a, b)$

$\tilde{g}$

$\boxed{\cancel{f(x)}\ f(x) = g(x) - i g(ix)}$

$f(x) = \underline{a + bi}$ $g(x) = \mathrm{Re} f = a$.

$i f(x) = f(ix)$ $g(ix) = \mathrm{Re} f(ix) = -b$

$= -b + ia$

$|\tilde{f}(x)| \le p(x)$

$\theta = \arg \tilde{f}(x)$.

$|\tilde{f}(x)| = |e^{-i\theta} \tilde{f}(x)| = |\tilde{f}(e^{-i\theta} x)| = \tilde{g}(e^{-i\theta} x) \le p(e^{-i\theta} x)$

$= |e^{-i\theta}| p(x)$

推论. 若 X 为赋范线性空间, V为子集.

f是V上的有界线性泛函. 则 f 可延拓

到 X上. 且满足: $\|f\|_{X^*} = \|f\|_{V^*}$.

$p(x) = \|f\| \|x\|$. $\qquad \|f(x)\| \le \underbrace{\|f\|}_{=1} \|x\|$. on V.

$$\|f(x)\| \le \|f\|_{V^*} \|x\|.$$

$$\|f\|_{X^*} \le \|f\|_{V^*}$$

$$\|f\|_A \le \|f\|_B \qquad A \subset B.$$

Zbc.

X为非零赋范线性空间

对 $\forall x_0 \in X$. $x_0 \neq 0$. $\exists X$上有界线性

泛函 f. 使得 $f(x_0) = \|x_0\|$. 且 $\|f\| = 1$

$\{x_0\}$ 张成的空间 $\qquad f(\lambda x_0) = |\lambda| \|x_0\|$

X为赋范线性空间. $x_0 \in X$. 对 $\forall X$上有界

线性泛函 f 均有 $f(x_0) = 0$ 则 $x_0 = 0$

对偶空间的例子.

Hilbert 空间 H的对偶空间

$\forall y \in H$. $l_y: x \mapsto (y, x)$

$y \mapsto l_y$

$H \mapsto H^* = \mathcal{L}(H, \mathbb{K})$

$\mathcal{L}(H, \mathcal{L}(H, \mathbb{K}))$

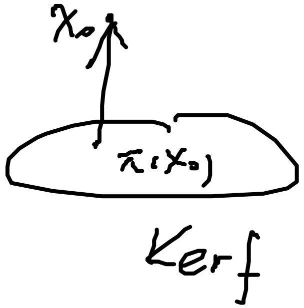
$\|l_y\|_{H^*} = \|y\|_H$

$\|l_y - l_z\|_{H^*} = \|y - z\|_H$

$H \mapsto H^*$!

Rieze 表示

不妨 $f \neq 0$. $\operatorname{Ker} f$



H. $\forall f \in H^*$. $\exists!\ y_f \in H$.

s.t. $\forall x \in H$. 有 $f(x) = (y, x)$

$f(y_f) = (y_f, y_f)$ $\quad f$

$\forall x_0 \in H \setminus V$

$\exists\ a \in \mathbb{K}$. $\quad v = a(x_0 - \pi(x_0))$

$f(v) = (v, v)$ a. $\quad \mathbb{K}$.

1 $\quad$ 2.

$\forall x \in H.$ $f(x) = (v, x)$ $(v, v) = f(v)$

$v \perp \ker f.$

$\underline{V^\perp}$ $\underline{\{v\}}$ $\underline{\mathbb{K}}$ 1.

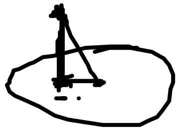
1维.

$\forall u \in V^\perp.$ $u \neq 0.$ $f(u) \neq 0 \in \mathbb{K}.$

$f(bu) = bf(u) = f(v)$ $f(bu - v) = 0$

$\underbrace{bu - v}_{\in V^\perp} \in \ker f = V$ $bu = v$

$\underline{\forall x.}$ $x = \pi(x) + \underbrace{(x - \pi(x))}_{\lambda v}$



$V^\perp$

$f(x) = f(x - \pi(x)) = f(\lambda v) = \lambda f(v)$

$= \lambda(v, v) = (v, \lambda v) = (v, \pi(x) + \lambda v) = (v, x)$

Lax - Milgram. $y^T A x = (Ax, y)$

$\boxed{x^T A \bar{y}}$

令 H 为 Hilbert 空间，$a(x, y)$ 是 H 上

共轭双线性函数，若：

1. $\exists M > 0.$ s.t.

$\forall x, y \in H.$ $\underline{|a(x, y)| \leq M \|x\| \|y\|}.$

2. $\underline{\exists \delta > 0.}$ s.t.

$\underline{\forall x \in H.}$ $\underline{a(x, x) \geq \delta \|x\|^2}$

则 H 上唯一有连续逆的连续线性算子 A.

s.t. $\forall x, y \in H.$ $a(x, y) = (Ax, y)$

$\forall x \in H$. $y \mapsto a(x,y)$ 有界线性泛函.

由 Rieze 表示定理

$\exists!$ $x' \in H$. $a(x,y) = (x', y)$

$Ax = x'$

$(A(x+y), z)$

$A(x+y) = (x+y)'$

$Ax = x'$

$Ay = y'$

$\forall z$ $((x+y)', z) = a(x+y, z)$

$= a(x,z) + a(y,z)$

$= (x', z) + (y', z)$

$= (x'+y', z)$

$= (Ax+Ay, z)$

$\forall z \in H$. $A(x+y) = Ax+Ay$

$\|Ax\| = \sup\limits_{\|y\|=1} |(Ax, y)| = \sup |a(x,y)|$

$\leq M\|x\|$.

A 为有界线性算子.

$\|Ax\| = \sup\limits_{\|y\|=1} |a(x,y)| \geq |a(x, \frac{x}{\|x\|})|$

$\geq \delta \|x\|$

$A(H) = H$. $H_0 = A(H)$

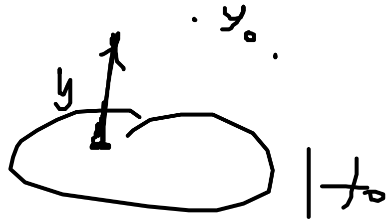
$Ax_n \to y_0$. 目标 $y_0 = Ax_0$

$A(H)$ $A(H)$ $\{Ax_n\}$ Cauchy 列.

$\|Ax\| \geq \delta\|x\|$. $\|x_n - x_m\| \leq \frac{1}{\delta}\|Ax_n - Ax_m\|$

$\{x_n\}$ 为 Cauchy 列

$H_0 \neq H$ $\quad y_0 \in H \setminus H_0$.



$(y \neq 0)$ $\quad y \perp H_0$.

$\Downarrow$

$(Ax, y) = 0 \quad \forall x \in H$.

$0 = (Ay, y) = a(y, y) \geq \delta \|y\|^2 > 0$.

$H_0 = H$.

$A: H \longrightarrow H$ 双射

$\delta \|x\| \leq \|Ax\|$

$\Updownarrow$

$\delta \|A^{-1}x\| \leq \|x\|$

$\|A^{-1}x\| \leq \frac{1}{\delta}\|x\|$

有界.

$\square$

$L^p(X, d\mu)$.

$\begin{cases} 1 \leq p < \infty. \\ p = \infty. \end{cases}$

$\left(\frac{1}{p'} + \frac{1}{p} = 1\right)$

(1) $1 \leq p < \infty$. $\quad \iota: L^{p'}(X, d\mu) \longmapsto (L^p(X, d\mu))^*$

$\forall f \in L^p$, $\quad g \in L^{p'}$

$[\iota(g)](f) = \int_X f(x) g(x)\, d\mu$.

(2) E.M. Stein and R. Shakarchi

Functional analysis

Thm 4.1

$L: \quad L^1(\mathbb{R}^n) \mapsto (L^\infty(\mathbb{R}^n))^*$ 不是满射.

令 $\varphi_n = e^{-n^2x^2} \in L^\infty(\mathbb{R}^n)$. $\{\varphi_n\}$ 是 L^∞ 一元素 线性

由 Banach 定理 $\exists$ $L^\infty(\mathbb{R}^n)$ 有界线性泛函

f. $f(\varphi_n) = 1$

Span $\{\varphi_n\}$

$\|\varphi_n\| = 1$

$h = \sum a_j \varphi_j$

$|f(h)| = |\sum a_j| = |h(0)| \le \|h\|_{L^\infty}$

$1 = f(\varphi_m) = \int_{\mathbb{R}^n} \varphi_m(x) g(x) dx$

$m \to \infty$

$= 0$ LDC.

$e^{-m^2x^2}$

第二对偶空间 $(X^*)^* = X$

$\langle x^*, x \rangle = f(x) = x^*(x)$

$(x \in X. \quad x^* \in X^*)$

x 定义 X^* 上的线性泛函

$(X^*)^* = X$

对 ∀ 赋范线性空间 X.

$\exists\ L: X \longrightarrow X^{**}$ 等距

$\forall x \in X.\quad x^* \in X^*$

$\langle L(x), x^* \rangle_{X^{**} \times X^*} = \langle x^*, x \rangle_{X^* \times X}$

$\langle L(x+y), x^* \rangle = \langle x^*, x+y \rangle$

$= \langle x^*, x \rangle + \langle x^*, y \rangle$

$= \langle L(x), x^* \rangle + \langle L(y), x^* \rangle$

$\| L(x) \|_{X^{**}} = \| x \|$.

$X \subseteq X^{**}$.

$\| L(x) \|_{X^{**}} = \sup_{\|x^*\| = 1} | \langle L(x), x^* \rangle |$

$= \sup_{\|x^*\| = 1} | \langle x^*, x \rangle |$

Banach. $\|x^*\| = 1$

$| \langle x^*, x \rangle | \le \| x \|$.

$x^* \in X^*$. $\langle x^*, x \rangle = \|x\|$ 且 $\|x^*\|_{X^*} = 1$

$\| L(x) \| = \| x \|$.

若 $\iota: X \longrightarrow X^{**}$ 是满射，则

称其为<u>自反的</u>.

$\to \mathbb{K}$

$\mathbb{R}/\mathbb{C}$.

$X = X^{**}$

$X \neq X^{**}$.