# CSC236H Exercise 2

Sample Solutions

Winter 2016

1. Let $F$ be a set defined as follows:

   - any tree consisting of a single node is an element of $F$;
   - if $t_1, t_2 \in F$, so is a binary tree consisting of a new root with $t_1$ and $t_2$ as subtrees;
   - nothing else belongs to $F$.

   Use structural induction to prove that every $t \in F$ has exactly one more leaf than interior nodes.

   **Solution:** $P(t)$ : the tree $t$ has exactly one more leaf than interior nodes.

   The goal is to prove for all $t \in F$, $P(t)$.

   **Base Case:** Let $t$ be a single node.
   Then $t$ has one leaf and no interior node. Therefore, $t$ has exactly one more leaf than interior nodes, and so $P(t)$.

   **Induction Step:** Let $t_1, t_2 \in F$. By definition, $F$ includes a binary tree $t$ consisting of a root with $t_1$ and $t_2$ as subtrees.
   Suppose $P(t_1)$ and $P(t_2)$, i.e., both $t_1$ and $t_2$ have exactly one more leaf than interior nodes. **[IH]**
   **WTP:** $P(t)$, i.e., $t$ has exactly one more leaf than interior nodes.

   Let $l_1$ denote the number of leaves in $t_1$, $l_2$ denote the number of leaves in $t_2$, and $l$ denote the number of leaves in $t$.
   Let $i_1$ denote the number of interior nodes in $t_1$, $i_2$ denote the number of interior nodes in $t_2$, and $i$ denote the number of interior nodes in $t$.
   Then $l = l_1 + l_2$ and $i = i_1 + i_2 + 1$.
   By IH, $l_1 = i_1 + 1$ and $l_2 = i_2 + 1$.
   Then, $l = i_1 + 1 + i_2 + 1 = i_1 + i_2 + 2$.
   Therefore $l = i + 1$, and so $P(t)$ holds.

2. Let $G$ be a set defined as follows:

   - if $x$ is a propositional variable, then $x \in G$;
   - if $f_1, f_2 \in G$, then $\neg f_1 \in G$, $(f_1 \vee f_2) \in G$, and $(f_1 \wedge f_2) \in G$;
   - nothing else belongs to $G$.

   Use structural induction to prove that for every $f \in G$, there exists $f' \in G$ such that $f$ and $f'$ are logically equivalent, and $f'$ does not contain the $\wedge$ symbol.

   (Recall that propositional formulas $f_1$ and $f_2$ are *logically equivalent* if $f_1$ and $f_2$ evaluate to the same value, no matter how their variables are set.)

**Solution:** $P(f)$ : There exists $f' \in G$ such that $f'$ and $f$ are logically equivalent, and $f'$ does not contain the $\wedge$ symbol.
We use structural induction (on the set $G$) to prove that for all $f \in G$, $P(f)$ holds.

**Base Case:** Let $f = x$, where $x$ is a propositional variable.
Let $f' = x$. Then $f'$ is in $G$, $f'$ does not contain the $\wedge$ symbol, and $f'$ and $f$ are logically equivalent.
Therefore $P(f)$ holds.

**Induction Step:** Assume $f_1, f_2 \in G$. By definition, $\neg f_1 \in G$, $(f_1 \vee f_2) \in G$, and $(f_1 \wedge f_2) \in G$.
Suppose $P(f_1)$ and $P(f_2)$, i.e., there exist $f_1', f_2' \in G$ such that $f_1'$ is logically equivalent to $f_1$, and $f_2'$ is logically equivalent to $f_2$, and $f_1'$ and $f_2'$ do not contain the $\wedge$ symbol. **[IH]**
**WTP:** (A) $P(\neg f_1)$, (B) $P((f_1 \vee f_2))$, (C) $P((f_1 \wedge f_2))$.

Case (A): For $f = \neg f_1$, let $f' = \neg f_1'$.
By IH, $f_1' \in G$, and therefore $f' \in G$.
By IH, $f_1'$ does not contain the $\wedge$ symbol and therefore neither does $f'$.
By IH, $f_1$ is logically equivalent to $f_1'$, and therefore $f$ is logically equivalent to $f'$.
Therefore $P(f)$.

Case (B): For $f = (f_1 \vee f_2)$, let $f' = (f_1' \vee f_2')$.
By IH, $f_1', f_2' \in G$, and therefore $f' \in G$.
By IH, $f_1'$ and $f_2'$ do not contain the $\wedge$ symbol, and therefore neither does $f'$.
By IH, $f_1$ and $f_2$ are logically equivalent to $f_1'$ and $f_2'$ respectively. Therefore, $f$ is logically equivalent to $f'$.
Therefore $P(f)$.

Case (C): For $f = (f_1 \wedge f_2)$, let $f' = \neg(\neg f_1' \vee \neg f_2')$.
Since $f_1', f_2'$ are in $G$, then so are $\neg f_1'$ and $\neg f_2'$, as well as $(\neg f_1' \vee \neg f_2')$, and $\neg(\neg f_1' \vee \neg f_2')$. Thus $f' \in G$.
Also, $f'$ and $f$ are logically equivalent.
By IH, $f_1'$ and $f_2'$ do not contain the $\wedge$ symbol.
Then $f'$ does not contain the $\wedge$ symbol.
Therefore $P(f)$.