Algorithm Design, Analysis & Complexity

Lecture 8 - Linear Programming

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Pivot I

procedure $\textsc{Pivot}(N, B, A, \boldsymbol{b}, \boldsymbol{c}, v, \ell, e)$

  # Compute the coefficients of the equations for the new basic variable x_e

  Let $\hat{A}$ be a new $m \times n$ matrix

  $\hat{b}_e = \dfrac{b_\ell}{a_{\ell e}}$

  for $(j \in N \setminus \{\ell\})$ **do**

    $\hat{a}_{ej} = \dfrac{a_{\ell j}}{a_{\ell e}}$

  $\hat{a}_{e\ell} = \dfrac{1}{a_{\ell e}}$

  # Compute the coefficients of the remaining constraints

  for $(i \in B \setminus \{\ell\})$ **do**

    $\hat{b}_i = b_i - a_{ie}\hat{b}_e$

    for $(j \in N \setminus \{\ell\})$ **do**

      $\hat{a}_{ij} = a_{ij} - a_{ie}\hat{a}_{ej}$

    $\hat{a}_{i\ell} = -a_{ie}\hat{a}_{e\ell}$

Pivot II

\# Compute the objective function
$\hat{v} = v + c_e \hat{b}_e$
for $(j \in N \setminus \{\ell\})$ **do**
    $\hat{c}_j = c_j - c_e \hat{a}_{ej}$
$\hat{c}_\ell = -c_e \hat{a}_{e\ell}$
\# Compute new sets of basic and nonbasic variables
$\hat{N} = N \cup \{\ell\} \setminus \{e\}$
$\hat{B} = B \cup \{e\} \setminus \{\ell\}$
return $(\hat{N}, \hat{B}, \hat{A}, \hat{\boldsymbol{b}}, \hat{\boldsymbol{c}}, \hat{v})$

Simplex I

procedure $\textsc{Simplex}(A, \boldsymbol{b}, \boldsymbol{c})$
    $(N, B, A, \boldsymbol{b}, \boldsymbol{c}, v) = \textsc{Initialize-Simplex}\ (A, \boldsymbol{b}, \boldsymbol{c})$
    Let Δ be a new vector of length m
    while some index $j \in N$ has $c_j > 0$ **do**
        Choose an index $e \in N$ for which $c_e > 0$
        for each index $i \in B$ **do**
            if $a_{ie} > 0$ **then**
                $\Delta_i = \frac{b_i}{a_{ie}}$
            else
                $\Delta_i = \infty$
        Choose an index $\ell \in B$ that minimizes Δ_ℓ
        if $\Delta_\ell == \infty$ **then**
            return "unbounded"
        else
            $(N, B, A, \boldsymbol{b}, \boldsymbol{c}, v) = \textsc{Pivot}\ (N, B, A, \boldsymbol{b}, \boldsymbol{c}, v, \ell, e)$

Simplex II

for $i = 1 \ldots n$ **do**
    if $i \in B$ **then**
        $\bar{x}_i = b_i$
    else
        $\bar{x}_i = 0$
return $(\bar{x}_1, \ldots, \bar{x}_n)$

The subroutine Initialize-Simplex checks if the original problem is infeasible. If yes, it returns "infeasible" and halts. If not, it returns a slack form for which the basic solution is feasible.

Proof of correctness

We will prove the correctness of the SIMPLEX Algorithm by proving the following in order:

1. If the original problem is infeasible, SIMPLEX returns "infeasible".
2. If SIMPLEX has an initial feasible solution and eventually terminates, then it either returns a feasible solution or determines that the linear program is unbounded.
3. SIMPLEX terminates (after breaking ties appropriately).
4. Finally, we show that the feasible solution returned by SIMPLEX is indeed optimal.

SIMPLEX returns a feasible solution

Lemma

Given an LPP $(A, \boldsymbol{b}, \boldsymbol{c})$, suppose that INITIALIZE-SIMPLEX *returns a slack form for which the basic solution is feasible. Then either the LPP is unbounded and* SIMPLEX *returns "unbounded", or else* SIMPLEX *returns a feasible solution.*

Proof.

We prove the lemma by proving the following claims by mathematical induction on the i^{th} iteration of the while loop:

1. the slack form in every iteration of the while loop is equivalent to the slack form returned by the call to INITIALIZE-SIMPLEX;
2. for each $i \in B$, we have $b_i \geq 0$; and
3. the basic solution associated with the slack form is feasible.

Proof

Base Case: At the first iteration of the while loop, statements 1 and 3 are obvious because the slack form is the one that is returned by INITIALIZE-SIMPLEX. Consequently, statement 2 is also true since $x_i = b_i$ for each $i \in B$ (and $x_j = 0$ for each $j \in N$) and all the x_i's are nonnegative.
Induction Hypothesis: Assume the three statements hold at iteration i of the while loop.
Induction Step: We want to show that the three statements hold at iteration $i + 1$ of the while loop.
Note that going from iteration i to $i + 1$, we basically exchange the role of a basic and nonbasic variable by calling the PIVOT subroutine. Consequently, the slack form at iteration $i + 1$ is equivalent to the slack form at iteration i. By the induction hypothesis and by transitivity of "equivalence", the slack form at iteration $i + 1$ is equivalent to the one returned by the call to INITIALIZE-SIMPLEX.

Proof

For the second statement, observe that $\hat{b}_e = \dfrac{b_\ell}{a_{\ell e}}$ by the $\textsc{Pivot}$ subroutine. By induction hypothesis $b_\ell \geq 0$, while $a_{\ell e} > 0$ by the $\textsc{Simplex}$ algorithm. Hence, $\hat{b}_e \geq 0$.

For the remaining indices $i \in B \setminus \{\ell\}$, it follows by the $\textsc{Pivot}$ subroutine that

$$\hat{b}_i = b_i - a_{ie}\hat{b}_e = b_i - a_{ie}\frac{b_\ell}{a_{\ell e}}.$$

If $a_{ie} \leq 0$, then $\hat{b}_i \geq 0$ since $b_i, b_\ell \geq 0$ (by ind. hyp.) and $a_{\ell e} > 0$ (by the $\textsc{Simplex}$ algorithm). If $a_{ie} > 0$, then since we choose ℓ s.t.

$$\frac{b_\ell}{a_{\ell e}} \leq \frac{b_i}{a_{ie}} \text{ for all } i \in B,$$

we obtain

$$\hat{b}_i = b_i - a_{ie}\frac{b_\ell}{a_{\ell e}} \geq b_i - a_{ie}\frac{b_i}{a_{ie}} = b_i - b_i = 0.$$

Thus, $\hat{b}_i \geq 0$ for all $i \in B$.

Proof

Finally, for the third statement, observe that the basic solution sets $x_j = 0$ for all $j \in N$, which then implies that $x_i = b_i$ for all $i \in B$. By the second statement, it follows that all the variables are nonnegative, and hence the basic solution is feasible.

Coming back to the proof of the Lemma, note that the while loop in $\textsc{Simplex}$ can terminate in two ways. Either the condition in the while loop fails, or it terminates by returning "unbounded".

- In the first case, when the loop ends, the current basic solution is feasible (by what we proved above) and is being returned by $\textsc{Simplex}$.
- In the second case, we obtain that $a_{ie} \leq 0$ for each $i \in B$. Consider the following solution $\bar{\boldsymbol{x}}$ defined as

$$\bar{x}_i = \begin{cases} \infty & \text{if } i = e \\ 0 & \text{if } i \in N \setminus \{e\} \\ b_i - \sum_{j \in N} a_{ij}\bar{x}_j & \text{if } i \in B. \end{cases}$$

Proof

Check that this solution $\bar{\boldsymbol{x}}$ is feasible.

Also, observe that the objective value at $\bar{\boldsymbol{x}}$ is unbounded:

$$z = v + \sum_{j \in N} c_j \bar{x}_j = v + c_e \bar{x}_e = \infty \text{ (since } c_e > 0 \text{ and } \bar{x}_e = \infty\text{).}$$

Hence, the LPP is unbounded.

Thus, either the LPP is unbounded and SIMPLEX returns "unbounded", or else SIMPLEX returns a feasible solution. $\square$

Simplex terminates

Recall that in the Pivot subroutine, we have

$$\hat{v} = v + c_e \hat{b}_e.$$

Since $c_e > 0$ and $\hat{b}_e \geq 0$, it follows that $\hat{v} \geq v$.

Thus, the objective value is nondecreasing in every iteration of the while loop. But there is a degenerate case when it might not be increasing (e.g., when $b_\ell = 0$).

Degeneracy can lead to **cycling**, which might prevent Simplex from terminating. Fortunately, that's the only way Simplex can fail to terminate. We make this statement concrete in what follows.

$\textsc{Simplex}$ terminates

Lemma

Let I be a set of indices. For each $j \in I$, let α_j and β_j be real numbers, and let x_j be a real-valued variable. Let γ be any real number. Suppose that for all values of x_j for $j \in I$, we have

$$\sum_{j \in I} \alpha_j x_j = \gamma + \sum_{j \in I} \beta_j x_j .$$

Then, $\alpha_j = \beta_j$ for each $j \in I$, and $\gamma = 0$.

Proof.

Exercise! □

SIMPLEX terminates

Claim

Let $(A, \boldsymbol{b}, \boldsymbol{c})$ be an LPP. Given a set B of basic variables, the associated slack form is unique.

Proof.

Assume for a contradiction that L and L' are two slack forms with the same set of basic variables:

$$
\begin{array}{rrcl}
L: & z & = & v + \sum_{j \in N} c_j x_j \\
 & x_i & = & b_i - \sum_{j \in N} a_{ij} x_j \text{ for } i \in B \\
L': & z & = & v' + \sum_{j \in N} c'_j x_j \\
 & x_i & = & b'_i - \sum_{j \in N} a'_{ij} x_j \text{ for } i \in B.
\end{array}
$$

By $L - L'$, we obtain that for any value of x_j for $j \in N$, we have

$$\sum_{j \in N} a_{ij} x_j = (b_i - b'_i) + \sum_{j \in N} a'_{ij} x_j \text{ for } i \in B.$$

Consequently, by the lemma, $b_i = b'_i$ and $a_{ij} = a'_{ij}$ for all i, j.
Hence, $L = L'$. $\square$

SIMPLEX terminates

Lemma

If SIMPLEX *fails to terminate in at most* $\binom{n+m}{m}$ *iterations, then it cycles.*

Proof.

By previous claim, the set B of basic variables uniquely determines a slack form.

There are $n+m$ variables and $|B| = m$. Therefore, there are at most $\binom{n+m}{m}$ ways to choose B.

Thus, there are at most $\binom{n+m}{m}$ unique slack forms.

Therefore, if SIMPLEX runs for more than $\binom{n+m}{m}$ iterations, it must cycle. $\square$

SIMPLEX terminates

Fact

There are ways to break ties so that cycling doesn't happen. One option is **Bland's Rule** *where the entering and leaving variables are chosen with the smallest index.*

Dual LPP

Definition

Given an LPP $(A, \boldsymbol{b}, \boldsymbol{c})$,

$$\begin{array}{rl} \text{maximize} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & \\ & \begin{array}{rcl} A\boldsymbol{x} & \leq & \boldsymbol{b} \\ \boldsymbol{x} & \geq & \mathbf{o} \end{array} \end{array}$$

we define the **dual** linear program as follows:

$$\begin{array}{rl} \text{minimize} & \boldsymbol{b}^T\boldsymbol{y} \\ \text{subject to} & \\ & \begin{array}{rcl} A^T\boldsymbol{y} & \geq & \boldsymbol{c} \\ \boldsymbol{y} & \geq & \mathbf{o}. \end{array} \end{array}$$

We then refer to the original LPP as the **primal** linear program.

Example

Compute the dual linear program for the following LPP:

$$\begin{array}{lrcl} \text{maximize} & 5x_1 + 3x_2 & & \\ \text{subject to} & & & \\ & 3x_1 + 6x_2 & \leq & 7 \\ & 4x_1 + 9x_2 & \leq & 8 \\ & 7x_1 + x_2 & \leq & 9 \\ & x_1, x_2 & \geq & 0. \end{array}$$

Solution.

$$\begin{array}{lrcl} \text{minimize} & 7y_1 + 8y_2 + 9y_3 & & \\ \text{subject to} & & & \\ & 3y_1 + 4y_2 + 7y_3 & \geq & 5 \\ & 6y_1 + 9y_2 + y_3 & \geq & 3 \\ & y_1, y_2, y_3 & \geq & 0. \end{array}$$

Weak Linear Programming Duality

Lemma (Weak Linear Programming Duality)

Let $\bar{\boldsymbol{x}}$ be any feasible solution to the primal and let $\bar{\boldsymbol{y}}$ be any feasible solution to the dual LPP. Then, we have

$$\sum_{j=1}^{n} c_j \bar{x}_j \leq \sum_{i=1}^{m} b_i \bar{y}_i.$$

Proof.

$$\begin{aligned}
\sum_{j=1}^{n} c_j \bar{x}_j &\leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} \bar{y}_i \right) \bar{x}_j && \text{(since } \bar{\boldsymbol{y}} \text{ is a solution to the dual)} \\
&= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} \bar{x}_j \right) \bar{y}_i \\
&\leq \sum_{i=1}^{m} b_i \bar{y}_i && \text{(since } \bar{\boldsymbol{x}} \text{ is a solution to the primal).}
\end{aligned}$$

□

Corollary

Corollary

Let $\bar{\boldsymbol{x}}$ be a feasible solution to the primal and let $\bar{\boldsymbol{y}}$ be a feasible solution to the corresponding dual LPP. If

$$\sum_{j=1}^{n} c_j \bar{x}_j = \sum_{i=1}^{m} b_i \bar{y}_i,$$

then $\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{y}}$ are optimal solutions to the primal and dual LPPs, respectively.

Strong Linear Programming Duality

Theorem (Strong Linear Programming Duality)

Suppose that SIMPLEX *returns values* $\bar{\boldsymbol{x}} = (\bar{x}_1, \ldots, \bar{x}_n)^T$ *for the LPP* $(A, \boldsymbol{b}, \boldsymbol{c})$*. Let* N *and* B *denote the nonbasic and basic variables for the final slack form, let* $\bar{c}'$ *denote the coefficients in the final slack form, and let* $\bar{\boldsymbol{y}} = (\bar{y}_1, \ldots, \bar{y}_m)^T$ *be defined as follows:*

$$\bar{y}_i = \begin{cases} -c'_{n+i} & \text{if } n+i \in N \\ 0 & \text{otherwise.} \end{cases}$$

Then, $\bar{\boldsymbol{x}}$ *is an optimal solution to the primal,* $\bar{\boldsymbol{y}}$ *is an optimal solution to the dual, and*

$$\sum_{j=1}^{n} c_j \bar{x}_j = \sum_{i=1}^{m} b_i \bar{y}_i.$$

Furthermore, if the primal is unbounded, then the dual is infeasible.

Proof.

Exercise! $\square$

Initial Basic Feasible Solution

Lemma

Let L be an LPP in the standard form. Let x_0 be a new variable, and let L_{aux} be the following LPP with $n+1$ variables:

$$\begin{array}{lrcll} \textit{maximize} & & -x_0 & & \\ \textit{subject to} & & & & \\ & \sum_{j=1}^{n} a_{ij}x_j - x_0 & \leq & b_i & \text{for } i = 1, \ldots, m \\ & x_j & \geq & 0 & \text{for } j = 0, 1, \ldots, n. \end{array}$$

Then L is feasible $\iff$ the optimal objective value of L_{aux} is 0.

Proof

Proof.

Suppose L has a feasible solution $\bar{\boldsymbol{x}} = (\bar{x}_1, \ldots, \bar{x}_n)^T$. Then the solution of $\bar{x}_0 = 0$ combined with $\bar{\boldsymbol{x}}$ is a feasible solution to L_{aux} with objective value 0. Since $x_0 \geq 0$ is a constraint of L_{aux} and the objective function is to maximize $-x_0$, this solution must be optimal for x_0.

Conversely, suppose that the optimal objective value of L_{aux} is 0. Then $\bar{x}_0 = 0$, and the remaining solution values of $\bar{\boldsymbol{x}}$ satisfy the constraints of L. $\square$

Initialize-Simplex I

procedure Initialize-Simplex$(A, \boldsymbol{b}, \boldsymbol{c})$
    Let k be the index of the minimum b_i
    if $b_k \geq 0$ **then**
        return $(\{1, 2, \ldots, n\}, \{n+1, \ldots, n+m\}, A, \boldsymbol{b}, \boldsymbol{c}, 0)$
    Form L_{aux} by adding $-x_0$ to the L.H.S. of each constraint and setting the objective function to $-x_0$.
    Let $(N, B, A, \boldsymbol{b}, \boldsymbol{c}, v)$ be the resulting slack form for L_{aux}.
    Let $\ell = n + k$
    $(N, B, A, \boldsymbol{b}, \boldsymbol{c}, v) = \text{Pivot}\ (N, B, A, \boldsymbol{b}, \boldsymbol{c}, v, \ell, 0)$
    # The basic solution is now feasible for L_{aux}
    Iterate the while loop of Simplex until an optimal solution to L_{aux} is found
    if the optimal solution to L_{aux} sets $\bar{x}_0$ to 0 **then**
        if $\bar{x}_0$ is basic **then**
            perform one (degenerate) Pivot operation to make it nonbasic

Initialize-Simplex II

from the final slack form of L_{aux}, remove x_0 from the constraints and restore the original objective function of L, but replace each basic variable in this objective function by the R.H.S. of its associated constraint

return the modified final slack form

else

return "Infeasible"

Example

Solve the following LPP:

$$\begin{aligned} \text{maximize} \quad & x_1 + 3x_2 \\ \text{subject to} \quad & \end{aligned}$$

$$\begin{aligned} x_1 - x_2 &\leq 8 \\ -x_1 - x_2 &\leq -3 \\ -x_1 + 4x_2 &\leq 2 \\ x_1, x_2 &\geq 0. \end{aligned}$$

Solution. By Initialize-Simplex, we first solve the following LPP L_{aux}:

$$\begin{aligned} \text{maximize} \quad & -x_0 \\ \text{subject to} \quad & \end{aligned}$$

$$\begin{aligned} x_1 - x_2 - x_0 &\leq 8 \\ -x_1 - x_2 - x_0 &\leq -3 \\ -x_1 + 4x_2 - x_0 &\leq 2 \\ x_0, x_1, x_2 &\geq 0. \end{aligned}$$

Example

Converting this to the slack form, we obtain

$$\begin{aligned} z &= -x_0 \\ x_3 &= 8 - x_1 + x_2 + x_0 \\ x_4 &= -3 + x_1 + x_2 + x_0 \\ x_5 &= 2 + x_1 - 4x_2 + x_0 \\ x_0, x_1, x_2, x_3, x_4, x_5 &\geq 0. \end{aligned}$$

Here the minimum b_i is $b_4 = -3$. So, we interchange x_0 and x_4:

$$x_0 = 3 - x_1 - x_2 + x_4.$$

Example

Substituting this into the other equations, we obtain

$$\begin{aligned}
z &= -3 + x_1 + x_2 - x_4 \\
x_3 &= 8 - x_1 + x_2 + (3 - x_1 - x_2 + x_4) \\
&= 11 - 2x_1 + x_4 \\
x_5 &= 2 + x_1 - 4x_2 + (3 - x_1 - x_2 + x_4) \\
&= 5 - 5x_2 + x_4 \\
x_0 &= 3 - x_1 - x_2 + x_4 \\
x_0, x_1, x_2, x_3, x_4, x_5 &\geq 0.
\end{aligned}$$

Check that the basic solution $x_1 = x_2 = x_4 = 0, x_3 = 11, x_5 = 5$ and $x_0 = 3$ is now feasible.

For the next round, entering variable: x_1, leaving variable: x_0. Interchanging x_1 and x_0, we obtain

$$x_1 = 3 - x_0 - x_2 + x_4.$$

Example

Substituting this into the other equations, we obtain

$$\begin{aligned} z &= -x_0 \\ x_1 &= 3 - x_0 - x_2 + x_4 \\ x_3 &= 5 + 2x_0 + 2x_2 - x_4 \\ x_5 &= 5 - 5x_2 + x_4 \\ x_0, x_1, x_2, x_3, x_4, x_5 &\geq 0. \end{aligned}$$

Since all the coefficients in the objective function are negative, simplex stops.

This slack form is the final solution to L_{aux} with solution: $x_0 = x_2 = x_4 = 0, x_1 = 3, x_3 = 5$ and $x_5 = 5$.

The objective value at this solution $= -x_0 = 0$.

Example

Hence, our original LPP is feasible.

We obtain the following equivalent slack form for the original LPP by removing x_0 from the equations (since $x_0 = 0$) in the last slack form obtained above:

$$\begin{aligned} x_1 &= 3 - x_2 + x_4 \\ x_3 &= 5 + 2x_2 - x_4 \\ x_5 &= 5 - 5x_2 + x_4 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0. \end{aligned}$$

Here the nonbasic variables are x_2 and x_4. Simplifying the original objective function in terms of the nonbasic variables, we obtain

$$z = x_1 + 3x_2 = (3 - x_2 + x_4) + 3x_2 = 3 + 2x_2 + x_4.$$

Example

Thus, the full slack form obtained is

$$\begin{aligned}
z &= 3 + 2x_2 + x_4 \\
x_1 &= 3 - x_2 + x_4 \\
x_3 &= 5 + 2x_2 - x_4 \\
x_5 &= 5 - 5x_2 + x_4 \\
x_1, x_2, x_3, x_4, x_5 &\geq 0.
\end{aligned}$$

Note that this slack form has a feasible basic solution.

We now continue to solve this slack form using Simplex. By Bland's Rule, we choose x_2 as the entering variable, and we obtain x_5 as the leaving variable.

Interchanging the two, we obtain

$$x_2 = 1 + \frac{x_4}{5} - \frac{x_5}{5}.$$

Example

Substituting this into the other equations, we obtain

$$
\begin{aligned}
z &= 3 + 2\left(1 + \frac{x_4}{5} - \frac{x_5}{5}\right) + x_4 = 5 + \frac{7x_4}{5} - \frac{2x_5}{5} \\
x_1 &= 3 - \left(1 + \frac{x_4}{5} - \frac{x_5}{5}\right) + x_4 = 2 + \frac{4x_4}{5} + \frac{x_5}{5} \\
x_2 &= 1 + \frac{x_4}{5} - \frac{x_5}{5} \\
x_3 &= 5 + 2\left(1 + \frac{x_4}{5} - \frac{x_5}{5}\right) - x_4 = 7 - \frac{3x_4}{5} - \frac{2x_5}{5} \\
x_1, x_2, x_3, x_4, x_5 &\geq 0.
\end{aligned}
$$

Now entering variable is x_4 and leaving variable is x_3.
Interchanging the two, we obtain

$$x_4 = \frac{35}{3} - \frac{5x_3}{3} - \frac{2x_5}{3}.$$

Example

Substituting this into the other equations, we obtain

$$
\begin{aligned}
z &= 5 + \frac{7}{5}\left(\frac{35}{3} - \frac{5x_3}{3} - \frac{2x_5}{3}\right) - \frac{2x_5}{5} \\
&= \frac{64}{3} - \frac{7x_3}{3} - \frac{4x_5}{3} \\
x_1 &= 2 + \frac{4}{5}\left(\frac{35}{3} - \frac{5x_3}{3} - \frac{2x_5}{3}\right) + \frac{x_5}{5} \\
&= \frac{34}{3} - \frac{4x_3}{3} - \frac{x_5}{3} \\
x_2 &= 1 + \frac{1}{5}\left(\frac{35}{3} - \frac{5x_3}{3} - \frac{2x_5}{3}\right) - \frac{x_5}{5} \\
&= \frac{10}{3} - \frac{x_3}{3} - \frac{x_5}{3} \\
x_4 &= \frac{35}{3} - \frac{5x_3}{3} - \frac{2x_5}{3} \\
x_1, x_2, x_3, x_4, x_5 &\geq 0.
\end{aligned}
$$

Example

Since there are no more variables with positive coefficients left in the objective function, simplex ends with the (basic) solution:

$$x_3 = x_5 = 0, x_1 = \frac{34}{3}, x_2 = \frac{10}{3}, x_4 = \frac{35}{3},$$

and objective value $= z = \dfrac{64}{3}$.

Define y_1, y_2, y_3 as follows (here $n = 2$):

$$y_i = \begin{cases} -c_{n+i} & \text{if } n+i \in N \\ 0 & \text{otherwise.} \end{cases}$$

We obtain

$$y_1 = \frac{7}{3}, \quad y_2 = 0, \quad y_3 = \frac{4}{3}.$$

Example

Verify that $(y_1, y_2, y_3)^T$ is a feasible solution to the corresponding dual problem.

Moreover, the objective value of the dual program at this point is

$$\sum_{i=1}^{3} b_i y_i = 8 \times \frac{7}{3} + (-3) \times 0 + 2 \times \frac{4}{3} = \frac{56}{3} + \frac{8}{3} = \frac{64}{3}.$$

Hence, by the **Corollary** to the **Weak Duality Theorem**,
$(x_1, x_2) = \left(\frac{34}{3}, \frac{10}{3}\right)$ is an optimal solution to the primal, and
$(y_1, y_2, y_3) = \left(\frac{7}{3}, 0, \frac{4}{3}\right)$ is an optimal solution to the dual.

This is what the **Strong Duality Theorem** states as well.

Complementary Slackness

Definition

For an inequality constraint, the constraint has slack if the slack variable is positive; otherwise it is binding.
For a variable constrained to be nonnegative, there is slack if the variable is positive.

Theorem (Complementary Slackness)

Assume the primal P has a solution $\boldsymbol{x}^*$ and the dual D has a solution $\boldsymbol{y}^*$.

1. *If $x_j^* > 0$, then the j^{th} constraint in D is binding.*
2. *If the j^{th} constraint in D is not binding, then $x_j^* = 0$.*
3. *If $y_i^* > 0$, then the i^{th} constraint in P is binding.*
4. *If the i^{th} constraint in P is not binding, then $y_i^* = 0$.*

Complementary Slackness

The original primal and its corresponding dual are

$$
\begin{array}{ll}
\text{maximize} & x_1 + 3x_2 \\
\text{subject to} &
\end{array}
\qquad
\begin{array}{rcl}
x_1 - x_2 & \leq & 8 \\
-x_1 - x_2 & \leq & -3 \\
-x_1 + 4x_2 & \leq & 2 \\
x_1, x_2 & \geq & 0.
\end{array}
$$

$$
\begin{array}{ll}
\text{minimize} & 8y_1 - 3y_2 + 2y_3 \\
\text{subject to} &
\end{array}
\qquad
\begin{array}{rcl}
y_1 - y_2 - y_3 & \geq & 1 \\
-y_1 - y_2 + 4y_3 & \geq & 3 \\
y_1, y_2, y_3 & \geq & 0.
\end{array}
$$

Want to check: Is $(x_1, x_2) = \left(\frac{34}{3}, \frac{10}{3}\right)$ an optimal solution to the primal?

First, check that it is feasible for the primal. If it is not feasible, it cannot be optimal.

Now, observe that these values make the first and the third (but not the second) constraints in the primal binding.

Complementary Slackness

By the CS Theorem, this implies that the corresponding solution to the dual problem must have $y_2 = 0$ and $y_1, y_3 > 0$.

Moreover, since both $x_1, x_2 > 0$, it implies that the two linear constraints in the dual problem are binding. Thus,

$$\begin{aligned} y_1 - y_3 &= 1 \\ -y_1 + 4y_3 &= 3. \end{aligned}$$

Solving these two equations, we get $y_1 = \dfrac{7}{3}$ and $y_3 = \dfrac{4}{3}$.

Check that $(y_1, y_2, y_3) = \left(\dfrac{7}{3}, 0, \dfrac{4}{3}\right)$ is feasible for the dual.

Moreover, $x_1 + 3x_2 = \dfrac{64}{3} = 8y_1 - 3y_2 + 2y_3$.

Hence, $(x_1, x_2) = \left(\dfrac{34}{3}, \dfrac{10}{3}\right)$ and $(y_1, y_2, y_3) = \left(\dfrac{7}{3}, 0, \dfrac{4}{3}\right)$ are **optimal** solutions to the primal and dual problems, respectively.

Complementary Slackness Vector Form

Given the original primal and its corresponding dual

$$
\begin{array}{llll}
\text{maximize} & \boldsymbol{c}^T\boldsymbol{x} & \text{minimize} & \boldsymbol{b}^T\boldsymbol{y} \\
\text{subject to} & & \text{subject to} & \\
& \begin{aligned} A\boldsymbol{x} + \boldsymbol{w} &= \boldsymbol{b} \\ \boldsymbol{x}, \boldsymbol{w} &\geq \mathbf{0} \end{aligned} & & \begin{aligned} A^T\boldsymbol{y} - \boldsymbol{z} &= \boldsymbol{c} \\ \boldsymbol{y}, \boldsymbol{z} &\geq \mathbf{0}, \end{aligned}
\end{array}
$$

the complementary slackness conditions are

$$
\begin{aligned}
x_j z_j &= 0 \quad j = 1, 2, \ldots, n \\
w_i y_i &= 0 \quad i = 1, 2, \ldots, m.
\end{aligned}
$$

In vector notation, we can't write $\boldsymbol{x}\boldsymbol{z} = \mathbf{0}$, since the product $\boldsymbol{x}\boldsymbol{z}$ is not defined.

Complementary Slackness Vector Form

So we introduce a new notation:

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \Longrightarrow \quad X = \begin{bmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{bmatrix}$$

With that notation, we can write the complementarity conditions as

$$\begin{aligned} XZ\boldsymbol{e} &= \mathbf{0} \\ WY\boldsymbol{e} &= \mathbf{0}, \end{aligned}$$

where $\boldsymbol{e}$ is the vector of all 1's.

Complementary Slackness Vector Form

Thus, the final optimality conditions are

$$\begin{aligned} A\boldsymbol{x} + \boldsymbol{w} &= \boldsymbol{b} \\ A^T\boldsymbol{y} - \boldsymbol{z} &= \boldsymbol{c} \\ XZ\boldsymbol{e} &= \mathbf{0} \\ WY\boldsymbol{e} &= \mathbf{0} \\ \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} &\geq \mathbf{0}. \end{aligned}$$

These are $2n + 2m$ equations in $2n + 2m$ variables. Solve them using Newton's method.

But not all equations are linear!

It is the nonlinearity of the complementarity conditions that makes LP fundamentally harder than solving systems of equations.

Barrier Function

The trick is to put a barrier (penalty) function which discourages the solution to get to negative values.

Suppose instead of the original linear program, we look at a slight variant:

$$
\begin{array}{ll}
\text{maximize} & \boldsymbol{c}^T\boldsymbol{x} + \mu \sum_{j=1}^{n} \ln x_j + \mu \sum_{j=1}^{n} \ln w_j \\
\text{subject to} & \\
& A\boldsymbol{x} + \boldsymbol{w} \;=\; \boldsymbol{b}
\end{array}
$$

Barrier Function

1. μ is some constant and not a variable for this optimization program. As μ tends to zero, the optimization program gets closer to the original LP (assuming the solution is positive).
2. The objective function is changed such that whenever x_i goes close to zero, a huge penalty is imposed on the solution. Remember that $\ln x_i$ is a very big negative number for x_i close to zero.
3. The constraint $\boldsymbol{x} \geq \mathbf{o}$ is removed because of the new objective function.

Analog of complementary slackness conditions for this variant is:

$$
\begin{aligned}
A\boldsymbol{x} + \boldsymbol{w} &= \boldsymbol{b} \\
A^T\boldsymbol{y} - \boldsymbol{z} &= \boldsymbol{c} \\
XZ\boldsymbol{e} &= \mu\boldsymbol{e} \\
WY\boldsymbol{e} &= \mu\boldsymbol{e}.
\end{aligned}
$$

Interior Point Method

procedure $\textsc{InteriorPointMethod}(A, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$
    Choose $\rho \in (0, 1), \mu_0 \geq 0$
    Start with a feasible solution of the primal and dual, $\boldsymbol{w_0}, \boldsymbol{x_0}, \boldsymbol{y_0}, \boldsymbol{z_0}$
    for $i = 1, \ldots$ **do**
        if Desired accuracy is not reached **then**
            $\mu_i := \rho \mu_{i-1}$
            Compute Newton step directions $\delta w, \delta x, \delta y, \delta z$
            For a suitable α, set $(w_i, x_i, y_i, z_i) = (w_{i-1}, x_{i-1}, y_{i-1}, z_{i-1}) + \alpha(\delta w, \delta x, \delta y, \delta z)$
        else
            break