CSC373 Summer '22
Tutorial 6 Solutions
July 21, 2022

Q1 P vs NP vs co-NP

Are the following decision problems in P, NP, or co-NP? Give the strongest possible answer (i.e., if you can show that the decision problem is in P, use that instead of NP or co-NP).

1. **TRIANGLE**
Input: An undirected graph $G = (V, E)$.
Question: Does G contain a "triangle" (i.e., a subset of three vertices such that there is an edge between any two of them)?

2. **CLIQUE**
Input: An undirected graph $G = (V, E)$ and a positive integer k.
Question: Does G contain a k-clique (i.e., a subset of k vertices such that there is an edge between any two of them)?

3. **NON-ZERO**
Input: A set of integers S.
Question: Does every non-empty subset of S have non-zero sum?

4. **HAMILTONIAN-PATH (HP)**
Input: An undirected graph $G = (V, E)$.
Question: Does G contain a simple path that includes every vertex?

Solution to Q1

- P: One can brute-force and check all triplets of vertices for triangles in $O(n^3)$ time.
- NP: Given a k-clique as advice, a TM can verify in polynomial time whether all $\binom{k}{2}$ pairs of vertices have an edge. (Think why $O(k^2)$ is polynomial time.)
- co-NP: If the answer is NO (i.e. there is a non-empty subset of S with zero sum), then given such a subset as advice, a TM can verify in polynomial time that its sum is indeed zero and thus the answer to the problem is NO.
- NP: Given a Hamiltonian path as advice, a TM can verify that it includes every vertex exactly once and there is an edge between every pair of adjacent vertices (i.e. it is indeed a path).

Q2 NP-Completeness I

Consider the Hamiltonian Cycle (HC) problem, which is similar to the HP problem described above.

HAMILTONIAN-CYCLE (HC)
Input: An undirected graph $G = (V, E)$.
Question: Does G contain a simple cycle that includes every vertex?

(a) The textbook CLRS shows that HC is is NP-complete (Subsection 34.5.3). Give a reduction from HC to HP (i.e., HC $\leq_p$ HP) to prove HP is also NP-complete.

(b) Suppose instead that we knew HP is NP-complete and wanted to use it to show that HC is NP-complete. Give a reduction from HP to HC (i.e., HP $\leq_p$ HC).

Solution to Q2

(a) Given $G = (V, E)$ for the HC problem, create $G' = (V', E')$ for the HP problem as follows.

- Start with $G' = G$.
- Choose an arbitrary vertex $v \in V$ and add a copy of it (say v') to G': that is, for every $(v, u) \in E$, we also add $(v', u) \in E'$.
- Add a new "start" vertex s and a new "end" vertex t. Add edges (s, v) and (t, v') to G'.

The construction of G' from G is shown in the figure below.

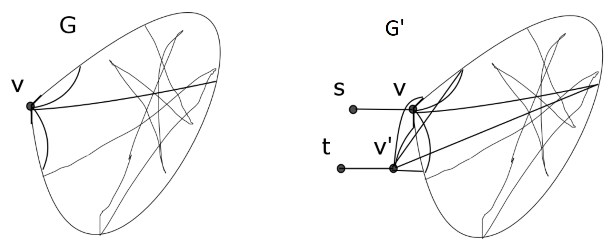

Figure 1: Citation: https://math.stackexchange.com/a/1290804

We prove that G has an HC if and only if G' has an HP.

First, suppose G has an HC. Then, we can construct an HP in G' as follows: we start from s, then visit v, then follow the HC in G, and when the HC is about to return to v from some vertex (say w), we instead go from w to v', and then v' to t.

Next, suppose G' has an HP. Then, because s and t have degree 1 each, they must be the two endpoints of the HP. Then, the second and the second to last vertices in the HP must v and v'. Then, a HC in G can be constructed by starting at v, following the HP in G', and instead of reaching v' from some vertex (say w), going from w to v to complete the cycle.

(b) Given $G = (V, E)$ for the HP problem, create $G' = (V', E')$ for the HC problem as follows.

- Start with $G' = G$.
- Add a new vertex u and add edges (u, v) for every $v \in V$.

Note that there is a 1-1 correspondence between HPs in G and HCs in G': $(v_1, \ldots, v_n)$ is an HP of G if and only if $(u, v_1, \ldots, v_n)$ is an HC of G' (where we return from v_n to u at the end).

Q3 NP-Completeness II

Consider the following problem. A multiset allows repeated elements.

PARTITION

Input: A multiset S containing positive integers.

Question: Is there a partition of S into two multisets (i.e. $S_1, S_2 \subseteq S$ such that $S_1 \cap S_2 = \emptyset$ and $S_1 \cup S_2 = S$) whose elements have equal sum?

(a) Prove that PARTITION is in NP.

(b) Prove that PARTITION is NP-hard through a reduction from SUBSET-SUM.

SUBSET-SUM

Input: A multiset S containing positive integers and an integer W.

Question: Is there a subset $S' \subseteq S$ whose elements sum to W?

Solution to Q3

(a) If the answer to the PARTITION problem is YES, then we can provide as advice a partition (S_1, S_2) of S with the two multisets having equal sum. Trivially, one can verify in polynomial time that (i) (S_1, S_2) is indeed a partition of S (i.e. the number of times each element appears in S is the sum of the number of times it appears in S_1 and S_2) and (ii) the sum of elements of S_1 is equal to the sum of elements of S_2.

(b) Take an instance (S, W) of SUBSET-SUM and construct an instance of PARTITION as follows. Let T be the sum of all elements of S. Then, we let the multiset of the PARTITION instance be $S^* = S \cup \{|T - 2W|\}$. Let us now prove that this is a valid reduction. First, the reduction clearly takes polynomial time to construct.

Next, suppose the answer to the SUBSET-SUM instance is YES. That is, there exists $S' \subseteq S$ with its sum of elements being W. Then, the sum of elements of $S \setminus S'$ must be $T - W$. Note that the absolute difference between the two sums is $|T - 2W|$. Hence, adding the new element $|T - 2W|$ to the set with the smaller sum makes the sums of the two sets equal, i.e., it constructs a partition of S^*, implying that the answer to the PARTITION instance is also YES.

Conversely, suppose the answer to the PARTITION instance is YES. That is, there is a partition of S^* into two subsets with equal sum of $\frac{T+|T-2W|}{2}$. If $T \leq 2W$, then both parts have sum W. In this case, choosing the part that doesn't contain the added element $|T - 2W|$ gives a YES answer to SUBSET-SUM instance. If $T > 2W$, then both parts have sum $T - W$. In this case, choosing the part that has the added element $T - 2W$ and removing this element gives a subset of S with sum $(T - W) - (T - 2W) = W$, again implying that the SUBSET-SUM instance has answer YES.