In solving the questions in this assignment, I worked together with my classmate Yi Yun Ding & 1004795214

I confirm that I have written the solutions/code/report in my own words.

Part I: Theoretical Problems

Q1: LTI Systems

$$\delta(n) = \begin{cases} 1, & \text{if } n=0 \\ 0, & \text{else} \end{cases}$$

Let $x(n) = \sum_{k} \delta(n-k)\, x(k)$

$T[\delta(n)] = h(n)$

By time-invariant property

$T[\delta(n-a)] = h(n-a)$

Since a is constant $\Rightarrow x(a)$ is constant

By linearity,

$$T[x(a)\,\delta(n-a)] = x(a)\, T(\delta(n-a)) = x(a)\, h(n-a)$$

By linearity, $T[\sum_{a} x(a)\,\delta(n-a)] = \sum_{a} x(a)\, T(\delta(n-a))$

$$= \sum_{a} x(a)\, h(n-a)$$

$$= h(n) * x(n)$$

$\Rightarrow T[x(n)] = h(n) * x(n)$

Q2

Let $u(x) = \sum_{m=0}^{J} u_m x^m \quad v(x) = \sum_{n=0}^{L} v_n x^n$

Their product

$$u(x)\, v(x) = \sum_{m=0}^{J} \sum_{n=0}^{L} u_m v_n\, x^m x^n$$

if we want to write it as the single power of x. then we need to make $m+n=k$, rewrite equation

$$u(x)\, v(x) = \sum_{m=0}^{J} \sum_{n=0}^{L} u_{k-n} v_n\, x^k$$

Then we can assume that padding U_{k-n}, V_n with zero for indices where they are not defined.

$u_m = 0 \quad \forall m < 0$

$V_n = 0 \quad \forall n < 0$

$u_m = 0 \quad \forall m > J$

$V_n = 0 \quad \forall n > L$

Then we can rewrite equation as

$$u(x)\, v(x) = \sum_{k=0}^{J+L} \left(\sum_{n=0}^{k} u_{k-n} V_n \right) x^n$$

Let $h(n) = \sum_{n=0}^{k} u_{k-n} V_n$

$$h(n) = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix} = u * V$$

Q3:

The minimum information required from the Guassian pyramid is the image of level n in Guassian pyramid (I_n)

Let F de ote the upsampling

$$I_k = \begin{cases} I_n, & k=n \\ F(I_{k+1}) + L_k, & 0 \le k < n \end{cases}$$

$$\begin{aligned} I_0 &= F(I_1) + L_0 \\ &= F(F(I_2) + L_1) + L_0 \\ &= F(F(F(I_3) + L_2) + L_1) + L_0 \\ &= \underbrace{F(F(\cdots(F}_{\#n \text{ of } F}(I_n) + L_{n-1}) + L_{n-2}) + \cdots) + L_1) + L_0 \end{aligned}$$

Q4

Let $x = r\cos\theta \quad y = r\sin\theta$

since we change variable,

by Jacobian matrix

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow u = x\cos\theta + y\sin\theta$$

$$v = -xr\sin\theta + yr\cos\theta$$

since r is just scale,

Let $v = -x\sin\theta + y\cos\theta$

$u = x\cos\theta + y\sin\theta$

First, WTP: $u \cdot v = 0$

$$u = x\cos\theta + y\sin\theta = r\cos^2\theta + r\sin^2\theta = r$$

$$v = -x\sin\theta + y\cos\theta = -r\cos\theta\sin\theta + r\sin\theta\cos\theta$$

$$= 0$$

$$\Rightarrow u \cdot v = 0$$

Then, we want to prove: $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}$

by chain rule:

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}$$

$$= \frac{\partial f}{\partial u}\sin\theta + \frac{\partial f}{\partial v}\cos\theta$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial u}\sin\theta + \frac{\partial f}{\partial v}\cos\theta\right)$$

$$= \frac{\partial}{\partial y}\frac{\partial f}{\partial u}\sin\theta + \frac{\partial}{\partial y}\frac{\partial f}{\partial v}\cos\theta$$

Since $\frac{\partial}{\partial y}\frac{\partial f}{\partial u} = \frac{\partial}{\partial u}\frac{\partial f}{\partial y}$ $\quad \frac{\partial}{\partial y}\frac{\partial f}{\partial v} = \frac{\partial}{\partial v}\frac{\partial f}{\partial y}$

$$\frac{\partial}{\partial u}\frac{\partial f}{\partial y} = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u}\sin\theta + \frac{\partial f}{\partial v}\cos\theta\right)$$

$$= \frac{\partial^2 f}{\partial u^2}\sin\theta + \frac{\partial^2 f}{\partial u \partial v}\cos\theta$$

$$\frac{\partial}{\partial v}\frac{\partial f}{\partial y} = \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial u}\sin\theta + \frac{\partial f}{\partial v}\cos\theta\right)$$

$$= \frac{\partial^2 f}{\partial u \partial v}\sin\theta + \frac{\partial^2 f}{\partial v^2}\cos\theta$$

plug those in back to $\frac{\partial^2 f}{\partial y^2}$

$$\Rightarrow \frac{\partial^2 f}{\partial y^2} = \left(\frac{\partial^2 f}{\partial u^2}\sin\theta + \frac{\partial^2 f}{\partial u \partial v}\cos\theta\right)\sin\theta +$$

$$\left(\frac{\partial^2 f}{\partial u \partial v}\sin\theta + \frac{\partial^2 f}{\partial v^2}\cos\theta\right)\cos\theta$$

$$= \frac{\partial^2 f}{\partial u^2}\sin^2\theta + \frac{\partial^2 f}{\partial v^2}\cos^2\theta + 2\frac{\partial^2 f}{\partial u \partial v}\sin\theta\cos\theta$$

Second → Compute $\frac{\partial^2 f}{\partial x^2}$

by chain rule

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$$

$$= \frac{\partial f}{\partial u}\cos\theta - \frac{\partial f}{\partial v}\sin\theta$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}\cos\theta - \frac{\partial f}{\partial v}\sin\theta\right)$$

$$= \frac{\partial}{\partial x}\frac{\partial f}{\partial u}\cos\theta - \frac{\partial}{\partial x}\frac{\partial f}{\partial v}\sin\theta$$

Similar to above

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial u} = \frac{\partial}{\partial u}\frac{\partial f}{\partial x} \qquad \frac{\partial}{\partial x}\frac{\partial f}{\partial v} = \frac{\partial}{\partial v}\frac{\partial f}{\partial x}$$

$$\frac{\partial}{\partial u}\frac{\partial f}{\partial x} = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u}\cos\theta - \frac{\partial f}{\partial v}\sin\theta\right)$$

$$= \frac{\partial^2 f}{\partial u^2}\cos\theta - \frac{\partial^2 f}{\partial u \partial v}\sin\theta$$

$$\frac{\partial}{\partial v}\frac{\partial f}{\partial x} = \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial u}\cos\theta - \frac{\partial f}{\partial v}\sin\theta\right)$$

$$= \frac{\partial^2 f}{\partial u \partial v}\cos\theta - \frac{\partial^2 f}{\partial v^2}\sin\theta$$

plug those in back

$$\frac{\partial^2 f}{\partial x^2} = \left(\frac{\partial^2 f}{\partial u^2}\cos\theta - \frac{\partial^2 f}{\partial u \partial v}\sin\theta\right)\cos\theta - \left(\frac{\partial^2 f}{\partial u \partial v}\cos\theta - \frac{\partial^2 f}{\partial v^2}\sin\theta\right)\sin\theta$$

$$= \frac{\partial^2 f}{\partial u^2}\cos^2\theta + \frac{\partial^2 f}{\partial v^2}\sin^2\theta - 2\frac{\partial^2 f}{\partial u \partial v}\sin\theta\cos\theta$$

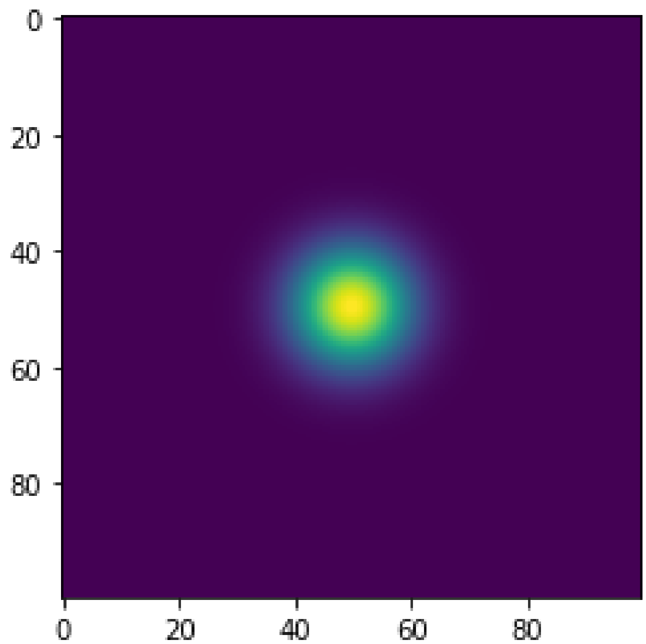
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \left(\frac{\partial^2 f}{\partial u^2}\cos^2\theta + \frac{\partial^2 f}{\partial u^2}\sin^2\theta\right) + \left(\frac{\partial^2 f}{\partial v^2}\sin^2\theta + \frac{\partial^2 f}{\partial v^2}\cos^2\theta\right) + 2\frac{\partial^2 f}{\partial u \partial v}\sin\theta\cos\theta - 2\frac{\partial^2 f}{\partial u \partial v}\sin\theta\cos\theta$$

$$= \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}$$

Presentation

Step 1:

gkern(100,100,20)



gkern(100,100,30)

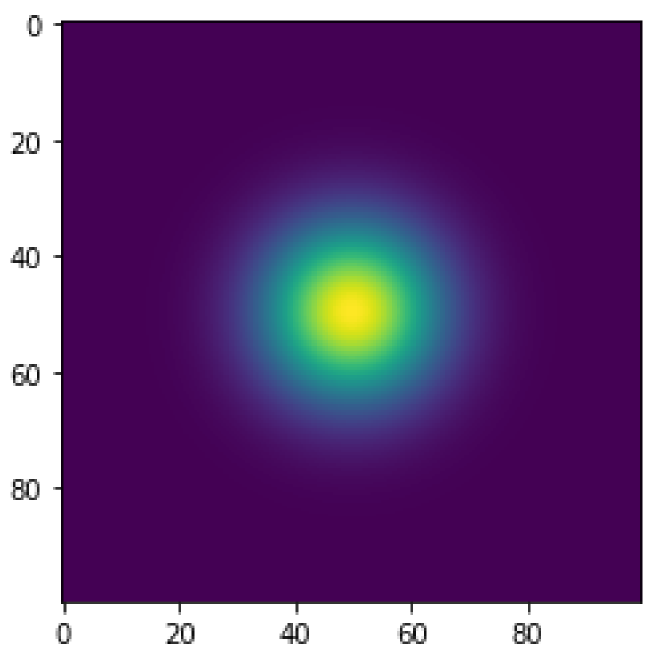
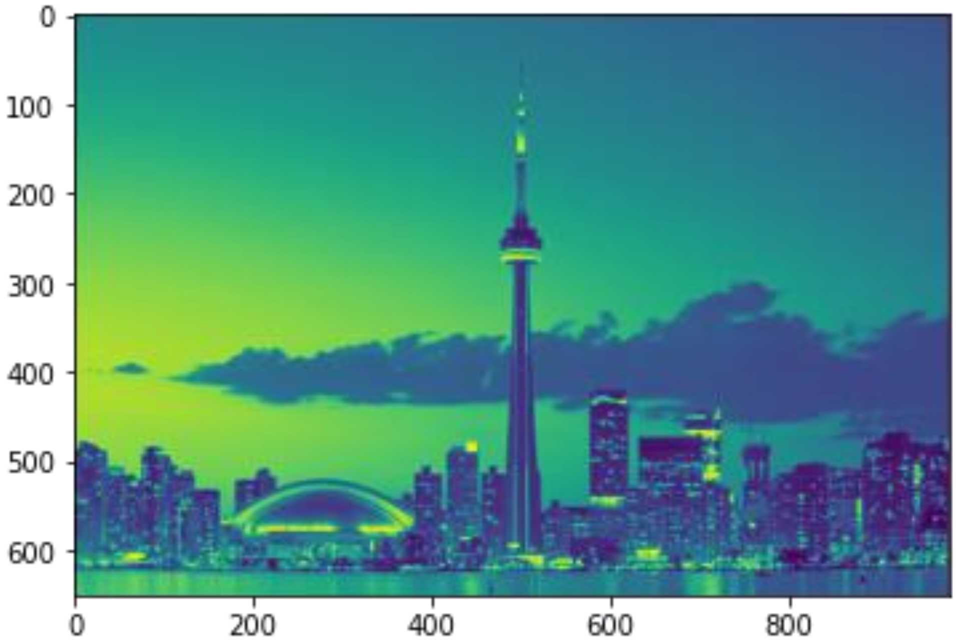


Image 1(convert to grayscale)



Edge detection image1

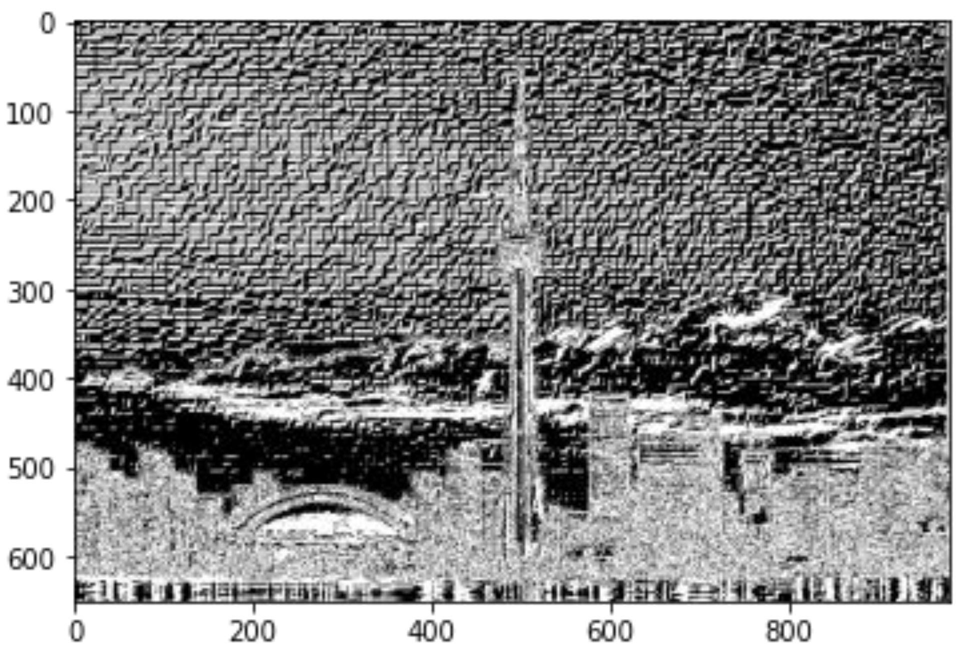
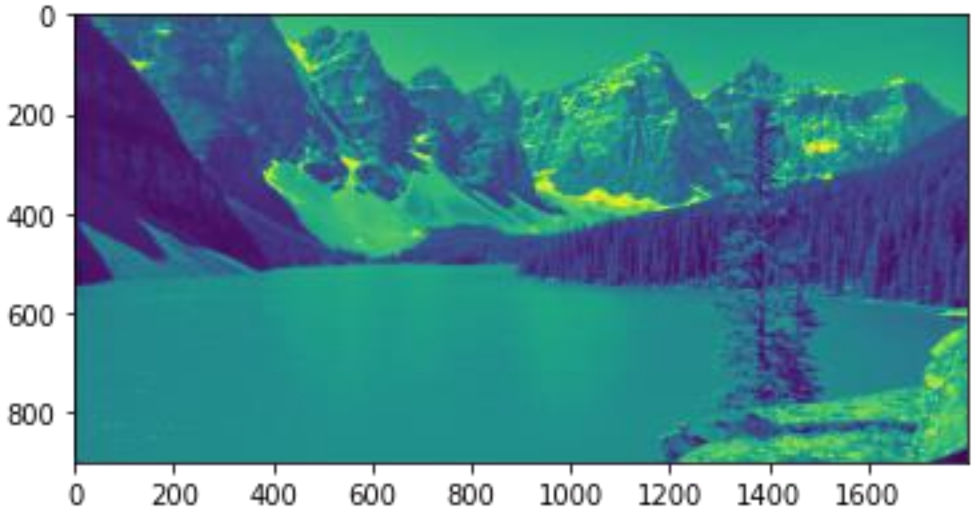


Image 2(convert to grayscale)



Edge detection image2

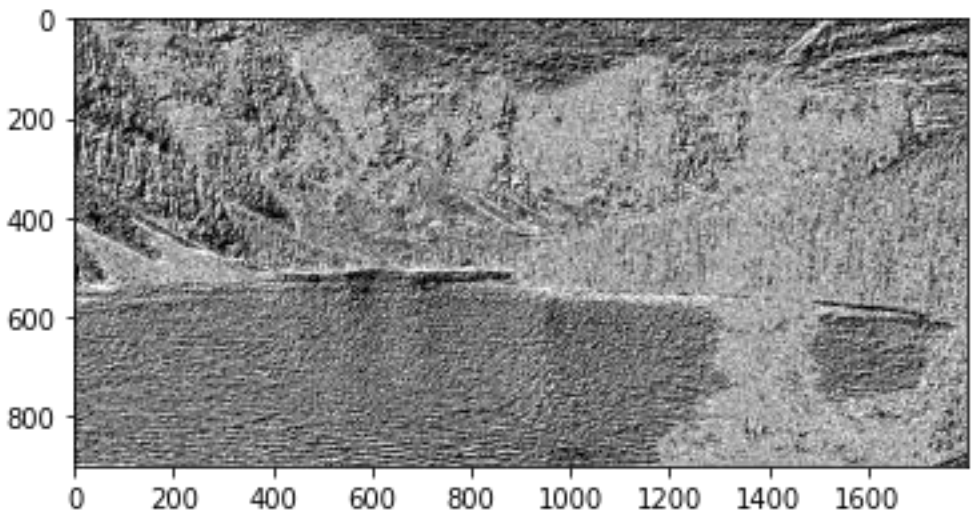
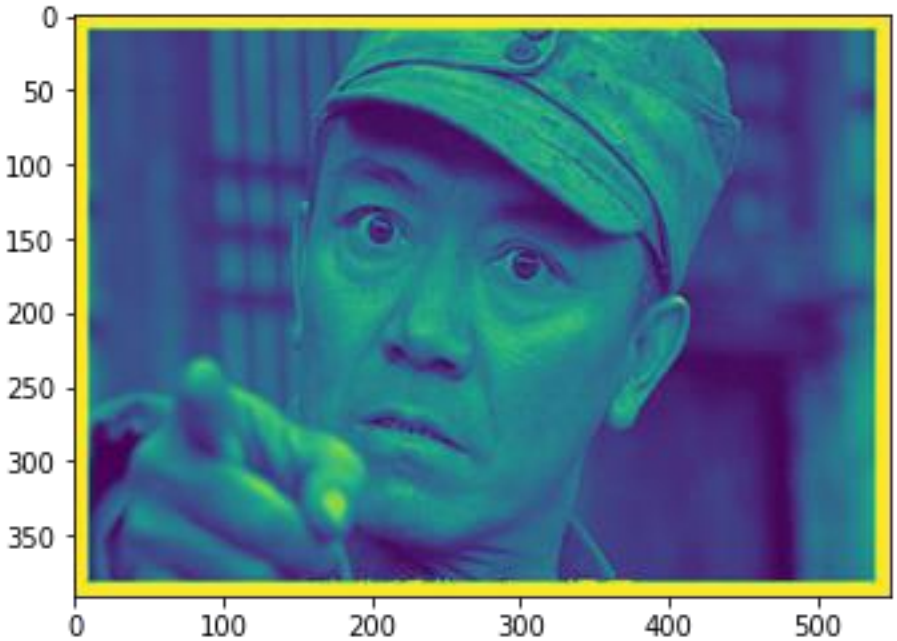
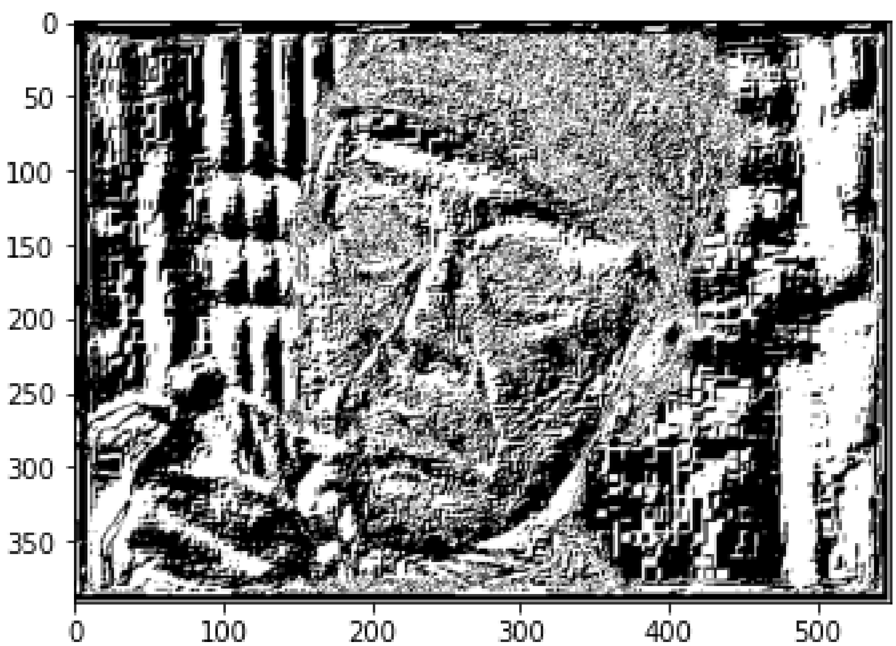


Image 3(grayscale)



Edge detection image3



Discussion

Weakness:

1. When the color contrast in the specific area is not high enough, in such a case, the edge detection output becomes blurred. The edges cannot be shown clearly. For example, in the bottom area of image1, edges of the building are not clear to separate the building since the original image does not have high contrast. For image2, the upper area, the edges of the sky and mountain does not separate clearly.
2. The time consuming, it would spend some time to do the iterations in the algorithm and every pixel needs to have comparisons, so it takes long.

Strength:

1. When the color contrast is high enough, the edges are very clear, for example the CN tower in the image1.
2. The threshold algorithm would not change the original image and just operated on the original image to find the edge detection output
3. The threshold value is found automatically so we don’t need to determine by ourselves.