COMP40018.2 Logic
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Contents

1 Propositional Logic

Definition 1.1 (Propositional Formula)

- *Any propositional atom (p, q, r, etc.) is a formula*
- $\top$ *and* $\bot$ *are formulas*
- *If* ϕ *is a formula, so is* $(\neg\phi)$
- *If* ϕ *and* ψ *are formulas, so are* $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \to \psi)$ *and* $(\phi \leftrightarrow \psi)$

Aside 1.1 (Binding Convention)

(strongest) $\neg,\ \wedge,\ \vee,\ \to,\ \leftrightarrow$ *(weakest)*

Aside 1.2 (Overall Logical Form) *The connective at the root of the **formation tree** is the **principal connective** which gives rise to the overall logical form.*

Aside 1.3 (Subformulas) *The subformulas of a formula are the formulas built in the stages on the way to building the formula as in Definition 1.1. Subformulas are repeated if they appear more than once.*

Definition 1.2 (Logical Forms)

- $\top$, $\bot$, p *are called **atomic***
- $\neg\phi$ *is a **negated formula** and* $\neg\top$, $\neg\bot$ *and* $\neg p$ *are called negated-atomic*
- $\phi \wedge \psi$ *is a **conjunction** of conjuncts*
- $\phi \vee \psi$ *is a **disjunction** of disjuncts*
- $\phi \to \psi$ *is an **implication** from antecedent to consequent*
- $\phi \leftrightarrow \psi$ *is a **bidirectional implication***

Definition 1.3 (Literal & Clauses)

- *A formula that is either atomic or negated-atomic is a **literal***
- *A clause is a disjunction* $(\vee)$ *of one or more literals*

Literals and clauses are repeated if they appear more than once.

Definition 1.4 (Atomic Evaluation Function) *Let A be a set of propositional atoms. An atomic evaluation function* $v{:}A \to \{tt, ff\}$ *assigns truth-values to each atom in A.*

Definition 1.5 (Evaluation Functions) *Let A be a set of propositional atoms and v an atomic evaluation function for A. The evaluation function* $|...|_v$ *assigns the truth value (tt) or (ff) to the formulas as follows*

$\|p\|_v$=tt	iff	$v(p)$=tt	$\|\phi \wedge \psi\|_v$=tt	iff	$\|\phi\|_v$=tt and $\|\psi\|_v$=tt
$\|\neg\phi\|_v$=tt	iff	$\|\phi\|_v$=ff	$\|\phi \vee \psi\|_v$=tt	iff	$\|\phi\|_v$=tt or $\|\psi\|_v$=tt
$\|\top\|_v$=tt			$\|\phi \to \psi\|_v$=tt	iff	$\|\phi\|_v$=ff or $\|\psi\|_v$=tt
$\|\bot\|_v$=ff			$\|\phi \leftrightarrow \psi\|_v$=tt	iff	$\|\phi\|_v = \|\psi\|_v$

Definition 1.6 (Functional Completeness) *A set of Boolean connectives C is functionally complete if any connection of any arity can be defined just in terms of connectives in C. Examples include $\{\uparrow\}$ and $\{\neg, \wedge\}$.*

Aside 1.4 (Translation)

p but q	$p \wedge q$	*p sufficient for q*	$p \to q$	*p unless q*	$p \leftrightarrow (\neg q)$
p only if q	$p \to q$	*p necessary for q*	$q \to p$	*p unless q*	$p \vee q$

Interjections, commands and questions cannot be translated.
Time, permission and obligation are also poorly translated.

Definition 1.7 (Valid Argument) *"$\phi_1, \phi_2, .., \phi_n$ therefore ψ is valid if ϕ is true in every situation where $\phi_1, \phi_2, ..., \phi_n$ is true. We write $\phi_1, \phi_2, ..., \phi_n \models \psi$. $\models$ reads logically entails/implies or semantically entails.*

Definition 1.8 (Valid Formula) *A propositional formula is logically valid if it is true in every situation*

Definition 1.9 (Satisfiable Formula) *A propositional formula is satisfiable if it is true in at least one situation*

Definition 1.10 (Equivalent Formulas) *Two propositional formulas are logically equivalent if they are true in exactly the same situations*

Definition 1.11 (Corresponding Implication Formula) *Let $\phi_1, \phi_2, ..., \phi_n \models \psi$ be an argument. $\phi_1, \phi_2, ..., \phi_n \to \psi$ is its corresponding implication formula.*

Theorem 1.12 *$\phi_1, \phi_2, ..., \phi_n \models \psi$ is a valid formula iff its corresponding implication formula $\phi_1, \phi_2, ..., \phi_n \to \psi$ is a valid formula*

1.1 Truth Tables

1.2 Direct Argument

Valid Formula: Take any situation. To be a valid formula then it must be true in this situation. ... Since in any situation the formula is true, it is a valid formula.
Equivalence: Take any situation. ϕ is true if and only if .... This is so if and only if ψ is true. So ϕ and ψ have the same truth value in this situation. Since the situation was arbituary, they are logically equivalent.
OR So ϕ is true in exactly the same situation as ψ. Hence the two formulas are logically equivalent.

1.3 Equivalences

1.3.1 Equivalences involving $\wedge$

1. $\phi \wedge \psi \equiv \psi \wedge \phi$ (commutativity of $\wedge$)
2. $\phi \wedge \phi \equiv \phi$ (idempotence of $\wedge$)
3. $\phi \wedge \top \equiv \phi$ and $\top \wedge \phi \equiv \phi$
4. $\phi \wedge \bot \equiv \bot$, $\bot \wedge \phi \equiv \bot$, $\phi \wedge \neg\phi \equiv \bot$ and $\neg\phi \wedge \phi \equiv \bot$
5. $(\phi \wedge \psi) \wedge \rho \equiv \phi \wedge (\psi \wedge \rho)$ (associativity of $\wedge$)

1.3.2 Equivalences involving $\vee$

6. $\phi \vee \psi \equiv \psi \vee \phi$ (commutativity of $\vee$)
7. $\phi \vee \phi \equiv \phi$ (idempotence of $\vee$)
8. $\phi \vee \top \equiv \top$, $\top \vee \phi \equiv \top$, $\phi \vee \neg\phi \equiv \top$ and $\neg\phi \vee \phi \equiv \top$
9. $\phi \vee \bot \equiv \phi$ and $\bot \vee \phi \equiv \phi$
10. $(\phi \vee \psi) \vee \rho \equiv \phi \vee (\psi \vee \rho)$ (associativity of $\vee$)

1.3.3 Equivalences involving $\neg$

11. $\neg\top \equiv \bot$
12. $\neg\bot \equiv \top$
13. $\neg\neg\phi \equiv \phi$

1.3.4 Equivalences involving $\to$

14. $\phi \to \phi \equiv \top$
15. $\top \to \phi \equiv \phi$
16. $\phi \to \top \equiv \top$
17. $\bot \to \phi \equiv \top$
18. $\phi \to \bot \equiv \neg\phi$
19. $\phi \to \psi \equiv \neg\phi \vee \psi \equiv \neg(\phi \wedge \neg\psi)$
20. $\neg(\phi \to \psi) \equiv \phi \wedge \neg\psi$

1.3.5 Equivalances involving $\leftrightarrow$

21. $\phi \leftrightarrow \psi \equiv$
    - $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$
    - $(\phi \wedge \psi) \vee (\neg\phi \wedge \neg\psi)$
    - $\neg\phi \leftrightarrow \neg\psi$
22. $\neg(\phi \leftrightarrow \psi) \equiv$
    - $\phi \leftrightarrow \neg\psi$
    - $\neg\phi \leftrightarrow \psi$
    - $(\phi \wedge \neg\psi) \vee (\neg\phi \wedge \psi)$

1.3.6 De Morgan's Law

23. $\neg(\phi \wedge \psi) \equiv \neg\phi \vee \neg\psi$
24. $\neg(\phi \vee \psi) \equiv \neg\phi \wedge \neg\psi$

1.3.7 Distributivity of $\wedge, \vee$

25. $\phi \wedge (\psi \vee \rho) \equiv (\phi \wedge \psi) \vee (\phi \wedge \rho)$ and $(\psi \vee \rho) \wedge \phi \equiv (\psi \wedge \phi) \vee (\rho \wedge \psi)$
26. $\phi \vee (\psi \wedge \rho) \equiv (\phi \vee \psi) \wedge (\phi \vee \rho)$ and $(\psi \wedge \rho) \vee \phi \equiv (\psi \vee \phi) \wedge (\rho \vee \psi)$

1.3.8 Absorption

27. $\phi \wedge (\phi \vee \psi) \equiv \phi \vee (\phi \wedge \psi) \equiv \phi$

1.3.9 Normal Forms

Definition 1.13 (DNF) *A formula is in disjunctive normal form if it is a disjunction* ($\vee$) *of literals and is not further simplifiable without leaving this form*

- *From truth table, take the disjunction of conjunctive formulas where the formula evaluates to tt*
- *A DNF is unsatisfiable iff each of its conjunctions contains some literal and its negation*

Definition 1.14 (CNF) *A formula is in conjunctive normal form if it is a conjunction* ($\wedge$) *of literals and is not further simplifiable without leaving this form*

- *From truth table, take the conjunction of clauses where the formula evaluates to tt*
- *A CNF is valid iff each of its conjunctions contains some literal and its negation*

1.4 Proof Systems (Natural Deduction)

In the sequent $\phi_1, ..., \phi_n \vdash \psi$, $\phi_1, ..., \phi_n$ are premises, and ψ is the conclusion.

$\wedge$ introduction 1. ϕ 2. ψ 3. $\phi \wedge \psi$ $\quad \wedge I(1,2)$	$\wedge$ elimination 1. $\phi \wedge \psi$ 2. ϕ $\quad \wedge E(1)$ 3. ψ $\quad \wedge E(1)$
$\vee$ introduction 1. ϕ 2. $\phi \vee \psi$ $\quad \vee I(1)$	$\vee$ elimination 1. $\phi \vee \psi$ 2. ϕ $\quad ass$ $\qquad$ 4. ψ $\quad ass$ 3. ρ $\qquad\qquad$ 5. ρ 6. ρ $\quad \vee E(1, 2-3, 4-5)$
$\rightarrow$ introduction 1. ϕ $\quad ass$ 2. ψ 3. $\phi \rightarrow \psi$ $\quad \rightarrow I(1,2)$	$\rightarrow$ elimination 1. $\phi \rightarrow \psi$ 2. ϕ 3. ψ $\quad \rightarrow E(1,2)$
$\neg$ introduction 1. ϕ $\quad ass$ 2. $\bot$ 3. $\neg\phi$ $\quad \neg I(1,2)$	$\neg$ elimination 1. $\neg\phi$ 2. ϕ 3. $\bot$ $\quad \neg E(2,1)$
$\neg\neg$ introduction 1. ϕ 2. $\neg\neg\phi$ $\quad \neg\neg I(1)$	$\neg\neg$ elimination 1. $\neg\neg\phi$ 2. ϕ $\quad E(1)$
$\leftrightarrow$ introduction 1. $\phi \rightarrow \psi$ 2. $\psi \rightarrow \phi$ 3. $\psi \leftrightarrow \psi$ $\quad \leftrightarrow I(1,2)$	$\leftrightarrow$ elimination 1. $\phi \leftrightarrow \psi$ 2. ϕ 3. ψ $\quad \leftrightarrow E(1,2)$
$\bot$ introduction 1. ϕ 2. $\neg\phi$ 3. $\bot$ $\quad \bot I(1,2)$	$\bot$ elimination 1. $\bot$ 2. ϕ $\quad \bot E(1)$

Modus Tollens	Proof by Contradiction
1. $\phi \to \psi$ 2. $\neg\psi$ 3. $\neg\phi$ $\quad MT(1,2)$	$\boxed{\begin{array}{ll} 1.\ \neg\phi & ass \\ 2.\ \bot & \end{array}}$ 3. ϕ $\quad PC(1,2)$

Definition 1.15 (Natural Deduction Proof) *$\phi_1, ..., \phi_n \vdash \psi$ means there is a (natural deduction) proof of ψ starting with $\phi_1, ..., \phi_n$ as premises.*

- *ϕ is a theorem if $\vdash \phi$*

Definition 1.16 (Soundness and Completeness) *A proof system is sound if every theorem is valid and complete if every valid formula is a theorem*

Theorem 1.17 (Soundness of Natural Deduction) *If $\phi_1, ..., \phi_n \vdash \psi$, then $\phi_1, ..., \phi_n \models \psi$*

Theorem 1.18 (Completeness of Natural Deduction) *If $\phi_1, ..., \phi_n \models \psi$, then $\phi_1, ..., \phi_n \vdash \psi$*

Definition 1.19 (Consistency) *A formula ϕ is consistent if $\nvdash \neg\phi$. A collection $\phi_1, ..., \phi_n$ is consistent if $\nvdash \bigwedge_{1 \leq i \leq n} \phi_i$*

Theorem 1.20 *A formula is consistent iff it is satisfiable*

Definition 1.21 (Provable Equivalence) *Two formulas ϕ and ψ are probably equivalent if $\phi \vdash \psi$ and $\psi \vdash \phi$, denoted $\phi \dashv\vdash \psi$*

Theorem 1.22 *Two formulas are provably equivalent iff they are semantically equivalent*

2 Predicate Logic

Definition 2.1 (Signature) *The signature, L, of a predicate logic is a triple $\langle K, F, P\rangle$, where K is a (possibly empty) set of constants, F is a (possibly empty) set of function symbols, and P is a set of predicate symbols. Function and predicate symbols have specific arities.*

Definition 2.2 (Term) *For a signature L,*

1. *Any constant in L is an L-term*
2. *Any variable in L is an L-term*
3. *If f is an n-ary function sumbol in L and $t_1, ..., t_n$ are L-terms, then $f(t_1, ..., t_n)$ is an L-term*

A closed term or ground term is one that doesn't involve a variable

Definition 2.3 (Formula) *For a signature L,*

1. *If R is an n-ary predicate symbol in L, and $t_1, ..., t_n$ are L-terms, then $R(t_1, ..., t_n)$ is an atomic L-formula*
2. *If t, t' are L-terms then $t = t'$ is an atomic L-formula*
3. *$\top$ and $\bot$ are atomic L-formulas*
4. *If ϕ and ψ are L-formulas then so are $(\neg\phi)$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$ and $(\phi \leftrightarrow \psi)$*
5. *If ϕ is an L-formula and x is a variable, then $(\forall x\psi)$ and $(\exists x\psi)$ are L-formulas*

A closed term or ground term is one that doesn't involve a variable.
An atomic formula is a predicate symbol with arguments filled in with terms.
A literal is an atomic formula or its negation.

Definition 2.4 (L-structure) *Let L be a signature $\langle K, F, P\rangle$. An L-structure, M, is a pair $M = \langle D, I\rangle$, where*

- *D is a domain of discourse of M, a non-empty set of objects that M 'knows about'. It's also called universe of M, and sometimes written as dom(M).*
- *I is an interpretation that specifies the meaning of each symbol in L in terms of the objects in D:*
  - *for each constant c in K, $I(c) = c_M \in D$*
  - *for each n-ary function symbol f in F, $I(f) = f_M : D^n \mapsto D$*
  - *for each n-ary predicate symbol P in P, $I(P) = P_M \subseteq D^n$*

Aside 2.1 (Binding Convention) *$\forall$, $\exists$ and $\neg$ same strength*

Definition 2.5 (Free and Bound Variables) *Let ϕ be a formula*

1. *An occurrence of a variable x in ϕ is said to be bound if it occurs in the scope of a quantifier $\forall x$ or $\exists x$*
2. *Variables that are not bound are said to be free*
3. *The free variables of ϕ are those variables with free occurrences in ϕ*

Definition 2.6 (Assignment) *Let M be a structure. An assignment into M is a function that assigns an object in dom(M) to each variable. That is, $h : V \mapsto dom(M)$ is an assignment, where V is the set of variables.*
For an assignment h and a variable x, we write h(x) to denote the object in dom(M) assigned to x by h.

Definition 2.7 (Value of term) *Let L be a signature, $M = \langle D, I \rangle$ an L-structure, and h an assignment into M . Then for any L-term t, the value of t in M under h, denoted as $v^h_M(t)$, is the object in D allocated to t by*

- *M, if t is a constant — that is, $v^h_M(t) = I(t) = t_M$*
- *h, if t is a variable — that is, $v^h_M(t) = h(t)$*
- *M and h, if t is a term $f(t_1, ..., t_n)$ — that is, $v^h_M(t) = f_M(v^h_M(t_1), ..., v^h_M(t_n))$*

Definition 2.8

1. *Let $v^h_M(t_i) = a_i$ be the value of t_i in M under h for each i = 1,...,n. $M, h \models R(t_1, ..., t_n)$ if $(a_1, ..., a_n) \in R_M$*
2. *$M, h \models t = t'$ if $v^h_M(t) = v^h_M(t')$*
3. *$M, h \models \top$ and $M, h \not\models \bot$*

Aside 2.2 (Variable Equivalent) *We say that two functions are x-equivalent, written $g =_x h$, if they differ at most in the assignment of x.*

Definition 2.9

- *$M, h \models \exists x\phi$ if $M, g \models \phi$ for some assignment g into M that is $g =_x h$*
- *$M, h \models \forall x\phi$ if $M, g \models \phi$ for every assignment g into M that is $g =_x h$*

Definition 2.10 *A sentence is a formula with no free variables*

Definition 2.11 (Valid Argument) *Let L be a signature and $A_1, ..., A_n, B$ be L-formulas. An argument '$A_1, ..., A_n$, therefore B' is valid if for any L-structure M and assignment h into M, if $M, h \models A_1$, $M, h \models A_2$, ..., and $M, h \models A_n$, then $M, h \models B$. We write $A_1, ..., A_n \models B$ in this case.*

Definition 2.12 (Valid Formula) *An L-formula A is valid if for every L-structure M and assignment h into M, we have $M, h \models A$. We write '$\models A'$ (as above) if A is valid.*

Definition 2.13 (Satisfiable Formula) *An L-formula A is satisfiable if for some L-structure M and assignment h into M, we have $M, h \models A$.*

Definition 2.14 (Equivalent Formula) *L-formulas A,B are logically equivalent if for every L-structure M and assignment h into M, we have $M, h \models A$ if and only if $M, h \models B$.*

2.1 Direct Reasoning

To show $\phi \to \forall x \psi$. Take any M such that ϕ. Then, we take arbituary a in dom(M) and show $M \models \psi(a)$

2.2 Equivalences

2.2.1 Equivalences involving predicate logic

1. $\forall x \forall y A \equiv \forall y \forall x A$
2. $\exists x \exists y A \equiv \exists y \exists x A$
3. $\neg \forall x A \equiv \exists x \neg A$
4. $\neg \exists x A \equiv \forall x \neg A$
5. $\forall x(A \wedge B) \equiv \forall x A \wedge \forall x B$
6. $\exists x(A \vee B) \equiv \exists x A \vee \exists x B$

2.2.2 Equivalences involving variables not free in A

7. $\forall x A \equiv \exists x A \equiv A$
8. $\exists x(A \wedge B) \equiv A \wedge \exists x B$ and $\forall x(A \vee B) \equiv A \vee \forall x B$
9. $\forall x(A \to B) \equiv A \to \forall x B$ and $\exists x(A \to B) \equiv A \to \exists x B$
10. $\forall x(B \to A) \equiv \exists x B \to A$ and $\exists x(B \to A) \equiv \forall x B \to A$

2.2.3 Rename bound variables

11. $A \equiv B$ if B is got from A by replacing all bound occurances of any variable x by a variable y that does not occur in A

2.2.4 Equivalances involving equality

12. $t = t$
13. $t = u \equiv u = t$
14. (Leibniz Principle) If A is a formula, y not occur in A, B is got from A by replacing one or more free occurances of x by y, then $x = y \to (A \leftrightarrow B)$

2.2.5 Non-equivalances

15. $\forall x(A \to B) \models \forall xA \to \forall xB$
16. $\exists x(A \wedge B) \models \exists xA \wedge \exists xB$
17. $\forall xA \vee \forall xB \models \forall x(A \vee B)$

2.3 Natural Deduction

For a formula A, a variable x, and a term t, we write A(t/x) for the formula got from A by replacing all free occurrences of x in A by t.

$\exists$ introduction 1. $A(t/x)$ 2. $\exists xA$ $\exists I(1)$	$\exists$ elimination (Skolem's constant) 1. $\exists xA$ 2. $A(c/x)$ ass 3. B 4. B $\exists E(1, 2, 3)$
$\forall$ introduction 1. c $\forall I$ $const$ 2. $A(c/x)$ 3. $\forall xA$ $\forall I(1, 2)$	$\forall$ elimination (for a closed term t) 1. $\forall xA$ 2. $A(t/x)$ $\forall E(1)$
$=$ reflexivity 1. $t = t$ refl	$=$ substitution 1. $A(t/x)$ 2. $t = u$ 3. $A(u/x)$ $= sub(1, 2)$
$\forall \to$ elimination 1. $\forall x(A(x) \to B(x))$ 2. $A(t/x)$ 3. $B(t/x)$ $\forall \to E(2, 1)$	$=$ symmetry 1. $c = d$ 2. $d = c$ $= sym(1)$

Soundness and completeness same as defined in propositional logic