COMP40018.2 Logic
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1 Propositional Logic

Definition 1.1 (Propositional Formula)

- *Any propositional atom (p, q, r, etc.) is a formula*
- $\top$ *and* $\bot$ *are formulas*
- *If* ϕ *is a formula, so is* $(\neg\phi)$
- *If* ϕ *and* ψ *are formulas, so are* $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \to \psi)$ *and* $(\phi \leftrightarrow \psi)$

Aside 1.1 (Binding Convention)

$$\textit{(strongest)}\ \neg,\ \wedge,\ \vee,\ \to,\ \leftrightarrow\ \ \textit{(weakest)}$$

Aside 1.2 (Overall Logical Form) *The connective at the root of the **formation tree** is the **principal connective** which gives rise to the overall logical form.*

Aside 1.3 (Subformulas) *The subformulas of a formula are the formulas built in the stages on the way to building the formula as in Definition 1.1. Subformulas are repeated if they appear more than once.*

Definition 1.2 (Logical Forms)

- $\top$, $\bot$, p *are called* ***atomic***
- $\neg\phi$ *is a* ***negated formula*** *and* $\neg\top$, $\neg\bot$ *and* $\neg p$ *are called negated-atomic*
- $\phi \wedge \psi$ *is a* ***conjunction*** *of conjuncts*
- $\phi \vee \psi$ *is a* ***disjunction*** *of disjuncts*
- $\phi \to \psi$ *is an* ***implication*** *from antecedent to consequent*
- $\phi \leftrightarrow \psi$ *is a* ***bidirectional implication***

Definition 1.3 (Literal & Clauses)

- *A formula that is either atomic or negated-atomic is a* ***literal***
- *A clause is a disjunction* ($\vee$) *of one or more literals*

Literals and clauses are repeated if they appear more than once.

Definition 1.4 (Atomic Evaluation Function) *Let A be a set of propositional atoms. An atomic evaluation function* $v{:}A \to \{tt, ff\}$ *assigns truth-values to each atom in A.*

Definition 1.5 (Evaluation Functions) *Let A be a set of propositional atoms and v an atomic evaluation function for A. The evaluation function* $|...|_v$ *assigns the truth value (tt) or (ff) to the formulas as follows*

$\lvert p\rvert_v = tt$	iff	$v(p) = tt$	$\lvert \phi \wedge \psi\rvert_v = tt$	iff	$\lvert\phi\rvert_v = tt$ and $\lvert\psi\rvert_v = tt$
$\lvert\neg\phi\rvert_v = tt$	iff	$\lvert\phi\rvert_v = ff$	$\lvert \phi \vee \psi\rvert_v = tt$	iff	$\lvert\phi\rvert_v = tt$ or $\lvert\psi\rvert_v = tt$
$\lvert\top\rvert_v = tt$			$\lvert \phi \to \psi\rvert_v = tt$	iff	$\lvert\phi\rvert_v = ff$ or $\lvert\psi\rvert_v = tt$
$\lvert\bot\rvert_v = tt$			$\lvert \phi \leftrightarrow \psi\rvert_v = tt$	iff	$\lvert\phi\rvert_v = \lvert\psi\rvert_v$

Definition 1.6 (Functional Completeness) *A set of Boolean connectives C is functionally complete if any connection of any arity can be defined just in terms of connectives in C. Examples include* $\{\uparrow\}$ *and* $\{\neg, \wedge\}$.

Aside 1.4 (Translation)

p but q	$p \wedge q$	p unless q	$p \vee q$	p (strong) unless q	$p \leftrightarrow (\neg q)$
p only if q	$p \to q$	p necessary for q	$q \to p$	p sufficient for q	$p \to q$

Interjections, commands and questions cannot be translated. Time, permission and obligation are also poorly translated.

Definition 1.7 (Valid Argument) *"*$\phi_1, \phi_2, .., \phi_n$ *therefore* ψ *is valid if* ϕ *is true in every situation where* $\phi_1, \phi_2, ..., \phi_n$ *is true. We write* $\phi_1, \phi_2, ..., \phi_n \models \psi$. $\models$ *reads logically entails/implies or semantically entails.*

Definition 1.8 (Valid Formula) *A propositional formula is logically valid if it is true in every situation*

Definition 1.9 (Satisfiable Formula) *A propositional formula is satisfiable if it is true in at least one situation*

Definition 1.10 (Equivalent Formulas) *Two propositional formulas are logically equivalent if they are true in exactly the same situations*

Definition 1.11 (Corresponding Implication Formula) *Let* $\phi_1, \phi_2, ..., \phi_n \models \psi$ *be an argument.* $\phi_1, \phi_2, ..., \phi_n \to \psi$ *is its corresponding implication formula.*

Theorem 1.12 $\phi_1, \phi_2, ..., \phi_n \models \psi$ *is a valid formula iff its corresponding implication formula* $\phi_1, \phi_2, ..., \phi_n \to \psi$ *is a valid formula*

1.1 Truth Tables

1.2 Direct Argument

Valid Formula: Take any situation. To be a valid formula then it must be true in this situation. ... Since in any situation the formula is true, it is a valid formula.
Equivalence: Take any situation. ϕ is true if and only if .... This is so if and only if ψ is true. So ϕ and ψ have the same truth value in this situation. Since the situation was arbituary, they are logically equivalent.
OR So ϕ is true in exactly the same situation as ψ. Hence the two formulas are logically equivalent.

1.3 Equivalences

1.3.1 Equivalences involving $\wedge$

1. $\phi \wedge \psi \equiv \psi \wedge \phi$ (commutativity of $\wedge$)
2. $\phi \wedge \phi \equiv \phi$ (idempotence of $\wedge$)
3. $\phi \wedge \top \equiv \phi$ and $\top \wedge \phi \equiv \phi$
4. $\phi \wedge \bot \equiv \bot$, $\bot \wedge \phi \equiv \bot$, $\phi \wedge \neg\phi \equiv \bot$ and $\neg\phi \wedge \phi \equiv \bot$
5. $(\phi \wedge \psi) \wedge \rho \equiv \phi \wedge (\psi \wedge \rho)$ (associativity of $\wedge$)

1.3.2 Equivalences involving $\vee$

6. $\phi \vee \psi \equiv \psi \vee \phi$ (commutativity of $\vee$)
7. $\phi \vee \phi \equiv \phi$ (idempotence of $\vee$)
8. $\phi \vee \top \equiv \top$, $\top \vee \phi \equiv \top$, $\phi \vee \neg\phi \equiv \top$ and $\neg\phi \vee \phi \equiv \top$
9. $\phi \vee \bot \equiv \phi$ and $\bot \vee \phi \equiv \phi$
10. $(\phi \vee \psi) \vee \rho \equiv \phi \vee (\psi \vee \rho)$ (associativity of $\vee$)

1.3.3 Equivalences involving $\neg$

11. $\neg\top \equiv \bot$
12. $\neg\bot \equiv \top$
13. $\neg\neg\phi \equiv \phi$

1.3.4 Equivalences involving $\to$

14. $\phi \to \phi \equiv \top$
15. $\top \to \phi \equiv \phi$
16. $\phi \to \top \equiv \top$

17. $\bot \to \phi \equiv \top$
18. $\phi \to \bot \equiv \neg\phi$
19. $\phi \to \psi \equiv \neg\phi \vee \psi \equiv \neg(\phi \wedge \neg\psi)$
20. $\neg(\phi \to \psi) \equiv \phi \wedge \neg\psi$

1.3.5 Equivalances involving $\leftrightarrow$

21. $\phi \leftrightarrow \psi \equiv$
    - $(\phi \to \psi) \wedge (\psi \to \phi)$
    - $(\phi \wedge \psi) \vee (\neg\phi \wedge \neg\psi)$
    - $\neg\phi \leftrightarrow \neg\psi$
22. $\neg(\phi \leftrightarrow \psi) \equiv$
    - $\phi \leftrightarrow \neg\psi$
    - $\neg\phi \leftrightarrow \psi$
    - $(\phi \wedge \neg\psi) \vee (\neg\phi \wedge \psi)$

1.3.6 De Morgan's Law

23. $\neg(\phi \wedge \psi) \equiv \neg\phi \vee \neg\psi$
24. $\neg(\phi \vee \psi) \equiv \neg\phi \wedge \neg\psi$

1.3.7 Distributivity of $\wedge, \vee$

25. $\phi \wedge (\psi \vee \rho) \equiv (\phi \wedge \psi) \vee (\phi \wedge \rho)$ and $(\psi \vee \rho) \wedge \phi \equiv (\psi \wedge \phi) \vee (\rho \wedge \psi)$
26. $\phi \vee (\psi \wedge \rho) \equiv (\phi \vee \psi) \wedge (\phi \vee \rho)$ and $(\psi \wedge \rho) \vee \phi \equiv (\psi \vee \phi) \wedge (\rho \vee \psi)$

1.3.8 Absorption

27. $\phi \wedge (\phi \vee \psi) \equiv \phi \vee (\phi \wedge \psi) \equiv \phi$

1.3.9 Normal Forms

Definition 1.13 (DNF) *A formula is in disjunctive normal form if it is a disjunction ($\vee$) of literals and is not further simplifiable without leaving this form*

- *From truth table, take the disjunction of conjunctive formulas where the formula evaluates to tt*
- *A DNF is unsatisfiable iff each of its conjunctions contains some literal and its negation*

Definition 1.14 (CNF) *A formula is in conjunctive normal form if it is a conjunction ($\wedge$) of literals and is not further simplifiable without leaving this form*

- *From truth table, take the conjunction of clauses where the formula evaluates to tt*
- *A CNF is valid iff each of its conjunctions contains some literal and its negation*

1.4 Proof Systems

In the sequent $\phi_1, ..., \phi_n \vdash \psi$, $\phi_1, ..., \phi_n$ are premises, and ψ is the conclusion.

$\wedge$ introduction 1. ϕ 2. ψ 3. $\phi \wedge \psi$ $\wedge I(1,2)$	$\wedge$ elimination 1. $\phi \wedge \psi$ 2. ϕ $\wedge E(1)$ 3. ψ $\wedge E(1)$
$\vee$ introduction 1. ϕ 2. $\phi \vee \psi$ $\vee I(1)$	$\vee$ elimination 1. $\phi \vee \psi$ 2. ϕ ass 4. ψ ass 3. ρ 5. ρ 6. ρ $\vee E(1, 2-3, 4-5)$
$\rightarrow$ introduction 1. ϕ ass 2. ψ 3. $\phi \rightarrow \psi$ $\rightarrow I(1,2)$	$\rightarrow$ elimination 1. $\phi \rightarrow \psi$ 2. ϕ 3. ψ $\rightarrow E(1,2)$
$\neg$ introduction 1. ϕ ass 2. $\perp$ 3. $\neg\phi$ $\neg I(1,2)$	$\neg$ elimination 1. $\neg\phi$ 2. ϕ 3. $\perp$ $\neg E(2,1)$
$\neg\neg$ introduction 1. ϕ 2. $\neg\neg\phi$ $\neg\neg I(1)$	$\neg\neg$ elimination 1. $\neg\neg\phi$ 2. ϕ $E(1)$

$\leftrightarrow$ introduction 1. $\phi \to \psi$ 2. $\psi \to \phi$ 3. $\psi \leftrightarrow \psi$ $\quad \leftrightarrow I(1,2)$	$\leftrightarrow$ elimination 1. $\phi \leftrightarrow \psi$ 2. ϕ 3. ψ $\quad \leftrightarrow E(1,2)$
$\perp$ introduction 1. ϕ 2. $\neg\phi$ 3. $\perp$ $\quad \perp I(1,2)$	$\perp$ elimination 1. $\perp$ 2. ϕ $\quad \perp E(1)$
Modus Tollens 1. $\phi \to \psi$ 2. $\neg\psi$ 3. $\neg\phi$ $\quad MT(1,2)$	Proof by Contradiction 1. $\neg\phi$ $\quad ass$ 2. $\perp$ 3. ϕ $\quad PC(1,2)$

Definition 1.15 (Natural Deduction Proof) *$\phi_1, ..., \phi_n \vdash \psi$ means there is a (natural deduction) proof of ψ starting with $\phi_1, ..., \phi_n$ as premises.*

- *ϕ is a theorem if $\vdash \phi$*

Definition 1.16 (Soundness and Completeness) *A proof system is sound if every theorem is valid and complete if every valid formula is a theorem*

Theorem 1.17 (Soundness of Natural Deduction) *If $\phi_1, ..., \phi_n \vdash \psi$, then $\phi_1, ..., \phi_n \models \psi$*

Theorem 1.18 (Completeness of Natural Deduction) *If $\phi_1, ..., \phi_n \models \psi$, then $\phi_1, ..., \phi_n \vdash \psi$*

Definition 1.19 (Consistency) *A formula ϕ is consistent if $\nvdash \neg\phi$. A collection $\phi_1, ..., \phi_n$ is consistent if $\nvdash \bigwedge_{1 \le i \le n} \phi$*

Theorem 1.20 *A formula is consistent iff it is satisfiable*

Definition 1.21 (Provable Equivalence) *Two formulas ϕ and ψ are probably equivalent if $\phi \vdash \psi$ and $\psi \vdash \phi$, denoted $\phi \dashv\vdash \psi$*

Theorem 1.22 *Two formulas are provably equivalent iff they are semantically equivalent*