Certificate of Optimality

- Suppose you design a state-of-the-art LP solver that can solve very large problem instances

- You want to convince someone that you have this new technology without showing them the code
  - Idea: They can give you very large LPs and you can quickly return the optimal solutions
  - Question: But how would they know that your solutions are optimal, if they don't have the technology to solve those LPs?

Certificate of Optimality

$$\begin{aligned} \max\ x_1 + 6x_2 & \\ x_1 &\le 200 \\ x_2 &\le 300 \\ x_1 + x_2 &\le 400 \\ x_1, x_2 &\ge 0 \end{aligned}$$

- Suppose I tell you that $(x_1, x_2) = (100{,}300)$ is optimal with objective value 1900
- How can you check this?
  - Note: Can easily substitute (x_1, x_2), and verify that it is feasible, and its objective value is indeed 1900

Certificate of Optimality

$$2x_1 + 7x_2 \leq 2 \times 200 + 7 \times 300$$

$$\begin{aligned} \max\ \ & x_1 + 6x_2 \\ x_1 &\leq 200 \\ x_2 &\leq 300 \\ x_1 + x_2 &\leq 400 \\ x_1, x_2 &\geq 0 \end{aligned}$$

- Claim: $(x_1, x_2) = (100{,}300)$ is optimal with objective value 1900

- Any solution that satisfies these inequalities also satisfies their positive combinations
  - E.g. 2*first_constraint + 5*second_constraint + 3*third_constraint
  - Try to take combinations which give you $x_1 + 6x_2$ on LHS

Certificate of Optimality

$$\begin{aligned} \max\ x_1 + 6x_2 & \\ x_1 &\leq 200 \\ x_2 &\leq 300 \\ x_1 + x_2 &\leq 400 \\ x_1, x_2 &\geq 0 \end{aligned}$$

- Claim: $(x_1, x_2) = (100{,}300)$ is optimal with objective value 1900

- first_constraint + 6*second_constraint
  - $x_1 + 6x_2 \leq 200 + 6 * 300 = 2000$
  - This shows that no feasible solution can beat 2000

Certificate of Optimality

$$\begin{aligned} \max \ & x_1 + 6x_2 \\ x_1 &\leq 200 \\ x_2 &\leq 300 \\ x_1 + x_2 &\leq 400 \\ x_1, x_2 &\geq 0 \end{aligned}$$

- Claim: $(x_1, x_2) = (100{,}300)$ is optimal with objective value 1900

- 5*second_constraint + third_constraint
  - $5x_2 + (x_1 + x_2) \leq 5 * 300 + 400 = 1900$
  - This shows that no feasible solution can beat 1900
    - No need to proceed further
    - We already know one solution that achieves 1900, so it must be optimal!

Is There a General Algorithm?

- Introduce variables y_1, y_2, y_3 by which we will be multiplying the three constraints
  - Note: These need not be integers. They can be reals.

Multiplier	Inequality
y_1	$x_1 \leq 200$
y_2	$x_2 \leq 300$
y_3	$x_1 + x_2 \leq 400$

- After multiplying and adding constraints, we get:

$$(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 200y_1 + 300y_2 + 400y_3$$

$1x_1 + 6x_2$

Is There a General Algorithm?

Multiplier	Inequality
y_1	$x_1 \leq 200$
y_2	$x_2 \leq 300$
y_3	$x_1 + x_2 \leq 400$

- We have:

  $(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq \boxed{200y_1 + 300y_2 + 400y_3}$

- What do we want?
  - $y_1, y_2, y_3 \geq 0$ because otherwise direction of inequality flips
  - LHS to look like objective $x_1 + 6x_2$
    - In fact, it is sufficient for LHS to be an upper bound on objective
    - So, we want $y_1 + y_3 \geq 1$ and $y_2 + y_3 \geq 6$

Is There a General Algorithm?

Multiplier	Inequality
y_1	$x_1 \leq 200$
y_2	$x_2 \leq 300$
y_3	$x_1 + x_2 \leq 400$

- We have:
  $(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 200y_1 + 300y_2 + 400y_3$

- What do we want?
  - $y_1, y_2, y_3 \geq 0$
  - $y_1 + y_3 \geq 1, \;\; y_2 + y_3 \geq 6$
  - Subject to these, we want to minimize the upper bound $200y_1 + 300y_2 + 400y_3$

Is There a General Algorithm?

Multiplier	Inequality
y_1	$x_1 \leq 200$
y_2	$x_2 \leq 300$
y_3	$x_1 + x_2 \leq 400$

- We have:
  $$(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 200y_1 + 300y_2 + 400y_3$$

- What do we want?
  - This is just another LP!
  - Called the dual
  - Original LP is called the primal

$$\begin{aligned} \min\ & 200y_1 + 300y_2 + 400y_3 \\ & y_1 + y_3 \geq 1 \\ & y_2 + y_3 \geq 6 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

Is There a General Algorithm?

PRIMAL

$$\begin{aligned} \max\ x_1 + 6x_2 \\ x_1 \le 200 \\ x_2 \le 300 \\ x_1 + x_2 \le 400 \\ x_1, x_2 \ge 0 \end{aligned}$$

DUAL

$$\begin{aligned} \min\ 200y_1 + 300y_2 + 400y_3 \\ y_1 + y_3 \ge 1 \\ y_2 + y_3 \ge 6 \\ y_1, y_2, y_3 \ge 0 \end{aligned}$$

- The problem of verifying optimality is another LP
  - For any (y_1, y_2, y_3) that you can find, the objective value of the dual is an upper bound on the objective value of the primal
  - If you found a specific (y_1, y_2, y_3) for which this dual objective becomes equal to the primal objective for the (x_1, x_2) given to you, then you would know that the given (x_1, x_2) is optimal for primal (and your (y_1, y_2, y_3) is optimal for dual)

Is There a General Algorithm?

PRIMAL

$$\begin{aligned} \max\ x_1 + 6x_2 \\ x_1 \leq 200 \\ x_2 \leq 300 \\ x_1 + x_2 \leq 400 \\ x_1, x_2 \geq 0 \end{aligned}$$

DUAL

$$\begin{aligned} \min\ 200y_1 + 300y_2 + 400y_3 \\ y_1 + y_3 \geq 1 \\ y_2 + y_3 \geq 6 \\ y_1, y_2, y_3 \geq 0 \end{aligned}$$

- The problem of verifying optimality is another LP
  - Issue 1: But…but…if I can't solve large LPs, how will I solve the dual to verify if optimality of (x_1, x_2) given to me?
    - You don't. Ask the other party to give you both (x_1, x_2) and the corresponding (y_1, y_2, y_3) for proof of optimality
  - Issue 2: What if there are no (y_1, y_2, y_3) for which dual objective matches primal objective under optimal solution (x_1, x_2)?
    - As we will see, this can't happen!

Is There a General Algorithm?

Primal LP

$$\max \ \mathbf{c}^T\mathbf{x}$$
$$\mathbf{A}\mathbf{x} \leq \mathbf{b} \cdot y$$
$$\mathbf{x} \geq 0$$

y_1
y_2
$\vdots$
y_m

Dual LP

$$\min \ \mathbf{y}^T\mathbf{b}$$
$$\boxed{\mathbf{y}^T\mathbf{A} \geq \mathbf{c}^T}$$
$$\mathbf{y} \geq 0$$

- General version, in our standard form for LPs

Is There a General Algorithm?

Primal LP	**Dual LP**
$\max\ \mathbf{c}^T\mathbf{x}$	$\min\ \mathbf{y}^T\mathbf{b}$
$\mathbf{A}\mathbf{x} \leq \mathbf{b}$	$\mathbf{y}^T\mathbf{A} \geq \mathbf{c}^T$
$\mathbf{x} \geq 0$	$\mathbf{y} \geq 0$

- $c^T x$ for any feasible $x \;\leq\; y^T b$ for any feasible y
- $\max\limits_{\text{primal feasible } x} c^T x \;\leq\; \min\limits_{\text{dual feasible } y} y^T b$
- If there is (x^*, y^*) with $c^T x^* = (y^*)^T b$, then both must be optimal
- In fact, for optimal (x^*, y^*), we claim that this must happen!
  - Does this remind you of something? Max-flow, min-cut…

Weak Duality

Primal LP	**Dual LP**
$\max\ \mathbf{c}^T\mathbf{x}$	$\min\ \mathbf{y}^T\mathbf{b}$
$\mathbf{A}\mathbf{x} \leq \mathbf{b}$	$\mathbf{y}^T\mathbf{A} \geq \mathbf{c}^T$
$\mathbf{x} \geq 0$	$\mathbf{y} \geq 0$

- From here on, assume primal LP is feasible and bounded
- Weak duality theorem:
  - For any primal feasible x and dual feasible y, $c^T x \leq y^T b$
- Proof:

$$c^T x \leq (y^T A)x = y^T(Ax) \leq y^T b$$

Strong Duality

$x : n \times 1$ $A : m \times n$

$b : m \times 1$

$c : n \times 1$

$c^T : 1 \times n$

$y : m \times 1$

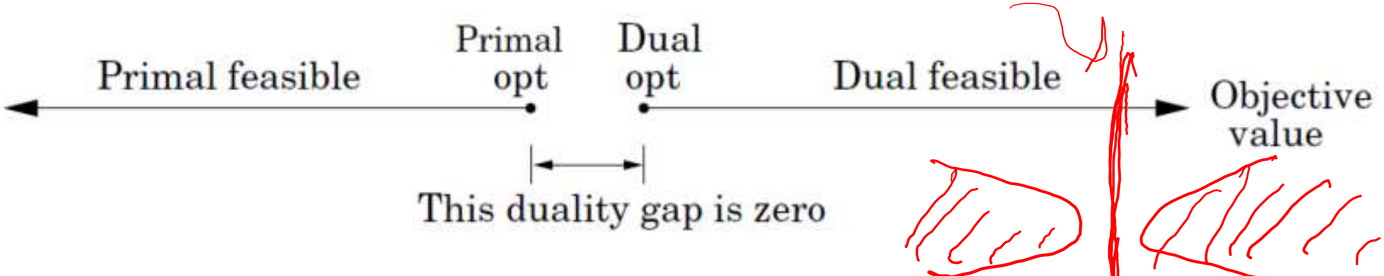
y^T A

$[y_1 \cdots y_m]$ $1 \times m$ $m \times n$

Primal LP	Dual LP
$\max\ \boxed{\mathbf{c}}^T \mathbf{x}$	$\min\ \mathbf{y}^T \mathbf{b}$
$\mathbf{A}\mathbf{x} \le \mathbf{b}$	$\mathbf{y}^T \mathbf{A} \ge \mathbf{c}^T$
$\mathbf{x} \ge 0$	$\mathbf{y} \ge 0$

$1 \times n$ $n \times 1$ $1 \times n$

- Strong duality theorem:
  - For any primal optimal x^* and dual optimal y^*, $c^T x^* = (y^*)^T b$



Strong Duality Proof

This slide is not in the scope of the course

- Farkas' lemma (one of many, many versions):
  - Exactly one of the following holds:
  1) There exists x such that $Ax \leq b$
  2) There exists y such that $y^T A = 0,\ y \geq 0,\ y^T b < 0$

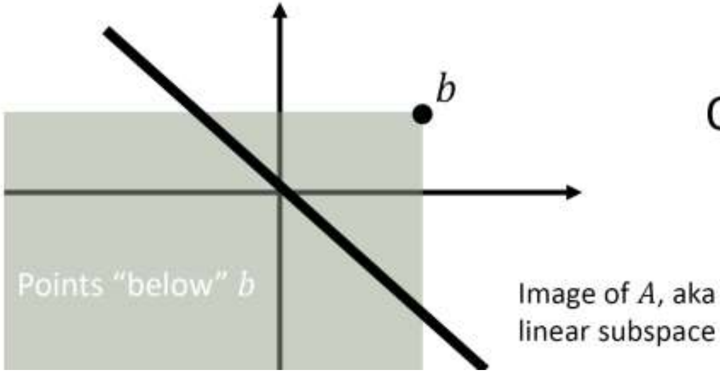
- Geometric intuition:
  - Define image of A = set of all possible values of Ax
  - It is known that this is a "linear subspace" (e.g., a line in a plane, a line or plane in 3D, etc)

Strong Duality Proof

This slide is not in the scope of the course

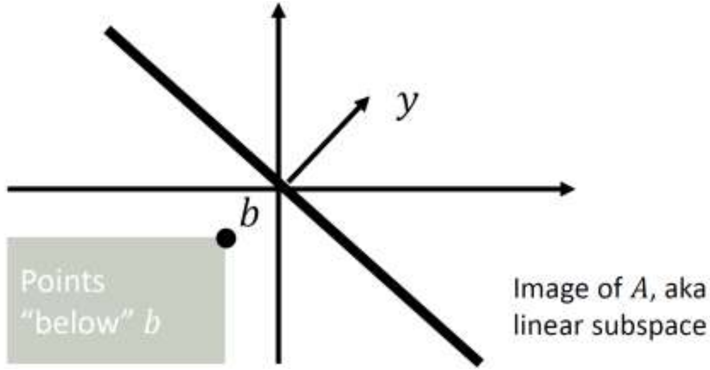
- Farkas' lemma: Exactly one of the following holds:
  1) There exists x such that $Ax \leq b$
  2) There exists y such that $y^T A = 0,\ y \geq 0,\ y^T b < 0$

1) Image of A contains a point "below" b



OR

2) The region "below" b doesn't intersect image of A
this is witnessed by normal vector to the image of A



Strong Duality

This slide is not in the scope of the course

Primal LP

$$\max \ \mathbf{c}^T\mathbf{x}$$
$$\mathbf{A}\mathbf{x} \le \mathbf{b}$$
$$\mathbf{x} \ge 0$$

Dual LP

$$\min \ \mathbf{y}^T\mathbf{b}$$
$$\mathbf{y}^T\mathbf{A} \ge \mathbf{c}^T$$
$$\mathbf{y} \ge 0$$

- Strong duality theorem:
  - For any primal optimal x^* and dual optimal y^*, $c^T x^* = (y^*)^T b$
  - Proof (by contradiction):
    - Let $z^* = c^T x^*$ be the optimal primal value.
    - Suppose optimal dual objective value $> z^*$
    - So, there is no y such that $y^T A \ge c^T$ and $y^T b \le z^*$, i.e.,

$$\begin{pmatrix} -A^T \\ b^T \end{pmatrix} y \le \begin{pmatrix} c \\ z^* \end{pmatrix}$$

Strong Duality

This slide is not in the scope of the course

- There is no y such that $\begin{pmatrix} -A^T \\ b^T \end{pmatrix} y \leq \begin{pmatrix} c \\ z^* \end{pmatrix}$
- By Farkas' lemma, there is x and λ such that

$$(x^T \quad \lambda) \begin{pmatrix} -A^T \\ b^T \end{pmatrix} = 0, x \geq 0, \lambda \geq 0, -x^T c + \lambda z^* < 0$$

- Case 1: $\lambda > 0$
  - Note: $c^T x > \lambda z^*$ and $Ax = 0 = \lambda b$.
  - Divide both by λ to get $A\left(\frac{x}{\lambda}\right) = b$ and $c^T\left(\frac{x}{\lambda}\right) > z^*$
    - Contradicts optimality of z^*
- Case 2: $\lambda = 0$
  - We have $Ax = 0$ and $c^T x > 0$
  - Adding x to optimal x^* of primal improves objective value beyond $z^* \Rightarrow$ contradiction

Exercise: Formulating LPs

- A canning company operates two canning plants (A and B).
- Three suppliers of fresh fruits:
  - S1: 200 tonnes at $11/tonne
  - S2: 310 tonnes at $10/tonne
  - S3: 420 tonnes at $9/tonne
- Shipping costs in $/tonne:

From: \ To:	Plant A	Plant B
S1	3	3.5
S2	2	2.5
S3	6	4

- Plant capacities and labour costs:

	Plant A	Plant B
Capacity	460 tonnes	560 tonnes
Labour cost	$26/tonne	$21/tonne

- Selling price: $50/tonne, no limit
- Objective: Find which plant should get how much supply from each grower to maximize profit

$x_{i,j}$ = # tonnes of fruit purchased from supplier i and sent to plant j

$i \in \{1, 2, 3\}$

$j \in \{A, B\}$

max: $50\left(\sum_{\substack{i \in \{1,2,3\} \\ j \in \{A,B\}}} x_{ij}\right) - \left(3x_{1A} - 3.5x_{1B} - 2x_{2A} - 2.5x_{2B} - 6x_{3A} - 4x_{3B}\right)$

Purchase cost

$+ 26(x_{1A} + x_{2A} + x_{3A})$

$+ 2\left(\cancel{x_{1B} + x_{2B} + x_{3B}}\right)$

Constraints:

$$x_{1A} + x_{1B} \leq 200, \quad x_{2A} + x_{2B} \leq 310,$$

$$x_{3A} + x_{3B} \leq 420$$

$$x_{1A} + x_{2A} + x_{3A} \leq 410$$

$$x_{1B} + x_{2B} + x_{3B} \leq 560$$

$$x_{ij} \geq 0.$$

$x_A, x_B, x_C \geq 0 \rightarrow x_A + x_B = \max\{x_A, x_B\} \rightarrow x_A \cdot x_B = 0$

Exercise: Formulating LPs

- Similarly to the brewery example from earlier:
  - A brewery can invest its inventory of corn, hops and malt into producing three types of beer
  - Per unit resource requirement and profit are as given below
  - <u>The brewery cannot produce positive amounts of *both* A and B</u>
  - Goal: maximize profit

$x_A \geq 0$

$x_B \geq 0$

Beverage	Corn (kg)	Hops (kg)	Malt (kg)	Profit ($)
A	5	4	35	13
B	15	4	20	23
C	10	7	25	15
Limit	500	300	1000	

x_A, x_B, x_C

LP1 LP2

$x_A + x_B$ … $x_A = 0$ $x_B = 0$

$$\left\{ x \;\middle|\; \underline{x_A = 0} \text{ or } \underline{x_B = 0},\ x_A, x_B, x_C \geq 0 \right\}$$

$$(5, 0, 3) \quad \& \quad (0, 5, 3)$$

$$(\underline{\underline{2.5}}, 2.\underline{\underline{5}}, 3)$$

$$\{$$

Feasible region 1

$\{x \mid x_A = 0,\ x_A, x_B, x_C \geq 0\}$

Feasible region 2

$\{x \mid x_B = 0;\ \_\_\ \_\_\}$

$67.1 \le x_c \le 67.9$

Exercise: Formulating LPs

$x_c^* = 67.3$

- Similarly to the brewery example from the beginning:
  - A brewery can invest its inventory of corn, hops and malt into producing three types of beer
  - Per unit resource requirement and profit are as given below
  - The brewery can only produce C in integral quantities up to 100
  - Goal: maximize profit

$\lfloor x_c^* \rfloor = 67$

$x_c = 67$

$x_c \le 68$

Beverage	Corn (kg)	Hops (kg)	Malt (kg)	Profit ($)
A	5	4	35	13
B	15	4	20	23
C	10	7	25	15
Limit	500	300	1000	

LP 1

$x_c = 0$

LP 2

$x_c = 1$

. . .

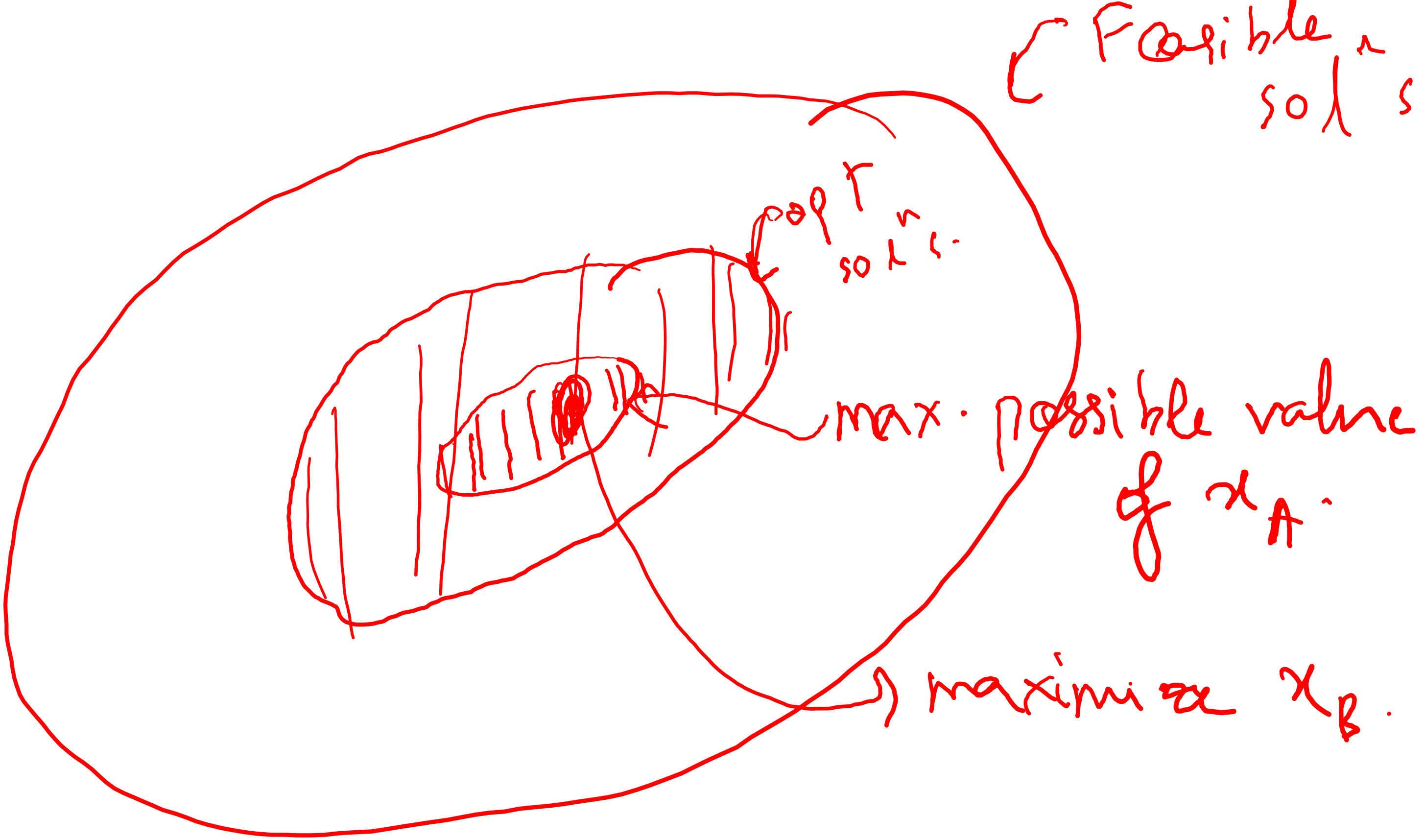
LP 100

$x_c = 100$

Exercise: Formulating LPs

- Similarly to the brewery example from the beginning:
  - A brewery can invest its inventory of corn, hops and malt into producing three types of beer
  - Per unit resource requirement and profit are as given below
  - Goal: maximize profit, but if there are multiple profit-maximizing solutions, then…
    - Break ties to choose those with the largest quantity of A
    - Break any further ties to choose those with the largest quantity of B

Beverage	Corn (kg)	Hops (kg)	Malt (kg)	Profit ($)
A	5	4	35	13
B	15	4	20	23
C	10	7	25	15
Limit	500	300	1000	



Feasible solⁿs.
optⁿ solⁿs.
max. possible value of x_A.
maximize x_B.

LP1

max profit

constraints.

$\boxed{x^*}$

LP 2

max x_A

consts.

$\text{profit} \geq x^*$

x_A^*

LP3

max x_B

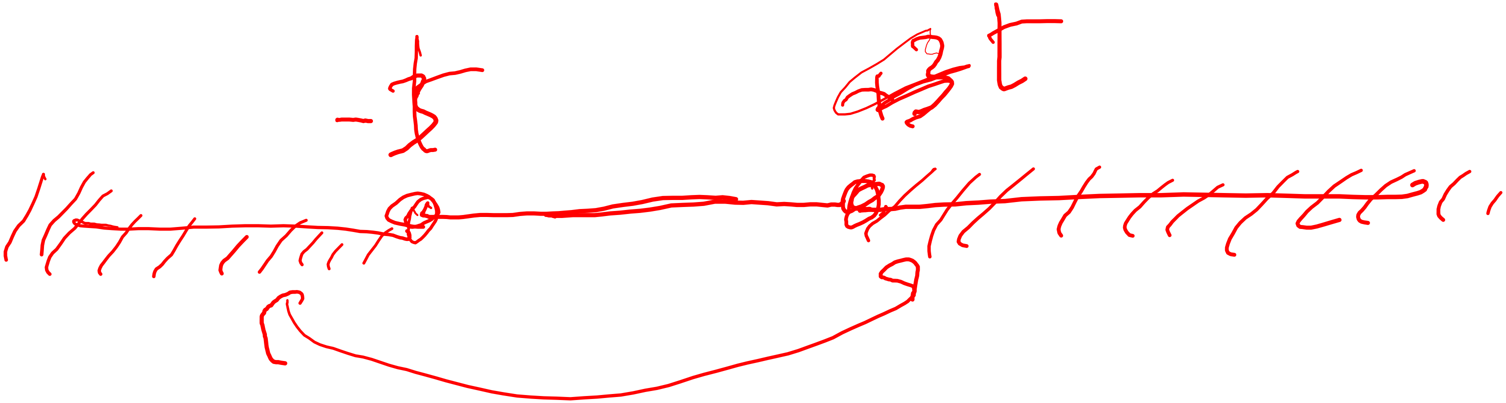
consts.

$x_A \geq x_A^*$

More Tricks

$x \geq 3$ $x \leq -3$

- Constraint: $|x| \leq 3$
  - Replace with constraints $x \leq 3$ and $-x \leq 3$
  - What if the constraint is $|x| \geq 3$?
- Objective: minimize $3|x| + y$
  - Add a variable t
  - Add the constraints $t \geq x$ and $t \geq -x$ (so $t \geq |x|$)
  - Change the objective to minimize $3t + y$
  - What if the objective is to *maximize* $3|x| + y$?
- Objective: minimize $\max(3x + y, x + 2y)$
  - Hint: minimizing $3|x| + y$ in the earlier bullet was equivalent to minimizing $\max(3x + y, -3x + y)$
- …



More Tricks

max min $3t + y$ | $3|x| + y$

s.t. $t \geq x$, $t \geq -x$, $t \geq |x|$

$t \leq |x|$

- Constraint: $|x| \leq 3$
  - Replace with constraints $x \leq 3$ and $-x \leq 3$
  - What if the constraint is $|x| \geq 3$?

constr.

- Objective: minimize $3|x| + y$
  - Add a variable t
  - Add the constraints $t \geq x$ and $t \geq -x$ (so $t \geq |x|$)
  - Change the objective to minimize $3t + y$
  - What if the objective is to *maximize* $3|x| + y$?

$(x, y) \xrightarrow{\text{wrts}} (|x|, x, y)$ is a sol^n to new LP.

- Objective: minimize $\max(3x + y, x + 2y)$
  - Hint: minimizing $3|x| + y$ in the earlier bullet was equivalent to minimizing $\max(3x + y, -3x + y)$
- …

min t

s.t. $t \geq 3x + y$

$t \geq x + 2y$


NOW I KNOW
LINEAR PROGRAMMING

Network Flow via LP

- Problem
  - Input: directed graph $G = (V, E)$, edge capacities $c: E \to \mathbb{R}_{\geq 0}$
  - Output: Value $v(f^*)$ of a maximum flow f^*

- Flow f is valid if:
  - Capacity constraints: $\forall (u, v) \in E: 0 \leq f(u, v) \leq c(u, v)$
  - Flow conservation: $\forall u \neq s, t: \sum_{(u,v) \in E} f(u, v) = \sum_{(v,u) \in E} f(v, u)$

Linear constraints

- Maximize $v(f) = \sum_{(s,v) \in E} f(s, v)$

Linear objective!

Network Flow via LP

maximize $\quad \sum_{(s,v)\in E} f_{sv}$

$0 \leq f_{uv} \leq c(u,v) \qquad$ for all $(u,v) \in E$

$\sum_{(u,v)\in E} f_{uv} = \sum_{(v,w)\in E} f_{v,w} \qquad$ for all $v \in V \setminus \{s,t\}$

Exercise: Write the dual of this LP.
What is the dual trying to find?

Shortest Path via LP

$d_t \le$ wt. of any path from s – t

- Problem
  - Input: directed graph $G = (V, E)$, edge weights $w: E \to \mathbb{R}_{\geq 0}$, source vertex s, target vertex t
  - Output: weight of the shortest-weight path from s to t

- Variables: for each vertex v, we have variable d_v

Why max?

$$\boxed{\text{maximize} \quad d_t}$$

subject to

$$\boxed{\begin{aligned} d_v &\le d_u + w(u,v) \quad \text{for each edge } (u,v) \in E\,, \\ d_s &= 0\,. \end{aligned}}$$

Exercise: prove formally that this works!

If objective was min., then we could set all variables d_v to 0.

$\forall v$: $d_v \le$ the wt. of the shortest path from s to v.

Any path $s \rightsquigarrow v$

$d_{v_1} \le d_s + w(s, v_1)$. $\quad d_{v_2} \le w(s - v_1 - v_2)$

$d_{v_2} \le d_{v_1} + w(v_1, v_2)$

$\boxed{d_v} \le d_{v_l} + w(v_l, v)$.

But…but…

- For these problems, we have different combinatorial algorithms that are much faster and run in strongly polynomial time

- Why would we use LP?

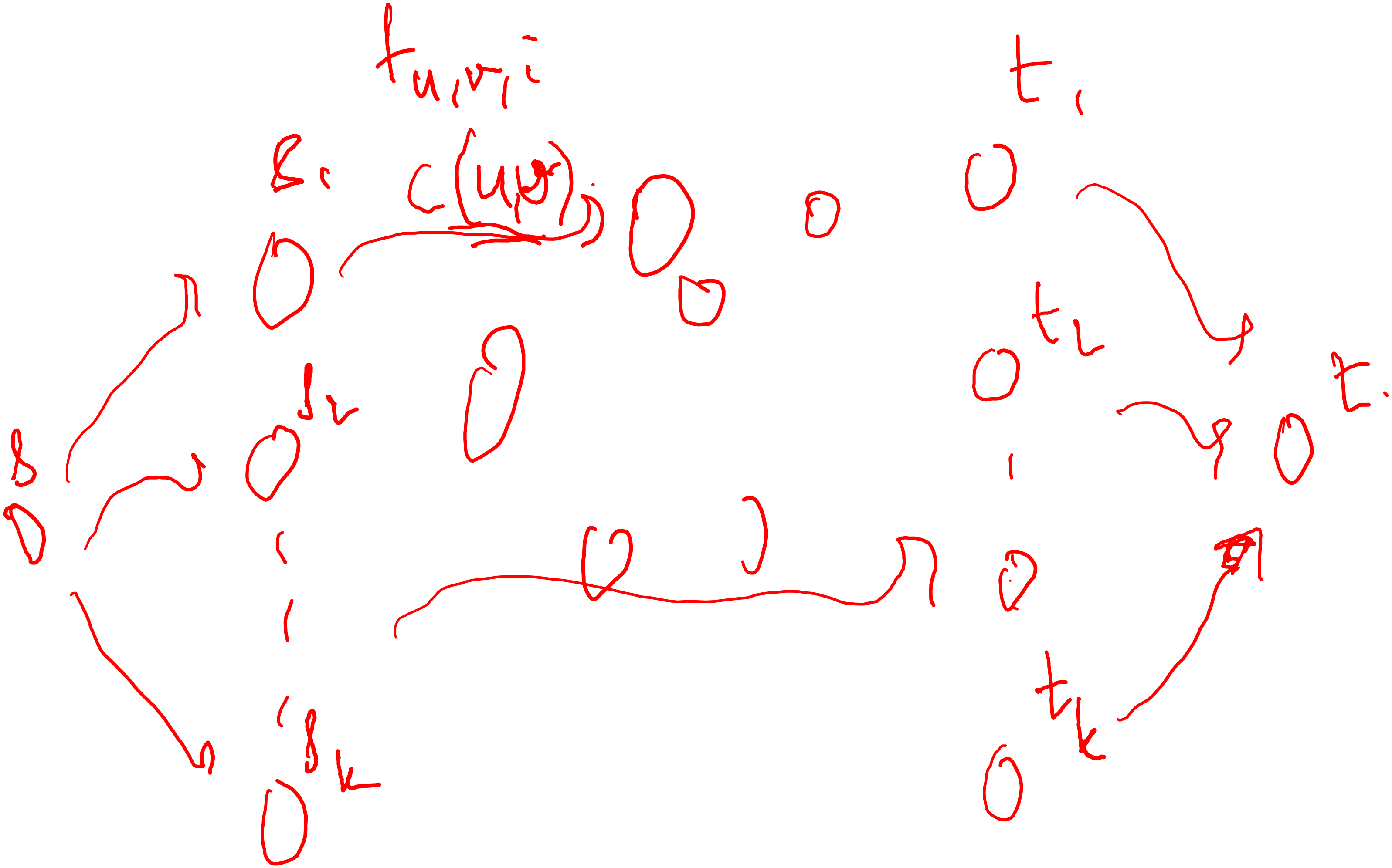
- For some problems, we don't have faster algorithms than solving them via LP

Multicommodity-Flow

- Problem:
  - Input: directed graph $G = (V, E)$, edge capacities $c: E \to \mathbb{R}_{\geq 0}$, k commodities (s_i, t_i, d_i), where s_i is source of commodity i, t_i is sink, and d_i is demand.
  - Output: valid multicommodity flow $(f_1, f_2, \dots, f_k)$, where f_i has value d_i and all f_i jointly satisfy the constraints

The only known polynomial time algorithm for this problem is based on solving LP!

$$
\begin{aligned}
\sum_{i=1}^{k} f_{iuv} &\leq c(u,v) && \text{for each } u, v \in V\,, \\
\sum_{v \in V} f_{iuv} - \sum_{v \in V} f_{ivu} &= 0 && \text{for each } i = 1, 2, \dots, k \text{ and} \\
& && \text{for each } u \in V - \{s_i, t_i\}\,, \\
\sum_{v \in V} f_{i,s_i,v} - \sum_{v \in V} f_{i,v,s_i} &= d_i && \text{for each } i = 1, 2, \dots, k\,, \\
f_{iuv} &\geq 0 && \text{for each } u, v \in V \text{ and} \\
& && \text{for each } i = 1, 2, \dots, k\,.
\end{aligned}
$$


$f_{u,v,i}$
$c(u,v)$
s_1
s_2
s_k
s
t_1
t_2
t_k
t

Integer Linear Programming

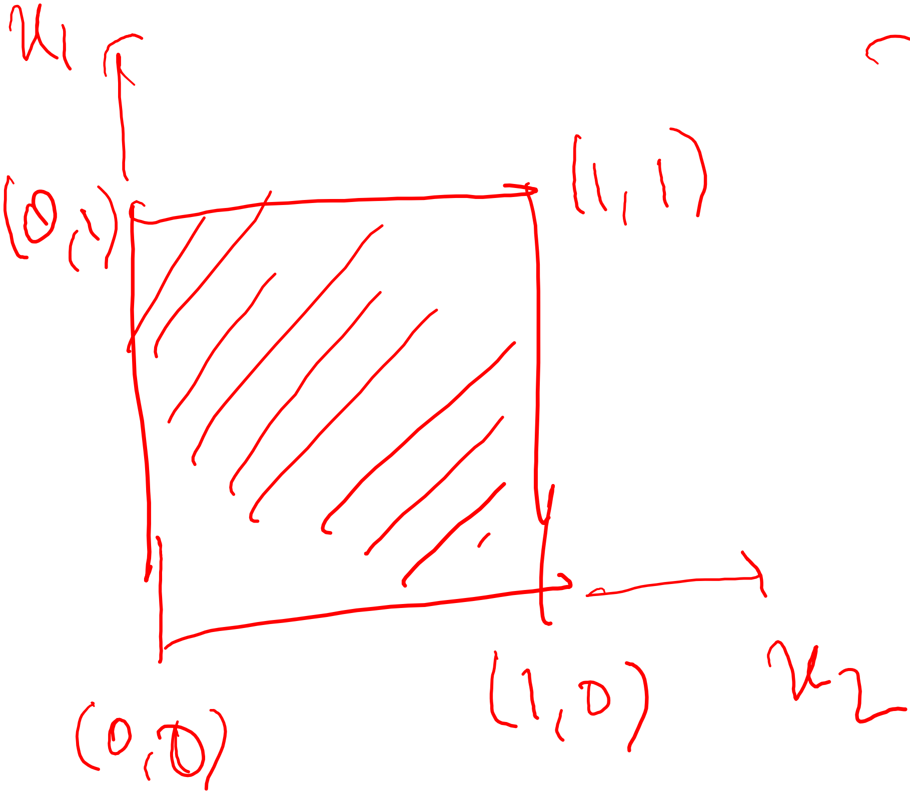
- Variable values are restricted to be integers
- Example:
  - Input: $c \in \mathbb{R}^n, b \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}$
  - Goal:

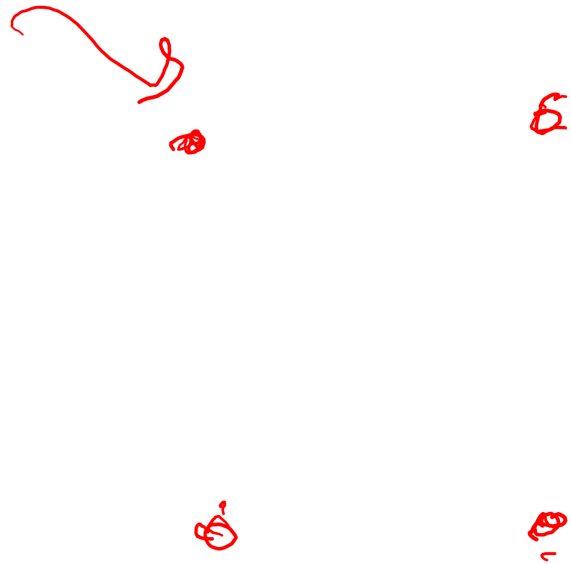
$$\begin{array}{ll} \text{Maximize} & c^T x \\ \text{Subject to} & Ax \leq b \\ & \boldsymbol{x} \in \{\boldsymbol{0}, \boldsymbol{1}\}^{\boldsymbol{n}} \end{array}$$

$\rightarrow x_i \in \{0,1\}$

$0 \leq x_i \leq 1 \; \forall i$

- Does this make the problem easier or harder?
  - Harder. We'll prove that this is "NP-complete".





$\{0,1\}^n$

LPs are everywhere…

- Microeconomics
- Manufacturing
- VLSI (very large scale integration) design
- Logistics/transportation
- Portfolio optimization
- Bioengineering (flux balance analysis)
- Operations research more broadly: maximize profits or minimize costs, use linear models for simplicity
- Design of approximation algorithms
- Proving theorems, as a proof technique
- …