MA104 BURTON W1ASSIGNMENT

FRACTIONS: ORDER OF OPERATIONS

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Abstract

This paper will include methods for solving expressions over the Field $\mathbb{Q}$, with attention payed to the order in which certain terms are evaluated, in particular the order of operations or the quintessential PEMDAS acronym.

Keywords: Fields, Algebra, Groups, Order of Operations.

A Note To My Professor

This is my first college course. I am so excited to have the chance at this experience. As my first college paper ever, I wanted to present a paper that I thought would be a proper paper, even if on such a simple subject. This is not meant to be the proverbial apple left by me on the professor's desk, only a manifestation of my excitement of actually being in college. I am the first person in my family to do so, so this really means something. As you will see, this is not my first shot at math since High School. I actually study math as a hobby. I know you have a lot of papers to grade and check so I will not dally any longer. I don't expect you to read this whole paper, but I hope it is satisfactory. Please be strict with me, with out which I cannot become the best I can be.

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FRACTIONS: ORDER OF OPERATIONS

Definition: 1.0.0

Given a set A, and a function f, we define a binary operation as the arrow $f\colon A \times A \to A$ such that we have a mapping denoted by $(a, b) \mapsto c$. If for all $a, b \in A |\, f(a, b) \in A$ we say that A is closed over f.

Definition: 1.0.1

Given the set of rational numbers $\mathbb{Q}$, and two binary operations + and * called addition and multiplication, we define a Field $(\mathbb{Q}, +, *)$ as a algebraic structure that satisfies the following axioms.

We will define the multiplicative and additive axioms in sequence instead of separately, and when it does not make sense to do so, it will be clear from the context what axiom is being defined. For all a, b, c of $\mathbb{Q}$.

- **A1** – Associativity

$$(a + b) + c = a + (b + c)$$

$$(ab)c = a(bc)$$

- **A2** – Commutativity

$$a + b = b + a$$

$$ab = ba$$

- **A3** – Identity

$$\exists 0\ \forall a \in \mathbb{Q} :\ a + 0 = 0 + a = a$$

$$\exists 1\ \forall a \in \mathbb{Q} :\ a1 = 1a = a$$

- A4 – Inverses

$$\forall a \in \mathbb{Q} \exists! -a : a + (-a) = (-a) + a = 0$$

$$\forall a \in \mathbb{Q}^{+} \; \exists! \, a^{-1} \neq 0 : aa^{-1} = a^{-1}a = 1$$

- A5 – Distributivity

$$a(b + c) = (b + c)a = ab + ac = ba + ca$$

These are not typical field axioms, but they are important properties of any Field. Quickly, for any $a, b, c, d \in \mathbb{Q}$, so for some ab^{-1} and cd^{-1} such that $d^{-1} \neq b^{-1}$, we say $ab^{-1} + cd^{-1} = (ab^{-1}d + cd^{-1}b)(b^{-1}d^{-1})^{-1}$

Problem Set 1

$$\frac{5(-6) \div 2 + (4-2)^3}{2(-1)}$$

Algebraic Proof:

Given the problem set…

$$\frac{5(-6) \div 2 + (4-2)^3}{2(-1)} = 5(-6)(2 + (4-2)^3)^{-1}\big(2(-1)\big)^{-1} =$$

The easiest way to start is to start with the inner most parenthesis. The three outer products are independent of each other and can be done in any order by A1 and A2. However, the second term contains an inner scope, and math is eagerly evaluated. So Lets handle the second term first.

Since $\mathbb{Q}$ is closed over addition and multiplication

$$5(-6)(2 + (4-2)^3)^{-1}\big(2(-1)\big)^{-1} =$$

$$5(-6)(2+(2)^3)^{(-1)}\big(2(-1)\big)^{(-1)} =$$

$$5(-6)(2 \;+\; 2^3)^{-1}\big(2(-1)\big)^{-1} \;=$$

$$5(-6)(2+8)^{-1}\big(2(-1)\big)^{-1}=$$

$$5(-6)(10)^{-1}\big(2(-1)\big)^{-1}=$$

Since $a(-b) = \,-ab$

$$(-30)(10)^{-1}\big(2(-1)\big)^{-1}=$$

$$(-30)(10)^{-1}(-2)^{-1}=$$

Since $a^{-1}b^{-1} = (ab)^{-1}$

$$(-30)(10)^{-1}(-2)^{-1}=$$

$$(-30)(10 \times -2)^{-1}= (-30)(-20)^{-1}=$$

Since $ab^{-1} = \frac{a}{b}$ and $-a(-b^{-1}) = \frac{-a}{-b} = ab^{-1}$

$$(-30)(-20)^{-1} = \frac{-30}{-20} = \frac{3\times 10}{2\times 10} = \frac{3\times \cancel{10}}{2\times \cancel{10}} = \frac{3}{2}$$

□

We introduce some notation in this problem set that we have not taken the time to explain, and that is the absolute value of a number e.g. $|a-b|$. We can define the absolute value of a number in two ways: One as a metric or the distance of a value from the origin of a frame of reference. Two as a subset of the Field, in particular as an interval $[0,a] = |a|$. Because we are assuming the closure of the Field $\mathbb{Q}$, and the term $|a|$ is merely the magnitude of the value of the element of the Field, then defining the axioms of this metric would only add complexity to the topic. We will skip this step. I will say that since $|a|$ is the magnitude of a, then it would follow that $|a|$ is always positive and $|a\ | = \ 0$ if and only if $a\ =\ 0$. Hopefully this quick examination of this topic of metrics and magnitude is enough to clear any ambiguity in the problem set.

Problem Set 2

$$\frac{6-3(-5-3^2)}{-2|2-5|}$$

Algebraic Proof:

Given the problem set, the next steps follow from the fact that $\mathbb{Q}$ is closed over addition and multiplication.

$$\left(6-3(-5-3^2)\right)(2|2-5|)^{-1} =$$

Since we have not defined the binary operator of subtraction, I want to quickly disambiguate the notion the problem might introduce for some readers. Given a term $-a\ -\ b^n$ is the same as writing $-a+\left(-(b^n)\right)$. However, it is not the same as the expression $-a\ +$ $(-b)^n$ where $n \in \mathbb{Z}$. The former expression's second term would result in a positive number whereas the latter expression's second term would result in a negative number. So let us rewrite the problem set so as to clear this confusion.

$$\left(6-3(-5+(-(2^3)))\right)(2|2-5|)^{-1} =$$

$$\left(6-3(-5-9)\right)(2|2-5|)^{-1} =$$

Since $-a(-b)=ab$ and $-a-b=-a+(-b)$

$$\left(6+\left((-3)(-5)+(-3)(-9)\right)\right)(2|2-5|)^{-1} =$$

$$\left(6+(15+27)\right)(2|2-5|)^{-1} =$$

$$(6 + 42)(2|2 - 5|)^{-1} =$$

$$(48)(2|2 - 5|)^{-1} =$$

$$(48)(2|-3|)^{-1} =$$

$$(48)(2 * 3)^{-1} = \frac{48}{6} = 8$$

□

That concludes this topic. Hope you enjoyed this paper, because I enjoyed writing it. Hopefully it is clear that the order of operations, or PEMDAS is a vital system in reducing an expression to a simpler form. That algebra and math is amazing, that the rules they taught us in middle school and high school were not out of spite or to make the work harder, every rule serves a purpose and is a direct result of the underlying structure of the numbers we use every day. Thank you.

References

Dummit, David S 1999 Abstract Algebra Englewood Cliffs : Prentice Hall

Footnotes