PS3

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1 Proofs by induction

a) Proof: We want to prove that the closed form of $a_n = x^{2^n}$ where x $\in \mathbb{R}$, n $\in \mathbb{N}$
Let x $\in \mathbb{R}$, define P(n) as $a_n = x^{2^n}$ where n $\in \mathbb{N}$

Base Case: Let x $\in \mathbb{R}$, let n = 0, we want to prove that P(0) is True, that is $a_0 = x^{2^0} = x$
By recursive sequence defined in question, $a_0 = x$
let n = 1, we want to prove that P(1) is True, that is $a_1 = x^{2^1} = x^2$
By definition, since $n \neq 0 \Rightarrow a_1 = x \cdot \prod_{i=0}^{1-1} a_i = x \cdot a_0 = x \cdot x = x^2$
Thus, base case is proved.

Inductive Step: Let x $\in \mathbb{R}$, let k $\in \mathbb{N}$ and let $k \geq 1$, assume P(k) is True, $a_k = x^{2^k}$,
we will prove P(k+1) is True, that is $a_{k+1} = x^{2^{k+1}}$
Since $k > 0$, so $k + 1 > 0$, by definition, $a_{k+1} = x \cdot \prod_{i=0}^{(k+1)-1} a_i = x \cdot \prod_{i=0}^{k} a_i = (x \cdot \prod_{i=0}^{k-1} a_i) \cdot a_k = a_k \cdot a_k = (a_k)^2$
Since $k > 0$, by induction hypothesis, $a_{k+1} = (a_k)^2 = (x^{2^k})^2 = x^{2^{k+1}}$
Thus, P(k+1) is hold, $a_{k+1} = x^{2^{k+1}}$ is proved. Inductive step is proved.

The closed form of a_n is x^{2^n} where n $\in \mathbb{N}, x \in \mathbb{R}$.

b) Proof: We want to prove that $\forall$n $\in \mathbb{N}, E_n = \frac{3^n+1}{2} \wedge O_n = \frac{3^n-1}{2}$
Define P(n) as $E_n = \frac{3^n+1}{2} \wedge\ O_n = \frac{3^n-1}{2}$ where n $\in \mathbb{N}$

Base Case: Let n = 0, we want to prove P(0) is True. That is $E_0 = \frac{3^0+1}{2} = 1 \wedge\ O_0 = \frac{3^0-1}{2} = 0$
By definition, digit sum of empty ternary string ϵ is 0, so $E_0 = 1$,
since in total there are $3^0 = 1$ number of ternary string of length 0, their digit sums are either even or odd
so $O_0 + E_0 = 3^0 = 1$
so $O_0 = 1 - E_0 = 1 - 1 = 0$
Thus, base case is proved.

Inductive Step: let k $\in \mathbb{N}$, assume P(k) is True, that is $E_k = \frac{3^k+1}{2} \wedge O_k = \frac{3^k-1}{2}$
We will prove P(k+1) is True. That is $E_{k+1} = \frac{3^{k+1}+1}{2} \wedge O_{k+1} = \frac{3^{k+1}-1}{2}$
E_{k+1} is number of even digit sum ternary strings with length k+1.
Denote x as the last character of ternary string of length k+1, so x $\in (0, 1, 2)$
Let $S_k =$ the digit sum of any ternary string of length k
so $S_{k+1} =$ the digit sum of any ternary string of length k +1
By Fact, $Even(S_{k+1}) \Leftrightarrow (Even(S_k) \wedge Even(x)) \vee (Odd(S_k) \wedge Odd(x))$
Since x can be 0 or 2 to be even, x can be 1 to be odd.

- Case: by fact,$Even(S_{k+1})$ can consist of $(Even(S_k) \wedge Even(x))$
  if x is even, x = 0 or x = 2

By Induction Hypothesis, we assume P(K) is right. Therefore E_k=$\frac{3^k+1}{2}$.
In this case, there are $2 \cdot E_k$ ternary strings of length k+1 that the digit sum is even

- Case: by fact,$Even(S_{k+1})$ can consist of $(Odd(S_k) \wedge Odd(x)$
  if x is odd, x=1
  By Induction Hypothesis, we assume P(K) is right. Therefore O_k=$\frac{3^k-1}{2}$.
  In this case, there are $1 \cdot O_k$ ternary strings of length k+1 that the digit sum is even

$\Rightarrow E_{k+1} = 2 \cdot E_k + 1 \cdot O_k = 2 \cdot \frac{3^k+1}{2} + \frac{3^k-1}{2} = 3^k + 1 + \frac{3^k-1}{2} = \frac{3 \cdot 3^k+1}{2} = \frac{3^{k+1}+1}{2}$
Since in total there are 3^{k+1} number of ternary string of length k+1, their digit sums are either even or odd
$\Rightarrow E_{k+1} + O_{k+1} = 3^{k+1}$
$\Rightarrow O_{k+1} = 3^{k+1} - E_{k+1} = 3^{k+1} - \frac{3^{k+1}+1}{2} = \frac{3^{k+1}-1}{2}$

Thus the inductive step is proved.
$\forall n \in \mathbb{N}, E_n = \frac{3^n+1}{2} \wedge O_n = \frac{3^n-1}{2}$

2 Binary representations

a) We want to prove: $\forall n \in \mathbb{Z}^+, 4 \mid n \Rightarrow 0.2 - z_n = \frac{1}{5 \cdot 2^n}$
let n $\in \mathbb{Z}^+$, assume that 4 | n $\Rightarrow$ n = 4k for some k $\in \mathbb{Z}^+$, prove by induction on k.
Define P(k) as 0.2 - z_{4k} = $\frac{1}{5 \cdot 2^{4k}}$ where k $\in \mathbb{Z}^+$

Base Case: let k =1, so n =4k = 4. We want to prove P(1) is True, that is 0.2 - z_4 = $\frac{1}{5 \cdot 2^4} = \frac{1}{80}$
By question, z_4 = 0.1875 $\Rightarrow$ 0.2 - z_4 = 0.0125 = $\frac{1}{80} = \frac{1}{5 \cdot 2^4}$
Thus, base case is proved.

Inductive Step: let k $\in \mathbb{Z}^+$, assume P(k) is True, that is 0.2 - z_{4k} = $\frac{1}{5 \cdot 2^{4k}}$.
We want to prove P(k+1) is True, that is 0.2 - $z_{4(k+1)}$ = $\frac{1}{5 \cdot 2^{4(k+1)}}$ $\Rightarrow$ 0.2 - z_{4k+4} = $\frac{1}{5 \cdot 2^{4k+4}}$
We can write $z_{4k} = (b_0 b_1 ... b_{4k})_2$ *and* $z_{4k+4} = (b_0 ... b_{4k} b_{4k+1} b_{4k+2} b_{4k+3} b_{4k+4})_2$
By question, we know that 0.2 using infinite sequence of bits, and z_{4k}, z_{4k+4} are approximation of 0.2
$\Rightarrow b_{4k+1} = 0,\ b_{4k+2} = 0,,\ b_{4k+3} = 1,\ b_{4k+4} = 1$
$\Rightarrow z_{4k+4} = z_{4k} + \frac{1}{2^{4k+1}} \times 0 + \frac{1}{2^{4k+2}} \times 0 + \frac{1}{2^{4k+3}} \times 1 + \frac{1}{2^{4k+4}} \times 1 = z_{4k} + \frac{1}{2^{4k+3}} + \frac{1}{2^{4k+4}}$
$\Rightarrow 0.2 - z_{4k+4} = 0.2 - z_{4k} - \frac{1}{2^{4k+3}} - \frac{1}{2^{4k+4}}$
(by Induction hypothesis) $= \frac{1}{5 \cdot 2^{4k}} - \frac{1}{2^{4k} . 2^3} - \frac{1}{2^{4k} . 2^4} = \frac{1}{5 \cdot 2^{4k}} - \frac{1}{2^{4k}} \cdot \frac{1}{8} - \frac{1}{2^{4k}} \cdot \frac{1}{16} = \frac{1}{2^{4k}} \cdot (\frac{1}{5} - \frac{1}{8} - \frac{1}{16})$
$= \frac{1}{2^{4k}} \cdot \frac{1}{80} = \frac{1}{2^{4k} . 5 \cdot 16} = \frac{1}{2^{4k} . 5 \cdot 2^4} = \frac{1}{5 \cdot 2^{4k+4}}$

Thus, the inductive step is proved.$\forall n \in \mathbb{Z}^+, 4 \mid n \Rightarrow 0.2 - z_n = \frac{1}{5 \cdot 2^n}$

b) We want to prove $\forall n \in \mathbb{Z}^+, \forall x \in S, \exists x_1 \in S,\ FB(n, x_1) \wedge 0 \leq x - x_1 \leq \frac{1}{2^n}$, prove by induction on n.
Define P(n) as $\forall x \in S, \exists x_1 \in S,\ FB(n, x_1) \wedge 0 \leq x - x_1 \leq \frac{1}{2^n}$ where n $\in \mathbb{Z}^+$

Base Case: let n = 1. Let x $\in$ S = $\{a \in \mathbb{R} | 0 \leq a < 1\}$

- Case 1: x $\in [0, \frac{1}{2})$, take $x_1 = 0 = (0.0)_2 \Rightarrow FB(1, X_1)$ is True
  $\Rightarrow x - x_1 \in [0, \frac{1}{2}) \Rightarrow 0 \leq x - x_1 < \frac{1}{2} \Rightarrow 0 \leq x - x_1 \leq \frac{1}{2}$
  $\Rightarrow FB(1, X_1) \wedge 0 \leq x - x_1 \leq \frac{1}{2}$
- Case 2: $x \in [\frac{1}{2}, 1)$, take $x_1 = \frac{1}{2} = (0.1)_2 \Rightarrow FB(1, X_1)$ is True
  $\Rightarrow x \geq \frac{1}{2} = x_1 \Rightarrow x - x_1 \geq 0$
  since $x < 1 \Rightarrow x - x_1 < \frac{1}{2} \Rightarrow x - x_1 \leq \frac{1}{2}$
  $\Rightarrow 0 \leq x - x_1 \leq \frac{1}{2}$
  $\Rightarrow FB(1, X_1) \wedge 0 \leq x - x_1 \leq \frac{1}{2}$

  Thus, base case is proved

Induction Hypothesis: let k $\in \mathbb{Z}^+$, assume P(k) is True
that is $\forall x \in S, \exists x_k \in S,\ FB(k, x_k) \wedge 0 \leq x - x_k \leq \frac{1}{2^k}$ where $k \in \mathbb{Z}^+$
Want to prove P(k+1) is True
that is $\forall x \in S, \exists x_{k+1} \in S,\ FB(k+1, x_{k+1}) \wedge 0 \leq x - x_{k+1} \leq \frac{1}{2^{k+1}}$ where k $\in \mathbb{Z}^+, x, x_{k+1} \in S$
By Induction Hypothesis, $\exists\ x_k$ such that $FB(k, x_k) \wedge 0 \leq x - x_k \leq \frac{1}{2^k}$

- Case1: if $0 \leq x - x_k \leq \frac{1}{2^{k+1}}$
  By Inductive Hypothesis, $FB(k, x_k)$ is True
  $\Rightarrow$ we can write $x_k = (0.b_1 b_2 ... b_k)_2$ where $b_1, ..., b_k \in \{0, 1\}$
  let $x_{k+1} = (0.b_1 b_2 ... b_k b_{k+1})_2\ with\ b_1, ..., b_k$ equals to bits of x_k and $b_{k+1} = 0$
  By Inductive Hypothesis, $FB(k, x_k)$ is True $\Rightarrow FB(k+1, x_{k+1})$ is True
  so $x_{k+1} = x_k,\ x - x_{k+1} = x - x_k \in [0, \frac{1}{2^{k+1}}]$
  $\Rightarrow 0 \leq x - x_{k+1} \leq \frac{1}{2^{k+1}}$
  $\Rightarrow FB(k+1, x_{k+1}) \wedge 0 \leq x - x_{k+1} \leq \frac{1}{2^{k+1}}$
- Case2: $\frac{1}{2^{k+1}} < x - x_k \leq \frac{1}{2^k}$
  By Inductive Hypothesis, $FB(k, x_k)$ is True
  $\Rightarrow$ we can write $x_k = (0.b_1 b_2 ... b_k)_2$ where $b_1, ..., b_k \in \{0, 1\}$
  let $x_{k+1} = (0.b_1 b_2 ... b_k b_{k+1})_2\ with\ b_1, ..., b_k$ equals to bits of x_k and $b_{k+1} = 1$
  By Inductive Hypothesis, $FB(k, x_k)$ is True $\Rightarrow FB(k+1, x_{k+1})$ is True
  Since $b_{k+1} = 1 \Rightarrow x_{k+1} = x_k + \frac{1}{2^k+1} \times 1$
  Since $x - x_k > \frac{1}{2^{k+1}} \Rightarrow x - x_{k+1} = x - x_k - \frac{1}{2^k+1} > \frac{1}{2^{k+1}} - \frac{1}{2^{k+1}} = 0 \Rightarrow x - x_{k+1} > 0$
  $\Rightarrow x - x_{k+1} \geq 0$
  Similarly, since $x - x_k \leq \frac{1}{2^k} \Rightarrow x - x_{k+1} = x - x_k - \frac{1}{2^{k+1}} \leq \frac{1}{2^k} - \frac{1}{2^{k+1}} = \frac{1}{2^{k+1}}$
  $\Rightarrow x - x_{k+1} \leq \frac{1}{2^{k+1}}$
  $\Rightarrow 0 \leq x - x_{k+1} \leq \frac{1}{2^{k+1}}$
  $\Rightarrow FB(k+1, x_{k+1}) \wedge 0 \leq x - x_{k+1} \leq \frac{1}{2^{k+1}}$
  $\Rightarrow$ P(k+1) holds

Thus, the statement is proved.

3 Asymptotic notation

a) WTP: $n^4 + 165n^3 \in O(n^4 - n^2)$
WTS:$\exists c \in \mathbb{R}^+, \exists n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow n^4 + 165n^3 \leq c \cdot (n^4 - n^2)$
Proof: let n_0= 170, c= 2.Let n $\in \mathbb{N}$, assume n $\geq n_0$, we want to prove $n^4 + 165n^3 \leq c \cdot (n^4 - n^2)$
$2n^2 \leq 2n^3 \Rightarrow 2n^2 + 165n^3 \leq 2n^3 + 165n^3 = 168n^3$
$\Rightarrow 2n^2 + 165n^3 \leq 170n^3$
$Since\ n \geq n_0 = 170 \Rightarrow 2n^2 + 165n^3 \leq n^4$
$\Rightarrow 165n^3 \leq n^4 - 2n^2$
$\Rightarrow n^4 + 165n^3 \leq 2n^4 - 2n^2 = 2 \cdot (n^4 - n^2) = c \cdot (n^4 - n^2)$

Thus, $\exists c \in \mathbb{R}^+, \exists n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow n^4 + 165n^3 \leq c \cdot (n^4 - n^2)$
And then, by definition, $n^4 + 165n^3 \in O(n^4 - n^2)$

b) We want to disprove the statement, so we prove its negation.

WTP: $\forall f : \mathbb{N} \to \mathbb{R}^+, \exists g : \mathbb{N} \to \mathbb{R}_{\geq 0}, g \notin O(f) \wedge g \notin \Omega(f)$
$Let\ f : \mathbb{N} \to \mathbb{R}^+$,choose:

$$g(n) = \begin{cases} 0, if\ n\ is\ odd \\ (n+1)f(n), if\ n\ is\ even \end{cases} \tag{1}$$

First, we want to prove g $\notin$ O(f), that is

$\forall c_1 \in \mathbb{R}^+, \forall n_1 \in \mathbb{R}^+, \exists n \in \mathbb{N}, n \geq n_1 \wedge g(n) > c_1 f(n)$
let $c_1 \in \mathbb{R}^+$, let $n_1 \in \mathbb{R}^+$, let $n = 2 \cdot max(\lceil c_1 \rceil, \lceil n_1 \rceil)$, so n $\in \mathbb{N}$, we want to prove n $\geq n_1 \wedge g(n) > c_1 f(n)$
Since n $= 2 \cdot$ max($\lceil c_1 \rceil$, $\lceil n_1 \rceil$), so n $\geq n_1$
also, n is even, so g(n)= (n+1)f(n)
by definition of n, $c_1 \leq$ n $\Rightarrow c_1$f(n) $\leq$ nf(n)
Since codomain of f is $\mathbb{R}^+ \Rightarrow c_1$f(n) $\leq$ (n+1)f(n) = g(n)
so g $\not\in$ O(f) is proved.

Second, we want to prove g $\notin \Omega(f)$, that is $\forall c_2 \in \mathbb{R}^+, \forall n_2 \in \mathbb{R}^+, \exists n' \in \mathbb{N}, n' \geq n_2 \wedge g(n') < c_2 f(n')$
let $c_2 \in \mathbb{R}^+$, let $n_2 \in \mathbb{R}^+$, let $n' = 2\lceil n_2 \rceil + 1$, so $n' \in \mathbb{N}$, we want to prove $n' \geq n_2 \wedge g(n') < c_2 f(n'))$
by definition of ceil function, $n' \geq n_2$
also, n is odd, so $g(n') = 0$
by definition of f, the codomain of f is $\mathbb{R}^+$,so $f(n') > 0$, also $c_2 \in \mathbb{R}^+$, so $c_2 > 0$
$\Rightarrow c_2 f(n') > 0 = g(n')$

so g $\notin \Omega(f)$ is proved.
$\forall f : \mathbb{N} \to \mathbb{R}^+, \exists g : \mathbb{N} \to \mathbb{R}_{\geq 0}, g \notin O(f) \wedge g \notin \Omega(f)$

4 Little-Oh

b) We want to prove $\forall f, g : \mathbb{N} \to \mathbb{R}^+$, if g $\in$ o(f) then f $\notin$ O(g)
Let $f : \mathbb{N} \to \mathbb{R}^+$, let $g : \mathbb{N} \to \mathbb{R}^+$, Assume g $\in$ o(f)
that is $\forall c \in \mathbb{R}^+, \exists n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \leq cf(n)$

WTS: f $\notin$ O(g), i.e. $\forall c_0 \in \mathbb{R}^+, \forall k_0 \in \mathbb{R}^+, \exists k \in \mathbb{N}, k \geq k_0 \wedge f(k) > c_0 g(k)$

Let c $\in \mathbb{R}^+ \Rightarrow c + 1 \in \mathbb{R}^+ \Rightarrow \frac{1}{c+1} \in \mathbb{R}^+$
by assumption, since $\frac{1}{c+1} \in \mathbb{R}^+$, we can fix n_0 such that $\forall n \in \mathbb{N}, n \geq n_0 \wedge g(n) \leq \frac{f(n)}{c+1}$
Let $c_0 \in \mathbb{R}^+$, let $k_0 \in \mathbb{R}^+$ fix k $= max[n_0, k_0]$
By assumption, since k $\geq n_0$, we can fix k such that g(k) $\leq \frac{f(k)}{c_0+1}$
since $c_0 \in \mathbb{R}^+ \Rightarrow c_0 + 1 \in \mathbb{R}^+ \Rightarrow g(k)(c_0 + 1) \leq f(k)$
Since the codomain of g is greater than 0, so g(k) > 0
$\Rightarrow c_0 g(k) \ < \ f(k)$
$\Rightarrow k \geq k_0 \wedge c_0 g(k) < f(k)$
Hence, f $\notin$ O(g).

The statement is proved.