Question 1. Disaster Relief [15 MARKS]

In the "disaster relief" problem, you are given a set of locations of injured people and a set of locations of medical facilities, along with the number of new patients that each facility can accept. You would like to transport every injured person to a nearby facility so that every person gets treated – without overloading any facility.

More precisely, given a list of ℓ injured persons $p_1, \ldots, p_\ell$ and a list of k medical facilities $m_1, \ldots, m_k$, the distances $d(p_i, m_j)$ between any person and facility, a distance bound D, and the "vacancy" $v(m_i)$ for each facility,

(a) (10 marks) use network flow techniques to determine an assignment of persons to facilities such that each person is assigned to a facility no more than distance D away, each facility m_j is assigned no more than $v(m_j)$ new patients, and the number of treated persons is maximized.

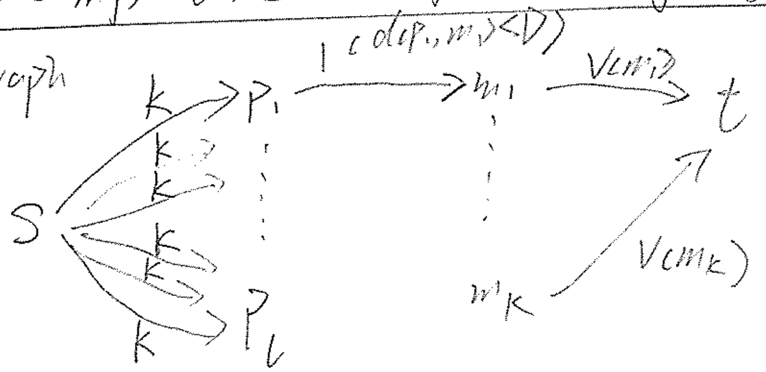
(b) (5 marks) Briefly argue that your solution is correct.

a) Construct a network N with source s and tink t, s connects all the injured persons $(p_1, \cdots, p_l)$. injured person p_i connects to medical facilities $(m_1, \cdots, m_k)$, finally, medical facilities connect to tink t.

capacity:

$$\begin{cases} C(s, p_i) = k & \forall i \in \{1, \cdots, l\} \\ C(p_i, m_j) = 1 & \forall i \in \{1, \cdots, l\}, \forall j \in \{1, \cdots, k\}, d(p_i, m_j) \le D \\ C(p_i, m_j) = 0 & \text{if } d(p_i, m_j) > D. \\ C(m_j, t) = v(m_j) & \forall j \in \{1, \cdots, k\} \end{cases}$$

Rough Graph



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Clearly label each such solution with the appropriate question and part number.

b) The question asks for the assignment of persons to facilities while satisfied the constraints. By the network in part a) so the extreme condition is that all the $C(s, p_i) = k$ medical facilities $\{m_1, \cdots, m_k\}$ is in distance D of injured person p_i, so p_i can connects to all facilities with edge capacity 1. Since the constraint is that injured person can only be sent to facilities in the distance D of person, so in network N, $C(p_i, m_j) = 1$ iff $d(p_i, m_j) \leq D$, $C(p_i, m_j) = 0$ if $d(p_i, m_j) > D$, in such a case, the person p_i will only be considered sent to facility that in distance D.

By get the maxflow of the network N, since $C(m_i, t) = v(m_i)$ the "vacancy" number of each facility, so maxflow number $|f|$ represents the largest number of treated persons which is the same as question wants.

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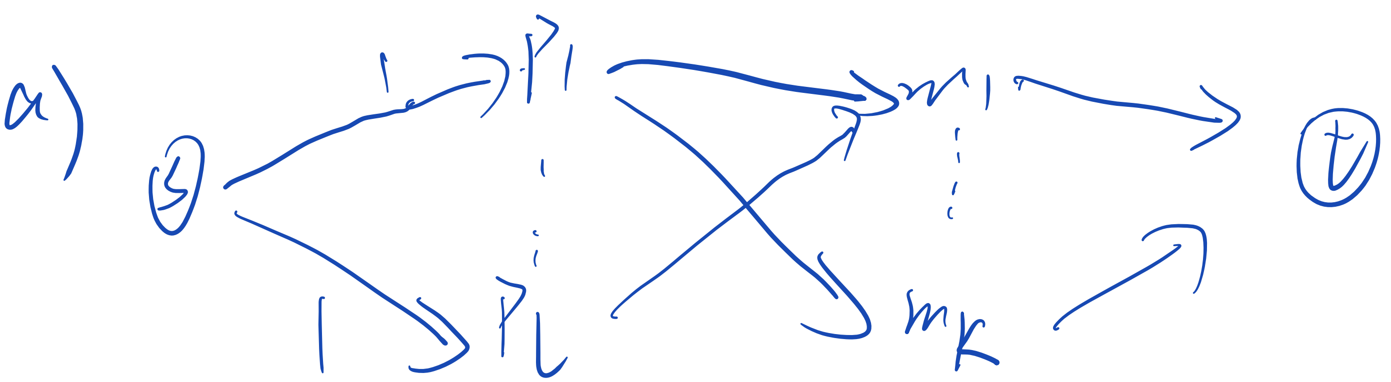
(b) (5 marks) Briefly argue that your solution is correct. $<D$ connect

a)



the maxflow would find the number of treated persons. , $m_i \to t$ is $v(m_i)$

max $\sum s_i$
constraints. $\sum f_i = \sum s_i$
$f_i \le p_i$
$s_i \le q_i$

$y_{i,j} = 1$ $c_{i,j} > 0$

$y_{i,j}$ – represents the if truck on highway $c_{i,j}$

Question 2. Selling Products [20 MARKS]

(a) (10 marks) Company A has factories in n cities to make its products. It also runs one shop in each of these cities to sell its products. For $1 \leqslant i \leqslant n$, the maximum number of products that the factory can produce at city i in one day is $p_i \in \mathbb{Z}^+$ (the set of *positive* integers). The maximum number of products that the shop at city i can sell in one day is $q_i \in \mathbb{Z}^+$. In addition, there are *one-way* highways connecting some pairs of cities i.e., if there is a highway to go from city i to j, then there is no highway to go from j to i. You are given values $c_{i,j} \in \{0\} \cup \mathbb{Z}^+$ such that (i) there is no highway to go from city i to city j if and only if $c_{i,j} = 0$, and (ii) if there is a highway to go from city i to city j, it allows a maximum of $c_{i,j} \in \mathbb{Z}^+$ products to be transferred from city i to city j every day. Your goal is to maximize the sales of the product while maintaining the balance between the factory production and the shop consumption – at the end of the day, every product that has been produced during that day must be sold. What is the maximum number of products Company A can produce and sell per day, and how will it achieve this goal?

For example, suppose $n = 3$, $p_1 = 20$, $p_2 = 0$, $p_3 = 1$, $q_1 = 2$, $q_2 = 4$, $q_3 = 10$, $c_{1,2} = 7$, and $c_{2,3} = 10$ (the remaining $c_{i,j} = 0$). Note, for example, that $c_{2,3} > 0$ implies $c_{3,2} = 0$. Then the optimal solution produces and sells 10 products every day: city 1 produces 9 products, sells 2, and sends 7 to city 2; city 2 sells 4 products and sends 3 to city 3; city 3 produces 1 product and sells 4.

Design an algorithm to solve this problem using network flow techniques. Make sure that your algorithm outputs how much each factory will produce, how much each shop will sell, and how much will be shipped between any two cities. **For this part, you do not need to discuss the correctness of your construction.**

(b) (10 marks) Company A bought some new trucks! Each new truck **doubles** the transportation capacity of a one-way highway between two cities, if it is deployed on that highway. Suppose the company can deploy such new trucks to a maximum of m highways (for some $m \in \mathbb{Z}^+$). What is the maximum number of products Company A can produce and sell per day under these new conditions?

Formulate an integer linear program to solve this problem. You only need to explain the meaning of the objective function, each variable, and each constraint in your program. **For this part, you do not need to discuss the correctness of your construction.**

Note: Your program does **not** have to be a *binary* integer linear program i.e., it is okay for the variables of your program to take integer values other than 0 and 1.

a) Construct network N with source s and sink t, s connects to each city, and each city connects to t, between cities, there are edges connected if $c_{i,j} > 0$ for city i and j. (Let t_i represents city i)

Capacity: $c(s, t_i) = p_i$ $\forall i \in \{1, \cdots, n\}$

$c(t_i, t) = q_i$ $\forall i \in \{1, \cdots, n\}$

$c(t_i, t_j) = c_{i,j}$ $\forall i,j \in \{1, \cdots, n\}, i \ne j$

(in this condition, $c(t_j, t_i) = 0$)

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$c_{i,j}$

Use the space on this "blank" page for scratch work, or for any solution that did not fit elsewhere.
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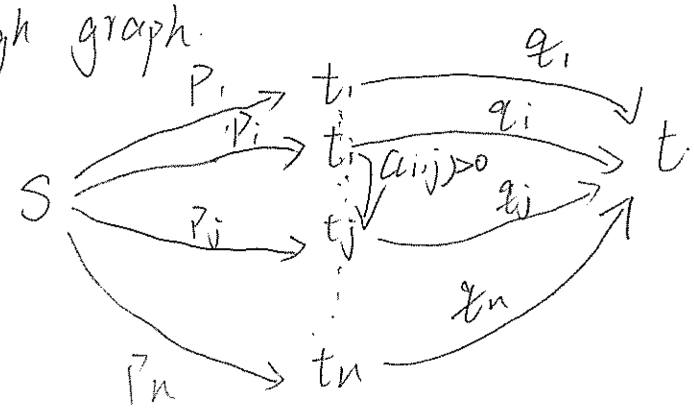
Rough graph.



b) Define in city i, the produce number each day is f_i, the sale number in shop is s_i, let x_{ij} be the integer variable represents if truck used on the one-way highway road from city i to city j. $x_{ij}=2$ if truck used, $x_{ij}=0$ if it is not used.

$x_{i,j} \in \{0, 2\}$ $s_i, f_i \in \mathbb{Z}_{\geq 0}$

Maximize $\sum_{i=1}^{n} s_i$

subject to ① $s_i \leq q_i$ $\forall i \in \{1, \dots, n\}$

② $\sum_{i=1}^{n} s_i = \sum_{i=1}^{n} f_i$

③ $f_j \leq p_j$ $\forall j \in \{1, \dots, n\}$

④ $\sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij} \leq 2m$

⑤ $s_j \leq f_j + \sum_{i=1}^{n} c(i,j) \cdot x_{i,j}$

⑥ $c_{i,j} \cdot x_{i,j} > 0$

⑦ $x_{i,j} \in \{0, 2\}$, $s_i, f_i \geq 0$

next page

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For objective function, it is to maximize the ~~total~~ sum of sales at each city i.

variable: s_i — sale number at shop in city i

f_i — produce number at factory in city i

$x_{i,j}$ — represents if truck used on one way highway from city i to j. if used, $x_{i,j}=2$, otherwise, $x_{i,j}=0$

constraint:

① $s_i \le q_i \quad \forall i \in \{1, \dots, n\}$
the sale number at city i must be no more than q_i (maximum number can sell)

② $\sum_{i=1}^{n} s_i = \sum_{i=1}^{n} f_i$ — number of sales of each day must equal to number of produced, so no waste.

③ $f_j \le p_j \quad \forall j \in \{1, \dots, n\}$ — similar to ①, the production number at each city j must be no more than p_j (the maximum number)

④ $\sum_{i=1}^{n} \sum_{j=1}^{n} x_{i,j} \le 2m$ — since $x_{i,j}$ is at most 2, and trucks can be used in at most m highways. so the maximum value of all $x_{i,j}$ is $2m$

⑤ $s_j \le f_j + \sum_{i=1}^{n} c_{i,j} \cdot x_{i,j}$ — the maximum number sells at city j cannot be more than the sum of produced number at city j and the transport number from other city to j.

⑥ $c_{i,j} \cdot x_{ij} \ge 0$ — so truck can only be used in the highway

Question 3. Min Cost Flow [15 MARKS]

In the "minimum cost flow problem", you are given a network with a *price* $p(e)$ for each edge e (where $p(e)$ represents the cost per unit flow, i.e., sending x units of flow across edge e will cost $p(e) \cdot x$), and you want to find a flow in the network that meets a certain target while keeping the cost as low as possible (where the cost for the entire network is the sum of the costs for each edge in the network).

More precisely, given a network $N = (V, E)$ with non-negative integer capacity $c(e)$ and non-negative integer price $p(e)$ for each edge $e \in E$, together with a non-negative integer demand d,

(a) (10 marks) formulate a linear program to find a flow f in N (i.e., an assignment of flow value $f(e)$ for each edge $e \in E$) such that the total flow in N is at least d (i.e., $|f| = f^{\text{out}}(s) = \sum_{(s,u)\in E} f(s,u) \geqslant d$) and the total cost of the flow, $\text{cost}(f) = \sum_{e\in E} p(e) \cdot f(e)$, is minimum.

(b) (5 marks) Briefly argue that your solution is correct.

a). Minimum $\sum_{e\in E} p(e)\cdot f(e)$

subject to $f(e)+p(e) \le c(e) \quad \forall e \in E$

$\sum_{(s,u)\in E} f(s,u) \geqslant d$

$$f^{in}(u) = \sum_{w \in V} f(w,u) = \sum_{\substack{v \\ \text{where } (u,v)\in E}} f(u,v)\cdot p(e_{uv}) + \sum_{\substack{v \\ \text{where } (u,v)\in E}} f(u,v)$$

$p(e), f(e) \geqslant 0$

b). Since we want to minimize the total cost of the flow. for each edge, the ~~sum~~ of flow and cost should be smaller than capacity value, and the total flow out of s is greater than d by constraint. Also, because of cost, so the total flow out of u is equal to total flow into v. plus the cost times flow out of u

$g: \xrightarrow{f(u)} (u) \xrightarrow{f(v)} (v)$

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P_i — max produce #

q_i : max sell number

n cities

1 shop/city

$C_{ij} = 0 \Leftarrow$ no highway from i to j

$C_{ij} > 0$

↳ max products

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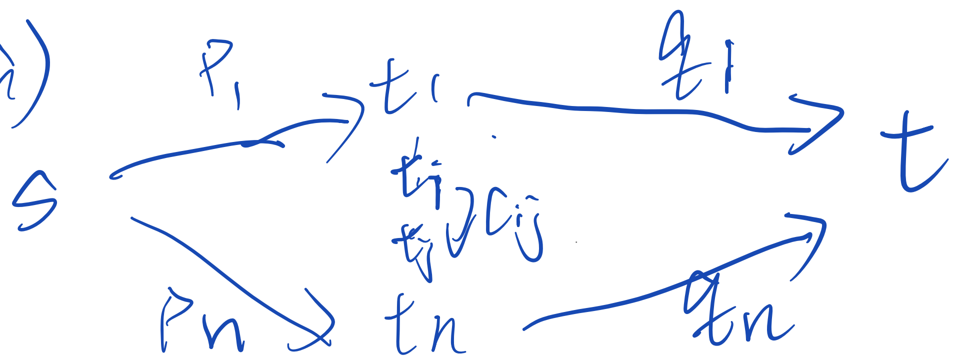
a)



max flow

ford fulkerson algorithm

b) $t_{ij} = \begin{cases} 1 & \text{when company uses truck on h-w} \\ 0 & \text{when doesn't use} \end{cases}$

$\sum t_{ij} \le m$

$t_{ij} = 0$ when $c_{ij} = 0$

$t_{ij} \in \{0, 1\}$

Max $\sum f_{it}$

$f_{si} \le p_i$

$f_{it} \le q_i$

$\sum a_i = \sum_i b$

$f_{ij} \le (t_{ij} + 1) \cdot c_{ij}$

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a) $\min \sum p(e) \cdot f(e)$

$\sum_{(s,u)\in E} f(s,u) \geqslant d$

$f(e_{u,v}) \leq c(e_{u,v})$

$\sum f(e_{a,u}) = \sum f(e_{u,b})$ except s,t