泛函分析讨论班

第五次

inversioner

$\alpha = (\alpha_1, \cdots, \alpha_n)$

$|\alpha| = \sum \alpha_i \qquad D^\alpha u = \prod_i \frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}} u.$

$Du = (D^{e_1}u, \cdots D^{e_n}u)^T = (u_{x_1} \cdots u_{x_n})$

$\alpha! = \alpha_1! \cdots \alpha_n!$

Hölder. $|u(x) - u(y)| \leq C|x-y|^r \qquad r \in (0, 1]$

u 有界连续. $\|u\|_{C(\bar U)} = \sup_{x \in U} |u(x)|$.

$[u]_{C^{0,r}(\bar U)} = \sup_{\substack{x \neq y \\ x, y \in U}} \frac{|u(x) - u(y)|}{|x-y|^r}$

$C^{k,r}(\bar U) \qquad u \in C^k(\bar U). \quad \|u\|_{C^{k,r}(\bar U)} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{C(\bar U)} + \sum_{|\alpha| = k} [D^\alpha u]_{C^{0,r}(\bar U)}$

弱导数. $u \in C^1(U)$. $\phi \in C_c^\infty(U)$

$\int u \phi_{x_i} dx = -\int u_{x_i} \phi dx$

$|\alpha| = k$. $u \in C^k$ $\quad \int u D^\alpha \phi \, dx = (-1)^\alpha \int v \phi \, dx$

$v = D^\alpha u$

Sobolev空间 $\quad W^{k,p}(U) \qquad u \in L^p(U)$

$\forall |\alpha| \leq k$. $\exists D^\alpha u$ 弱导数. 且在 L^p 中.

$$\|u\|_{W^{k,p}(U)} = \begin{cases} \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(U)}^p \right)^{\frac{1}{p}}, & 1 \leq p < \infty \\ \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty}, & p = \infty \end{cases}$$

$W^{0,p}(U) = L^p(U)$

$\boxed{H^k = W^{k,2}}$

弱导数. $u, v \in W^{k,p}(U)$, $|\alpha| \le k$.

(1) $D^\alpha u \in W^{k-|\alpha|,p}(U)$ $\forall \beta, \gamma$, $|\beta|+|\gamma| \le k$.

有 $D^{\beta+\gamma} u = D^\beta (D^\gamma u) = D^\gamma (D^\beta u)$

(2) $\lambda u + \mu v \in W^{k,p}(U)$. 且 $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$.

3) 如果 $V \subset U$ 为开集. 则 $u \in W^{k,p}(V)$

Thm. Sobolev 空间是 Banach 空间

u_m 是 $W^{k,p}(U)$ 的 Cauchy 列

$|\alpha| \le k$. $\{D^\alpha u_m\}$ 是 Cauchy 列. L^p

$D^\alpha u_m \to u_\alpha$ $L^p(U)$

$\alpha = (0, \cdots, 0)$ $u_m \to u_{\alpha_0}$ L^p.

$\phi \in C_c^\infty(U)$ 则

$$\int u D^\alpha \phi \, dx = \lim_{m\to\infty} \int u_m D^\alpha \phi \, dx$$

$$= (-1)^{|\alpha|} \lim_{m\to\infty} \int D^\alpha u_m \phi \, dx$$

$$= (-1)^{|\alpha|} \int u_{\alpha_0} \phi \, dx.$$

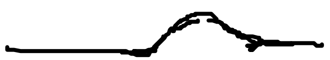
$D^\alpha u_m \to D^\alpha u_{\alpha_0}$ in L^p

$L^2(U)$ 是 Hilbert 空间 $W^{k,2} \subset L^2$

$\parallel$

H^k.

[illegible].
$$\eta(x) = \begin{cases} C\exp\left(\frac{1}{|x|^2-1}\right), & |x|<1 \\ 0 & |x| \geq 1 \end{cases}$$



$\eta_\varepsilon = \varepsilon^{-n}\eta\left(\frac{x}{\varepsilon}\right)$

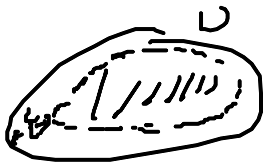
$\int \eta_\varepsilon \, dx = 1.$

$\underline{f^\varepsilon = \eta_\varepsilon * f}$ $\quad f^\varepsilon \to f. \quad L^p_{loc}$

$x+\varepsilon \quad \varepsilon\varphi + x$

① 设 $u \in W^{k,p}(U)$ $1 \leq p < \infty$. 记 $u^\varepsilon = \eta_\varepsilon * u$.

则 (1) $u^\varepsilon \in C^\infty(U_\varepsilon)$. $U_\varepsilon = \{x \in U \mid d(x, \partial U) > \varepsilon\}$



(2) $u^\varepsilon \to u$ in $W^{k,p}_{loc}(U)$. $\varepsilon \to 0$.

$D^\alpha u^\varepsilon = (D^\alpha u)^\varepsilon$

② u 有界. $u \in W^{k,p}$ $1 \leq p < \infty$

$u_m \in C^\infty(U) \cap W^{k,p}(U)$.

$u_m \to u$ in $W^{k,p}(U)$

V_i. $\quad U = \bigcup_{i=0}^\infty V_i \quad \zeta_i \in C_c^\infty(V_i)$

$\sum_{i=0}^\infty \zeta_i = 1 \qquad 0 \leq \zeta_i \leq 1$

$u = \underline{\sum_{i=0}^\infty \zeta_i u}$ $\qquad u^i = \eta_{\varepsilon_i} * (\zeta_i u)$

$\|u^i - \zeta_i u\|_{W^{k,p}(U)} \leq \frac{\delta}{2^{i+1}}$

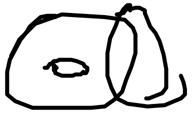
δ

Thm. u 是有界 且 $u\in W^{k,p}(U)$ $1\le p<\infty$

$\underline{\partial U \text{ 是 } C^1}$. 则 $\exists$ 函数列 $u_m\in C^\infty(\bar U)$. $u_m \to u$ in $W^{k,p}(U)$

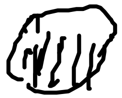
Δ.

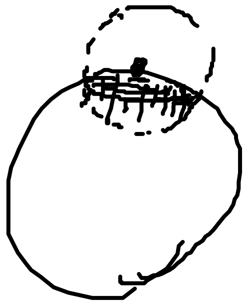
$\underline{\underline{U}}$: $\underline{\partial U\, C^k}$. $\forall x^0\in\partial U$. $\exists r>0$ $\gamma\in C^k: \mathbb{R}^{n-1}\to\mathbb{R}$.



$\underline{U\cap B(x^0;r)} = \{x\in B(x^0,r) \mid x_n > \gamma(x_1,\dots,x_{n-1})\}$

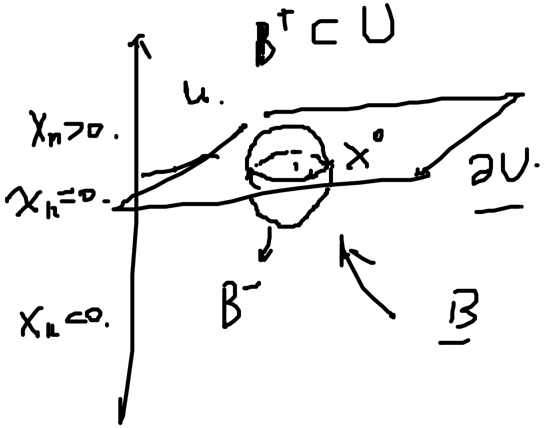
Δ.





$x^2+y^2<3$

$\underline{u\in C^\infty(\bar U)}$.



$\underline{扩张}$. 设 U是有界. ∂U 是 C^1 的. 取有界开集 V

s.t. ~~V~~ $U\subset\subset V$. 则 $\exists$ 有界线性算子

$E: W^{1,p}(U)\to W^{1,p}(\mathbb{R}^n)$

满足 (1) $Eu=u$ a.e. $\underline{\text{on } U}$

(2). $spt(Eu)\subset V$.

(3) $\exists C$. s.t. $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(U)}$.

$$\bar u(x) = \begin{cases} u(x) & x\in B^+ \\ -3u(x', -x_n) + 4u(x', -\frac{x_n}{2}). & x\in B^- \end{cases} \qquad x' = (x_1,\dots,x_{n-1})$$

$3 \quad -2$

$$\|\bar{u}\|_{W^{1,p}(B)} \le C\|u\|_{W^{1,p}(B^+)}$$

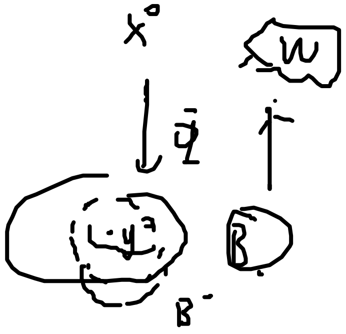
x^0

$U \cap B(x^0, r) = \{x_n > \gamma(x_1, \dots, x_{n-1})\}$

$x \in B(x^0, r)$

$y = \Phi(x)$

$$\begin{cases} y_i = x_i & i < n \\ y_n = x_n - \gamma(x_1, \dots, x_{n-1}) \end{cases}$$



$x = \Psi(y)$

$$\begin{cases} x_i = y_i & i < n \\ x_n = y_n + \gamma(y_1, \dots, y_{n-1}). \end{cases}$$

$\bar{u}' \in C^1(B)$

$$\|\bar{u}'\|_{W^{1,p}(B)} \le C_1\|u'\|_{W^{1,p}(B^+)}$$

$W = \Psi(B)$ $\quad \bar{u}(x) = u'(\Phi(x))$

$$\|\bar{u}\|_{W^{1,p}(W)} \le C_2\|u\|_{W^{1,p}(B)}$$

可以是 [illegible]..

$u \leftarrow u_m$. 逼近

C^∞

$$\frac{1}{p} + \frac{1}{q} = 1$$

u 是 n 元函数.

p. n.

$$\boxed{\|u\|_{L^q(\mathbb{R}^n)} \le C\|Du\|_{L^p(\mathbb{R}^n)}} \quad \forall u \in C_c^\infty(\mathbb{R}^n).$$

$\|u(x)\| \to u(\lambda x)$ $\qquad u_\lambda(x) = u(\lambda x)$

$$\left(\int u^q(x)\,dx\right)^{\frac{1}{q}} \le C\left(\int |Du(x)|^p dx\right)^{\frac{1}{p}}$$

$$\lambda^{-\frac{n}{q}}\left(\int u^q(\lambda x)\,d(\lambda x)\right)^{\frac{1}{q}} \le \lambda^{-\frac{n}{p}}\lambda\, C(\ldots)$$

$\lambda^{\frac{n}{p}}$

u_λ.

$$\lambda^{-\frac{n}{q}}\|u\|_{L^q(\mathbb{R}^n)} \le \lambda^{1-\frac{n}{p}} C\|u\|_{L^p(\mathbb{R}^n)}$$

$$-\frac{n}{q} = 1 - \frac{n}{p} \qquad p^* := q = \frac{np}{n-p}. \quad \text{Soblev 共轭.}$$

$1 \le p < n$ (GNS. - Gagliardo - Nirenberg - Soblev 不等式]

$C = C_{p,n}$.

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C\|Du\|_{L^p(\mathbb{R}^n)}, \quad \forall u \in C_c^1(\mathbb{R}^n)$$

① . $p=1$. $\quad p^* = \frac{n}{n-1}$

$C=1$

$$\int |u|^{\frac{n}{n-1}} dx \le \left(\int |Du| \, dx \right)^{\frac{n}{n-1}}$$

$$|u(x)| = \left| \int_{-\infty}^{x_i} u_{x_i}(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n) \, dy_i \right|$$

$$\le \int_{-\infty}^{\infty} |Du(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)| \, dy_i$$

$$= \int_{-\infty}^{\infty} |Du| \, dy_i$$

$$\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1 \le \int_{-\infty}^{\infty} \left(\prod_{i=1}^{n} \int_{-\infty}^{\infty} |Du| \, dy_i \right)^{\frac{1}{n-1}} dx_1$$

$$= \left(\int_{-\infty}^{\infty} |Du| \, dy_1 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |Du| \, dy_i \right)^{\frac{1}{n-1}} dx_1$$

Hölder

$$\le \left(\int_{-\infty}^{\infty} |Du| \, dy_1 \right)^{\frac{1}{n-1}}$$

$$\left(\prod_{i=2}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| \, dx_1 \, dy_i \right)^{\frac{1}{n-1}}$$

$k=n$

$k=1$ $\quad \boxed{i=n}$

$$\int_{\mathbb{R}^k} |u|^{\frac{n}{n-1}} dx_1 \cdots dx_k \le \cancel{\prod_{i=1}^{k} \left(\int_{\mathbb{R}^k} |Du| \, dx_1 \cdots dx_k \, dy_i \right)^{\frac{1}{n-1}}}$$

$$\prod_{i=1}^{k} \left(\int_{\mathbb{R}^k} |Du| \, dx_1 \cdots dy_i \cdots dx_k \right)^{\frac{1}{n-1}}$$

$$\prod_{i=k+1}^{n} \left(\int_{\mathbb{R}^{k-1}} |Du| \, dx_1 \cdots dx_k \, dy_i \right)^{\frac{1}{n-1}}$$

$v = |u|^r$. r待定. $(r>1)$

$$\left(\int |u|^{\frac{rn}{n-1}}\right)^{\frac{n-1}{n}} dx \le \int |Du^r|\, dx$$

$$= r\int \underline{\underline{|u|^{r-1}}}\, \underline{\underline{|Du|}}\, dx$$

$$\le r\left(\int |u|^{\frac{(r-1)p}{p-1}}\right)^{\frac{p-1}{p}} \left(\int |Du|^p\, dx\right)^{\frac{1}{p}}$$

$$\frac{rn}{n-1} = \frac{(r-1)p}{p-1} = \frac{np}{n-p} = p^*$$

$$r = \frac{p(n-1)}{n-p}$$

$$\|u\|_{L^{p^*}} \le \|Du\|_{L^p}$$

设U是有界集. ∂U是C^1的.

令 $u \in W^{1,p}(U)$. 则 $u \in L^{p^*}(U)$

且满足

$$\|u\|_{L^{p^*}(U)} \le C\|u\|_{W^{1,p}(U)}.$$

$$\underline{W}^{1,p} \hookrightarrow \underline{\underline{L}}^{p^*}$$

$W \hookrightarrow$ Hölder Space

$$\left(\sum_{|\alpha|\le 1}\|D^\alpha u\|^p_{L^p}\right)^{\frac{1}{p}} = \left(\underline{\underline{\|u\|^p_{L^p}}} + \underbrace{\sum_{i=1}^{n}\|u_{x_i}\|^p_{L^p}}_{\|Du\|_{L^p}}\right)^{\frac{1}{p}}$$

补充.

$C_c^\infty(U) \subset W^{k,p}(U)$

$W_0^{k,p}$ 为 $C_c^\infty(U)$ 在 $W^{k,p}(U)$ 的闭包.

$C^\infty(\bar{\Omega})$ 在 $H^k(\Omega)$ 中稠密 $(k\in\mathbb{N}^+)$

Thm. U是有界开集. 设 $u\in W_0^{1,p}(U)$

则对 $\forall\ q\in[1,p^*]$ 有

$\|u\|_{L^q(U)} \le C\|Du\|_{L^p(U)}$

Sobolev.

Hölder

范数.

$p<n$. $p^*=\frac{np}{n-p}$

$p=n$ $\quad u\in W^{1,n} \not\hookrightarrow L^\infty$

$p>n$.

$\log(\log(1+\frac{1}{|x|}))$

Morrey. $\quad n<p<\infty. \quad u\in W^{1,p}.$

Thm. $\quad n<p<\infty$ $\quad \|\cdot\|_A \le C\|\cdot\|_B$

$A \leftarrow B$

$\|u\|_{C^{0,r}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\mathbb{R}^n)}$ $\quad \forall u\in C^1(\mathbb{R}^n)$

$r=1-\frac{n}{p}$

$\|u\|_{W^{k,p}(U)} = \left(\sum_{|\alpha|\le k}\|D^\alpha u\|^p_{L^p(U)}\right)^{\frac{1}{p}}$

$\|u\|_{C^{k,r}(\mathbb{R}^n)} = \sum_{|\alpha|\le k}\|D^\alpha u\|_{C(\bar{U})} + \sum_{|\alpha|=k}[D^\alpha u]_{C^{0,r}(\bar{U})}$

1. $$\boxed{\int_{B(x,r)} |u(y)-u(x)|\,dy \le C\int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{n-1}}\,dy.}$$

$C = C_{n,r}.$

$w\in\partial B(0,1).$ $\quad 0<s<r.$

$$|u(x+sw)-u(x)| = \left|\int_0^s \frac{d}{dt}u(x+tw)\,dt\right|.$$

$$= \left|\int_0^s Du(x+tw)\cdot w\,dt\right|$$

$$\le \int_0^s |Du(x+tw)|\,dt.$$

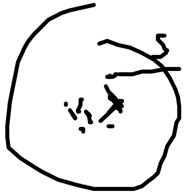
$$\underline{\int_{\partial B(0,1)} |u(x+sw)-u(x)|\,dS \le \int_0^s\int_{\partial B(0,1)} |Du(x+tw)|\,dS\,dt.}$$

$$\text{RHS} = \int_0^s\int_{\partial B(0,1)} |Du(\underline{\underline{x+tw}})|\,dS\,dt$$

$$= \int_0^s \int_{\partial B(x,t)} \frac{|Du(y)|}{t^{n-1}}\,dS\,dt$$

$$= \int_{B(0,s)} \frac{|Du(y)|}{|y-x|^{n-1}}\,dy$$

$$\underline{\le \int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{n-1}}\,dy.}$$



$$\underline{\text{LHS} = \frac{1}{s^{n-1}}\left(\int_{\partial B(x,s)} |u(z)-u(x)|\,dS(z)\right) =}$$

$$\int_{B(x,r)} |u(y)-u(x)|\,dy \le \frac{r^n}{n}\int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{n-1}}\,dy$$

$$|u(x)| \le \frac{1}{m(B(x,1))}\left(\int_{B(x,1)} \overset{u(x)-u(y)}{|u(x)-u(y)|}\,dy + \int_{B(x,1)} |u(y)|\,dy\right)$$

$$\le C\int_{B(x,1)} \frac{|Du(y)|}{|y-x|^{n-1}}\,dy + C\|u\|_{L^p(B(x,1))}$$

$$\le C\left(\int |Du|^p\,dy\right)^{\frac{1}{p}}\left(\int_{B(x,1)} \frac{1}{|y-x|^{\frac{(n-1)p}{p-1}}}\,dy\right)^{\frac{p-1}{p}} + C\|u\|_{L^p(B(x,1))}$$

$$\le C\|u\|_{W^{1,p}(\mathbb{R}^n)} \qquad \frac{(n-1)p}{p-1} < n.$$

$$\|u\|_{C(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

$$\sup_{x\neq y}\frac{|u(x)-u(y)|}{|x-y|^{\gamma}} \qquad W. \qquad \boxed{Cr^{\gamma}}$$

$$|u(x)-u(y)| \le \frac{1}{m(W)}\left(\int_W |u(x)-u(z)|\,dz + \int_W |u(y)-u(z)|\,dz\right)$$

$$\int_W |u(x)-u(z)|\,dz \le \int_{B(x,r)} |u(x)-u(z)|\,dz$$

$$\le \frac{r^n}{n}\int_{B(x,r)} \frac{Du(z)}{|x-z|^{n-1}}\,dz$$

$$\le C\frac{r^n}{n}\left(\int_{B(x,r)} |Du|^p\,dz\right)^{\frac{1}{p}}\left(\int_{B(x,r)} \frac{1}{|x-z|^{\frac{(n-1)p}{p-1}}}\right)^{\frac{p-1}{p}}$$

$$\le C\|Du\|_{L^p}\, r^{n+\gamma}.$$

$$n - \frac{(n-1)p}{p-1} = \gamma.$$

$$|u(x)-u(y)| \le C|x-y|^{\gamma}\|Du\|_{L^p}$$

$$[u]_{C^{0,\gamma}(\mathbb{R}^n)} \le C\|Du\|_{L^p}$$

U是有界，∂U是C^1的.

则对$\forall u \in W^{1,p}(U)$. $\exists$ u的一个版本u^*

$$\|u^*\|_{C^{0,\gamma}(\bar{U})} \le C\|u\|_{W^{1,p}(U)}$$

$\gamma = 1-\frac{n}{p}$ $u^* = u$ a.e.

紧性

$L^{p^*} \leftarrow W^{1,p} \rightarrow$ Hölder space

紧嵌入. X, Y是Banach空间. $X \subset Y$.

(1) $\forall u \in X$. $\|u\|_Y \le C\|u\|_X$.

(2). X中有界序列在Y中有收敛子列

则称X可以紧嵌入Y. $X \subset\subset Y$.

Rellich-Kondrachov定理 U是有界开集.

∂U是C^1的. $1 \le p < n$.

则 $W^{1,p}(U) \subset\subset L^q(U)$ $q \in [1, p^*)$

设 $\{u_m\}$ 是 $W^{1,p}(U)$ 中的有界序列，则

$\exists$ 子列 $\{u_{m_j}\}$ 在 $L^q(U)$ 收敛。

$U=\mathbb{R}^n$. $\quad spt(u_m) \subset V$. $\quad V$ 为有界开集

$\sup_m \|u_m\|_{W^{1,p}(V)} < \infty$

令. $u_m^\varepsilon = \eta_\varepsilon * u_m$, $\quad u_m^\varepsilon \to u_m \quad (\varepsilon \to 0)$

u_m^ε 在 $L^q(U)$ 中关于 m 一致收敛.

$$u_m^\varepsilon(x) - u_m(x) = \frac{1}{\varepsilon^n}\int_{B(x,\varepsilon)} \eta\left(\frac{x-z}{\varepsilon}\right)\left(u_m(z) - u_m(x)\right) dz$$

$$= -\varepsilon \int_{B(0,1)} \eta(y) \int_0^1 Du_m(x-\varepsilon t y)\cdot y \, dt\, dy$$

$$\int_V |u_m^\varepsilon(x) - u_m(x)|\, dx \le \varepsilon \int_V |Du_m(z)|\, dz.$$

$u_m \in W^{1,p}$ 也成立.

$$\|u_m^\varepsilon - u_m\|_{L^1(V)} \le \varepsilon \|Du_m\|_{L^1(V)} \le \varepsilon C \|Du_m\|_{L^p(V)}$$

$1 < q < p$. $\quad \underline{q=1}$

$$\|u_m^\varepsilon - u_m\|_{L^q(V)} \le \|u_m^\varepsilon - u_m\|_{L^1(V)}^{\theta} \|u_m^\varepsilon - u_m\|_{L^{p^*}(V)}^{1-\theta}$$

$$\frac{1}{q} = \theta + \frac{1-\theta}{p^*}$$

GNS.

$$\le \|\cdot\|^{\theta} \cdot \underline{C.\|u_m^\varepsilon - u_m\|_{W^{1,p}(V)}}$$

C.

$$\|u_m^\varepsilon - u_m\|_{L^q(V)} \le C\|u_m^\varepsilon - u_m\|^\theta_{L^1(V)}$$

$1 < q < p^*$

Arzelà-Ascoli 定理

对ε. $\{u_m^\varepsilon\}$ 有一致收敛子列

$$|u_m^\varepsilon(x)| \le \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)\, u_m(y)\, dy$$

$$\le \|\eta_\varepsilon\|_{L^\infty} \|u_m\|_{L^1(V)} \le C\frac{1}{\varepsilon^n}.$$

$$|Du_m^\varepsilon(x)| \le \int_{B(x,\varepsilon)} |D\eta_\varepsilon(x-y)|\, |u_m(y)|\, dy$$

$$\le \|D\eta_\varepsilon\|_{L^\infty} \|u_m\|_{L^1(V)} \le \frac{C}{\varepsilon^{n+1}}$$

u_m^ε 在 L^q 中关于 m 一致收敛

$\forall\, \delta>0$, 取 ε. 使

$$\|u_m^\varepsilon - u_m\|_{L^q(V)} \le \frac{\delta}{2}$$

$$\downarrow$$

$u_{m_j}^\varepsilon$. $\qquad \limsup\limits_{j,k\to\infty} \|u_{m_j}^\varepsilon - u_{m_k}^\varepsilon\|_{L^q(V)} = 0$

$\le \delta$.