Ex1

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1. We want to prove: $\forall n \in \mathbb{N}, (x^n + \frac{1}{x^n}) \in \mathbb{Z}$.
   Let x $\in \mathbb{R}$, assume $(x^n + \frac{1}{x^n}) \in \mathbb{Z}$, let n $\in \mathbb{N}$, prove by strong induction on n.
   Define P(n) as $(x^n + \frac{1}{x^n})$

   Base Case: Let k $= 0$.
   $P(0) = (x^0 + \frac{1}{x^0}) = 1 + 1 = 2$
   Since 2 $\in \mathbb{Z}$
   $\Rightarrow$ P(0) holds
   Let k = 1, $P(1) = (x^1 + \frac{1}{x^1}) \in \mathbb{Z}$(by assumption)
   $\Rightarrow$ P(1) holds

   Inductive Step: Let k $\in \mathbb{N},$ suppose for all j $\in \mathbb{N}, 0 \leq j \leq k$, P(j) is true,i.e., $(x^j + \frac{1}{x^j}) \in \mathbb{Z}$
   Also assume that k ≥ 1.
   WTP: P(k+1) holds, i.e., $(x^{k+1} + \frac{1}{x^{k+1}}) \in \mathbb{Z}$
   By induction hypothesis, since 1, k-1 and k are all natural numbers and they are smaller than k
   So P(1), P(k-1) and P(k) holds, so P(1), P(k-1) and P(k) are all integers
   Since $\mathbb{Z}$(the integer set) is a closed set, so $P(k) \times P(1)$ is integer.
   $\Rightarrow P(k) \times P(1) = (x^k + \frac{1}{x^k}) + (x^1 + \frac{1}{x^1}) = x^{k+1} + x^{k-1} + \frac{1}{x^{k-1}} + \frac{1}{x^{k+1}} = (x^{k+1} + \frac{1}{x^{k+1}}) + (x^{k-1} + \frac{1}{x^{k-1}})$
   $\Rightarrow P(k) \times P(1) = P(k+1) + P(k-1) \Rightarrow P(k+1) = P(k) \times P(1) - P(k-1)$
   Since $P(k) \times P(1)$ and P(k-1) are integers, therefore P(k+1) is integer.

   Therefore, P(k+1) holds