ETH
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Probabilistic Foundations of Artificial Intelligence

Probabilistic Planning

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Markov Decision Processes

- An MDP is specified by
  - A set of states $X = \{1, \ldots, n\}$ …
  - A set of actions $A = \{1, \ldots, m\}$
  - Transition probabilities

    $P(x' \mid x, a) = \text{Prob(Next state} = x' \mid \text{Action } a \text{ in state } x)$
  - A reward function $r(x, a)$

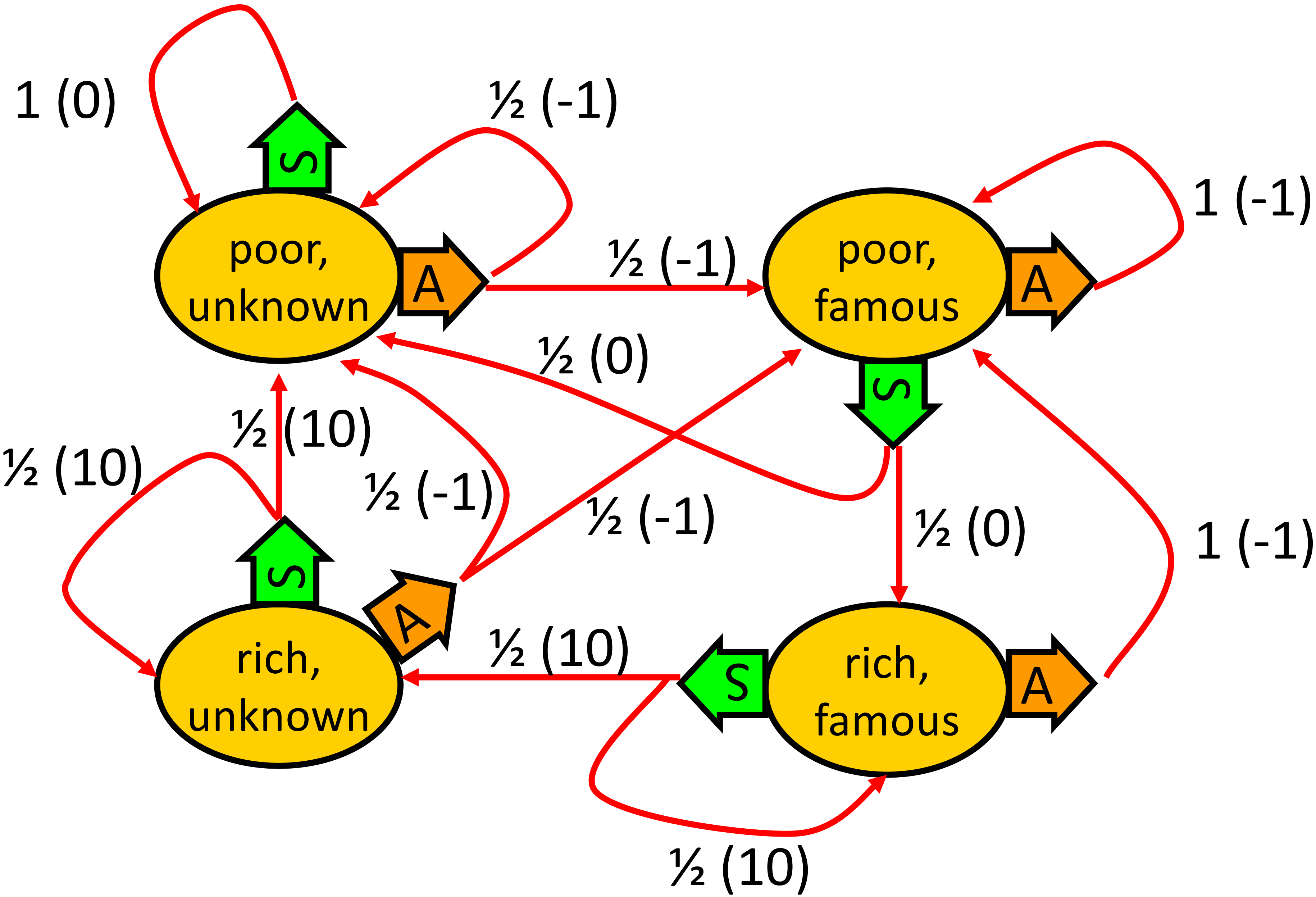
    Reward can be random with mean $r(x, a)$;

    Reward may depend on x only or (x, a, x') as well.
- For now assume r and P are known!
- Want to choose actions to maximize reward

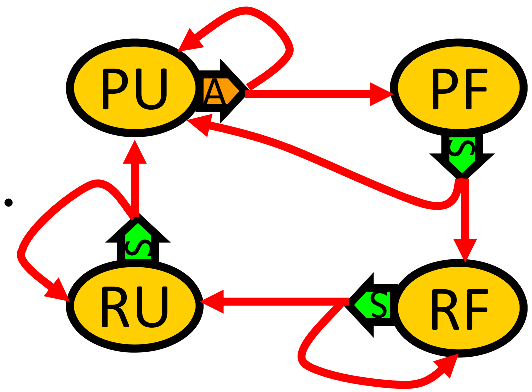
Applications of MDPs

- Robot action planning
- Elevator scheduling
- Manufactoring processes
- Network switching and routing
- AI in computer games
- Becoming rich and famous
- …

Becoming rich and famous



Planning in MDPs



- Deterministic policy $\pi: X \rightarrow A$
- Induces a **Markov chain**: $X_0, X_1, \ldots, X_t, \ldots$ with transition probabilities

$$P(X_{t+1}=x' \mid X_t=x) = P(x' \mid x, \pi(x))$$

- Expected value $J(\pi) = E[\ r(X_0,\pi(X_0)) + \gamma\, r(X_1,\pi(X_1)) + \gamma^2 r(X_2,\pi(X_2)) + \ldots\ ]$

Computing the value of a policy

For a fixed policy define value function

$$V^\pi(x) = J(\pi \mid X_0 = x) = \mathbb{E}\Big[\sum_{t=0}^{\infty} \gamma^t r(X_t, \pi(X_t)) \mid X_0 = x\Big]$$

Recursion:

$$V^\pi(x) = \mathbb{E}\Big[\gamma^0 r(X_0, \pi(X_0)) + \sum_{t=1}^{\infty} \gamma^t r(X_t, \pi(X_t)) \Big| X_0 = x\Big]$$

$$= r(x, \pi(x)) + \mathbb{E}\Big[\sum_{t=1}^{\infty} \gamma^t r(X_t, \pi(X_t)) \mid X_0 = x\Big]$$

$$= r(x, \pi(x)) + \mathbb{E}\Big[\sum_{t=0}^{\infty} \gamma^{t+1} r(X_{t+1}, \pi(X_{t+1})) \mid X_0 = x\Big]$$

$$= r(x, \pi(x)) + \gamma \sum_{x'} P(X_1 = x' \mid X_0 = x)\, \mathbb{E}\Big[\sum_{t=0}^{\infty} \gamma^t r(X_{t+1}, \pi(X_{t+1})) \mid X_1 = x'\Big]$$

$$\overset{\text{stat.}}{=} r(x, \pi(x)) + \gamma \sum_{x'} P(x' \mid x)\, \underbrace{\mathbb{E}\Big[\sum_{t=0}^{\infty} \gamma^t r(X_t, \pi(X_t)) \mid X_0 = x'\Big]}_{V^\pi(x')}$$

Solving for the value of a policy

$$V^\pi(x) = r(x,\pi(x)) + \gamma \sum_{x'} P(x' \mid x,\pi(x)) V^\pi(x')$$

$$V^\pi = \begin{bmatrix} V^\pi(1) \\ \vdots \\ V^\pi(n) \end{bmatrix}, \quad T^\pi = \begin{pmatrix} P(1 \mid 1,\pi(1)) & \cdots & P(n \mid 1,\pi(1)) \\ \vdots & & \\ P(1 \mid n,\pi(n)) & \cdots & P(n \mid n,\pi(n)) \end{pmatrix}, \quad r^\pi = \begin{bmatrix} r(1,\pi(1)) \\ \vdots \\ r(n,\pi(n)) \end{bmatrix}$$

$$V^\pi = r^\pi + \gamma T^\pi V^\pi \Rightarrow (I - \gamma T^\pi) V^\pi = r^\pi$$

$$\Rightarrow \boxed{V^\pi = (I - \gamma T^\pi)^{-1} r^\pi}$$

➔ **Can compute V^π exactly by solving linear system! ☺**

A simple algorithm

- For every policy π compute $J(\pi)$
- Pick $\pi^* = \text{argmax}\, J(\pi)$

Is this a good idea?

$\hookrightarrow$ m^n policies

Value functions and policies

Every value function induces a policy

Value function V^π

$$V^\pi(x) = r(x,\pi(x)) + \gamma \sum_{x'} P(x' \mid x,\pi(x))\, V^\pi(x')$$

Greedy policy w.r.t. V

$$\pi_V(x) = \arg\max_a r(x,a) + \gamma \sum_{x'} P(x' \mid x,a)\, V(x')$$

Every policy induces a value function

Theorem (Bellman):
Policy optimal ⇔ greedy w.r.t. its induced value function!

$$V^*(x) = \max_a \left[r(x,a) + \gamma \sum_{x'} P(x' \mid x, a)\, V^*(x') \right]$$

Policy iteration

- Start with an arbitrary (e.g., random) policy π
- Until converged do:

  Compute value function $V^{\pi}(x)$

  Compute greedy policy π_G w.r.t. V^{π}

  Set $\pi \leftarrow \pi_G$

- Guaranteed to
  - Monotonically improve $\forall x: V^{\pi_{t+1}}(x) \geq V^{\pi_t}(x)$
  - Converge to an optimal policy π^* in $O^*(n^2\, m\, /\, (1\text{-}\gamma))$ iterations! [Ye '10]

Alternative approach

- Recall (Bellman): For the optimal policy π^* it holds

$$\boxed{V^*(x) = \max_a r(x,a) + \gamma \sum_{x'} P(x' \mid x, a)\, V^*(x')}$$

- Compute V^* using fixed point / dynamic programming:

$V_t(x)$ = Max. expected reward when starting in state x and world ends in t time steps

$V_0(x) = \max_a r(x,a)$

$V_1(x) = \max_a r(x,a) + \gamma \sum_{x'} P(x' \mid x,a)\, V_0(x')$

$V_{t+1}(x) =$ ______ " ______ $V_t(x')$

Value iteration

- Initialize $V_0(x) = \max_a r(x, a)$
- For $t = 1$ to ∞

  For each x, a, let

  $$Q_t(x,a) = r(x,a) + \gamma \sum_{x'} P(x' \mid x,a)\, V_{t-1}(x')$$

  For each x let

  $$V_t(x) = \max_a Q_t(x,a)$$

  Break if $\|V_t - V_{t-1}\|_\infty = \max_x |V_t(x) - V_{t-1}(x)| \le \varepsilon'$

- Then choose greedy policy w.r.t. V_t

- **Guaranteed to converge to ε-optimal policy!**

Tradeoffs: Value vs Policy Iteration

- Policy iteration
  - Finds exact solution in polynomial # iterations!
  - Every iteration requires computing a value function
  - Complexity per iteration:
- Value iteration
  - Finds ε-optimal solution in polynomial # iterations
  - Complexity per iteration:
- In practice, which works better depends on application
- Can combine ideas of both algorithms

Recap: Ways for solving MDPs

- Policy iteration:
  - Start with random policy π
  - Compute exact value function V^{π} (matrix inversion)
  - Select greedy policy w.r.t. V^{π} and iterate
- Value iteration
  - Solve Bellman equation using dynamic programming
    $V_t(x) = \max_a r(x, a) + \gamma \sum_{x'} P(x' \mid x, a)\, V_{t-1}(x)$
- Linear programming