CSC373 Summer '22
Practice Problems on Linear Programming

Q1 [20 Points] Travel Planning is Hard

Anu is making holiday travel plans. There are 6 cities that she is interested in visiting. For each $k \in \{1, \ldots, 6\}$, it would cost c_k to incorporate city k into the itinerary but she would gain h_k units of happiness from visiting city k (both amounts are regardless of which other cities are on the itinerary). Unfortunately, she has a limited budget of B, so she may not be able to visit all the cities.

There are also other considerations. Cities 3 and 5 are too similar to each other, so Anu prefers to visit at most one of them. The same is true for cities 1, 5, and 6. Some cities can only be enjoyed in the fullest if visited together with some other cities. Anu doesn't want to visit cities 5 or 6, unless at least one of cities 3 and 4 is also on the itinerary. Finally, she wants to visit at least one and at most three among the cities $1, 2, 4, 5, 6$.

Anu obviously wants to find an itinerary (a subset of cities to visit) that will maximize her happiness, given the constraints listed above.

(a) [10 Points] Formulate this problem as a binary integer linear program. Here, "binary" means that every variable x_k in your program should be in $\{0, 1\}$.

(b) [10 Points] Imagine a "relaxation" of the program from part (a), where you allow each variable x_k to take any non-negative real value (i.e., $x_k \geq 0$). This relaxation becomes a linear program. Write the dual of this linear program.

Q2 [20 Points] Tricky Transportation

Anu has begun her much awaited journey. While she planned a bit before leaving, she didn't work out all the details in advance. She now faces the problem of planning her transportation.

There are n buses running along a road (think of it as the real axis). Each bus i starts at location s_i and goes to location t_i (where $t_i > s_i$). If she decides to use bus i, she can board at any point $x \in [s_i, t_i]$, drop off at any later point $y \in [x, t_i]$, and must pay a fixed cost c_i regardless of where she boards and drops off. Anu wants to get from point a to point b on the road (where $b > a$).

(a) [15 Points] Write a binary integer linear program that helps Anu achieve this goal while minimizing her total cost. Your program only needs to find the minimum total cost. It is OK if it does not find the locations at which Anu should board and drop off buses. Your program must have a finite (and ideally, a polynomial) number of constraints. Prove that your program is correct.

(b) [5 Points] Suppose each bus is operated by one of two companies. Let T_1 and T_2 be the indices of buses operated by companies 1 and 2, respectively ($T_1 \cap T_2 = \emptyset$ and $T_1 \cup T_2 = \{1, \ldots, n\}$). For some reason, Anu wants to make sure that she don't spend more than 70% of the total cost on any single company. Note that this constraint might raise the minimum total cost she needs to pay in order to get from point a to point b. How would you add this constraint to the program in part (a)?

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$h_1, \ldots h_6$

a) maximize $\sum_{k=1}^{6} h_k \cdot x_k$

subject to $x_3 + x_5 \leq 1$

$x_1 + x_5 + x_6 \leq 1$

$\sum_{k=1}^{6} c_k \cdot x_k \leq B$

$x_1 + x_2 + x_4 + x_5 + x_6 \leq 3$

~~$x_5 + x_6 \leq x_3$~~

~~$x_5 + x_6 \leq x_4$~~

$x_5 \leq x_3 + x_4$

$x_6 \leq x_3 + x_4$

$x_i \in \{0, 1\}$ $\quad h_i \geq 0$

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a) $x_i \in \{0,1\}$ take bus i or not

$$\min \sum_{i=1}^{n} x_i \cdot c_i$$

subject to: $\sum_{s_i \le a < t_i} x_i \ge 1$

~~$\sum_{s_i \le b < t_i} x_i \ge 1$~~

$\sum_{s_i \le t_j < t_i} x_i \ge 1 \quad \forall j : a \le t_j < b$