Asset Price Modeling Simulation

Application of the Black-Scholes Model to Real Market Data

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Project Overview
This project aims to bridge the gap between stochastic calculus theory and financial practice. By implementing the **Black-Scholes-Merton model** from scratch, we explore how a mathematical abstraction (Geometric Brownian Motion) translates into a computational tool for price forecasting. We analyze stock data (NVIDIA) to test the model's accuracy in short-term tracking versus long-term projection.

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1 Introduction Context

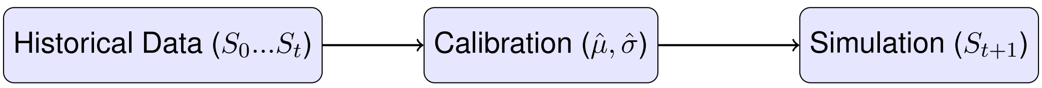
1.1 Why this project?

Financial markets are inherently unpredictable, yet financial engineering relies on mathematical models to price derivatives and manage risk. The Black-Scholes model is the cornerstone of this field. Understanding its mechanics—from the Stochastic Differential Equation (SDE) to its discrete simulation—is crucial for any quantitative analysis.

1.2 Objectives

This study focuses on three key steps:

1. **Mathematical Derivation:** Solving the SDE using Itô's Lemma to obtain the exact solution for the asset price S_t.
2. **Calibration:** Estimating the drift (μ) and volatility (σ) parameters from historical data using a rolling window approach.
3. **Simulation:** Using Monte Carlo methods to project future price paths and compare them with reality.



2 Mathematical Framework

2.1 Model Assumptions

To model the dynamics of a financial asset $S(t)$ over time, we place ourselves within the framework of the Black-Scholes-Merton model. We assume the market operates under the following conditions:

- **Continuous Trading:** The asset can be bought or sold at any time t.
- **Geometric Brownian Motion:** The asset price $S(t)$ follows a Geometric Brownian Motion (GBM), implying that log-returns are normally distributed.
- **Constant Parameters (Locally):** The drift μ and the volatility σ are assumed constant over the estimation window.
- **No Arbitrage:** The market is efficient and offers no risk-free profit opportunities.

2.2 Stochastic Differential Equation (SDE)

The dynamics of the asset price $S(t)$ are governed by the following Stochastic Differential Equation:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \quad (*) \tag{1}$$

Where:

- μ is the drift (expected return per unit of time).
- σ is the volatility (standard deviation of returns).
- W_t is a standard Wiener process (Brownian Motion).

2.3 Resolution via Itô's Lemma

Since the noise term dW_t is multiplicative (dependent on S_t), we cannot integrate directly. We linearize the equation using a logarithmic transformation. Let us define the function $f(S_t) = \ln(S_t)$.

According to **Itô's Lemma**, for a function $f(S_t, t)$, the differential is given by:

$$df(S_t) = f'(S_t)dS_t + \frac{1}{2}f''(S_t)(dS_t)^2 \tag{2}$$

Derivatives Calculation

We compute the partial derivatives required for the expansion:

$$
\begin{aligned}
f'(S_t) &= \frac{d}{dS_t}(\ln S_t) = \frac{1}{S_t} \\
f''(S_t) &= \frac{d}{dS_t}\left(\frac{1}{S_t}\right) = -\frac{1}{S_t^2}
\end{aligned}
$$

Quadratic Variation $(dS_t)^2$

To compute the second-order term, we apply the standard Itô multiplication rules:

$$
\begin{aligned}
dt \cdot dt &= 0 \\
dt \cdot dW_t &= 0 \\
(dW_t)^2 &= dt
\end{aligned}
$$

$$
\begin{aligned}
(dS_t)^2 &= (\mu S_t dt + \sigma S_t dW_t)^2 \\
&= \mu^2 S_t^2 (dt)^2 + 2\mu\sigma S_t^2 (dt)(dW_t) + \sigma^2 S_t^2 (dW_t)^2 \\
&= \sigma^2 S_t^2 dt
\end{aligned}
$$

$$
\boxed{(dS_t)^2 = \sigma^2 S_t^2 dt} \tag{3}
$$

Substitution and Simplification

Substituting these terms back into Itô's formula:

$$
\begin{aligned}
d(\ln S_t) &= \frac{1}{S_t}(\mu S_t dt + \sigma S_t dW_t) + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)(\sigma^2 S_t^2 dt) \\
&= (\mu dt + \sigma dW_t) - \frac{1}{2}\sigma^2 dt
\end{aligned}
$$

Grouping the deterministic dt terms, we obtain the dynamics of the log-price:

$$
\boxed{d(\ln S_t) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t} \tag{4}
$$

We observe that the Right-Hand Side is now independent of S_t.

2.4 Analytical Solution

We integrate the equation between time 0 and t:

$$
\int_0^t d(\ln S_u) = \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right) du + \int_0^t \sigma dW_u \tag{5}
$$

$$\ln(S_t) - \ln(S_0) = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma(W_t - W_0) \tag{6}$$

Using the definition $W_0 = 0$, and exponentiating both sides, we obtain the global solution for the Geometric Brownian Motion:

$$\boxed{S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right)} \tag{7}$$

2.5 Discrete-Time Simulation Model

To implement this model numerically, we transition from the global view ($0 \to t$) to a local incremental view ($t \to t + \Delta t$). Using the property of independent increments of the Brownian motion:

$$S(t + \Delta t) = S(t) \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) \Delta t + \sigma(W_{t+\Delta t} - W_t)\right) \tag{8}$$

The Brownian increment $\Delta W_t = W_{t+\Delta t} - W_t$ follows a Normal distribution $\mathcal{N}(0, \Delta t)$. For simulation purposes, we can express this as:

$$W_{t+\Delta t} - W_t = \sqrt{\Delta t} Z, \quad \text{where } Z \sim \mathcal{N}(0, 1) \tag{9}$$

This yields the recursive formula used for the Monte Carlo simulation:

$$\boxed{S_{i+1} = S_i \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) \Delta t + \sigma\sqrt{\Delta t} Z_i\right)} \tag{10}$$

2.6 Parameter Estimation (Rolling Window)

To apply the model to real market data, we must estimate μ and σ. We use a rolling window approach on historical data.

Let P_i be the asset price at day i. We calculate the daily log-returns:

$$x_i = \ln\left(\frac{P_i}{P_{i-1}}\right) \tag{11}$$

Over a window of size N, we compute the sample mean $\bar{x}$ and sample standard deviation s_x. We then annualize these statistics to obtain the model parameters.

1. Volatility (σ): Since variance scales linearly with time:

$$\boxed{\hat{\sigma} = \frac{s_x}{\sqrt{\Delta t}}} \tag{12}$$

2. Drift (μ): The arithmetic mean of log-returns corresponds to the term $(\mu - \frac{1}{2}\sigma^2)\Delta t$. To recover μ, we must add the convexity correction term (Itô's term):

$$\bar{x} = \left(\hat{\mu} - \frac{1}{2}\hat{\sigma}^2\right) \Delta t \quad \Longrightarrow \quad \boxed{\hat{\mu} = \frac{\bar{x}}{\Delta t} + \frac{1}{2}\hat{\sigma}^2} \tag{13}$$

3 Empirical Analysis: NVIDIA Case Study

3.1 Methodology

To test the validity of the Black-Scholes model, we applied our Python implementation to real market data.

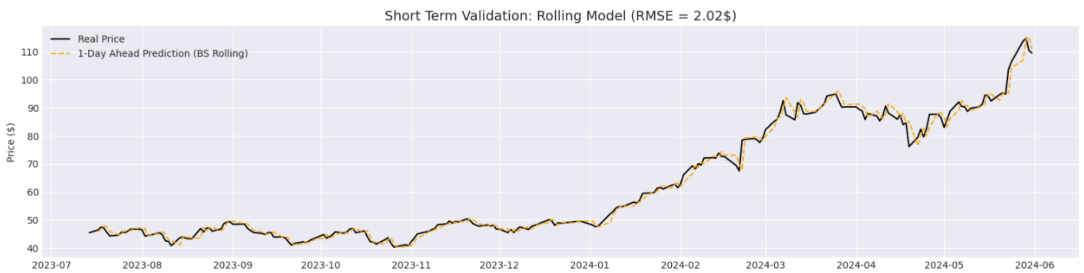
- **Asset:** NVIDIA Corp. (NVDA) - Chosen for its high volatility and strong trend.
- **Period:** Jan 1, 2020 to June 1, 2024.
- **Data Split:** 80% for Training (Calibration), 20% for Testing (Future).

We compare two simulation approaches:

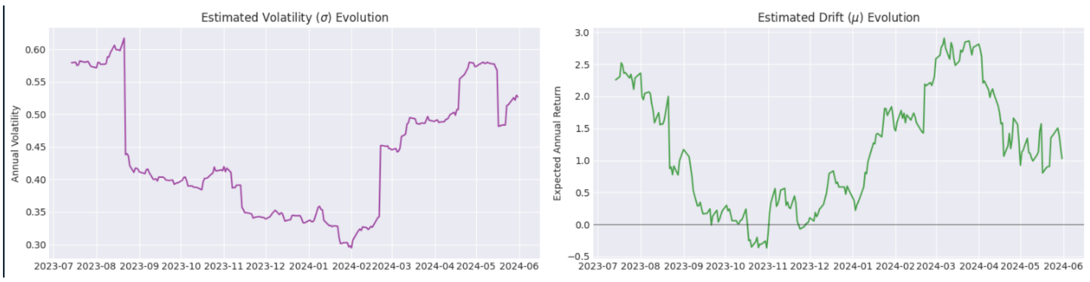
1. **Short-Term (Rolling):** Recalibrating parameters daily to predict $t + 1$.
2. **Long-Term (Static):** Freezing parameters at $t = 0$ and projecting 6 months ahead.

3.2 Short-Term Validation: The Rolling Backtest

In this approach, we estimate μ and σ every day using a 60-day moving window. We then predict the next day's price.



(a) One-Step-Ahead Forecasting



(b) Dynamic Evolution of μ and σ

Figure 1: The rolling model follows the real price very closely (RMSE $\approx$ \$2.02). This confirms that the GBM equation is locally valid when parameters are updated.

3.3 Long-Term Projection: The Monte Carlo Simulation

Here, we stop updating the parameters. We estimate them once and project 2000 possible futures.

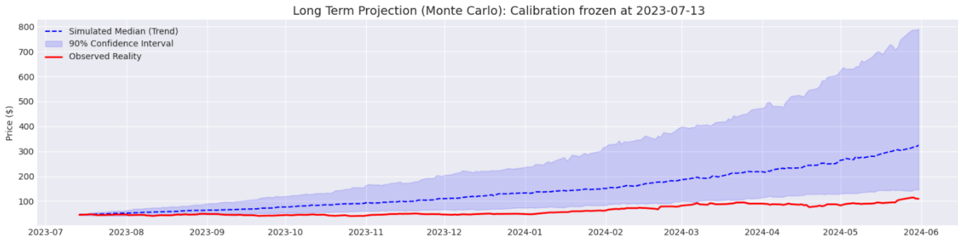


Figure 2: **Static Projection vs Reality.** The blue area represents the 90% confidence interval.

Observation: The real price (Red) quickly diverges from the simulated median (Blue Dotted). The model predicts an exponential rise ($\mu > 0$), but the market crashed.

3.4 Critical Analysis

The discrepancy highlights the fundamental flaws of the Black-Scholes model for long-term forecasting:

1. The Constant Drift Problem

The model assumes μ is constant. It extrapolates the past trend linearly. It cannot foresee market cycles or macro-economic news (e.g., interest rate hikes) that reverse the trend.

2. Volatility Clustering

As shown in Figure 1(b), the volatility σ is not constant. It fluctuates wildly. The static model underestimates risk during crises because it relies on an average historical volatility.

4 Conclusion

This project demonstrates that the **Geometric Brownian Motion** provides a solid theoretical foundation for asset pricing but has distinct limitations in practice.

- **Short Term:** It is highly effective for pricing and short-term risk management if recalibrated frequently (daily).
- **Long Term:** It is unsuitable for directional forecasting as it fails to capture changing market regimes and non-constant volatility.

Ultimately, the Black-Scholes model is a powerful tool for defining a **probabilistic range** of prices rather than predicting a specific future value.