CSC373 Summer '22
Assignment 3: Linear Programming

Due Date: July 21, 2022, 11:59pm ET

Instructions

1. Typed assignments are preferred (e.g., PDFs created using LaTeX or Word), especially if your handwriting is possibly illegible or if you do not have access to a good quality scanner. Either way, you need to submit a single PDF named "hwk3.pdf" on MarkUS at `https://markus.teach.cs.toronto.edu/2022-05`
2. You will receive 20% of the points for a (sub)question when you leave it blank (or cross off any written solution) and write "I do not know how to approach this problem." If you leave it blank but do not write this or a similar statement, you will receive 10%. This does not apply to any bonus (sub)questions.
3. You may receive partial credit for the work that is clearly on the right track. But if your answer is largely irrelevant, you will receive 0 points.

Q1 [30 Points] Max Flow with Losses

The *Maximum Flow with Losses* problem is similar to the maximum flow problem: you are given as input a directed graph $G = (V, E)$ with source s and sink t, except here, in addition to the graph, each vertex $u \in V - \{s, t\}$ has a real number called the *loss coefficient* $\varepsilon_u \in [0, 1]$ such that the total flow out of u must equal $(1 - \varepsilon_u)$ times the total flow into u. As before, we are looking for an assignment of flow values to every edge that maximizes the total flow out of s.

(a) [15 Points] Show how to solve the maximum flow with losses problem using linear programming. Give a detailed description of your linear program and justify clearly and carefully that it solves the problem.

(b) [15 Points] Convert the linear program above into the standard form, and describe how a solution to this modified linear program would lead you to a solution of your original linear program. More precisely, specify these quantities: n, m, an $n \times 1$ coefficient vector c, an $m \times n$ constraint matrix A, and an $m \times 1$ bound vector b (all numbers are real numbers) such that the linear program from part (a) can be "converted" to the linear program which maximizes $c^T x$ subject to the constraints $Ax \leq b$ and $x \geq 0$. Also, specify clearly which variable of your original linear program does each x_i (the i-th entry of x) play the role of.

Q2 [20 Points] Tasks and Tools

You have m different tasks to complete and to help you, n different software tools you could purchase, each with a positive integer cost c_i. You have no choice about which tasks to complete (you must complete them all), but you get to choose which tools you will purchase.
Tools are not necessary to complete tasks; however, certain tasks have additional costs if they are completed without the use of specific tools. Information about these additional costs is provided through non-negative integer *dependencies*: for all pairs i, j, $d_{i,j}$ is the additional cost of completing

$$\sum f_{vu} \le c(1-\varepsilon_u)\sum_w f_{uw}$$

$$\sum f_{vu} \ge c(1-\varepsilon_u)\sum_w f_{uw}$$

$$-\sum f_{vu} \le (\varepsilon_u - 1)\sum_w f_{uw}$$

$$\left\{ \sum f_{vu} \le c(1-\varepsilon_u)\sum_w f_{uw} \right.$$

task i *without* tool j – a dependency can be equal to zero to indicate that there is no additional cost.

Finally, there are known *incompatibilities* between certain tools: for each tool i, you have a list of all the other tools with which tool i is incompatible. Obviously, it is not possible to install incompatible tools at the same time.

Formulate a linear program (or a binary integer program – i.e., an optimization problem with a linear objective, linear constraints, but with each variable restricted to taking a value in $\{0,1\}$) to determine which tools to purchase in order to minimize your total cost (from the purchase of tools and the dependencies). Also, remember to justify the correctness of your solution – this is important!

Q3 [10 Points] LP and IP Exercise

Consider the following linear program L in the standard form:

$$\begin{array}{ll} \text{Maximize} & x_2 \\ \text{Subject to} & -3x_1 + 5x_2 \leq 8 \\ & 7x_1 + 3x_2 \leq 12 \\ & x_1, x_2 \geq 0 \end{array}$$

We define the corresponding integer program I as follows:

$$\begin{array}{ll} \text{Maximize} & x_2 \\ \text{Subject to} & -3x_1 + 5x_2 \leq 8 \\ & 7x_1 + 3x_2 \leq 12 \\ & x_1, x_2 \in \{0,1\} \end{array}$$

Plot the feasible region of L. *Note: You do not need to submit this with the assignment, but it will be helpful to plot the feasible region.* You can use any online graphing programs such as desmos, fooplot, etc.

(a) [2.5 Points] What are the vertices of the feasible region of L? (No explanation is needed.)

(b) [2.5 Points] What are the optimal solutions of L and I? What are the corresponding optimal objective values? (No explanation is needed.)

(c) [2.5 Points] Provide the dual linear program (which we will call L') of L. Clearly indicate which dual variable in your formulation corresponds to which primal constraint.

(d) [2.5 Points] What are the optimal solutions of L' and its corresponding integer program I' (i.e., the program you obtain upon restricting all of the variables of L' to $\{0,1\}$)? What are the corresponding optimal objective values? Does strong duality hold for this particular pair of primal and dual integer programs i.e., for I and I'?

a)

$$-21x_1 + 35x_2 \le 56$$
$$21x_1 + 9x_2 \le 36$$
$$\downarrow$$
$$44x_2 \le 92$$
$$0 \le x_2 \le \frac{92}{44} = \frac{23}{11}$$

$$-9x_1 + 15x_2 \le 24$$
$$35x_1 + 15x_2 \le 60$$
$$44x_1 \le 36$$
$$0 \le x_1 \le \frac{36}{44} = \frac{9}{11}$$

b) $\left(\frac{9}{11}, \frac{23}{11}\right) \to \frac{23}{11}$

$(1, 1) \to 1$

c)

Consider the following linear program L in the standard form:

$$
\begin{array}{ll}
\text{Maximize} & x_2 \\
\text{Subject to} & -3x_1 + 5x_2 \leq 8 \\
& 7x_1 + 3x_2 \leq 12 \\
& x_1, x_2 \geq 0
\end{array}
$$

$y_1 \qquad -3x_1 + 5x_2 \leq 8$

$y_2 \qquad 7x_1 + 3x_2 \leq 12$

$$y_1(-3x_1 + 5x_2) + y_2(7x_1 + 3x_2) \leq 8y_1 + 12y_2$$

$$\Downarrow$$

$$(-3y_1 + 7y_2)x_1 + (5y_1 + 3y_2)x_2$$

$$\leq 8y_1 + 12y_2$$

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minimize

$$Z_y = 8y_1 + 12y_2$$

$$-3y_1 + 7y_2 \geq 0$$

$$5y_1 + 3y_2 \geq 1$$

$$y_1, y_2 \geq 0$$

$d)$