Problem Set 2

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1 Question 1

b)
Translation: $\forall P \in \mathbb{N}, \forall k \in \mathbb{Z}^+, \forall n \in \mathbb{Z}^+, Prime(p) \wedge p^k < n < p^k + p \implies gcd(p^k, n) = 1$
$Let\ p\ be\ prime\ numbers,\ let\ k \in \mathbb{Z}^+, let\ n \in \mathbb{Z}^+$
$Assume\ p^k < n < p^k + p$
$WTS : gcd(p^k, n) = 1$
by definition of prime numbers, $p \in \mathbb{Z}$
Hence, $p^k \in \mathbb{Z}$
by assumption, $p^k < n < p^k + p$
$\implies p * p^{k-1} < n < (p^{k-1} + 1) * p$
Part a) states that: $\forall x, d \in \mathbb{Z}, \forall a \in \mathbb{Z}, xd < a < (x+1)d \Rightarrow d \nmid a$
Choose $x = p^{k-1}, d = p$ Since p and p^{k-1} are all integers
By result from part a) $\implies p \nmid n$
Fact 3 states that: $\forall p \in \mathbb{N}, Prime(p) \Rightarrow (\forall k, d \in \mathbb{Z}^+, d \mid p^k \Rightarrow d = 1 \vee p \mid d)$
by fact 3, all the number divides p^k must be 1 or multiple of p
By contrapositive, since $p \nmid n$ and $n \neq 1 \Rightarrow n \nmid p^k$
Use contrapositive of fact 2: $\forall a, b, c \in \mathbb{Z}, a \nmid c \Rightarrow a \nmid b \vee b \nmid c$
$Let\ a = p,\ c = n,\ b = d\ which\ is\ multiple\ of\ p\ so\ p \mid d,\ since\ p \nmid n\ and\ p \mid d \Rightarrow d \nmid n$
$all\ the\ multiple\ of\ p\ cannot\ divides\ n,\ so\ all\ the\ factors\ of\ p^k\ cannot\ divides\ n$
$\Rightarrow The\ common\ divisor\ divides\ both\ n\ and\ p^k\ is\ 1$
$\implies gcd(p^k, n) = 1$ ■

c)
Translation: $\forall m \in \mathbb{Z}^+, \forall n_0 \in \mathbb{N}, \exists n \in \mathbb{N}, n > n_0 \wedge gcd(n, n+m) = 1$
$Let\ m \in \mathbb{N}$
Fact 4 states that: $\forall n_0 \in \mathbb{N}, \exists n \in \mathbb{N}, n > n_0 \wedge Prime(n)$
$Fix\ n\ to\ be\ prime\ numbers\ and\ n > n_0$
$WTS : gcd(n, n+m) = 1$
by Fact 4, for every integer m, we can find infinitely many prime number n such that $n > m$
$\implies n \nmid m$
$\implies n \nmid n + m$
Since n is a prime number, by definition of prime number, only 1 and n divides n, since $n \nmid n + m$
$\implies$ the integer divides both n and n + m is 1
$\implies gcd(n, n+m) = 1$ ■

d)
definition of Prime Gaps: let a $\in \mathbb{Z}^+, \exists p \in \mathbb{Z}^+$, a is prime gap if $Prime(p) \wedge Prime(p+a) \wedge (\forall n \in \mathbb{N}, (p < n < p + a) \Rightarrow \neg Prime(n))$
$Let\ a \in \mathbb{Z}^+,\ Let\ a\ be\ Prime\ gap,\ Fix\ p\ to\ be\ a\ prime\ number.$
Case 1: $p = 2,\ WTS:\ a = 1$
$Fix\ p = 2\ is\ prime,\ p + 1 = 3\ is\ also\ prime, in\ such\ case, prime\ gap\ a = 3 - 2 = 1$
Case 2: $p \neq 2\ WTS : 2 \mid a$
Fact 1 states that: $\forall a, b \in \mathbb{Z}, 2 \nmid a \wedge\ 2 \nmid b \Rightarrow 2 \mid a - b$

Since p and p +a is prime by definition of prime gap
$\Rightarrow 2 \nmid p \wedge\ 2 \nmid (p+a)$
By Fact 1 $\Rightarrow 2 \mid (p+a) - p$
$\Rightarrow 2 \mid a$ ■

2 Question 2

a)

$$\begin{aligned}
\text{We want to prove the statement} & \\
nx - \lfloor nx \rfloor &= nx - \lfloor nx \rfloor - n\lfloor x \rfloor + n\lfloor x \rfloor \\
&= n(x - \lfloor x \rfloor) - (\lfloor nx \rfloor - n\lfloor x \rfloor) \\
\text{by Fact 1, } \forall x \in \mathbb{R}, 0 \leq x - \lfloor x \rfloor < 1 & \\
&\Rightarrow 0 \leq n(x - \lfloor x \rfloor) < n \\
&\Rightarrow 0 \leq nx - \lfloor nx \rfloor < 1 \\
\text{Since } 0 \leq nx - \lfloor nx \rfloor & \\
&\Rightarrow \lfloor nx \rfloor - n\lfloor x \rfloor \leq n(x - \lfloor * \rfloor x) < n \\
&\Rightarrow \lfloor nx \rfloor - n\lfloor x \rfloor \leq n \\
\text{Let n} \in \mathbb{N}, \text{let k = n} & \\
&\Rightarrow \lfloor nx \rfloor - n\lfloor x \rfloor \leq k \ ■
\end{aligned}$$

b)

$$\begin{aligned}
\text{We want to disprove it, so we want to prove its negation} & \\
\text{The negation is } \forall k \in \mathbb{N}, \exists n \in \mathbb{N}, \exists x \in \mathbb{R}, \lfloor nx \rfloor - n\lfloor x \rfloor > k & \\
\text{Let k} \in \mathbb{N}, \text{choose n = 4k + 4, choose x = 0.5} & \\
\Rightarrow \lfloor x \rfloor = \lfloor 0.5 \rfloor = 0 & \\
&\Rightarrow \lfloor nx \rfloor - n\lfloor x \rfloor = \lfloor (4k+4) * 0.5 \rfloor - n\lfloor 0.5 \rfloor \\
&\Rightarrow \lfloor 2k+2 \rfloor - 0 = \lfloor 2k+2 \rfloor \\
\text{by Fact 2, since 2 is integer, k is natural number, 2k is integer} & \\
&\Rightarrow \lfloor 2k+2 \rfloor = 2k + \lfloor 2 \rfloor = 2k+2 \\
\text{Since k is natural number, } 2k+2 > k & \\
\Rightarrow \text{The negation is proved} & \\
\Rightarrow \text{The statement is disproved} \ ■ &
\end{aligned}$$

c)

$$
\begin{aligned}
\text{We want to prove the statement}& \\
\text{Let } y \in \mathbb{R}_{\geq 0}, \text{Let } n \in \mathbb{Z}^{+}, \text{assume } n > y& \\
\text{Choose } \epsilon = \sqrt{n^2+y} - n& \\
\text{Since y is non-negative real number, }& \Rightarrow n^2 < n^2 + y \\
& \Rightarrow n \leq \sqrt{n^2+y} \\
& \Rightarrow \epsilon = \sqrt{n^2+y} - n \geq 0 \\
\text{Since } n > y, n^2 + y < n^2 + n < n^2 + 2n + 1& \\
& \Rightarrow \sqrt{n^2+y} < \sqrt{n^2+2n+1} = n+1 \\
& \Rightarrow \epsilon = \sqrt{n^2+y} - n < 1 \\
& \Rightarrow 0 \leq \epsilon = \sqrt{n^2+y} - n < 1 \\
& \Rightarrow \epsilon \in \mathbb{R}_{\geq 0}
\end{aligned}
$$

$$
\begin{aligned}
\text{Next we want to prove } y &= (\epsilon + n)^2 - n^2 \\
\epsilon &= \sqrt{n^2+y} - n \\
\Rightarrow \epsilon + n &= \sqrt{n^2+y} \\
\Rightarrow \epsilon^2 + n^2 + 2n\epsilon &= n^2 + y \\
\Rightarrow (\epsilon + n)^2 &= n^2 + y \\
\Rightarrow y &= (\epsilon + n)^2 - n^2
\end{aligned}
$$

$$\Rightarrow \text{The statement is proved } \blacksquare$$

d)

$$
\begin{aligned}
\text{We want to prove the statement} \\
\text{The definition of f(x) is onto is } \forall y \in \mathbb{R}_{\geq 0}, \exists x \in \mathbb{R}_{\geq 0}, f(x) = y \\
\text{Consider } n = \lceil y \rceil + 1 \Rightarrow n > y \\
\text{use the statement from question c)} \\
\forall f(x) \in \mathbb{R}_{\geq 0}, \forall n \in \mathbb{Z}^{+}, n > f(x) \Rightarrow (\exists \epsilon \in \mathbb{R}_{\geq 0}, 0 \leq \epsilon < 1 \wedge f(x) = (n+\epsilon)^2 - n^2) \\
\text{For every positive integer n, there exists } \epsilon, \text{let } x = n + \epsilon \\
\text{by Fact 2, Since } 0 \leq \epsilon < 1, n \in \mathbb{Z}^{+} \Rightarrow \lfloor x \rfloor = \lfloor n + \epsilon \rfloor = n + \lfloor \epsilon \rfloor = n \\
\text{We can rewrite statement from c as} \\
\forall f(x) \in \mathbb{R}_{\geq 0}, \forall n \in \mathbb{Z}^{+}, n > f(x) \Rightarrow (\exists \epsilon \in \mathbb{R}_{\geq 0}, 0 \leq \epsilon < 1 \wedge f(x) = x^2 - (\lfloor x \rfloor)^2) \\
\text{For every nonnegative real number y = f(x),there exist } x = n + \epsilon \\
\text{such that } f(x) = x^2 - (\lfloor x \rfloor)^2 = (n+\epsilon)^2 - n^2 \\
\Rightarrow f(x) \text{ satisfies the definition of onto} \\
\Rightarrow \text{We have proved that f(x) is onto } \blacksquare
\end{aligned}
$$

3 Question 3

1. a) $\forall f : \mathbb{R} \to \mathbb{R}, \forall x \in \mathbb{R}, (Even(f) \wedge Odd(f)) \iff f(x) = 0$
   $\equiv \forall f : \mathbb{R} \to \mathbb{R}, \forall x \in \mathbb{R}, ((Even(f) \wedge Odd(f)) \Rightarrow f(x) = 0) \wedge (f(x) = 0 \Rightarrow (Even(f) \wedge Odd(f))$

   - $Even(f) \wedge Odd(f) \Rightarrow f(x) = 0$
     Assume $Even(f) \wedge Odd(f)$, so $\forall x \in \mathbb{R}, f(x) = f(-x) = -f(-x)$

Since the only real number that its positive equals its negative is 0,therefore f(x)=0

- $f(x) = 0 \Rightarrow Even(f) \wedge Odd(f)$
  $Assume f(x) = 0 :$

  $\forall x \in \mathbb{R}, f(x) = 0 \Rightarrow \forall x, f(-x) = 0 = -f(-x) = 0$
  $0 = f(x) = -f(x)$
  $f(x) = f(-x) \Rightarrow Even(f(x))$
  $Odd : f(x) = -f(-x)$
  $f(x) = 0 = -f(-x)$
  $f(x) = -f(-x) \Rightarrow Odd(f(x))$

  $Therefore,\ \forall f : \mathbb{R} \to \mathbb{R}, \forall x \in \mathbb{R}, (Even(f) \wedge Odd(f)) \iff f(x) = 0$ ■

2. b) $\forall f : \mathbb{R} \to \mathbb{R}, \exists f_1, f_2 : \mathbb{R} \Rightarrow \mathbb{R}, \forall x \in \mathbb{R}, Even(f_1) \wedge Odd(f_2) \wedge f(x) = f_1(x) + f_2(x)$

   Let f(x) = f_1(x) + f_2(x), let f_1(x) = $\frac{f(x)+f(-x)}{2}$, let f_2(x) = $\frac{f(x)-f(-x)}{2}$
   $\forall x \in \mathbb{R}, Even(f_1(x)) :$
   $f_1(x) = \frac{f(x)+f(-x)}{2}$
   $= \frac{f(-(-x))+f(-(x))}{2}$
   $= f_1(-x)$
   $\Rightarrow Even(f_1(x))$

   $\forall x \in \mathbb{R}, Odd(f_2(x)) :$
   $f_2(x) = \frac{f(x)-f(-x)}{2}$
   $= -\frac{f(-x)-f(x)}{2}$
   $= -\frac{f(-x)-f(-(-x))}{2}$
   $= -f_2(-x)$
   $\Rightarrow Odd(f_2(x))$

   $f(x) = f_1 + f_2$
   $f_1 + f_2 = \frac{f(x)+f(-x)+f(x)-f(-x)}{2}$
   $= \frac{2f(x)}{2}$
   $= f(x)$

   Therefore the statement holds ■

3. c) $\forall f : \mathbb{R} \to \mathbb{R}, \exists f_1, f_2 : \mathbb{R} \to \mathbb{R}, \forall x \in \mathbb{R}, Even(f_1) \wedge Odd(f_2) \wedge f(x) = f_1(x) \times f_2(x)$
   The negation is $\exists f : \mathbb{R} \to \mathbb{R}, \forall f_1, f_2 : \mathbb{R} \to \mathbb{R}, \exists x \in \mathbb{R}, \neg Even(f_1) \vee \neg Odd(f_2) \vee f(x) \neq f_1(x) \times f_2(x)$

   We want to prove the negation is right
   Since there are three or statement, we only need to prove one of them is right,
   we want to prove that $\exists f : \mathbb{R} \to \mathbb{R}, f(x) \neq f_1(x) \times f_2(x)$
   Prove by contradiction:

   Let f(x)= f_1(x)$\times$ $f_2(x)$:
   let $f_1(x)$ be even function, $f_1(x)$=$f_1(-x)$ and $f_2(x)$ be odd function, $f_2(x)$=-$f_2(-x)$
   $f(-x) = f_1(-x) \times f_2(-x)$
   $= f_1(x) \times -f_2(x)$
   $= -(f_1(x) \times f_2(x))$
   $= -f(x)$
   $\Rightarrow$ the product of $f_1(x)$ and $f_2(x)$, f(x) can only be odd functions
   $Choose\ f(x) = x^2$
   $f(-x) = x^2 = f(x) \Rightarrow f(x)\ is\ even\ function$

$\Rightarrow$ f(x) cannot be product of an even and odd function since it is even function, not odd function
$\Rightarrow$ the negation is right
$\Rightarrow$ We have disproved the statement $\blacksquare$