Subset Sum

- Problem
  - Input: Set of integers $S = \{w_1, \dots, w_n\}$, integer W
  - Question: Is there $S' \subseteq S$ that adds up to exactly W?

- Example
  - $S = \{1, 4, 16, 64, 256, 1040, 1041, 1093, 1284, 1344\}$, $W = 3754$?
  - Yes!
    - $1 + 16 + 64 + 256 + 1040 + 1093 + 1284 = 3754$

Subset Sum

- Claim: Subset Sum is in NP

- Recall: We need to show that there is a polynomial-time algorithm which
  - Can accept every YES instance with the right polynomial-size advice
  - Will not accept a NO instance with any advice

- Advice: the actual subset S'
- Algorithm: check that S' is indeed a subset of S and sums to W
- Simple!

Subset Sum

- Claim: Exact 3SAT $\leq_p$ Subset Sum

- Given a formula φ of Exact 3SAT, we want to construct (S, W) of Subset Sum with the same answer
- In the table in the following slide:
  - Columns are for variables and clauses
  - Each row is a number in S, represented in decimal
  - Number for literal ℓ : has 1 in its variable column and in the column of every clause where that literal appears
    - Number selected = literal set to TRUE
  - "Dummy" rows: can help make the sum in a clause column 4 if and only if at least one literal is set to TRUE

Subset Sum

- Claim: Exact 3SAT $\leq_p$ Subset Sum

$$C_1 = \bar{x} \vee y \vee z$$
$$C_2 = x \vee \bar{y} \vee z$$
$$C_3 = \bar{x} \vee \bar{y} \vee \bar{z}$$

Decimal representation

	x	y	z	C_1	C_2	C_3
x	1	0	0	0	1	0
¬ x	1	0	0	1	0	1
y	0	1	0	1	0	0
¬ y	0	1	0	0	1	1
z	0	0	1	1	1	0
¬ z	0	0	1	0	0	1
	0	0	0	1	0	0
	0	0	0	2	0	0
	0	0	0	0	1	0
	0	0	0	0	2	0
	0	0	0	0	0	1
	0	0	0	0	0	2
W	1	1	1	4	4	4

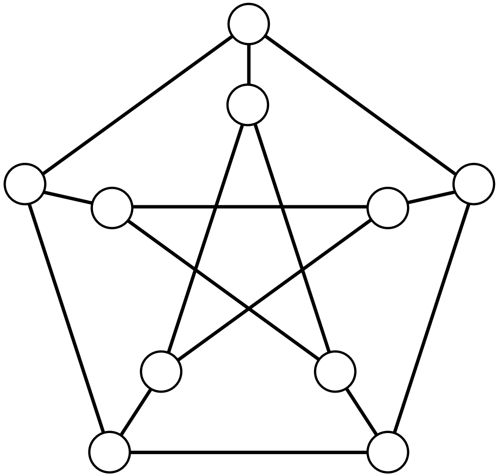
dummies to get clause columns to sum to 4

Subset Sum

- Note
  - The Subset Sum instance we constructed has "large" numbers
    - Their values are exponentially large ($\sim 10^{\#variables+\#clauses}$)
    - But the number of bits required to write them is polynomial
  - Can we hope to construct Subset Sum instance with numbers whose values are only $poly(\#variables, \#clasuses)$ large?
    - Unlikely, as that would prove $P = NP$!
    - Like Knapsack, Subset Sum can be solved in pseudo-polynomial time
      - That is, in polynomial time if the numbers are only polynomially large in value

3-Coloring

- Problem
  - Input: Undirected graph $G = (V, E)$
  - Question: Can we color each vertex of G using at most three colors such that no two adjacent vertices have the same color?



3-Coloring

- Claim: 3-coloring is in NP

- Recall: We need to show that there is a polynomial-time algorithm which
  - Can accept every YES instance with the right polynomial-size advice
  - Will not accept a NO instance with any advice

- Advice: colors of the nodes in a valid 3-coloring
- Algorithm: check that this is a valid 3-coloring
- Simple!

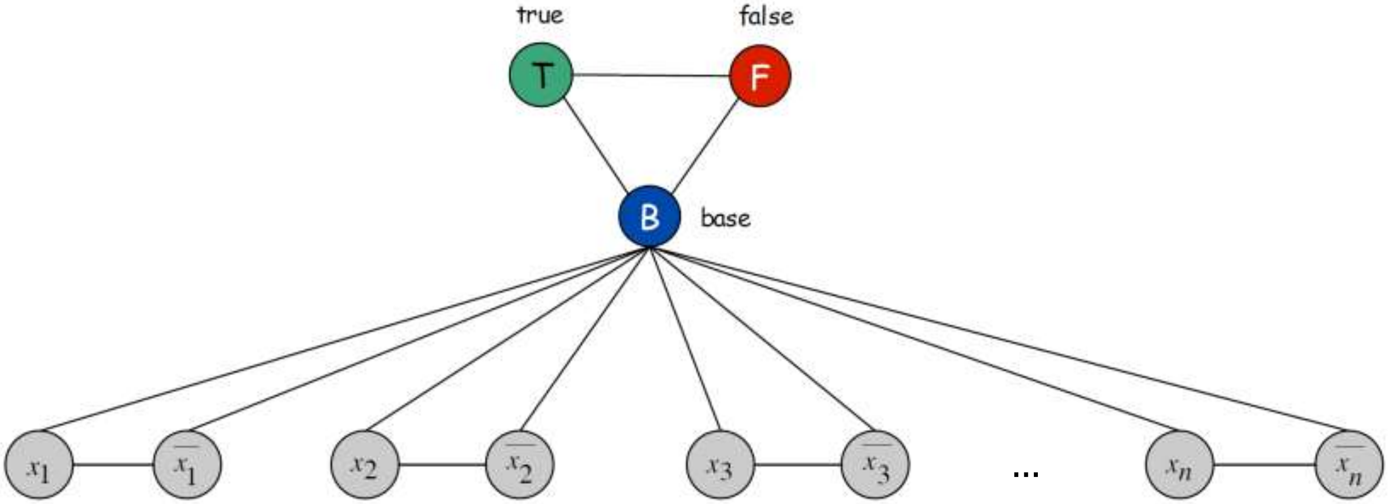
3-Coloring

- Claim: Exact 3SAT $\leq_p$ 3-Coloring

  - Given an Exact 3SAT formula φ, we want to construct a graph G such that G is 3-colorable if and only if φ has a satisfying assignment
  - G will have the following nodes:
    - Type 1: true, false, base, one for each x_i, one for each $\overline{x_i}$
    - Type 2: additional nodes for each clause C_j
  - 1-1 correspondence between valid 3-colorings of type 1 nodes and valid truth assignments:
    - All literals with the same color as "true" node are set to true
    - All literals with the same color as "false" node are set to false
  - Claim: Fix any colors of type 1 nodes. There exists a valid 3-coloring of G giving these colors to type 1 nodes if and only if the corresponding truth assignment is satisfying for φ.

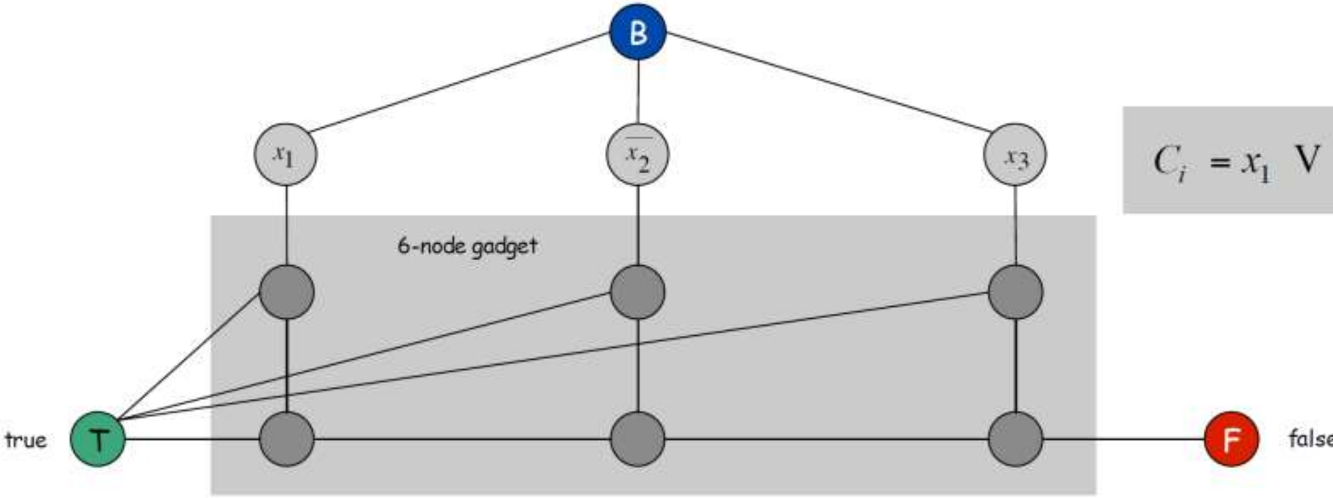
3-Coloring

- Create 3 new nodes T, F, and B, and connect them in a triangle
- Create a node for each literal, connect it to its negation and to B
- T-F-B must have different colors, and so must B-x_i-$\bar{x}_i$
  - Each literal has the color of T or F; its negation has the other color
  - Valid 3-coloring ⇔ valid truth assignment (set all with color T to true)



3-Coloring

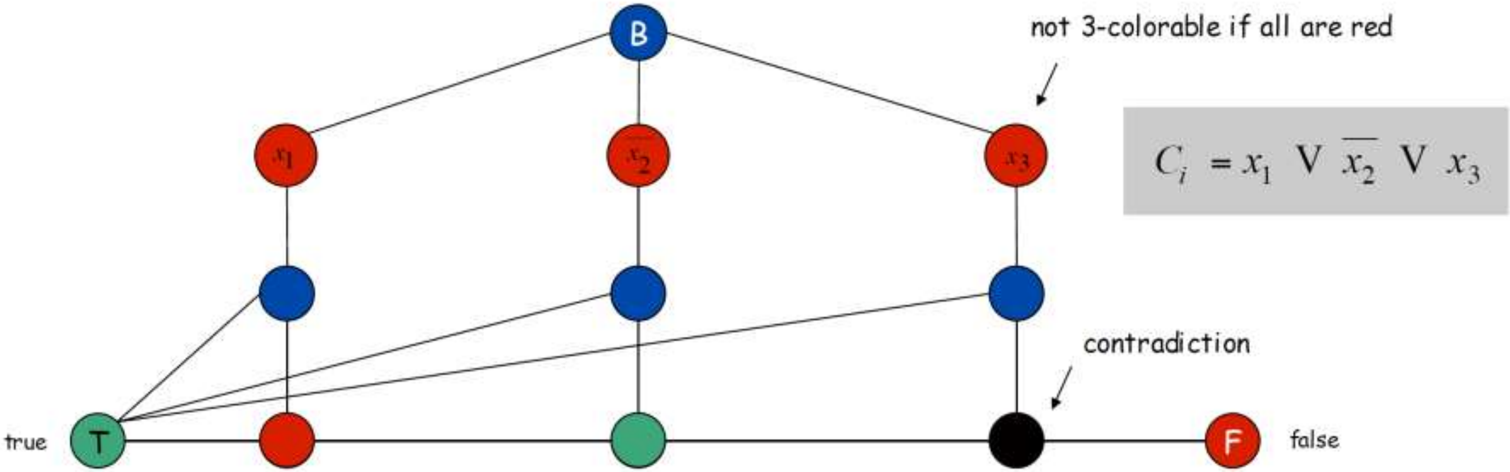
- We also need valid 3-coloring ⇔ *satisfying* truth assignment
  - For each clause, add the following gadget with 6 nodes and 13 edges
  - Claim: Clause gadget is 3-colorable ⇔ at least one of the nodes corresponding to the literals in the clause is assigned color of T



$C_i = x_1 \vee \overline{x_2} \vee x_3$

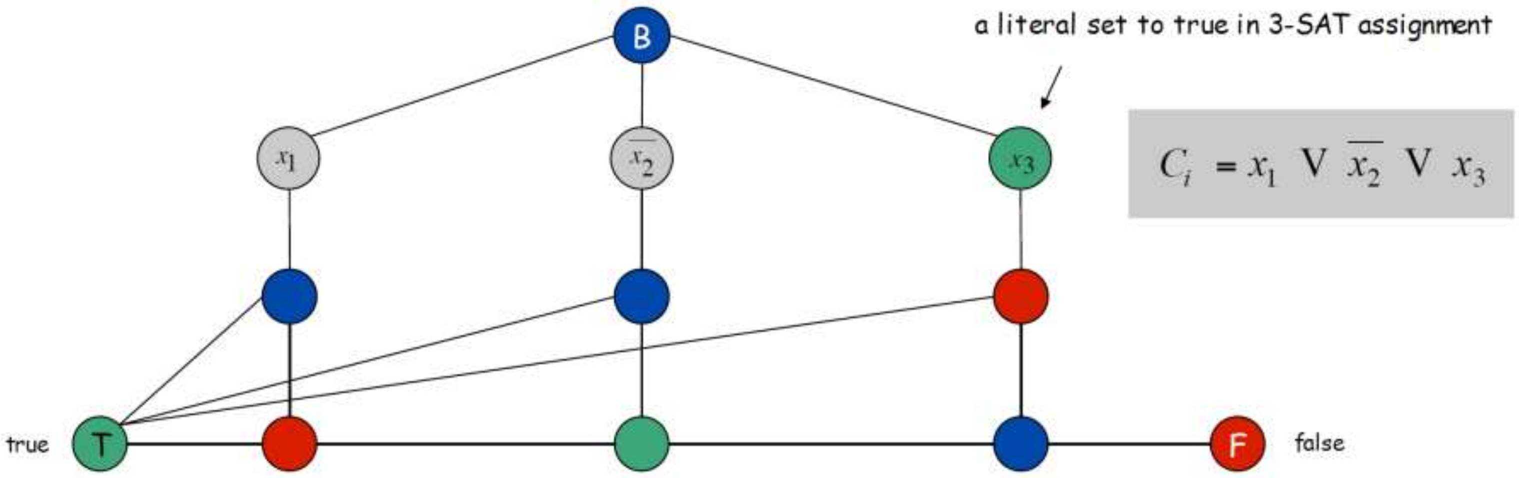
3-Coloring

- Claim: Valid 3-coloring ⇒ truth assignment satisfies φ
  - Suppose a clause C_i is not satisfied, so all its three literals must be F
  - Then the 3 nodes in top layer must be B
  - Then the first two nodes in bottom layer must be F and T
  - No color left for the remaining node ⇒ contradiction!



3-Coloring

- We just proved: valid 3-coloring ⇒ satisfying assignment
- Claim: satisfying assignment ⇒ valid 3-coloring
  - Each clause has at least one literal with color T
  - Exercise: Regardless of which literal has color T and which color (T/F) the other literals have, the clause widget can always be 3-colored



Review of Reductions

- If you want to show that problem B is NP-complete
- Step 1: Show that B is in NP
  - Some polynomial-size advice should be sufficient to verify a YES instance in polynomial time
  - No advice should work for a NO instance
  - Usually, the solution of the “search version” of the problem works
    - But sometimes, the advice can be non-trivial
    - For example, to check LP optimality, one possible advice is the values of both primal and dual variables, as we saw in the last lecture

Review of Reductions

- If you want to show that problem B is NP-complete
- Step 2: Find a known NP-complete problem A and reduce it to B (i.e., show A $\leq_p$ B)
  - This means taking an arbitrary instance of A, and solving it in polynomial time using an oracle for B
    - Caution 1: Remember the direction. You are "reducing known NP-complete problem to your current problem".
    - Caution 2: The size of the B-instances you construct should be polynomial in the size of the original A-instance
  - This would show that if B can be solved in polynomial time, then A can be as well
  - Some reductions are trivial, some are notoriously tricky…

Binary Integer Linear Programming (BILP)

- Problem
  - Input: $c \in \mathbb{R}^n, b \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}, k \in \mathbb{R}$
  - Question: Does there exist $x \in \{0,1\}^n$ such that $c^T x \geq k$ and $Ax \leq b$?

- Decision variant of "maximize $c^T x$ subject to $Ax \leq b$" but instead of any $x \in \mathbb{R}^n$ with $x \geq 0$, we are restricting x to binary.

- Does restricting search space make the problem easier or harder?
  - This is actually NP-complete!

BILP Feasibility

- An even simpler problem
  - Special case where $c = k = 0$, so $c^T x \geq k$ is always true

- Problem
  - Input: $b \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}$
  - Question: Does there exist $x \in \{0,1\}^n$ such that $Ax \leq b$?

  - Just need to find a feasible solution
  - This is still NP-complete!

BILP Feasibility

- Claim: BILP Feasibility is in NP

  - Recall: We need to show that there is a polynomial-time algorithm which
    - Can accept every YES instance with the right polynomial-size advice
    - Will not accept a NO instance with any advice

  - Advice: simply a vector x satisfying $Ax \leq b$
  - Algorithm: Check if $Ax \leq b$
  - Simple!

BILP Feasibility

- Claim: Exact 3SAT $\leq_p$ BILP Feasibility

  - Take any formula φ of Exact 3SAT
  - Create a binary variable x_i for each variable x_i in φ
    - We'll represent its negation $\bar{x}_i$ with $1 - x_i$
  - For each clause C, we want at least one of its three literals to be TRUE
    - Just make sure their sum is at least 1
    - E.g., $C = x_1 \lor \bar{x}_2 \lor \bar{x}_3 \Rightarrow x_1 + (1 - x_2) + (1 - x_3) \geq 1$
  - Easy to check that
    - this is a polynomial reduction
    - Resulting system has a feasible solution if and only if φ is satisfiable

ILP Feasibility

- Problem
  - Input: $b \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}$
  - Question: Does there exist $x \in \mathbb{Z}^n$ such that $Ax \leq b$?

- To prove that this is NP-hard, there is an obvious reduction from BILP feasibility to ILP feasibility

- What about membership in NP?
- Advice: simply a vector x satisfying $Ax \leq b$
- Algorithm: Check if $Ax \leq b$
- Simple?
  - No, not clear if, in every YES instance, there's a polynomial-length "advice" vector x satisfying $Ax \leq b$

On the Complexity of Integer Programming

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ABSTRACT. A simple proof that integer programming is in $\mathcal{NP}$ is given. The proof also establishes that there is a pseudopolynomial-time algorithm for integer programming with any (fixed) number of constraints.

KEY WORDS AND PHRASES: integer linear programming, $\mathcal{P}$, $\mathcal{NP}$, pseudopolynomial algorithms

CR CATEGORIES: 5 25, 5.3, 5.4

So far…

- To establish NP-completeness of problem B, we always reduced Exact 3SAT to B
  - But we can reduce any other problem A that we have already established to be NP-complete
  - Sometimes this might lead to a simpler reduction because A might already be "similar" to B

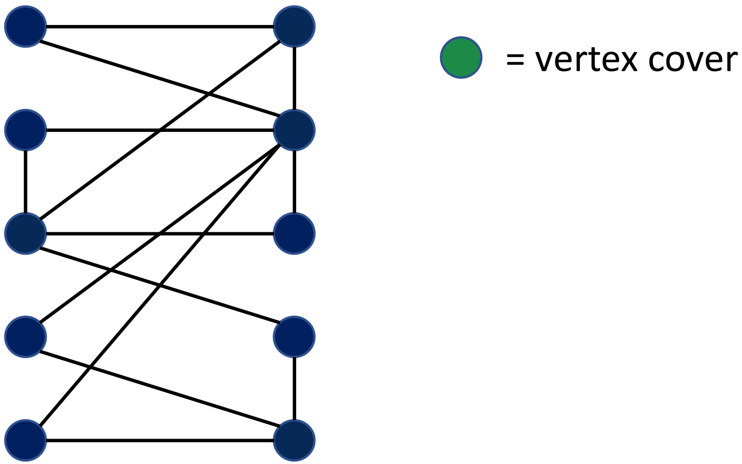
- Let's see an example!

Vertex Cover

- Problem
  - Input: Undirected graph $G = (V, E)$, integer k
  - Question: Does there exist a vertex cover of size k?
    - That is, does there exist $S \subseteq V$ with $|S| = k$ such that every edge is incident to at least one vertex in S?

Example:

- Does this graph have a vertex cover of size 4?
  - Yes!
- Does this graph have a vertex cover of size 3?
  - No!



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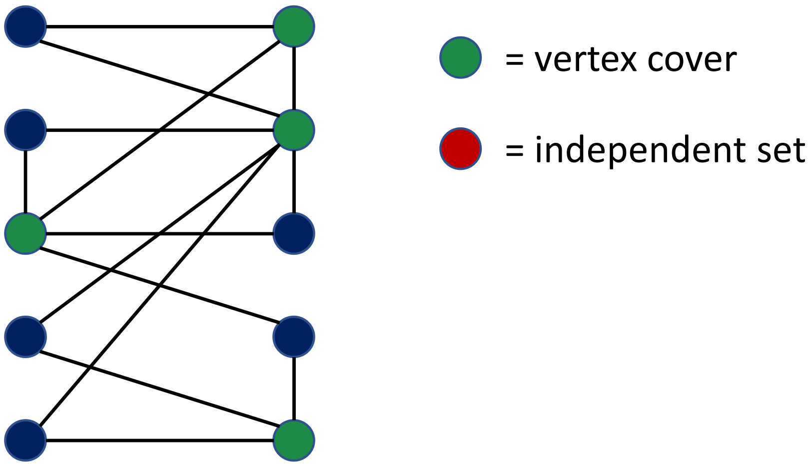
    - That is, does there exist $S \subseteq V$ with $|S| = k$ such that every edge is incident to at least one vertex in S?

Question:
- Did we see this graph in the last lecture?
  - Yes!
  - For independent set of size 6



Vertex Cover

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Question:

- Did we see this graph in the last lecture?
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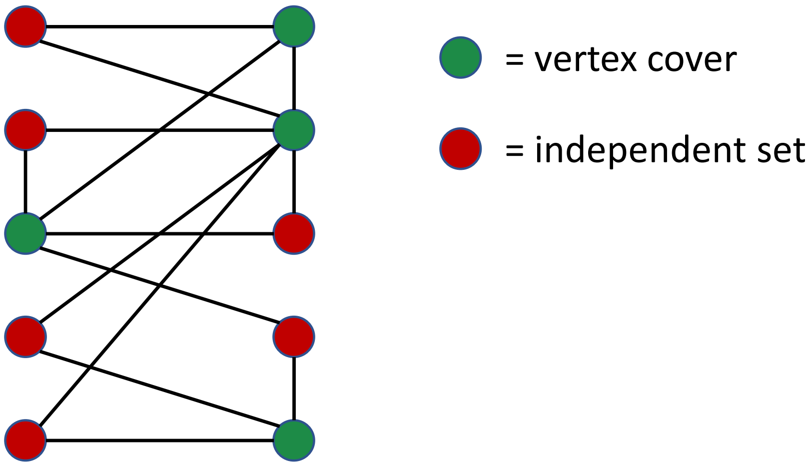


Vertex Cover

- Vertex cover and independent set are intimately connected!
- Claim: G has a vertex cover of size k if and only if G has an independent set of size $n - k$
- Stronger claim: S is a vertex cover if and only if $V \setminus S$ is an independent set

Vertex Cover

- Claim: S is a vertex cover if and only if $V \setminus S$ is an independent set
- Proof:
  - S is a vertex cover
  - IFF: For every $(u, v) \in E$, at least one of $\{u, v\}$ is in S
  - IFF: For every $(u, v) \in E$, at most one of $\{u, v\}$ is in $V \setminus S$
  - IFF: No two vertices of $V \setminus S$ are connected by an edge
  - IFF: $V \setminus S$ is an independent set ■

Vertex Cover

- Claim: Independent Set $\leq_p$ Vertex Cover

  - Take an arbitrary instance (G, k) of Independent Set
  - We want to check if there is an independent set of size k
  - Just convert it to the instance $(G, n - k)$ of Vertex Cover
  - Simple!
    - A reduction from Exact 3SAT would have basically repeated the reduction we already did for Exact 3SAT $\leq_p$ Independent Set

  - Note: I didn't argue that Vertex Cover is in NP
    - This is simple as usual. Just give the actual vertex cover as the advice.

Set Cover

- Problem
  - Input: A universe of elements U, a family of subsets S, and an integer k
  - Question: Do there exist k sets from S whose union is U?

- Example
  - $U = \{1,2,3,4,5,6,7\}$
  - $S = \{\{1,3,7\}, \{2,4,6\}, \{4,5\}, \{1\}, \{1,2,6\}\}$
  - $k = 3$? Yes! $\{\{1,3,7\}, \{4,5\}, \{1,2,6\}\}$
  - $k = 2$? No!

Set Cover

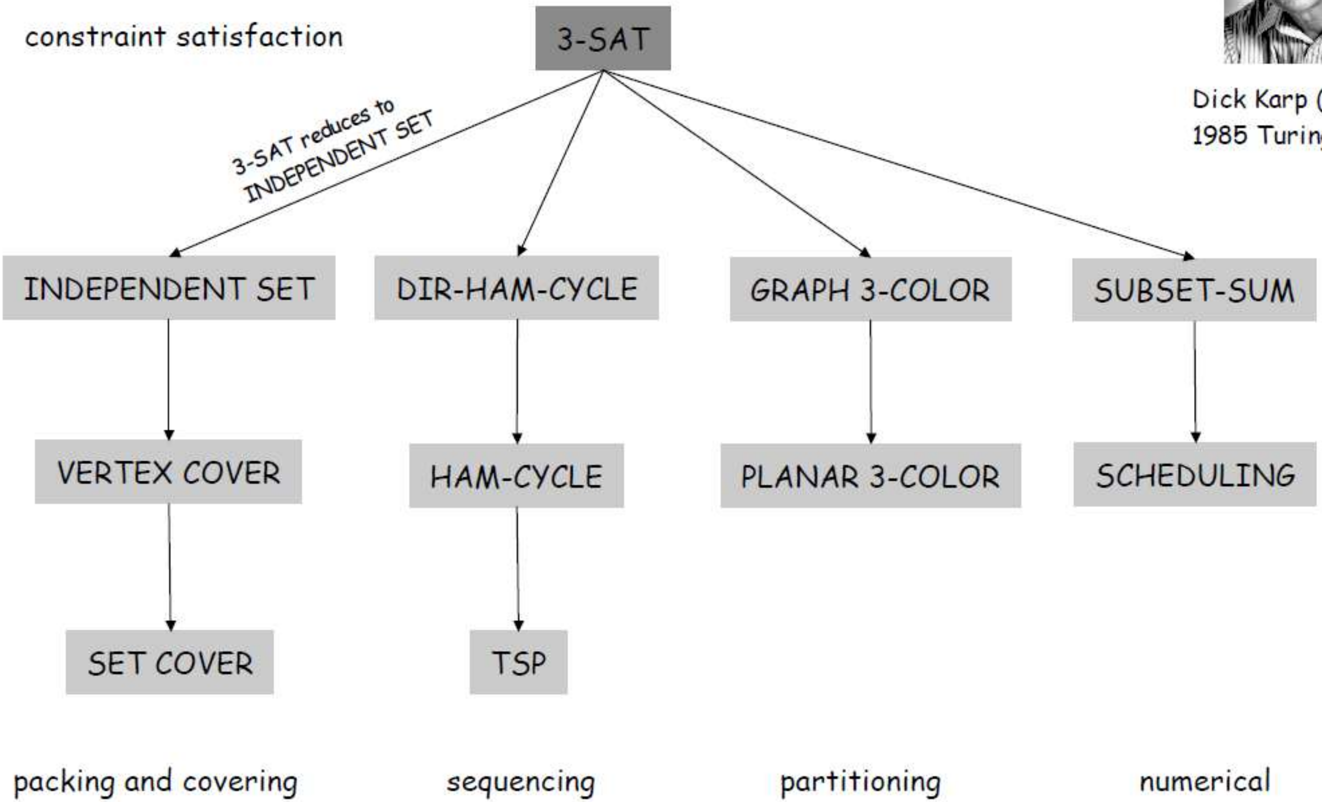
- Claim: Set Cover is in NP

  - Easy. Let the advice be the actual k sets whose union is U.

- Claim: Vertex Cover $\leq_p$ Set Cover

  - Given an instance of vertex cover with graph $G = (V, E)$ and integer k, create the following set cover instance
    - Set $U = E$
    - For each $v \in V$, S contains a set S_v of all the edges incident on v
    - Selecting k set whose union is U = selecting k vertices such that union of their incident edges covers all edges
    - Hence, the two problems obviously have the same answer

Polynomial-Time Reductions



Dick Karp (1972)
1985 Turing Award

Cook-Levin Theorem

- We did not prove "the first NP-completeness" result

- Theorem: Exact 3SAT is NP-complete
  - We need to prove this without using any other "known NP-complete" problem
  - We want to directly show that *every problem in NP* can be reduced to Exact 3SAT

- We will first reduce any NP problem to SAT, and then reduce SAT to Exact 3SAT

Cook-Levin Theorem

- We're not going to prove it in this class, but the key idea is as follows

  - If a problem is in NP, then ∃ Turing machine $T(x, y)$ which
    - takes as input a problem instance x and an advice y of size $p(|x|)$
    - verifies in $q(|x|)$ time whether x is a YES instance
    - both p and q are polynomials

  - x is a YES instance iff $\exists y\ T(x, y) = ACCEPT$

Cook-Levin Theorem

NOT IN SYLLABUS

- x is a YES instance iff $\exists y\, T(x, y) = ACCEPT$
  - We need to convert $\exists y\, T(x, y) = ACCEPT$ into whether a SAT formula φ is satisfiable
- Recall that a Turing machine T consists of a memory tape, a head pointer, a state, and a transition function
- What describes T at any given step of its computation?
  - What is written in each cell of its memory tape?
  - Which cell of the tape is the read/write head currently pointing to?
  - What state is the Turing machine in?

Cook-Levin Theorem

NOT IN SYLLABUS

- x is a YES instance iff $\exists y\ T(x, y) = ACCEPT$
  - We need to convert $\exists y\ T(x, y) = ACCEPT$ into $\exists z\ \varphi(z) = TRUE$, where z consists of Boolean variables and φ is a SAT formula

- Variables:
  - $T_{i,j,k}$ = True if machine's tape cell i contains symbol j at step k of the computation
  - $H_{i,k}$ = True if the machine's read/write head is at tape cell i at step k of the computation
  - $Q_{q,k}$ = True if machine is in state q at step k of the computation

  - Cell index i and computation step k only need to be polynomially large as T works in polynomial time

Cook-Levin Theorem

NOT IN SYLLABUS

- x is a YES instance iff $\exists y\ T(x, y) = ACCEPT$
  - We need to convert $\exists y\ T(x, y) = ACCEPT$ into $\exists z\ \varphi(z) = TRUE$, where z consists of Boolean variables and φ is a SAT formula

- Clauses:
  - Express how the variables must be related using the transition function
  - Express that the Turing machine must reach the state $ACCEPT$ at some step of the computation

- This establishes that SAT is NP-complete.
- Next: SAT $\leq_p$ Exact 3SAT.

Cook-Levin Theorem

- Claim: SAT $\leq_p$ Exact 3SAT
  - Take an instance $\varphi = C_1 \wedge C_2 \wedge \cdots$ of SAT
  - Replace each clause with multiple clauses with exactly 3 literals each

  - For a clause with one literal, $C = \ell_1$:
    - Add two variables z_1, z_2, and replace C with four clauses

      $(\ell_1 \vee z_1 \vee z_2) \wedge (\ell_1 \vee \bar{z}_1 \vee z_2) \wedge (\ell_1 \vee z_1 \vee \bar{z}_2) \wedge (\ell_1 \vee \bar{z}_1 \vee \bar{z}_2)$
    - Verify that this is logically equivalent to ℓ_1

  - For a clause with two literals, $C = (\ell_1 \vee \ell_2)$:
    - Add variable z_1 and replace it with the following:

$$(\ell_1 \vee \ell_2 \vee z_1) \wedge (\ell_1 \vee \ell_2 \vee \bar{z}_1)$$

    - Verify that this is logically equal to $(\ell_1 \vee \ell_2)$

Cook-Levin Theorem

- Claim: SAT $\leq_p$ Exact 3SAT
  - For a clause with three literals, $C = \ell_1 \vee \ell_2 \vee \ell_3$:
    - Perfect. No need to do anything!
  - For a clause with 4 or more literals, $C = (\ell_1 \vee \ell_2 \vee \cdots \vee \ell_k)$:
    - Add variables $z_1, z_2, \ldots, z_{k-3}$ and replace it with:

      $$(\ell_1 \vee \ell_2 \vee z_1) \wedge (\ell_3 \vee \bar{z}_1 \vee z_2) \wedge (\ell_4 \vee \bar{z}_2 \vee z_3) \wedge \cdots$$
      $$\wedge (\ell_{k-2} \vee \bar{z}_{k-4} \vee z_{k-3}) \wedge (\ell_{k-1} \vee \ell_k \vee \bar{z}_{k-3})$$
    - Check:
      - If any ℓ_i is TRUE, then there exists an assignment of z variables to make this TRUE
      - If all ℓ_i are FALSE, then no assignment of z variables will make this TRUE