Thought Questions:
The Geometric/Abstract Definition of Tangent Space

Jiaqi Tang

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Abstract

These are important exercises for understanding differential and tangent spaces.

Given an open set $\Omega \subseteq \mathbb{R}^n$ and $p \in \Omega$, let $\mathcal{C}(p)$ denote the set of all C^1 curves through p. By definition:

$$\mathcal{C}(p) := \left\{ \gamma \colon I \to \Omega \middle| I \subseteq \mathbb{R} \text{ is an open interval containing } 0,\ \gamma(0) = p,\ \gamma \in C^1 \right\},$$

we define an equivalence relation on $\mathcal{C}(p)$ (for $\gamma_1, \gamma_2 \in \mathcal{C}(p)$):

$$\gamma_1 \sim \gamma_2 \iff \gamma_1'(0) = \gamma_2'(0).$$

Consider the quotient set $\mathcal{C}(p)/\sim$ under this relation:

$$\mathcal{C}(p)/\sim = \{[\gamma] \mid \gamma \in \mathcal{C}(p)\},$$

where $[\gamma] = [\gamma']$ if $\gamma \sim \gamma'$. Geometrically, curves with identical tangent vectors at p are identified, justifying the name "tangent space" for this set.

Problem 1. Prove the map

$$\iota_p \colon \mathcal{C}(p)/\sim \to \mathbb{R}^n, \quad [\gamma] \mapsto \gamma'(0)$$

is well-defined and a bijection.

Solution. If $[\gamma_1] = [\gamma_2]$, then $\gamma_1 \sim \gamma_2$, so $\gamma_1'(0) = \gamma_2'(0)$. Thus, $\iota_p([\gamma_1]) = \gamma_1'(0) = \gamma_2'(0) = \iota_p([\gamma_2])$, so ι_p is well-defined.

We prove it is injective and surjective:

- **Injective.** If $\iota_p([\gamma_1]) = \iota_p([\gamma_2])$, then $\gamma_1'(0) = \gamma_2'(0)$, so $\gamma_1 \sim \gamma_2$, hence $[\gamma_1] = [\gamma_2]$.
- **Surjective.** For any $x \in \mathbb{R}^n$, we can easily get $\iota_p^{-1}(x) = [\gamma(t) = p + xt + o(t)]$

Problem 2. Prove $\mathcal{C}(p)/\sim$ can be given a $\mathbb{R}$-linear structure making ι_p a linear isomorphism. Denote this linear space (originally a set of curve equivalence classes) by $\mathrm{T}_p\,\Omega$

Solution. First, we define addition: For $[\gamma_1], [\gamma_2] \in \mathcal{C}(p)/\sim$, let $[\gamma_1] + [\gamma_2] := [\gamma_1 + \gamma_2]$.

Then we define scalar multiplication: For $c \in \mathbb{R}, [\gamma] \in \mathcal{C}(p)/\sim$, let $c \cdot [\gamma] := [c \cdot \gamma]$.

Now $\mathrm{T}_p\,\Omega = \mathcal{C}(p)/\sim$ is an $\mathbb{R}$-linear space. We have already know that the map ι_p is a bijection, hence we just need to show that ι_p is also a linear map:

- $\iota_p([\gamma_1]+[\gamma_2]) = \iota_p([\gamma_1+\gamma_2]) = (\gamma_1+\gamma_2)'(0) = (\gamma_1)'(0)+(\gamma_2)'(0) = \iota_p([\gamma_1])+\iota_p([\gamma_2])$.
- $\iota_p(c\cdot[\gamma]) = \iota_p([c\cdot\gamma]) = (c\cdot\gamma)'(0) = c\cdot\gamma'(0) = c\cdot\iota_p([\gamma])$.

Therefore, ι_p is a linear isomorphism.

Problem 3. Let $\Omega' = \mathbb{R}^m$ and $f\colon \Omega \to \Omega'$ be C^1 (i.e., its differential is continuous), with $p' = f(p)$. Prove the map

$$f_{\sharp,p}\colon \mathcal{C}(p) \to \mathcal{C}(p'), \quad \gamma \mapsto f\circ\gamma$$

is well-defined. Geometrically, composing with f pushes forward curves through p in Ω to curves through p' in Ω'.

Solution. $(f\circ\gamma)(0) = f(p) = p'$, $f\in C^1$, $\gamma\in C^1$, hence $f\circ\gamma\in\mathcal{C}(p')$.

For any $\gamma_1=\gamma_2\in\mathcal{C}(p)$, we have $f_{\sharp,p}(\gamma_1) = f\circ\gamma_1 = f\circ\gamma_2 = f_{\sharp,p}(\gamma_2)$, so $f_{\sharp,p}$ is well-defined.

Problem 4. Prove the map

$$f_{*p}\colon\ \mathrm{T}_p\,\Omega \to \mathrm{T}_{p'}\,\Omega', \quad [\gamma]\mapsto[f_{\sharp,p}(\gamma)]$$

is a well-defined linear map (called the **tangent map**).

Solution. For any $[\gamma_1]=[\gamma_2]$, then $\gamma_1\sim\gamma_2$, so $\gamma_1'(0)=\gamma_2'(0)$. Thus,

$$f_{*p}([\gamma_1]) = [f\circ\gamma_1] = [f\circ\gamma_2] = f_{*p}([\gamma_2])$$

The second equality is holds because we have $(f\circ\gamma_1)'(0) = \mathrm{d}\,f(p)(\gamma_1'(0)) = \mathrm{d}\,f(p)(\gamma_2'(0)) = (f\circ\gamma_2)'(0)$ for any C^1 map $f\colon\Omega\to\Omega'$. So f_{*p} is well-defined.

Now we prove its linearity:

- $f_{*p}([\gamma_1]+[\gamma_2]) = [f\circ(\gamma_1+\gamma_2)] = [f\circ\gamma_1+f\circ\gamma_2] = [f\circ\gamma_1]+[f\circ\gamma_2] = f_{*p}([\gamma_1])+f_{*p}([\gamma_2])$.
- $f_{*p}(c\cdot[\gamma]) = f_{*p}([c\cdot\gamma]) = [f\circ(c\cdot\gamma)] = [c\cdot(f\circ\gamma)] = c\cdot[f\circ\gamma] = c\cdot f_{*p}([\gamma])$.

So f_{*p} is a well-defined linear map.

Problem 5. Consider the curve on Ω:

$$l_k\colon(-1,1)\to\Omega,\quad t\mapsto p+\underbrace{(0,0,\cdots,0,t,0,\cdots,0)}_{\text{only the } k\text{-th component is non-zero}}.$$

Prove $\iota_p([l_k]) = \dfrac{\partial}{\partial x_k}$, where $\{x_1,\ldots,x_n\}$ is the coordinate system on $\mathbb{R}^n$.

Solution. Notice that $\iota_p([l_k]) = l_k'(0) = l_k'(t)|_{t=0} = \left.\dfrac{\partial}{\partial x_k}\right|_{t=0} = \dfrac{\partial}{\partial x_k}$.

Problem 6. Consider the curve on Ω':

$$l'_{k'}\colon(-1,1)\to\Omega',\quad t\mapsto p+\underbrace{(0,0,\cdots,0,t,0,\cdots,0)}_{\text{only the } k\text{-th component is non-zero}}.$$

Express $f_{*p}([l_k])$ using the partial derivatives of f and $[l'_{k'}]$.

Solution. $f_{*p}([l_k]) = [f\circ l_k] = \sum\limits_{k'=1}^{m}\left(\nabla_{\frac{\partial}{\partial x_k}} f_{k'}\right)(p)[l'_{k'}]$

The second equality is holds because

$$(f \circ l_k)'(0) = \mathrm{d}\, f(p) \left(\frac{\partial}{\partial x_k} \right) = \left(\nabla_{\frac{\partial}{\partial x_k}} f \right)(p) = \sum_{k'=1}^{m} \left(\nabla_{\frac{\partial}{\partial x_k}} f_{k'} \right)(p) \left(\frac{\partial}{\partial x_{k'}} \right).$$

Problem 7. Prove the commutative diagram:

$$\begin{tikzcd} \mathrm{T}_p\, \Omega \arrow[r, "\iota_p", "\cong"'] \arrow[d, "f_{*p}"] & \mathbb{R}^n \arrow[d, "\mathrm{d}\, f(p)"] \\ \mathrm{T}_{p'}\, \Omega' \arrow[r, "\iota_{p'}", "\cong"'] & \mathbb{R}^m \end{tikzcd}$$

For any $[\gamma] \in \mathrm{T}_p\, \Omega$, we have

$$f_{*p}([\gamma]) = \iota_{p'}^{-1}\Big(\mathrm{d}\, f(p)\big(\iota_p([\gamma])\big)\Big).$$

In this sense, f_{*p} coincides with $\mathrm{d}\, f(p)$.

Solution. For any $[\gamma] \in \mathrm{T}_p\, \Omega$, we have

$$\iota_{p'}(f_{*p}([\gamma])) = \iota_{p'}([f \circ \gamma]) = (f \circ \gamma)'(0) = \mathrm{d}\, f(p)(\gamma'(0)) = \mathrm{d}\, f(p)(\iota_p([\gamma])),$$

which shows that $\iota_{p'} \circ f_{*p} = \mathrm{d}\, f(p) \circ \iota_p$.

Problem 8. For non-zero $[\gamma_1], [\gamma_2] \in \mathrm{T}_p\, \Omega$, define their angle by:

$$\cos\big(\angle([\gamma_1], [\gamma_2])\big) = \frac{\gamma_1'(0) \cdot \gamma_2'(0)}{\|\gamma_1'(0)\|\, \|\gamma_2'(0)\|},$$

with the angle in $[0, \pi)$. Consider the inversion map:

$$f \colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}, \quad x \mapsto \frac{x}{\|x\|^2}.$$

Prove f is C^1 and for all $p \in \mathbb{R}^n \setminus \{0\}$, $f_{*p} \colon \mathrm{T}_p(\mathbb{R}^n \setminus \{0\}) \to \mathrm{T}_{f(p)}(\mathbb{R}^n \setminus \{0\})$ preserves angles.

Solution. $f = (f_1, f_2, \cdots, f_n)$. For all $1 \leqslant i, k \leqslant n$, we have

$$\nabla_{\frac{\partial}{\partial x_k}} f_i = \big(\delta_i^k \|x\|^2 - 2x_i x_k\big) / \|x\|^4 \in C(\mathbb{R}^n \setminus \{0\}).$$

So $f \in C^1$. For non-zero $[\gamma_1], [\gamma_2] \in \mathrm{T}_p(\mathbb{R}^n \setminus \{0\})$, we denote $\gamma_1'(0), \gamma_2'(0)$ by v_1, v_2 respectively.

$$\cos\big(\angle(f_{*p}([\gamma_1]), f_{*p}([\gamma_2]))\big) = \frac{(f \circ \gamma_1)'(0) \cdot (f \circ \gamma_2)'(0)}{\|(f \circ \gamma_1)'(0)\| \|(f \circ \gamma_2)'(0)\|} = \frac{\mathrm{d}\, f(p)(v_1) \cdot \mathrm{d}\, f(p)(v_2)}{\|\,\mathrm{d}\, f(p)(v_1)\| \|\,\mathrm{d}\, f(p)(v_2)\|}. \tag{1}$$

Denote $\mathrm{d}\, f(p)(v_1), \mathrm{d}\, f(p)(v_2)$ by w_1, w_2 respectively.

Notice that $w_1 \cdot w_2 = \det(\mathbf{J}^\top \mathbf{J}) v_1 \cdot v_2$, $\|w_i\| = \sqrt{w_i \cdot w_i} = \sqrt{\det(\mathbf{J}^\top \mathbf{J})} \|v_i\|, i = 1, 2$. Hence,

$$(1) = \frac{w_1 \cdot w_2}{\|w_1\| \|w_2\|} = \frac{\det(\mathbf{J}^\top \mathbf{J}) v_1 \cdot v_2}{\sqrt{\det(\mathbf{J}^\top \mathbf{J})} \|v_1\| \sqrt{\det(\mathbf{J}^\top \mathbf{J})} \|v_2\|} = \frac{v_1 \cdot v_2}{\|v_1\| \|v_2\|} = \cos\big(\angle([\gamma_1], [\gamma_2])\big).$$

Pure mathematics is, in its way, the poetry of logical ideas.

– Albert Einstein
