CSC373 Summer '22
Tutorial 5
July 7, 2022

Q1 Standard Form

Consider the following linear program.

max

$$\begin{aligned} \min\ & 4x + 3y - 6z \\ \text{s.t. } & y - 3z \geq 2x + 2 \\ & 3x + 2y + 5z = 10 \\ & x, z \geq 0 \end{aligned}$$

all variables on one side

only allow inequalities

y unconstrained

$\boxed{\text{let } y = \underbrace{y'}_{y \geq 0} - \underbrace{y''}_{y < 0}}$

Q1(a)

$$\underline{\max}\ -4x + 3(y'-y'') - 6z$$

$$\begin{aligned} \text{s.t. } & 2x - (y'-y'') + 3z \leq -2 && \gamma_1 \\ & 3x + 2(y'-y'') + 5z \leq 10 && \gamma_2 \\ & -3x - 2(y'-y'') - 5z \leq -10 && \gamma_3 \\ & \underline{x, z, y', y'' \geq 0} \end{aligned}$$

(a) Convert this LP into the standard form.

(b) Write the dual of the LP from Part (a).

Q2 Simple Scheduling with Prerequisites (SSP)

You are given n jobs with a list of durations $d_1, d_2, \ldots, d_n$. For every pair of jobs (i, j), you are also given a boolean $p_{i,j}$: if this is true, then job i must finish before job j can begin (i.e. job i is a prerequisite for job j).

Your goal is to find start times $s_1, s_2, \ldots, s_n$ for the jobs (no job can start earlier than time 0) such that the total time to complete all jobs is minimized while ensuring that the prerequisite constraints are met. Write a linear program to solve this problem.

Q3 Integer Linear Programming

Suppose you are writing down a binary integer linear program (i.e., an optimization problem with a linear objective, linear constraints, and each variable taking a value in $\{0, 1\}$). Three of the binary variables in your program are x, y, and z; you have already placed the constraint: $x, y, z \in \{0, 1\}$.

Now, you want to encode the following relationships between x, y, and z. Show how to do so using linear constraints. Briefly justify your answers.

(a) Logical AND, $z = x \wedge y$: You want z to be 1 whenever both x and y are 1, and 0 otherwise.

(b) Logical OR, $z = x \vee y$: You want z to be 1 whenever at least one of x and y is 1, and 0 otherwise.

(b) Logical NOT, $z = \neg x$: You want z to be 1 whenever x is 0, and 0 otherwise.

Q1 (a) $v = \begin{bmatrix} x \\ y' \\ y'' \\ z \end{bmatrix}$ $c = \begin{bmatrix} -4 \\ 3 \\ -3 \\ -6 \end{bmatrix}$ $A = \begin{bmatrix} 2 & -1 & 1 & 3 \\ 3 & 2 & -2 & 5 \\ -3 & -2 & 2 & -5 \end{bmatrix}$ $b = \begin{bmatrix} -2 \\ 10 \\ -10 \end{bmatrix}$

STD FORM LP: $\max \ \underline{c}^T v$

s.t. $Av \le \underline{b}$

(b) create variables for rows of A : $u = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix}$

STD DUAL LP $\min \ \underline{b}^T u$

s.t. $A^T u \ge \underline{c}$

$$\min \quad -2\gamma_1 + 10\gamma_4$$

$$\text{s.t.} \quad 2\gamma_1 + 3\gamma_4 \ge -4$$

$$\gamma_1 - 2\gamma_4 = 3$$

$$3\gamma_1 + 5\gamma_4 \ge 2 \qquad \gamma_{1,2,3} \ge 0$$

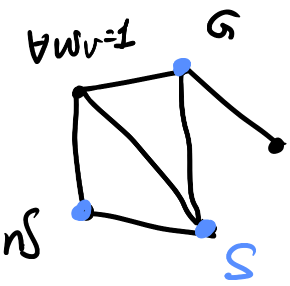
$\gamma_4 = \gamma_2 - \gamma_3$ unconstrained

LP relaxation of ILP(integer....)

1. VERTEX COVER $\in$ NP-complete

[IN] $G(V,E)$, $\forall u \in V$, $w_v \ge 0$ weight of v

[OUT] Find set $S \subseteq V$: $\forall uv \in E$ $|S \cap \{u,v\}| \ge 0$ and minimize sum of weights of vertices in S



ILP FOR VERTEX COVER (idea: let variable $x_v \in \{0,1\}$
0 if not in S
1 if in S)

OBJ: $\min \sum_{v \in V} w_v x_v$

ST: $x_u + x_v \ge 1 \quad \forall uv \in E \qquad x_v \in \{0,1\}$

$x_v \in [0,1]$ LP relaxation

LP Round/Relax

1. convert ILP $\to$ LP

2. Solve LP. Let $z = \begin{bmatrix} \\ \boxed{0.75} \\ \\ \end{bmatrix} z_v$

   (note: this is NOT a solution to the ILP yet)

   $\to$ If $z_v = 1/2$, then $y_v = 1$

3. round the entries of z $\quad y = \begin{bmatrix} \\ \boxed{0} \\ \\ \end{bmatrix} y_v$

   (this satisfies integrality constraint)

in particular, how do know $y_u + y_v \geq 1$?

we given $z_u + z_v \geq 1$ (because are LP solution)

WLOG, $z_u \geq z_v$ then $z_u \geq 1/2$

thus $y_u = \text{round}(z_u) = 1$

so regardless of y_v,

$$y_u + y_v \geq 1.$$

Why is y a valid solution of the ILP?

$\therefore$ y is a vertex cover

How good of a cover is y?

$OPT(ILP) \leq OBJ(y) = \sum_{v \in V} w_v \cdot y_v$

$\leq \sum_{v \in V} w_v \cdot (2 z_v)$

$= 2\,OPT(LP)$

$\leq 2\,OPT(ILP)$

$< 1/2$		$0 \leq 2z_v$
$z_v = 1/2$		$y_v = (1)/2$
$> 1/2$		$y_v = 1 \leq 2 \cdot (> \frac{1}{2})$

$\Rightarrow$ $OPT(ILP) \leq OBJ(y) \leq 2\,OPT(ILP)$

2-approximation