A1

Siwei Tang

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1 Principle of Well-Ordering

1. P(n): if the game is played with n players and n is odd natural number, there will be at least a survivor
   WTP: $\forall n \in \mathbb{N}$, if n is odd, then P(n) holds
   Assume the contradiction: $\exists n \in \mathbb{N},$ such that n is odd, P(n) does not hold.
   Let S be the set of all j $\in \mathbb{N}$, j is odd such that P(j) does not hold.
   By definition of S, S $\subseteq \mathbb{N}$, by assumption, S is not empty
   (By Principle of Well-Ordering) $\Rightarrow$ S has a minimum element c where c is odd natural number.
   By definition, since c is odd, so c -2 is odd,also c- 2 $\not\subset S$
   $\Rightarrow$ P(c-2) holds, i.e. there will always be at least one survivor when the game played with c-2 people.

   Define d(a,b) as the distance between a and b
   Assume there are c players with id 1, 2,..., c.
   Without loss of generality, assume d(1,2) is the least element of all possible distances between pairs
   That means 1 and 2 are closest to each other, 1 would throw water balloon to 2 and 2 throw water balloon to 1
   If we remove the pair which has the closest distance among all possible pairs of c players i.e. pair of 1 and 2
   For the rest c-2 players, there are two cases

   Case 1: there exists some of the rest c-2 players who originally throw water balloon to 1 or 2
   Since 1 and 2 are removed, so those players who throw to 1 or 2 before
   has to choose another players among c-2 players to throw.
   By assumption, since 1 and 2 throw water balloon to each other and P(c) does not hold
   $\Rightarrow$ P(c-2) should not hold since the rest c-2 players should also not have survivors.
   In such a case, since P(c-2) holds, it is contradiction.

   Case 2: No one in c-2 players throw water balloons to 1 or 2
   So the rest c-2 players throw water balloons among each other
   By assumption, since 1 and 2 throw water balloon to each other and P(c) does not hold
   $\Rightarrow$ P(c-2) should not hold since the rest c-2 players should also not have survivors.
   In such a case, P(c-2) holds i.e. there will be at least one survivor in c players which is a contradiction

   In conclusion, by two cases, if P(c) not hold, there is always contradiction
   So P(c) holds
   $\Rightarrow \forall n \in \mathbb{N},$ n is odd, P(n) holds i.e. there will be at least one survivor

2 Induction

P(x,y): if (x,y) = $(2^{k+1} + 1, 2^k + 1)$ where k $\in \mathbb{N}$, then (x,y) $\in$S

Base Case: (x,y) = (3,2).
Let k = 0, then $2^{k+1} + 1 = 2^1 + 1 = 3$
$2^k + 1 = 2^0 + 1 = 2.$

So for k = 0, P(x,y) holds.

Inductive Step: Let (x,y) $\in \mathbb{N}^2$. By definition, (3x-2y,x) $\in \mathbb{N}^2$.
Suppose P((x,y)) holds, i.e., (x,y) = $(2^{k+1}+1, 2^k+1)$ where k $\in \mathbb{N}$, then (x,y) $\in$S [IH]
WTP: P((3x-2y,x)) holds, i.e., (x,y) = $(2^{k'+1}+1, 2^{k\prime}+1)$ where k' $\in \mathbb{N}$, then (3x-2y,x) $\in$S
Let k' = k + 1. Since k $\in \mathbb{N}$, then k'$\in \mathbb{N}$.

$$\begin{aligned}
3x-2y &= 3(2^{k+1}+1) - 2(2^k+1)\textbf{(By IH)} \\
&= 3*2^{k+1} + 3 - 2^{k+1} - 2 \\
&= 2*2^{k+1} + 1 \\
&= 2^{k+2} + 1 \\
&= 2^{k'+1} + 1
\end{aligned}$$

By IH, x = $2^{k+1}+1 = 2^{k\prime}+1$.
$\Rightarrow$ (3x-2y,x) = $(2^{k'+1}+1, 2^{k\prime}+1)$.
Thus P((3x-2y,x)) holds.

$\forall k \in \mathbb{N}$, P((3x-2y,x)) holds
$\Rightarrow R \subset S$

3 Root Tree

P(t) : The root of t has the largest label of all nodes in the tree.

Base Case: Let t be a single node. Then t has only the root, so t is the largest label of t.
So P(t) holds.

Inductive Step: Let t $\in$ T. Let x be the root of a tree t. Let i $\in \mathbb{N}$
Assume that for all subtree t_i of t, P(t_i) holds, i.e., the root of t_i has the largest label in t_i. [IH]
WTP: P(t), i.e. the root of t has the largest label in t

let $x_1, ..., x_n$ be the root of all the subtrees of t, so $x_1, ..., x_n$ are the children of x
By IH, for all $1 \leq i \leq n$, x_i is the largest label in t_i.
Also, by the definition of T, for all $1 \leq i \leq n$,, since all x_i are children of x, so $a_{x_i} < a_x$,
Therefore, a_x is the largest label in t
$\Rightarrow$ P(t) holds i.e. the root of t has the largest label of all nodes in the tree.