- alternating process with Lévy walk and Brownian motion;
- counting process: Poisson process;
- Gaussian process: Brownian motion and fractional Brownian motion;
- Lévy process: symmetric α-stable process and β-stable subordinator;
- multiple internal states process: Lévy walk with multiple internal states and fractional compound Poisson process with multiple internal states;
- continuous-time random walks.

Contents

Appendix A: Models

In this appendix, we present a brief introduction to the concepts of stochastic processes involved in this paper. We provide theoretical insights about the normal and anomalous diffusion models considered in this classification, as well as the description of the pseudocode used for simulations (see Appendix B). The Python and MATLAB implementations of all the algorithms described below are available at https://github.com/tangxiangong/ClassTop/.

1. Stable distribution and power-law distribution

Before introducing the diffusion models, let us take a look at the stable distribution [1, 20, 25] and the power-law distribution. Let X_1 and X_2 be independent copies of a random variable X. Then X is said to be stable if for any positive constants a and b the random variable $aX_1 + bX_2$ has the same distribution as $cX + d$ for some constants $c > 0$ and d. Furthermore, a random variable X is stable if and only if its characteristic function $\mathbb{E}\left[\mathrm{e}^{\mathrm{i}kX}\right]$ can be written as [1, 20, 21]

$$\varphi(k;\alpha,\beta,\mu,\sigma) = \exp\left(-\sigma^\alpha |k|^\alpha \left(1 - \mathrm{i}\beta\omega(k;\alpha,\sigma)\,\mathrm{sgn}(k)\right) + \mathrm{i}\mu k\right) \tag{A1}$$

with

$$\omega(k;\alpha,\sigma) = \begin{cases} \left(|\sigma k|^{1-\alpha} - 1\right)\tan\left(\frac{\pi\alpha}{2}\right), & \alpha \neq 1, \\ -\frac{2}{\pi}\ln|\sigma k|, & \alpha = 1, \end{cases} \tag{A2}$$

where $\alpha \in (0, 2]$, is the index of stability, $\beta \in [-1, 1]$, called the skewness parameter, is a measure of asymmetry, $\mu \in \mathbb{R}$ is a shifted parameter, and $\sigma > 0$ is called the scaling parameter (without loss of generality, we will take $\sigma = 1$ in the following text).

When $\mu = \beta = 0$, it is called the symmetric α-stable distribution, and when $\alpha < 1$ and $\beta = 1$, it is called one-sided (or totally skewed) α-stable distribution. In this paper, we take $\mu = 0$ for totally skewed stable distribution, in this case, the distribution is supported by $[0, \infty)$.

When $\alpha = 2$, the stable distribution is normal distribution. If $\alpha < 2$, the stable distribution is power-law, that is, the probability density function (PDF) of symmetric α-stable distribution $L_\alpha(x)$, and the PDF of totally skewed α-stable distribution $S_\alpha(t)$ have the following asymptotic formulae [12]

$$L_\alpha(x) \propto \frac{1}{|x|^{1+\alpha}}, \quad |x| \to \infty, \tag{A3}$$

$$S_\alpha(t) \propto \frac{1}{t^{1+\alpha}}, \quad t \to \infty. \tag{A4}$$

To generate a symmetric α-stable random variable X, we can first generate a random variable V uniformly distributed on $(-\pi/2, \pi/2)$ and an exponential random variable W with mean 1, and then compute X as following [19, 39]

$$X = \frac{\sin(\alpha V)}{(\cos V)^{1/\alpha}} \cdot \left(\frac{\cos(V - \alpha V)}{W}\right)^{(1-\alpha)/\alpha}. \tag{A5}$$

Similarly, we can get a totally skewed α-stable S as below [19, 39],

$$S = c_1 \frac{\sin(\alpha(V + c_2))}{(\cos V)^{1/\alpha}} \cdot \left(\frac{\cos(V - \alpha(V + c_2))}{W}\right)^{(1-\alpha)/\alpha}, \tag{A6}$$

where $c_1 = (\cos(\pi\alpha/2))^{-1/\alpha}$ and $c_2 = \pi/2$.

The simulation algorithms for symmetric and totally skewed stable distribution are Algorithm 2 and Algorithm 3 respectively. In addition, one can also use the **Stable Distribution** in the MATLAB Statistic and Machine Learning Toolbox (https://www.mathworks.com/help/stats/stable-distribution.html) or **levy_stable** in the Python package SciPy (https://docs.scipy.org/doc/scipy/reference/generated/scipy.st to generate random numbers of stable distribution.

2. Continuous-time random walk

The continuous-time random walk (CTRW) model [10, 14, 21, 28] is based on the idea that the length of a given jump, as well as the waiting time elapsing between two successive jumps of a particle are drawn from a PDF $\psi(x, t)$. From $\phi(x, t)$, the jump length PDF $\lambda(x)$ and the waiting time PDF $\psi(t)$ can be deduced by marginal PDF. Here, we consider decoupled CTRW, that is, the waiting time and jump length are independent, $\phi(x, t) = \lambda(x)\psi(t)$. If $x(t)$ denotes the position of particle with initial position $x(0) = 0$, then

$$x(t) = \sum_{k=1}^{N(t)} \xi_k \tag{A7}$$

with

$$N(t) = \max\left\{n \in \mathbb{N} : \sum_{k=1}^{n} \tau_k \leqslant t\right\}, \tag{A8}$$

where waiting times $\{\tau_n\} \overset{\text{i.i.d}}{\sim} \psi(t)$ and jump lengths $\{\xi_n\} \overset{\text{i.i.d}}{\sim} \lambda(x)$.

Different types of CTRW processes can be categorised by the characteristic waiting time (mean-value of waiting time) and the jump length variance being finite or diverging[28], respectively. Therefore, we consider four cases of CTRW model. For finite and diverging characteristic waiting time, we choose exponential distribution and totally skewed β-stable distribution (power-law), respectively. For finite and diverging jump length variance, we choose symmetric α-stable distribution with $\alpha < 2$ and normal distribution ($\alpha = 2$), respectively.

For the finite characteristic waiting time and jump length variance CTRW model, it can model the normal diffusion in the sense of scaling limit with mean-squared displacement (MSD) $\langle x^2(t)\rangle \propto t$. For the finite characteristic waiting time and diverging jump length variance CTRW model, also called the Lévy flight, its MSD is diverging. But we can regard $\langle |x(t)|^\delta\rangle \propto t^{\delta/\alpha}$ as MSD for $0 < \delta < \alpha \leqslant 2$ [28]. For the diverging characteristic waiting and finite jump length variance CTRW model, its MSD $\langle x^2(t)\rangle \propto t^\beta$ [28]. However, the equations governing PDFs for these four situation have a unified form [6–8, 14, 25, 36], as below,

$$\frac{\partial}{\partial t}P(x,t) = K\,{}_0D_t^{1-\beta}\Delta^{\alpha/2}P(x,t), \tag{A9}$$

where K is the generalized diffusion coefficient, ${}_0D_t^{1-\beta}$ is the Riemann-Liouville fractional derivative operator [5, 29, 32] of oder $\beta \leqslant 1$,

$${}_0D_t^{1-\beta}f(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}f(s)\mathrm{d}s, \tag{A10}$$

and $\Delta^{\alpha/2}$ is the fractional Laplace operator [22, 30]. Therein, $\beta = 1$ corresponds to the exponential waiting time, and $\alpha = 2$ corresponds to normal jump length. Last, the simulation algorithm for these four CTRW models is given by Algorithm 5.

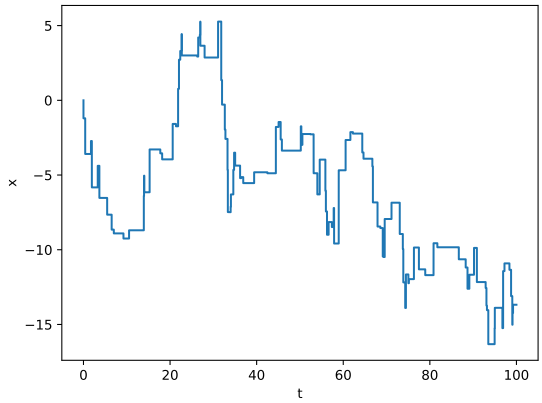

FIG. 1: CTRW with finite characteristic waiting time and jump length variance

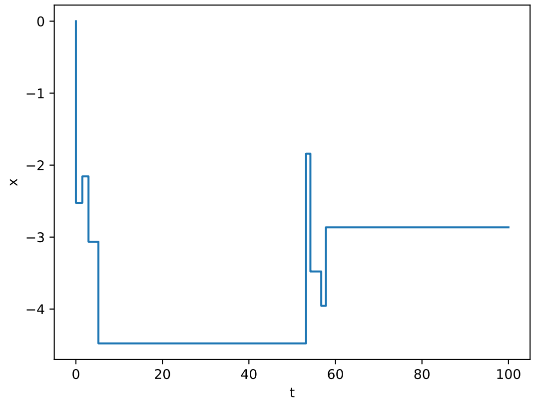

FIG. 2: CTRW with diverging characteristic waiting time and finite jump length variance

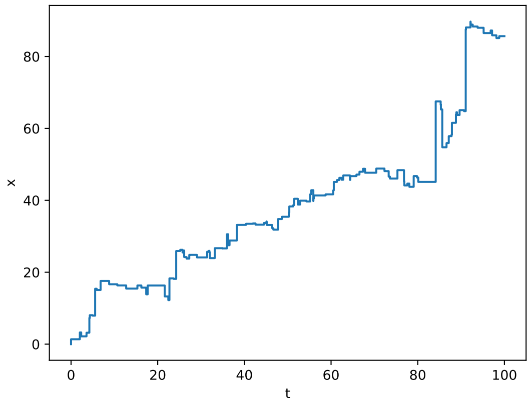

FIG. 3: CTRW with finite characteristic waiting time and diverging jump length variance

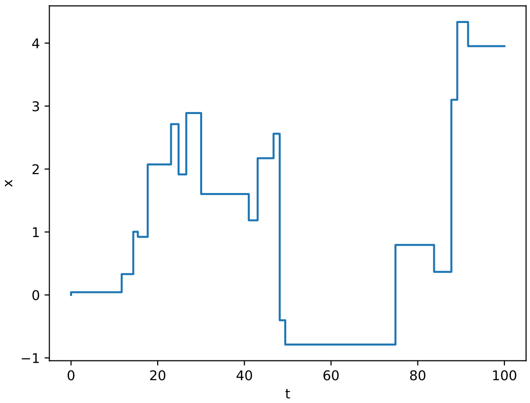

FIG. 4: CTRW with diverging characteristic waiting time and jump length variance

3. Lévy process

Let $\boldsymbol{X} = \boldsymbol{X}(t, \omega)$ be a d-dimensional stochastic process defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. We say that it has independent increments if for each $n \in \mathbb{N}$ and each $0 \leqslant t_1 < t_2 < \cdots < t_{n+1} < \infty$ the random variables $\{\boldsymbol{X}(t_{j+1}) - \boldsymbol{X}(t_j) : 1 \leqslant j \leqslant n\}$ are independent and that it has stationary increments if each $\boldsymbol{X}(t_{j+1}) - \boldsymbol{X}(t_j) \overset{\mathrm{d}}{=} \boldsymbol{X}(t_{j+1} - t_j) - \boldsymbol{X}(0)$, where the symbol $\overset{\mathrm{d}}{=}$ denotes identical distribution. Then $\boldsymbol{X}$ is a Lévy process [1, 20] if

1. $\boldsymbol{X}(0) = 0$, a.s.;

2. $\boldsymbol{X}$ has independent and stationary increments;

3. $\boldsymbol{X}$ is stochastically continuous, i.e., for all $a > 0$ and for all $t \geqslant 0$

$$\lim_{s \to t} \mathbb{P}\left(|\boldsymbol{X}(t) - \boldsymbol{X}(s)| > a\right) = 0.$$

Because of its property of independent increments, Lévy process is Markovian.

According to Lévy-khintchine theorem [1, 20, 24], we know that the characteristic function of Lévy process $\boldsymbol{X}$ has the following formula

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{X}(t)}\right] = \mathrm{e}^{t\phi(\boldsymbol{k})}, \tag{A11}$$

$$\phi(\boldsymbol{k}) = \mathrm{i}\boldsymbol{b}\cdot\boldsymbol{k} - \frac{1}{2}\boldsymbol{k}\cdot\boldsymbol{A}\boldsymbol{k} + \int_{\mathbb{R}^d\setminus\{0\}} \left(\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{y}} - 1 - \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{y}\mathbb{1}_{\{|\boldsymbol{x}|<1\}}(\boldsymbol{y})\right)\nu(\mathrm{d}\boldsymbol{y}), \tag{A12}$$

where $\boldsymbol{b}$ is a vector in $\mathbb{R}^d$, $\boldsymbol{A}$ is a $d \times d$ real symmetric semi-positive definite matrix, $\mathbb{1}$ is the indicator function of set, and ν is a Lévy measure, satisfying

$$\int_{\mathbb{R}^d\setminus\{0\}} \min\left\{1, |\boldsymbol{y}|^2\right\}\nu(\mathrm{d}\boldsymbol{y}) < \infty. \tag{A13}$$

a. isotropic α-stable Lévy process

We say Lévy process $\boldsymbol{L}_\alpha(t)$ is isotropic α-stable if for any fixed $t \geqslant 0$, $\boldsymbol{L}_\alpha(t)$ is an isotropic stable random vector with index of stability $0 < \alpha \leqslant 2$. And its characteristic function has the following specific formula

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{L}_\alpha(t)}\right] = \mathrm{e}^{-\sigma^\alpha|\boldsymbol{k}|^\alpha t}, \tag{A14}$$

where $\sigma > 0$ is the scaling parameter.

According to (A14), we can obtain that the PDF $P(\boldsymbol{x}, t)$ of $\boldsymbol{L}_\alpha(t)$ satisfies following equation

$$\frac{\partial}{\partial t}P(\boldsymbol{x}, t) = \Delta^{\alpha/2}P(\boldsymbol{x}, t), \tag{A15}$$

Here, we focus on the one-dimensional isotropic (or symmetric) α-stable Lévy process.

When $\alpha = 2$, the one-dimensional stable Lévy process $L_\alpha(t)$ is the Brownian motion $B(t)$. Brownian motion $B(t)$ is the only continuous Lévy process, and is also a Gaussian process with mean $\mathbb{E}[B(t)] = 0$ and covariance $\mathbb{E}[B(s)B(t)] = 2\min\{t, s\}$ for $t, s \geqslant 0$. And the PDF of $B(t)$ is

$$P(x, t) = \frac{1}{\sqrt{4\pi t}}\exp\left\{-\frac{x^2}{4t}\right\}. \tag{A16}$$

The algorithm used to simulate one-dimensional α-stable Lévy process trajectories is described in Algorithm 6. Here, we use the equation

$$L_\alpha(t_{n+1}) = L_\alpha(t_n) + (t_{n+1} - t_n)^{1/\alpha}\xi_n \tag{A17}$$

to simulate trajectories, where $\{\xi_n\}$ are independent random variables of symmetric α-stable distribution.

b. *β-stable subordinator*

A subordinator [1, 20] is a one-dimensional Lévy process that is non-decreasing (a.s.). And subordinator $T(t)$ is a β-stable subordinator if for any fixed $t \geqslant 0$, $T(t)$ is a totally skewed β-stable random variable with index of stability $0 < \beta < 1$. The Laplace transform of $T(t)$ is given by [1]

$$\mathbb{E}\left[\mathrm{e}^{-\lambda T(t)}\right] = \mathrm{e}^{-\lambda^\beta t}. \tag{A18}$$

According to above Laplace transform, one can get the PDF $P(T,t)$ of $T(t)$ satisfies

$$\partial_t P(T,t) = -\partial_T^\beta P(T,t) - \frac{T^{-\beta}}{\Gamma(1-\beta)}, \tag{A19}$$

where ∂_t^β is the Caputo fractional derivative operator [5, 30] of order $\beta < 1$,

$$\partial_t^\beta u(x,t) = \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\partial_s u(x,s)\mathrm{d}s. \tag{A20}$$

Similar to equation (A17), we use equation

$$T(t_{n+1}) = T(t_n) + (t_{n+1} - t_n)^{1/\beta}\zeta_n \tag{A21}$$

to simulate the trajectories of β-stable subordinator, where $\{\zeta_n\}$ are independent random variables of totally skewed β-stable distribution. The simulation algorithm is Algorithm 7.

c. *Poisson process*

The (time homogeneous) Poisson process of intensity $\lambda > 0$ is a Lévy process $N(t)$ taking values in $\mathbb{N}$, and

$$\mathbb{P}\left[N(t) = n\right] = \frac{(\lambda t)^n}{n!}\mathrm{e}^{-\lambda t} \tag{A22}$$

for each $n = 0,\ 1,\ 2,\ \cdots$. The probability distribution $P(n,t) = \mathbb{P}[N(t) = n]$ satisfies following equation

$$\frac{\partial}{\partial t}P(n,t) = \lambda\left(P(n,t) - P(n-1,t)\right). \tag{A23}$$

According to the properties of independent and stationary increments, the Poisson process can also be defined by stating that the time differences between events of the counting process are exponential variables with mean $1/\lambda$. So we can use independent random numbers of mean-$1/\lambda$ exponential distribution with $\{\pi_n\}$ to simulate Poisson process's trajectories, that is,

$$N(t_n) = n,\ t_n = t_{n-1} + \pi_n,\ t_0 = 0. \tag{A24}$$

The simulation algorithm is Algorithm 8.

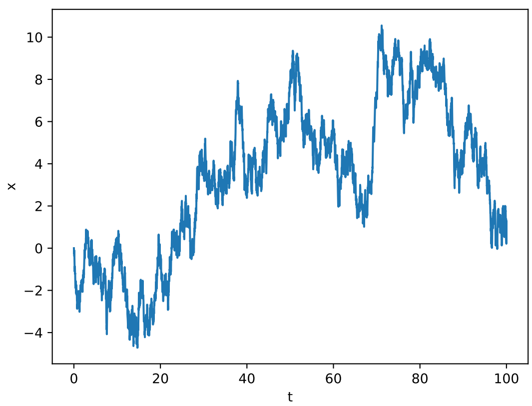

FIG. 5: Brownian motion

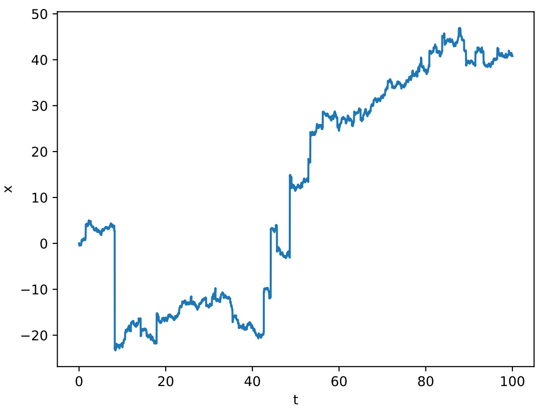

FIG. 6: α-stable Lévy process with $\alpha < 2$

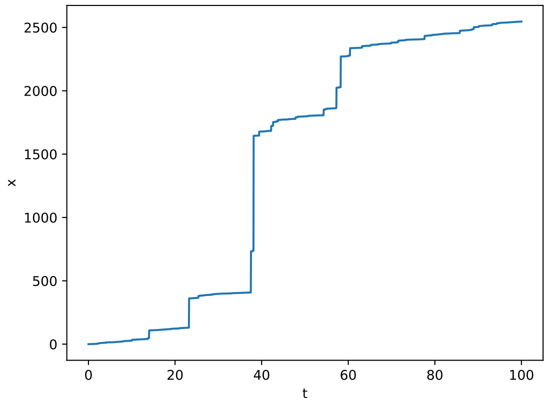

FIG. 7: stable subordinator

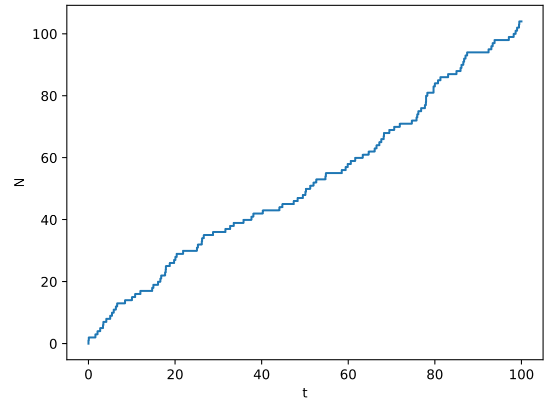

FIG. 8: Poisson process

4. Fractional Brownian motion

Fractional Brownian motion [4, 13, 15, 25, 31] $B_H(t)$ with Hurst index $0 < H < 1$ is a Gaussian process with stationary increments, and satisfies following properties:

$$B_H(0) = 0, \text{ a.s.}, \tag{A25}$$

$$\mathbb{E}\left[B_H(t)\right] = 0 \tag{A26}$$

for all $t \geqslant 0$, and

$$\mathbb{E}\left[B_H(t)B_H(s)\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right) \tag{A27}$$

for each $t,\ s \geqslant 0$. The mathematical definition of fractional Brownian motion is given by Mandelbrot and Ness [25], as following stochastic integral of Brownian motion $B(t)$

$$B_H(t) = \frac{1}{\Gamma(H+1/2)} \int_{\mathbb{R}} \left[(t-s)_+^{H-1/2} - (-s)_+^{H-1/2} \right] \mathrm{d}B(s), \tag{A28}$$

where $x_+ = \max\{x, 0\}$. Specially, $B_{1/2}(t) = B(t)$. In addition, in reference [18], the author proved that fractional Brownian motion is not Markov unless $H = 1/2$.

The mean-squared displacement of $B_H(t)$ is $\mathbb{E}\left[(B_H(t))^2\right] = t^{2H}$, so when $H < 1/2$, it is a subdiffusion, when $H > 1/2$, it is a superdiffusion. Since $B_H(t)$ is a Gaussian process, its characteristic function can be easily obtained by its mean and covariance,

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}kB_H(t)}\right] = \exp\left\{-\frac{1}{2}|k|^2 t^{2H}\right\}. \tag{A29}$$

Denote $P(x,t)$ as the PDF of $B_H(t)$. Then we can get

$$P(x,t) = \frac{1}{\sqrt{2\pi t^{2H}}} \exp\left\{-\frac{x^2}{2t^{2H}}\right\}, \tag{A30}$$

and

$$\frac{\partial}{\partial t} P(x,t) = H t^{2H-1} \frac{\partial^2}{\partial x^2} P(x,t). \tag{A31}$$

Various numerical approaches have been proposed to simulate fractional Brownian motion exactly. Here we use the Davies-Harte method [11, 15] and the Hosking method [15, 17] via the *stochastic* Python package (`https://pypi.org/project/stochastic/`). Details about the numerical implementations can be found in the associated references.

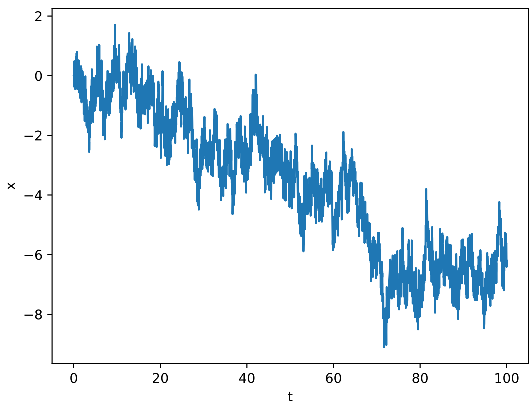

FIG. 9: fractional Brownian motion with Hurst index $H < 1/2$

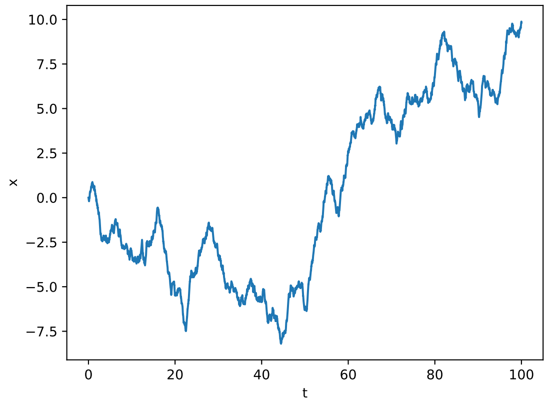

FIG. 10: fractional Brownian motion with Hurst index $H > 1/2$

5. Alternating process

A two-state process [37] serves as an intermittent search process, which alternates between Lévy walk [38, 42] and Brownian motion [3, 23], i.e., Lévy walk $\longrightarrow$ Brownian motion $\longrightarrow$ Lévy walk $\longrightarrow \cdots$. The searcher displays a slow active motion in the Brownian phase, during which the hidden target can be detected. While in Lévy walk phase, the searcher aims to relocate into some unvisited region to reduce oversampling. This kind of intermittent search process has wide applications in physical or biological problems [2, 35].

The sojourn time distributions of the two-state process switching between Lévy walk and Brownian phase are $\psi^+(t)$ and $\psi^-(t)$, respectively. The subscripts '+' and '-' are introduced to represent the Lévy walk and Brownian phase, respectively. This process can be explicitly described by means of the velocity process $v(t)$ which also consists of two states: $v^+(t)$ for Lévy walk and $v^-(t)$ for Brownian motion. The PDF of $v^+(t)$ is $\delta(|v| - v_0)/2$ for some constant velocity v_0, whereas $v^-(t) = \sqrt{2}\xi(t)$ with $\xi(t)$ being Gaussian white noise.

Let the sojourn time distributions in the two states be a power-law form with exponents $\alpha_\pm$, that is,

$$\psi_\pm(t) \propto t^{-(1+\alpha_\pm)} \tag{A32}$$

for large t. For $\alpha_\pm \in (0, 1)$, we know that one-sided $\alpha_\pm$-stable distribution has the above power-law form, so we can use equation (A6) to get such a random variable. In addition, we can assume $\psi_\pm(t)$ has the following expression

$$\psi_\pm(t) = \begin{cases} C(t+1)^{-(1+\alpha_\pm)}, & t \geqslant 0, \\ 0, & t < 0. \end{cases} \tag{A33}$$

By using the normalized property of the probability density function, one can get the parameter $C = \alpha_\pm$. Hence, the probability distribution function

$$\Psi_\pm(t) = \begin{cases} 1 - (t+1)^{-\alpha_\pm}, & t \geqslant 0, \\ 0, & t < 0. \end{cases} \tag{A34}$$

Therefore, if random variable $\xi \sim \Psi_\pm(t)$, then

$$\xi = (1 - X)^{-1/\alpha_\pm} - 1, \tag{A35}$$

where X is a random variable uniformly distributed on $(0, 1)$.

The MSD of this two-state process has the following expression [37]

$$\langle x^2(t) \rangle \sim K_1 t^{\nu_1} + K_2 t^{\nu_2}, \tag{A36}$$

where K_1 and K_2 are constants, the values ν_1 and ν_2 are given in the following table.

TABLE I: Parameters' values

Specific cases	ν_1	ν_2
$\alpha_+ = \alpha_- < 1$	2	1
$1 < \alpha_\pm < 2$	$3 - \alpha_+$	1
$\alpha_+ < \alpha_- < 1$	2	$\alpha_+ - \alpha_- + 1$
$\alpha_+ < 1 < \alpha_- < 2$	2	α_+
$\alpha_- < \alpha_+ < 1$	$\alpha_- - \alpha_+ + 2$	1
$\alpha_- < 1 < \alpha_+ < 2$	$\alpha_- - \alpha_+ + 2$	1

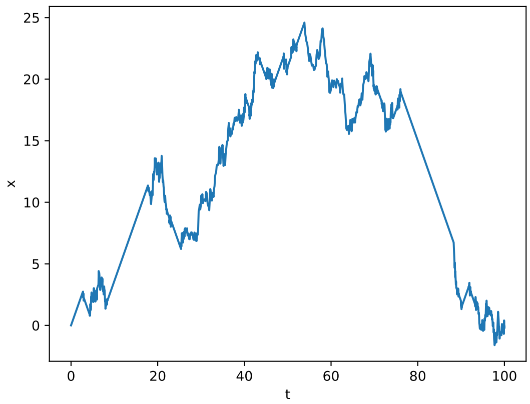

FIG. 11: alternating process with Lévy walk and Brownian motion

6. Multiple internal states process

a. fractional compound Poisson process with multiple internal states

Fractional Poisson process is a renewal process whose probability PDF of the holding/waiting times between two subsequent events has the asymptotic behavior $\phi(\tau) \sim 1/\tau^{1+\alpha}$, $0 < \alpha < 1$ when time is long enough. Let $N(t)$ be the fractional Poisson process, i.e., there exist independent identically distributed random variables $\{\tau_n\} \sim \phi(\tau)$ with $\phi(\tau) \sim 1/\tau^{1+\alpha}$ for $0 < \alpha < 1$ and large τ, and $\{\xi_n\}$ a sequence of independent identically distributed variables (jump lengths). Then $X(t) = \sum\limits_{n=1}^{N(t)} \xi_n$ is the fractional compound Poisson process.

Therefore, when the waiting time distribution is power-law with exponent $0 < \alpha < 1$, CTRW model (A7) is a fractional compound Poisson process. Furthermore, we can generalize the renewal processes to have multiple internal states, where the waiting times for different internal states are drawn from different distribution. Here, we focus on fractional compound Poisson process $X(t)$ with multiple internal states [40], that is, $N(t)$ of $X(t)$ is a fractional Poisson process with multiple internal states. Each internal state has an own distribution of holding time, but the distribution of the jump lengths are all simply taken as normal distribution. In other words, $X(t)$ is a kind of CTRW process with multiple internal states whose characteristic waiting time is diverging and jump length variance is finite.

Suppose that $X(t)$ has N internal states. The transition of the internal states is described by a Markov chain with transition matrix $M \in \mathbb{R}^{N\times N}$. And the element M_{ij} of M represents the transition probability from state i to state j. Here, the bras $\langle\cdot|$ and kets $|\cdot\rangle$ denote the row and column vectors respectively. Let $\Phi(t) = \operatorname{diag}\left(\phi^{(1)}(t), \phi^{(2)}(t), \cdots, \phi^{(N)}(t)\right)$ be the waiting time distribution matrix and $\Lambda(x) = \operatorname{diag}\left(\lambda^{(1)}(x), \lambda^{(2)}(x), \cdots, \lambda^{(N)}(x)\right)$ be the jump length one. And we use the notation $|\mathrm{init}\rangle$ to represent the column vector of initial distribution of the internal states. In the Laplace space, $\widehat{\Phi}(s) \sim I - \Phi^*(s)$ where $\Phi^*(s) = \operatorname{diag}\left(B_{\alpha_1}s^{\alpha_1}, \cdots, B_{\alpha_N}s^{\alpha_N}\right)$, $0 < \alpha_1, \cdots, \alpha_N < 1$. Then the Fokker-Planck equation for $X(t)$ is [40]

$$
\begin{aligned}
M^T\frac{\partial}{\partial t}|G(x,t)\rangle =& (M^T - I)\operatorname{diag}\left(B_{\alpha_1}^{-1}, \cdots, B_{\alpha_N}^{-1}\right)\operatorname{diag}\left(D_t^{1-\alpha_1}, \cdots, D_t^{1-\alpha_N}\right)|G(x,t)\rangle \\
&+ M^T\operatorname{diag}\left(K_{\alpha_1}, \cdots, K_{\alpha_N}\right)\operatorname{diag}\left(D_t^{1-\alpha_1}, \cdots, D_t^{1-\alpha_N}\right)\Delta|G(x,t)\rangle,
\end{aligned}
\tag{A37}
$$

where $|G(x,t)\rangle$ is the PDF of $X(t)$, $\{K_{\alpha_i}\}_{i=1}^N$ are generalized diffusion coefficient, $\left\{D_t^{1-\alpha_i}\right\}_{i=1}^N$ are Riemann-Liouville fractional derivative operator.

As for renewal process $N(t)$ of fractional compound Poisson process $X(t)$, after each update, a Markov chain is used to determine which internal state it is in. Therefore, we should know how to generate a non-uniform discrete distributed random number.

Suppose that Y is a finite discrete random variable taking values in $\{1, 2, \cdots, N\}$, and $\mathbb{P}[Y = j] = p_j$, $j = 1, 2, \cdots, N$. Here we use the inverse transform method to sample Y. The probability distribution function of Y is

$$
F_Y(x) = \begin{cases} 0, & x < 1 \\ \sum\limits_{k=1}^{j} p_k, & j \leqslant x < j+1,\ j = 1, \cdots, N-1, \\ 1, & x \geqslant N. \end{cases}
\tag{A38}
$$

The inverse of F_Y can be written as

$$
F_Y^{-1}(u) = j, \text{ if } \sum_{k=1}^{j-1} p_k < u \leqslant \sum_{k=1}^{j} p_k,\ j = 1, \cdots, N.
\tag{A39}
$$

If U is a random variable uniformly distributed on $(0, 1)$, then $F_Y^{-1}(U)$ is identically distributed with Y. Therefore, we can generate a sample ξ of U, then $F_Y^{-1}(\xi)$ is a sample of Y. The pseudocode is Algorithm 1.

After generating a sample of Y, we now can generate the trajectories of fractional compound Poisson process with multiple internal states $X(t)$. Firstly, we should determine the initial state by the initial distribution $|\text{init}\rangle$. In fact, we can choose initial state by generating a sample I of probability distribution $\mathbb{P}[Y = j] = |\text{init}\rangle_j$. Then the initial state is the I-th state. Next, after update in m-th internal state, we can generate a sample n of probability distribution $\mathbb{P}[Y = j] = M_{mj}$ to determine that next update is occurred is in the n-th internal state. The details is given by Algorithm 10.

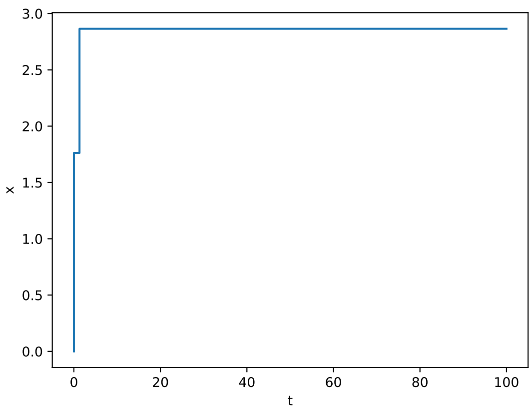

FIG. 12: fractional compound Poisson process with multiple internal states

b. *Lévy walks with multiple internal states*

As mentioned in Appendix A 2, the Lévy flight has divergent MSD and the particle's jump does not cost any time [28]. In order to make the model fitter for the real natural phenomena, Lévy walk is used to describe superdiffusion properly and naturally [34, 42]. The linearly-coupled Lévy walk with constant velocity v_0 is one of the most representative and important kinds of Lévy walks. Comparing to waiting time in CTRW, Lévy walk is given a random variable τ representing the duration of each step, and the total distance of each step of the particle's movement for linearly-coupled Lévy walk is then $v_0\tau$.

The Lévy walk with multiple internal states [41] is similar with fractional compound Poisson process with multiple internal states. The meanings of initial distribution vector $|\text{init}\rangle$ and the transition matrix M are same to the ones in Appendix A 6 a. However, the distributions of sojourn times $\{\phi_n(t)\}$ need not to be power-law, they can also be exponential. In addition, compared to fractional compound Poisson process with multiple internal states, Lévy walk with multiple internal states removes the jump length distribution, but has an additional feature, distributions of speed $\{\lambda_n(v_0)\}$. In Appendix B, we propose an algorithm (Algorithm 11) to simulate the trajectories of Lévy walk with multiple internal states.

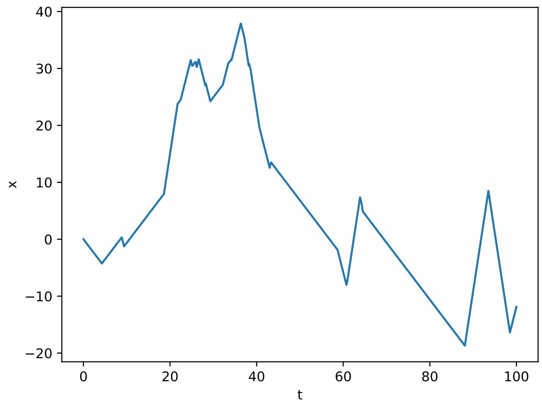

FIG. 13: Lévy walk with multiple internal states

Appendix B: Pseudocodes

1. Random number generator

Algorithm 1 Generating finite discrete distribution random number

Input: probability vector $\boldsymbol{p} = (p_1, \cdots, p_N)$ $\triangleright\ \mathbb{P}[Y = j] = p_j$

Output: a sample y of Y

1: Compute the cumulative sum $\boldsymbol{q} = (q_1, \cdots, q_N)$ of $\boldsymbol{p}$
2: $q_0 \leftarrow 0$
3: Generate a sample ξ uniformly distributed on $(0, 1)$
4: **for** $k \leftarrow 1$ to N **do**
5:     **if** $q_{k-1} < \xi \leqslant q_k$ **then**
6:         $y = k$
7:         **break**
8:     **end if**
9: **end for**
10: **return** y

Algorithm 2 Generating symmetric α-stable random number

Input: index of stability α

Output: symmetric α-stable random number ξ

1: Generate a random number V uniformly distributed on $(-\pi/2, \pi/2)$ and an exponential random number W with mean 1

2: $\xi \leftarrow \frac{\sin(\alpha V)}{(\cos V)^{1/\alpha}} \cdot \left(\frac{\cos(V-\alpha V)}{W}\right)^{(1-\alpha)/\alpha}$ ▷ Equation (A5)

3: **return** ξ

Algorithm 3 Generating totally skewed α-stable random number

Input: index of stability α

Output: totally skewed α-stable random number ξ

1: Generate a random number V uniformly distributed on $(-\pi/2, \pi/2)$ and an exponential random number W with mean 1

2: $c_1 \leftarrow (\cos(\pi\alpha/2))^{-1/\alpha}$ and $c_2 \leftarrow \pi/2$

3: $\xi \leftarrow c_1 \frac{\sin(\alpha(V+c_2))}{(\cos V)^{1/\alpha}} \cdot \left(\frac{\cos(V-\alpha(V+c_2))}{W}\right)^{(1-\alpha)/\alpha}$ ▷ Equation (A6)

4: **return** ξ

Algorithm 4 Generating power-law random number

Input: the exponent of power-law distribution α

Output: power-law random number ξ

1: Generate a random number X uniformly distributed on $(0, 1)$

2: $\xi \leftarrow (1 - X)^{-1/\alpha} - 1$ ▷ Equation (A35)

3: **return** ξ

2. CTRW

Algorithm 5 Generating CTRW trajectory

Input: length of trajectory T, index of waiting time β and index of jump length α, and initial position x_0 ▷ $\beta = 1$ means that the waiting time distribution is exponential

Output: time vector $\boldsymbol{t}$ and position vector $\boldsymbol{x}$

1: **if** $\beta = 1$ **then**

Set *random* as the random number generator of exponential distribution

else

Set *random* as the random number generator of totally skewed β-stable distribution ▷ Algorithm 3

end if

Generate empty vectors $\boldsymbol{t}$ and $\boldsymbol{x}$ ▷ Variable-length vectors

$\boldsymbol{t}(1) \leftarrow 0$ and $\boldsymbol{x}(1) \leftarrow x_0$ ▷ Initial value

$t_{tot} \leftarrow 0$ ▷ Total time

$x_c \leftarrow x_0$ ▷ Current position

$n \leftarrow 1$ ▷ Counter

while true do

Generate a random number τ_n by generator *random*

if $t_{tot} + \tau_n > T$ **then**

$\boldsymbol{t}(n+1) \leftarrow T$

$\boldsymbol{x}(n+1) \leftarrow x_c$

break

else

$t_{tot} \leftarrow t_{tot} + \tau_n$

$\boldsymbol{t}(n+1) \leftarrow t_{tot}$

Generate a random number ξ_n of symmetric α-stable distribution ▷ Algorithm 2

$x_c \leftarrow x_c + \xi_n$

$\boldsymbol{x}(n+1) \leftarrow x_c$

$n \leftarrow n + 1$

end if

end while

return $\boldsymbol{t}$ and $\boldsymbol{x}$

3. Lévy process

Algorithm 6 Generating α-stable Lévy process trajectory

Input: length of the trajectory T, index of stability α and time-stepping size τ, and initial position x_0

Output: time vector $\boldsymbol{t}$ and position vector $\boldsymbol{x}$

$N \leftarrow \lceil T/\tau \rceil$
Generate empty vectors $\boldsymbol{t}$ and $\boldsymbol{x}$ of length $N+1$
$\boldsymbol{t}(n) \leftarrow (n-1)\tau$ for $n = 1, \cdots, N+1$
$\boldsymbol{x}(1) \leftarrow x_0$ ▷ Initial position
$x_c \leftarrow x_0$ ▷ Current position
Generate N random numbers of symmetric α-stable distribution $\{\xi_n\}_{n=1}^{N}$ ▷ Algorithm 2
for $n \leftarrow 1$ to N **do**
    $x_c \leftarrow x_c + \tau^{1/\beta}\xi_n$
    $\boldsymbol{x}(n+1) \leftarrow x_c$ ▷ Equation (A17)
end for
return $\boldsymbol{t}$ and $\boldsymbol{x}$

Algorithm 7 Generating β-stable subordinator trajectory

Input: length of the trajectory T, index of stability β and time-stepping size τ

Output: time vector $\boldsymbol{t}$ and position vector $\boldsymbol{x}$

$N \leftarrow \lceil T/\tau \rceil$
Generate empty vectors $\boldsymbol{t}$ and $\boldsymbol{x}$ of length $N+1$
$\boldsymbol{t}(n) \leftarrow (n-1)\tau$ for $n = 1, \cdots, N+1$
$\boldsymbol{x}(1) \leftarrow 0$ ▷ Initial value
$x_c \leftarrow 0$ ▷ Current position
Generate N random numbers of totally skewed β-stable distribution $\{\zeta_n\}_{n=1}^{N}$ ▷ Algorithm 3
for $n \leftarrow 1$ to N **do**
    $x_c \leftarrow x_c + \tau^{1/\alpha}\zeta_n$
    $\boldsymbol{x}(n+1) \leftarrow x_c$ ▷ Equation (A21)
end for
return $\boldsymbol{t}$ and $\boldsymbol{x}$

Algorithm 8 Generating Poisson process trajectory

Input: length of the trajectory T and intensity λ

Output: time vector $\boldsymbol{t}$ and position vector $\boldsymbol{x}$

Generate empty vectors $\boldsymbol{t}$ and $\boldsymbol{x}$ ▷ Variable-length vectors
$\boldsymbol{t}(1) \leftarrow 0$ and $\boldsymbol{x}(1) \leftarrow 0$ ▷ Initial value
$t_{tot} \leftarrow 0$ ▷ Total time
$x_c \leftarrow 0$ ▷ Current position
$n \leftarrow 1$ ▷ Counter
while true do
    Generate a random number τ_n of exponential distribution with mean $1/\lambda$
    if $t_{tot} + \tau_n > T$ **then**
        $\boldsymbol{t}(n+1) \leftarrow T$
        $\boldsymbol{x}(n+1) \leftarrow x_c$
        break
    else
        $t_{tot} \leftarrow t_{tot} + \tau_n$
        $x_c \leftarrow x_c + 1$
        $\boldsymbol{t}(n+1) \leftarrow t_{tot}$
        $\boldsymbol{x}(n+1) \leftarrow x_c$
        $n \leftarrow n + 1$
    end if
end while
return $\boldsymbol{t}$ and $\boldsymbol{x}$

4. Alternating process

Algorithm 9 Generating alternating process trajectory

Input: length of trajectory T, sojourn time distributions' exponents α_+ and α_-, velocity of Lévy walk v_0 and initial position x_0

Output: time vector $\boldsymbol{t}$ and position vector $\boldsymbol{x}$

Generate empty vectors $\boldsymbol{t}$ and $\boldsymbol{x}$ ▷ Variable vectors
$\boldsymbol{t}(1) \leftarrow 0$ and $\boldsymbol{x}(1) \leftarrow x_0$ ▷ Initial position
$t_{tot} \leftarrow 0$ ▷ Total time
$x_c \leftarrow x_0$ ▷ Current position
$n \leftarrow 1$ ▷ Counter
while true do

Generate a random number ξ_n uniformly distributed on $(0, 1)$

if $\xi_n < 0.5$ **then**

  $d \leftarrow -1$ ▷ Direction of Lévy walk

else

  $d \leftarrow 1$

end if

Generate a power-law random number τ_{n+} with exponent α_+ ▷ Algorithm 4

if $t_{tot} + \tau_{n+} \geqslant T$ **then**

  $\boldsymbol{t} \leftarrow (\boldsymbol{t}, T)$

  $x_c \leftarrow x_c + dv_0(T - t_{tot})$

  $\boldsymbol{x} \leftarrow (\boldsymbol{x}, x_c)$

  break

else

  $t_{tot} \leftarrow t_{tot} + \tau_{n+}$

  $\boldsymbol{t} \leftarrow (\boldsymbol{t}, t_{tot})$

  $x_c \leftarrow x_c + dv_0\tau_{n+}$

  $\boldsymbol{x} \leftarrow (\boldsymbol{x}, x_c)$

end if

Generate a power-law random number τ_{n-} with exponent α_- ▷ Algorithm 4

if $t_{tot} + \tau_{n-} \geqslant T$ **then**

  Generate a Brownian motion trajectory $\hat{\boldsymbol{t}}_{n-}$ and $\hat{\boldsymbol{x}}_{n-}$ with length $T - t_{tot}$ and initial position x_c ▷ Algorithm 6

  $\boldsymbol{t}_{n-} \leftarrow t_{tot} + \hat{\boldsymbol{t}}_{n-}(2 : \mathbf{end})$

  $\boldsymbol{t} \leftarrow (\boldsymbol{t}, \boldsymbol{t}_{n-})$

  $\boldsymbol{x} \leftarrow (\boldsymbol{x}, \hat{\boldsymbol{x}}_{n-}(2 : \mathbf{end}))$

  break

else

  Generate a Brownian motion trajectory $\tilde{\boldsymbol{t}}_{n-}$ and $\tilde{\boldsymbol{x}}_{n-}$ with length τ_{n-} and initial position x_c ▷ Algorithm 6

  $\boldsymbol{t}_{n-} \leftarrow t_{tot} + \tilde{\boldsymbol{t}}_{n-}(2 : \mathbf{end})$

  $\boldsymbol{t} \leftarrow (\boldsymbol{t}, \boldsymbol{t}_{n-})$

  $\boldsymbol{x} \leftarrow (\boldsymbol{x}, \hat{\boldsymbol{x}}_{n-}(2 : \mathbf{end}))$

$t_{tot} \leftarrow t_{tot} + \tau_{n-}$
$x_{cur} \leftarrow \boldsymbol{x}(\mathbf{end})$
end if
$n \leftarrow n + 1$
end while
return $\boldsymbol{t}$ and $\boldsymbol{x}$

5. Multiple internal states process

Algorithm 10 Generating fractional compound Poisson process with multiple internal states trajectory

Input: length of trajectory T, index vector of waiting times $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_N)$, transition matrix M, initial state $\boldsymbol{I} = (I_1, \cdots, I_N)$ and initial position x_0

Output: time vector $\boldsymbol{t}$ and position vector $\boldsymbol{x}$

Set *random* as the random number generator of totally skewed stable distribution ▷ Algorithm 3
Generate empty vectors $\boldsymbol{t}$ and $\boldsymbol{x}$ ▷ Variable-length vectors
$\boldsymbol{t}(1) \leftarrow 0$ and $\boldsymbol{x}(1) \leftarrow x_0$ ▷ Initial value
$t_{tot} \leftarrow 0$ ▷ Total time
$x_c \leftarrow x_0$ ▷ Current position
$n \leftarrow 1$ ▷ Counter
Generate a sample *init* of probability distribution $\mathbb{P}[Y = j] = I_j$ ▷ Algorithm 1
$nums_S \leftarrow init$
while true do
Generate a random number τ_n by *random* with parameter α_{nums_S}
if $t_{tot} + \tau_n > T$ **then**
$\boldsymbol{t}(n+1) \leftarrow T$
$\boldsymbol{x}(n+1) \leftarrow x_c$
break
else
$t_{tot} \leftarrow t_{tot} + \tau_n$
$\boldsymbol{t}(n+1) \leftarrow t_{tot}$
Generate a random number ξ_n of standard normal distribution

$x_c \leftarrow x_c + \xi_n$

$\boldsymbol{x}(n+1) \leftarrow x_c$

Generate a sample num_{next} of probability distribution $\mathbb{P}[Y=j] = M_{num_S,j}$

$num_S \leftarrow num_{next}$

$n \leftarrow n+1$

end if

end while

return $\boldsymbol{t}$ and $\boldsymbol{x}$

Algorithm 11 Generating Lévy walk with multiple internal states trajectory

Input: length of trajectory T, vector of sojourn time distributions $\boldsymbol{w} = (w_1(t), \cdots, w_N(t))$, vector of velocity distributions $\boldsymbol{v}(x) = (v_1(x), \cdots, v_N(x))$, transition matrix M, initial state $\boldsymbol{I} = (I_1, \cdots, I_N)$ and initial position x_0

Output: time vector $\boldsymbol{t}$ and position vector $\boldsymbol{x}$

Generate empty vectors $\boldsymbol{t}$ and $\boldsymbol{x}$ ▷ Variable-length vectors

$\boldsymbol{t}(1) \leftarrow 0$ and $\boldsymbol{x}(1) \leftarrow x_0$ ▷ Initial value

$t_{tot} \leftarrow 0$ ▷ Total time

$x_c \leftarrow x_0$ ▷ Current position

$n \leftarrow 1$ ▷ Counter

Generate a sample *init* of probability distribution $\mathbb{P}[Y=j] = I_j$ ▷ Algorithm 1

$num_S \leftarrow init$

while true do

Generate a random number τ_n of distribution $w_{num_S}(t)$

Generate a sample ζ_n uniformly distributed on $(0,1)$

if $\zeta_n < 0.5$ **then**

$d \leftarrow -1$

else

$d \leftarrow 1$

end if

Generate a random number v_{0n} of distribution $v_{num_S}(x)$

if $t_{tot} + \tau_n > T$ **then**

$\boldsymbol{t}(n+1) \leftarrow T$

$x_c \leftarrow x_c + dv_{0n}(T - t_{tot})$

$\boldsymbol{x}(n+1) \leftarrow x_c$

break

else

$t_{tot} \leftarrow t_{tot} + \tau_n$

$x_c \leftarrow x_c + dv_{0n}\tau_n$

$\boldsymbol{t}(n+1) \leftarrow t_{tot}$

$\boldsymbol{x}(n+1) \leftarrow x_c$

Generate a sample num_{next} of probability distribution $\mathbb{P}[Y = j] = M_{num_S, j}$

$num_S \leftarrow num_{next}$

$n \leftarrow n + 1$

end if

end while

return $\boldsymbol{t}$ and $\boldsymbol{x}$

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