NUMERICAL METHOD FOR THE FOKKER-PLANCK EQUATION OF BROWNIAN MOTION SUBORDINATED BY INVERSE TEMPERED STABLE SUBORDINATOR WITH DRIFT *

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Abstract. Based on the complete Bernstein function, we propose a generalized regularity analysis for the Fokker–Planck equation, which governs the probability density function of Brownian motion subordinated by the inverse tempered stable subordinator with drift. We derive a generalized time–stepping finite element scheme based on the backward Euler convolution quadrature, and the optimal convergence order of the numerical solutions is established using the obtained regularity. Furthermore, the analysis is generalized to more general diffusion equations. Numerical experiments are provided to support the theoretical results.

2020 Mathematics Subject Classification. 65M12, 65M60, 60J60.

July 1, 2022.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, be a polygonal or polyhedral domain, and $T > 0$ be fixed. We deal with the following initial and homogeneous Dirichlet boundary value problem

$$\begin{cases} \kappa\partial_t u(x,t) + \partial_t^{\alpha,\mu} u(x,t) + Au(x,t) = f(x,t), & (x,t) \in \Omega \times (0,T], \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T], \end{cases} \tag{1.1}$$

where $\kappa > 0$ is a constant, $A = -\Delta$, $\partial_t^{\alpha,\mu}$ is the tempered fractional Caputo-type derivative operator, defined by

$$\partial_t^{\alpha,\mu} u(x,t) = \frac{\partial}{\partial t} \int_0^t w(t-s)(u(x,s) - u(x,0))\,\mathrm{d}s \tag{1.2}$$

with $\widehat{w}(z) = ((z+\mu)^\alpha - \mu^\alpha)/z$, $0 < \alpha < 1$, $\mu \geqslant 0$. Hereafter, we use the symbol "$\widehat{\ }$" to represent the Laplace transform of a function, i.e.,

$$\widehat{f}(z) = \int_0^\infty \mathrm{e}^{-zt} f(t)\,\mathrm{d}t.$$

In fact, Eq. (1.1) describes an anomalous diffusion, whose mean-squared displacement is proportional to the time t for both short and long times; see Appendix A for more details. In recent years, it is widely recognized that

Keywords and phrases: complete Bernstein function, backward Euler method, finite element method, generalized diffusion equation

* *This work was supported by National Natural Science Foundation of China under Grant No.12071195, AI and Big Data Funds under Grant No.2019620005000775, and Supercomputing Center of Lanzhou University.*

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anomalous diffusions are ubiquitous in diverse fields, including mathematics, physics, chemistry, engineering, biology, and so on, to name but a few; see, e.g., [1–4]. A special case ($\mu = 0$) of the diffusion equation (1.1) is the time-fractional diffusion equation, which describes subdiffusion. So far, several works and methods for time-fractional diffusion equation have been developed, such as finite difference methods [5–7], finite element methods [5, 8–10], spectral methods [11], et al.

To our knowledge, the L1-type scheme [7, 12–14] and k-step backward difference formula [15, 16] on uniform or graded meshes are two popular and predominant discretization techniques for the fractional derivatives in Caputo sense. As for the tempered time–fractional Caputo–type derivative we consider in this paper, its memory kernel function $w(t)$ is much more complex than its form in Laplace space. In fact, by the inverse Laplace transform, we have

$$w(t) = \frac{\alpha}{\Gamma(1-\alpha)} \int_t^\infty \mathrm{e}^{-\mu s} s^{-1-\alpha}\,\mathrm{d}s,$$

where $\Gamma(\cdot)$ is the Euler gamma function. By the way, in another our recent work [17], a modified L1 scheme on graded meshes has been employed to discrete the tempered time–fractional Caputo–type derivative. Therefore, we use the backward Euler convolution quadrature (BE–CQ) here. The BE–CQ has been widely used in the numerical methods for time–fractional diffusion equations; see, e.g., [18, 19]. For the discretization of spatial variable, we use the finite element method in this paper. It is worth mentioning that based on the complete Bernstein function, we propose a generalized regularity analysis for Eq. (1.1). Then, by using the obtained regularity, an optimal convergence order of the numerical solutions is established. Our analysis can be further extended to the general diffusion equations.

The rest of this paper is organized as follows. We introduce the complete Bernstein function and use some of its properties to get the regularity results for solution to Eq. (1.1) in Section 2. In Section 3, we propose the finite element scheme for Eq. (1.1) and make error estimates. Based on the BE–CQ, we construct a time–stepping method for Eq. (1.1) and then analyze its convergence in Section 4. Combining the two semi-discrete approximations, we get the full discretization in Section 5, and the numerical results and our discussions are left to Section 6 and Section 7, respectively. In addition, some explanations can be seen in Appendix A. Throughout this paper, the notation c denotes a generic constant, which may vary at different occurrences, but always be independent of the time t, mesh size h, and time-stepping size τ.

2. Preliminaries

In this section, we first introduce the complete Bernstein function and some of its useful properties. Then, we use these properties to study the mild solution to Eq. (1.1), and obtain some estimates.

2.1. Complete Bernstein function

Bernstein function and its important subclass, complete Bernstein function, have been widely used in various fields of mathematics, for example, probabilistic theory and harmonic analysis [20–22]. However, to the best of our knowledge, in numerical analysis, especially in the filed of numerical methods for partial differential equations, it seems that no researchers have used these functions in depth yet. So we give some related definitions at the beginning. For more details, we refer readers to the monograph [23].

Definition 2.1 (completely monotone function, [23, Definition 1.3, p. 2])**.** A function $f : (0, \infty) \to \mathbb{R}$ is a completely monotone function if f is of class C^∞ and

$$(-1)^n f^{(n)}(\lambda) \geqslant 0 \quad \text{for all } n \in \mathbb{N} \text{ and } \lambda > 0.$$

Definition 2.2 (Bernstein function, [23, Definition 3.1, p. 21])**.** A function $f : (0, \infty) \to \mathbb{R}$ is a Bernstein function if f is of class C^∞, $f(\lambda) \geqslant 0$ for all $\lambda > 0$, and

$$(-1)^n f^{(n)}(\lambda) \leqslant 0 \quad \text{for all } n \in \mathbb{N}^+ \text{ and } \lambda > 0.$$

For Bernstein function, we have the following alternative representation.

Lemma 2.3 ([23, Theorem 3.2, p. 21]). *A function $f:(0,\infty)\to\mathbb{R}$ is a Bernstein function if and only if it admits the representation*

$$f(\lambda) = a + b\lambda + \int_0^\infty \big(1-\mathrm{e}^{-\lambda t}\big)\mu(\mathrm{d}t), \tag{2.1}$$

where a, $b \geqslant 0$, and μ is a measure on $(0,\infty)$ satisfying $\int_0^\infty \min\{1,t\}\mu(\mathrm{d}t) < \infty$.

Using the above integral formula of Bernstein function, one can further define the complete Bernstein function.

Definition 2.4 (complete Bernstein function, [23, Definition 6.1, p. 69]). A Bernstein function f is called a complete Bernstein function if its Lévy measure μ in Eq. (2.1) has a completely monotone density $m(t)$ with respect to Lebesgue measure,

$$f(\lambda) = a + b\lambda + \int_0^\infty \big(1-\mathrm{e}^{-\lambda t}\big)m(t)\,\mathrm{d}t.$$

One can analytically extend the complete Bernstein function from $(0,\infty)$ to the cut complex plane $\mathbb{C}\backslash(-\infty,0]$ as follows.

Lemma 2.5 ([23, Theorem 6.2, Remark 6.4, Corolarry 6.5]). *A non-negative function f on $(0,\infty)$ is a complete Bernstein function if and only if f has an analytical continuation to the cut complex plane $\mathbb{C}\backslash(-\infty,0]$ such that $\operatorname{Im} z\cdot\operatorname{Im} f(z)\geqslant 0$, and the limit $f(0+)$ exists and is real. Moreover, for every complete Bernstein function f, there exists a measure σ on $(0,\infty)$ satisfying $\int_0^\infty (1+t)^{-1}\sigma(\mathrm{d}t)<\infty$, such that*

$$f(\lambda) = a + b\lambda + \int_0^\infty \frac{\lambda}{\lambda+t}\sigma(\mathrm{d}t).$$

The analytic continuation of f is of the form

$$f(z) = a + bz + \int_0^\infty \frac{z}{z+t}\sigma(\mathrm{d}t),$$

for $z\in\{z\in\mathbb{C} : \operatorname{Im} z>0\}$ and $f(z)=\overline{f(\bar{z})}$ for $z\in\{z\in\mathbb{C} : \operatorname{Im} z<0\}$.

The analytical continuation of complete Bernstein function has the following property.

Lemma 2.6 ([23, Corollary 6.6]). *Let f be the analytic extension of a complete Bernstein function. Then $f(S_{[0,\theta]})\subset S_{[0,\theta]}$ with $S_{[0,\theta]}=\{z\in\mathbb{C} : 0\leqslant \arg z\leqslant\theta\}$, $\theta\in(0,\pi)$.*

Remark 2.7. From Lemma 2.6 and Lemma 2.5, we obtain $f(S_{[-\theta,\theta]})\subset S_{[-\theta,\theta]}$ with $S_{[-\theta,\theta]}=\{z\in\mathbb{C} : -\theta\leqslant \arg z\leqslant\theta\}$ and $\theta\in(0,\pi)$ for all complete Bernstein function f.

2.2. **Regularity**

Assume that the function $f(t)=f(\cdot,t)$ in Eq. (1.1) is absolutely continuous. Then we can rewrite f as

$$f(t) = f(0) + \int_0^t \partial_s f(s)\,\mathrm{d}s =: f(0) + R(t).$$

Taking Laplace transform on the both sides of Eq. (1.1) leads to

$$(\kappa z + \phi(z) + A)\widehat{u}(z) = \big(\kappa + z^{-1}\phi(z)\big)u_0 + z^{-1}\Big(f(0) + \widehat{R}(z)\Big). \tag{2.2}$$

Note that the function $\phi(z)=(z+\mu)^\alpha-\mu^\alpha$ introduced in Appendix A is an analytical continuation of complete Bernstein function $\phi:(0,\infty)\to\mathbb{R}$. Denote the sector in complex plane

$$\Sigma_\theta=\{z\in\mathbb{C}\backslash\{0\}:|\arg z|\leqslant\theta\},\quad 0<\theta<\pi \tag{2.3}$$

and contour

$$\Gamma_{\delta,\varphi}=\{z\in\mathbb{C}:|z|=\delta,\ |\arg z|\leqslant\varphi\}\cup\big\{z\in\mathbb{C}:z=re^{\pm\mathbf{i}\varphi},\ r\geqslant\delta\big\}, \tag{2.4}$$

oriented with an increasing imaginary part.

Lemma 2.8. *For $z\in\Sigma_\theta$, $\theta\in(0,\pi)$, there exists $\phi(z)\in\Sigma_\theta$. Furthermore, by choosing $\theta\in(\pi/2,\pi)$ sufficiently close to $\pi/2$, and $\sigma>0$ sufficiently large (depending on μ), it holds*

$$c|z|^\alpha\leqslant|\phi(z)|\leqslant c|z|^\alpha, \tag{2.5}$$

for all $z\in\Sigma_\theta\cap\{z\in\mathbb{C}:|z|\geqslant\sigma\}$.

It is obvious that only when $z=0$, $\phi(z)=0$. Hence, according to Remark 2.7, we have $\phi(\Sigma_\theta)\subset\Sigma_\theta$. The rest of this lemma is a part of Proposition 3.1 in [18].

Lemma 2.9. *For $\eta(z)=\kappa z+\phi(z)=\kappa z+(z+\mu)^\alpha-\mu^\alpha$ and $0<\theta<\pi$, the following statements hold.*

(1) For all $z\in\Sigma_\theta$, we have

$$c|z|\leqslant|\eta(z)|. \tag{2.6}$$

(2) For all $z\in\Sigma_\theta$ with $|z|\geqslant\mu+1$, it holds

$$|\eta(z)|\leqslant c|z|. \tag{2.7}$$

(3) For $z\in\Sigma_\theta$, one can also get $\eta(z)\in\Sigma_\theta$.

Proof. By definition, we have

$$|\eta(z)|=|z|\big|\kappa+\phi(z)z^{-1}\big|. \tag{2.8}$$

For $z\in\Sigma_\theta$, according to Lemma 2.8 and Lemma 2.5, we know that $|\arg\phi(z)|\leqslant\theta$ and $(\arg\phi(z))(\arg z)\geqslant 0$. Therefore, $\big|\arg\phi(z)z^{-1}\big|=|\arg\phi(z)-\arg z|\leqslant\max\{|\arg\phi(z)|,|\arg z|\}\leqslant\theta$, i.e., $\phi(z)z^{-1}\in\Sigma_\theta$ for $z\in\Sigma_\theta$. Denoting $\phi(z)z^{-1}=re^{\mathbf{i}\varphi}$ with $|\varphi|\leqslant\theta$, we have

$$\big|\kappa+\phi(z)z^{-1}\big|^2=|\kappa+r\cos\varphi+\mathbf{i}r\sin\varphi|^2=r^2+2r\kappa\cos\varphi+\kappa^2\geqslant\begin{cases}\kappa^2, & 0<|\varphi|\leqslant\pi/2,\\ \kappa^2\sin^2\varphi, & \pi/2<|\varphi|<\pi.\end{cases} \tag{2.9}$$

Hence, we obtain $|\eta(z)|\geqslant\kappa\sin\theta|z|$ for $z\in\Sigma_\theta$. This implies that $\eta(z)\neq0$ for $z\in\Sigma_\theta$. Note that η is also a completely Bernstein function. Therefore, by Remark 2.7, we obtain $\eta(\Sigma_\theta)\subset\Sigma_\theta$. On the other hand, for $z\in\Sigma_\theta$ with $|z|\geqslant\mu+1$, by the triangular inequality, we have

$$|\phi(z)|=|(z+\mu)^\alpha-\mu^\alpha|\leqslant(|z|+\mu)^\alpha+\mu^\alpha\leqslant3|z|^\alpha\leqslant3(\mu+1)^{\alpha-1}|z|, \tag{2.10}$$

which leads to $|\eta(z)|\leqslant c|z|$. □

Then, we briefly introduce the Sobolev space used here. Denote space $\dot{\mathrm{H}}^r(\Omega)=\{v\in\mathrm{L}^2(\Omega):(-\Delta)^{\frac{r}{2}}v\in\mathrm{L}^2(\Omega)\}$ with the norm

$$\|v\|^2_{\dot{\mathrm{H}}^r(\Omega)}=\sum_{n=1}^\infty\lambda_n^r(v,\varphi_n)^2 \tag{2.11}$$

for $r \geqslant 0$, where $(\cdot,\cdot)$ denotes $\mathrm{L}^2(\Omega)$ inner product, and $\{\lambda_n, \varphi_n\}_{n=1}^{\infty}$ are the eigenvalues ordered non-decreasingly and the corresponding eigenfunctions normalized in the $\mathrm{L}^2(\Omega)$ norm $\|\cdot\|$ of operator $-\Delta$ subject to the homogeneous Dirichlet boundary condition on Ω. In addition, $\dot{\mathrm{H}}^0(\Omega) = \mathrm{L}^2(\Omega)$, $\dot{\mathrm{H}}^1(\Omega) = \mathrm{H}_0^1(\Omega)$, and $\dot{\mathrm{H}}^2(\Omega) = \mathrm{H}^2(\Omega) \cap \mathrm{H}_0^1(\Omega)$.

As for the operator $A = -\Delta : \dot{\mathrm{H}}^2(\Omega) \to \mathrm{L}^2(\Omega)$, there exists resolvent estimate [24, Theorem 3.7.11]

$$\left\|(z+A)^{-1}\right\|_{\mathrm{L}^2(\Omega)\to\mathrm{L}^2(\Omega)} \leqslant c|z|^{-1} \quad \forall z \in \Sigma_\theta \text{ with } \theta \in (\pi/2, \pi). \tag{2.12}$$

From Eq. (2.2) and Lemma 2.9, one can get the representation of mild solution to Eq. (1.1),

$$u(t) = E(t)u_0 + F(t)f(0) + \int_0^t F(t-s)\partial_s f(s)\,\mathrm{d}s, \tag{2.13}$$

where

$$E(t) = \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\delta,\theta}} \mathrm{e}^{zt} z^{-1} \eta(z)(\eta(z)+A)^{-1}\,\mathrm{d}z \tag{2.14}$$

and

$$F(t) = \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\delta,\theta}} \mathrm{e}^{zt} z^{-1} (\eta(z)+A)^{-1}\,\mathrm{d}z \tag{2.15}$$

with $\theta \in (\pi/2, \pi)$, $\delta > 0$. When the initial condition is smooth, that is, $u_0 \in \dot{\mathrm{H}}^2(\Omega)$, one can rewrite Eq. (1.1) as

$$\kappa\partial_t v(t) + \partial_t^{\alpha,\mu} v(t) + Av(t) = -Au_0 + f(0) + \int_0^t R(s)\,\mathrm{d}s, \tag{2.16}$$

where $v(t) = u(t) - u_0$ and $v(0) = 0$. In this case, the solution $u(t)$ is of the form

$$u(t) = v(t) + u_0 = u_0 - F(t)Au_0 + F(t)f(0) + \int_0^t F(t-s)\partial_s f(s)\mathrm{d}s. \tag{2.17}$$

Lemma 2.10. *For $0 \leqslant r \leqslant 1$ and $k \in \mathbb{N}$, we have*

$$\left\|A^r E^{(k)}(t)\right\|_{\mathrm{L}^2(\Omega)\to\mathrm{L}^2(\Omega)} \leqslant ct^{-r-k}, \tag{2.18}$$

and

$$\left\|A^r F^{(k)}(t)\right\|_{\mathrm{L}^2(\Omega)\to\mathrm{L}^2(\Omega)} \leqslant ct^{1-r-k}. \tag{2.19}$$

Proof. Firstly, thanks to the identity

$$A(\eta(z)+A)^{-1} = I - \eta(z)(\eta(z)+A)^{-1}, \tag{2.20}$$

where I denotes the identity operator on $\mathrm{L}^2(\Omega)$, we have for $m = 0, 1$, $z \in \Sigma_\theta$, $\theta \in (0, \pi)$,

$$\left\|A^m(\eta(z)+A)^{-1}\right\| \leqslant c|\eta(z)|^{m-1}. \tag{2.21}$$

Therefore, for $k \in \mathbb{N}$, one has

$$\begin{aligned}
\left\|A^m E^{(k)}(t)\right\|_{\mathrm{L}^2(\Omega)\to\mathrm{L}^2(\Omega)} &= \left\|\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{\mu+1,\theta}} \mathrm{e}^{zt} z^{k-1}\eta(z)A^m(\eta(z)+A)^{-1}\,\mathrm{d}z\right\|_{\mathrm{L}^2(\Omega)\to\mathrm{L}^2(\Omega)} \\
&\leqslant c\int_{\Gamma_{\mu+1,\theta}} \mathrm{e}^{t\,\mathrm{Re}\,z}|z|^{k-1}|\eta(z)|\left\|A^m(\eta(z)+A)^{-1}\right\|_{\mathrm{L}^2(\Omega)\to\mathrm{L}^2(\Omega)}|\mathrm{d}z| \\
&\leqslant c\int_{\Gamma_{\mu+1,\theta}} \mathrm{e}^{t\,\mathrm{Re}\,z}|z|^{k-1}|\eta(z)|^m\,|\mathrm{d}z| \\
&\leqslant c\int_{\Gamma_{\mu+1,\theta}} \mathrm{e}^{t\,\mathrm{Re}\,z}|z|^{m+k-1}\,|\mathrm{d}z| \\
&\leqslant c\left(\int_{\mu+1}^{\infty} \mathrm{e}^{tr\cos\theta}r^{m+k-1}\,\mathrm{d}r + \int_{-\theta}^{\theta}\mathrm{e}^{t(\mu+1)\cos\varphi}(\mu+1)^{m+k}\,\mathrm{d}\varphi\right).
\end{aligned} \tag{2.22}$$

Changing variable $s = rt$ leads to

$$\int_{\mu+1}^{\infty} \mathrm{e}^{tr\cos\theta}r^{m+k-1}\,\mathrm{d}r = t^{-m-k}\int_{(\mu+1)t}^{\infty}\mathrm{e}^{s\cos\theta}s^{m+k-1}\,\mathrm{d}r \leqslant ct^{-m-k}. \tag{2.23}$$

So substituting Eq. (2.23) into Eq. (2.22) yields

$$\left\|A^m E^{(k)}(t)\right\|_{\mathrm{L}^2(\Omega)\to\mathrm{L}^2(\Omega)} \leqslant ct^{-m-k}. \tag{2.24}$$

Similarly,

$$\begin{aligned}
\left\|A^m F^{(k)}(t)\right\|_{\mathrm{L}^2(\Omega)\to\mathrm{L}^2(\Omega)} &= \left\|\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{\mu+1,\theta}} \mathrm{e}^{zt} z^{k-1}A^m(\eta(z)+A)^{-1}\,\mathrm{d}z\right\|_{\mathrm{L}^2(\Omega)\to\mathrm{L}^2(\Omega)} \\
&\leqslant c\int_{\Gamma_{\mu+1,\theta}} \mathrm{e}^{t\,\mathrm{Re}\,z}|z|^{k+m-2}\,|\mathrm{d}z| \\
&\leqslant ct^{1-m-k}.
\end{aligned} \tag{2.25}$$

Using the property of interpolation space can end this proof. □

Theorem 2.11. *Let $u(t)$ be the solution to problem (1.1). Assume that $f(t)$ is absolutely continuous, $f(0) \in \mathrm{L}^2(\Omega)$ and $\int_0^t \|\partial_s f(s)\|\,\mathrm{d}s < \infty$ for all $t \in [0,T]$. Then the following statements hold.*

(1) If $u_0 \in \mathrm{L}^2(\Omega)$, then, for $0 \leqslant r \leqslant 1$, we have

$$\|u(t)\|_{\dot{\mathrm{H}}^{2r}(\Omega)} \leqslant c\left(t^{-r}\|u_0\| + t^{1-r}\|f(0)\| + t^{1-r}\int_0^t \|\partial_s f(s)\|\,\mathrm{d}s\right). \tag{2.26}$$

(2) If $u_0 \in \dot{\mathrm{H}}^2(\Omega)$, then there holds

$$\|u(t)\|_{\dot{\mathrm{H}}^2(\Omega)} + \|u'(t)\| \leqslant c\left(\|u_0\|_{\dot{\mathrm{H}}^2(\Omega)} + \|f(0)\| + \int_0^t \|\partial_s f(s)\|\,\mathrm{d}s\right). \tag{2.27}$$

Proof. (1). For $u_0 \in \mathrm{L}^2(\Omega)$, according to Eq. (2.13) and Lemma 2.10, we have

$$
\begin{aligned}
\|u(t)\|_{\dot{\mathrm{H}}^{2r}(\Omega)} &\leqslant \|A^r E(t)u_0\| + \|A^r F(t)f(0)\| + \int_0^t \|A^r F(t-s)\partial_s f(s)\| \,\mathrm{d}s \\
&\leqslant c\Big(t^{-r}\|u_0\| + t^{1-r}\|f(0)\| + t^{1-r}\int_0^t \|\partial_s f(s)\| \,\mathrm{d}s\Big).
\end{aligned} \tag{2.28}
$$

(2). For $u_0 \in \dot{\mathrm{H}}^2(\Omega)$, due to Eq. (2.17), we get

$$
\begin{aligned}
\|u(t)\|_{\dot{\mathrm{H}}^{2}(\Omega)} &\leqslant \|Au_0\| + \|AF(t)Au_0\| + \|AF(t)f(0)\| + \int_0^t \|AF(t)\partial_s f(s)\| \,\mathrm{d}s \\
&\leqslant c\Big(\|u_0\|_{\dot{\mathrm{H}}^{2}(\Omega)} + \|f(0)\| + \int_0^t \|\partial_s f(s)\| \,\mathrm{d}s\Big),
\end{aligned} \tag{2.29}
$$

and

$$
\begin{aligned}
\|u'(t)\| &\leqslant \|F'(t)u_0\| + \|F'(t)f(0)\| + \int_0^t \|F'(t-s)\partial_s f(s)\| \,\mathrm{d}s \\
&\leqslant c\Big(\|u_0\| + \|f(0)\| + \int_0^t \|\partial_s f(s)\| \,\mathrm{d}s\Big).
\end{aligned} \tag{2.30}
$$

This completes the proof. □

Remark 2.12. Under the conditions in Theorem 2.11, if $\kappa = 0$, we have

$$
\|u(t)\|_{\dot{\mathrm{H}}^{2r}(\Omega)} \leqslant c\Big(t^{-\alpha r}\|u_0\| + t^{1-\alpha r}\|f(0)\| + t^{1-\alpha r}\int_0^t \|\partial_s f(s)\| \,\mathrm{d}s\Big) \tag{2.31}
$$

in the case of $u_0 \in \mathrm{L}^2(\Omega)$, and

$$
\|u(t)\|_{\dot{\mathrm{H}}^{2}(\Omega)} + t^{1-\alpha}\|u'(t)\| \leqslant c\Big(\|u_0\|_{\dot{\mathrm{H}}^{2}(\Omega)} + \|f(0)\| + \int_0^t \|\partial_s f(s)\| \,\mathrm{d}s\Big) \tag{2.32}
$$

in the case of $u_0 \in \dot{\mathrm{H}}^2(\Omega)$. As one can see, in the case of $u_0 \in \dot{\mathrm{H}}^2(\Omega)$, i.e., the initial value is in the domain of operator A, the existence of the first order derivative ∂_t can eliminate the weak singularity of $u'(t)$ at origin $t = 0$.

3. Spatial semi-discrete approximation

In this section, we use the standard finite element method (FEM) to discretize the spatial variable in Eq. (1.1), and then estimate its error.

3.1. FEM scheme

Denote $V_h \subset H_0^1(\Omega)$ as continuous piecewise linear finite element space

$$
V_h = \big\{v_h \in \mathrm{C}\big(\bar{\Omega}\big) :\ v_h|_T \in \mathcal{P}^1(\mathcal{T}),\, \forall \mathcal{T} \in \mathcal{T}_h,\, v_h|_{\partial\Omega} = 0\big\}, \tag{3.1}
$$

where $\mathcal{P}^1(\mathcal{T})$ denotes the space of linear space on $\mathcal{T}$. To describe the FEM scheme, we need the $\mathrm{L}^2(\Omega)$ orthonormal projection $P_h : \mathrm{L}^2(\Omega) \to V_h$, defined by

$$
(P_h\varphi,\, \chi) = (\varphi,\, \chi), \quad \forall \chi \in V_h,\, \varphi \in \mathrm{L}^2(\Omega). \tag{3.2}
$$

Then the weak formula of problem (1.1) reads: find $u(t) \in H_0^1(\Omega)$ for $t \in (0,T]$ such that

$$((\kappa\partial_t + \partial_t^{\alpha,\mu})u,\, v) + (\nabla u,\, \nabla v) = (f,\, v), \quad u(0) = u_0, \quad \forall v \in \mathrm{H}_0^1(\Omega). \tag{3.3}$$

This implies the following FEM scheme: find $u_h(t) \in V_h$ for $t \in (0,T]$ such that

$$((\kappa\partial_t + \partial_t^{\alpha,\mu})u_h,\, v_h) + (A_h u_h, v_h) = (P_h f,\, v_h), \quad u_h(0) = P_h u(0), \quad \forall v_h \in V_h, \tag{3.4}$$

or equivalently,

$$(\kappa\partial_t + \partial_t^{\alpha,\mu})u_h(t) + A_h u_h(t) = P_h f(t) \tag{3.5}$$

with initial values $u_h(0) = P_h u_0$, where $A_h : V_h \to V_h$, defined as

$$(A_h u_h,\, v_h) = (\nabla u_h,\, \nabla v_h), \quad \forall u_h,\, v_h \in V_h. \tag{3.6}$$

Similarly, the solution to above FEM scheme is of form

$$u_h(t) = E_h(t) P_h u_0 + F_h(t) P_h f(0) + \int_0^t F_h(t-s) P_h(\partial_s f(s))\, \mathrm{d}s, \tag{3.7}$$

where

$$E_h(t) = \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\delta,\theta}} \mathrm{e}^{zt} z^{-1} \eta(z)(\eta(z) + A_h)^{-1}\, \mathrm{d}z, \tag{3.8}$$

and

$$F_h(t) = \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\delta,\theta}} \mathrm{e}^{zt} z^{-1} (\eta(z) + A_h)^{-1}\, \mathrm{d}z. \tag{3.9}$$

3.2. **Error analysis**

Lemma 3.1 ([25, Lemma 3.3])**.** *For any $\gamma_1,\, \gamma_2 \geqslant 0$ and $\theta \in (\pi/2, \pi)$, there holds*

$$\gamma_1 |z| + \gamma_2 \leqslant \frac{|\gamma_1 z + \gamma_2|}{\sin(\theta/2)}, \quad \forall z \in \Sigma_\theta. \tag{3.10}$$

Lemma 3.2. *Let $\varphi \in \mathrm{L}^2(\Omega)$, $z \in \Sigma_\theta$ with $\theta \in (\pi/2, \pi)$, $w = (\eta(z) + A)^{-1}\varphi$, and $w_h = (\eta(z) + A_h)^{-1} P_h \varphi$. Then*

$$\|w - w_h\| + h\|\nabla(w - w_h)\| \leqslant ch^2 \|\varphi\|. \tag{3.11}$$

Proof. By definition, we have

$$\eta(z)(w,\, \chi) + (\nabla w,\, \nabla \chi) = (\varphi,\, \chi),\ \forall \chi \in \mathrm{H}_0^1(\Omega), \tag{3.12}$$

and

$$\eta(z)(w_h,\, \chi_h) + (\nabla w_h,\, \nabla \chi_h) = (P_h \varphi,\, \chi_h),\ \forall \chi_h \in V_h. \tag{3.13}$$

Taking $\chi = \chi_h$ in Eq. (3.12), subtracting above two equations, denoting $e = w - w_h$, and using the orthonormal property of projection P_h lead to the following Galerkin orthogonality

$$\eta(z)(e,\, \chi_h) + (\nabla e,\, \nabla \chi_h) = 0,\ \forall \chi_h \in V_h. \tag{3.14}$$

With the help of Lemma 3.1, one has

$$\begin{aligned}
&|\eta(z)|\|e\|^2 + \|\nabla e\|^2 \\
\leqslant& c|\eta(z)(e,\, e) + (\nabla e,\, \nabla e)| \\
=& c|\eta(z)(e,\, w - \chi_h + \chi_h - w_h) + (\nabla e, \nabla(w - \chi_h + \chi_h - w_h))| \\
=& c|\eta(z)(e,\, w - \chi_h) + (\nabla e,\, \nabla(w - \chi_h))|
\end{aligned} \tag{3.15}$$

Taking $\chi_h = I_h w$, the Lagrange linear interpolate polynomial of w, and using the Cauchy-Schwartz inequality yield

$$|\eta(z)|\|e\|^2 + \|\nabla e\|^2 \leqslant ch^2\Big(|\eta(z)|\|\nabla w\|^2 + \|w\|^2_{\dot{\mathrm{H}}^2(\Omega)}\Big). \tag{3.16}$$

Besides, we have

$$\begin{aligned}
|\eta(z)|\|w\|^2 + \|\nabla w\|^2 &\leqslant c|((\eta(z) + A)w,\, w)| \\
&= c|(\varphi,\, w)| \\
&\leqslant c\|\varphi\|\|w\|.
\end{aligned} \tag{3.17}$$

Thus, one gets

$$\|w\| \leqslant c|\eta(z)|^{-1}\|\varphi\|, \tag{3.18}$$

and

$$\|\nabla w\| \leqslant c\|w\|^{1/2}\|\varphi\|^{1/2} \leqslant c|\eta(z)|^{-1/2}\|\varphi\|. \tag{3.19}$$

In addition,

$$\|w\|_{\dot{\mathrm{H}}^2(\Omega)} = \Big\|\Big(I - \eta(z)(\eta(z) + A)^{-1}\Big)\varphi\Big\| \leqslant c\|\varphi\|. \tag{3.20}$$

Substituting above three estimates into Eq. (3.16) leads to

$$|\eta(z)|\|e\|^2 + \|\nabla e\|^2 \leqslant ch^2\|\varphi\|^2. \tag{3.21}$$

Therefore, one has

$$h\|\nabla e\| \leqslant ch^2\|\varphi\|. \tag{3.22}$$

To end the proof, we will use the duality argument which is widely used in FEM error estimates. Suppose that there exists ψ such that $(\eta(z) + A)\psi = e$. The existence of ψ can be checked by the surjective property of $\eta(z) + A$ when $z \in \Sigma_\theta$. Then, we have

$$\begin{aligned}
\|e\|^2 &= |(e,\, (\eta(z) + A)\psi)| \\
&= |\eta(z)(e,\, \psi - I_h\psi) + (\nabla e,\, \nabla(\psi - I_h\psi))| \\
&\leqslant ch\Big(|\eta(z)|\|e\|\|\nabla\psi\| + \|\nabla e\|\|\psi\|_{\dot{\mathrm{H}}^2(\Omega)}\Big) \\
&\leqslant ch\Big(|\eta(z)|^{1/2}\|e\| + \|\nabla e\|\Big)\|e\| \\
&\leqslant ch^2\|\varphi\|\|e\|.
\end{aligned} \tag{3.23}$$

So we obtain the desired results. □

Denote

$$\mathcal{G}_h(t) = E(t) - E_h(t)P_h, \tag{3.24}$$

and

$$\mathcal{H}_h(t) = F(t) - F_h(t)P_h. \tag{3.25}$$

Lemma 3.3. *For $\varphi \in \mathrm{L}^2(\Omega)$, there hold*

$$\|\mathcal{G}_h(t)\varphi\| + h\|\nabla \mathcal{G}_h(t)\varphi\| \leqslant ct^{-1}h^2\|\varphi\|, \tag{3.26}$$

and

$$\|\mathcal{H}_h(t)\varphi\| + h\|\nabla \mathcal{H}_h(t)\varphi\| \leqslant ch^2\|\varphi\|. \tag{3.27}$$

Proof. According to Lemma 3.2, one has

$$\begin{aligned}
\|\mathcal{G}_h\varphi\| &= \left\| \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_{\mu+1,\theta}} \mathrm{e}^{zt} z^{-1}\eta(z)\big((\eta + A)^{-1} - (\eta + A_h)^{-1}P_h\big)\varphi \,\mathrm{d}z \right\| \\
&\leqslant c\int_{\Gamma_{\mu+1,\theta}} \mathrm{e}^{t\,\mathrm{Re}\,z}|z|^{-1}|\eta(z)|\big\|\big((\eta + A)^{-1} - (\eta + A_h)^{-1}P_h\big)\varphi\big\| \,|\mathrm{d}z| \\
&\leqslant ch^2\|\varphi\| \int_{\Gamma_{\mu+1,\theta}} \mathrm{e}^{t\,\mathrm{Re}\,z}|z|^{-1}|\eta(z)| \,|\mathrm{d}z| \\
&\leqslant ct^{-1}h^2\|\varphi\|,
\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}
\|\mathcal{H}_h\varphi\| &= \left\| \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_{\mu+1,\theta}} \mathrm{e}^{zt} z^{-1}\big((\eta + A)^{-1} - (\eta + A_h)^{-1}P_h\big)\varphi \,\mathrm{d}z \right\| \\
&\leqslant ch^2\|\varphi\| \int_{\Gamma_{\mu+1,\theta}} \mathrm{e}^{t\,\mathrm{Re}\,z}|z|^{-1} \,|\mathrm{d}z| \\
&\leqslant ch^2\|\varphi\|.
\end{aligned} \tag{3.29}$$

One can also prove the rest results in this way. □

Making use of above lemma, we obtain the following error estimates of FEM scheme Eq. (3.5).

Theorem 3.4. *Let $u(t)$ and $u_h(t)$ be the solutions to Eq. (1.1) and Eq. (3.5), respectively, and $e_h = u - u_h$. If $u_0 \in \mathrm{L}^2(\Omega)$, $f(t)$ is absolutely continuous, $f(0) \in \mathrm{L}^2(\Omega)$, and $\int_0^t \|\partial_s f(s)\| \,\mathrm{d}s < \infty$ for $t \in [0, T]$, then we have*

$$\|e_h\| + h\|\nabla e_h\| \leqslant ch^2\left(t^{-1}\|u_0\| + \|f(0)\| + \int_0^t \|\partial_s f(s)\| \,\mathrm{d}s\right). \tag{3.30}$$

Proof. Recall that the solutions to Eq. (1.1) and FEM approximation (3.5) are

$$u(t) = E(t)u_0 + F(t)f(0) + \int_0^t F(t-s)\partial_s f(s) \,\mathrm{d}s, \tag{3.31}$$

and

$$u_h(t) = E_h(t)P_h u_0 + F_h(t)P_h f(0) + \int_0^t F_h(t-s)P_h\partial_s f(s) \,\mathrm{d}s, \tag{3.32}$$

respectively. Therefore, making use of Lemma 3.3, we obtain

$$\begin{aligned}
\|e_h\| &= \left\| \mathcal{G}_h(t)u_0 + \mathcal{H}_h(t)f(0) + \int_0^t \mathcal{H}_h(t-s)\partial_s f(s) \,\mathrm{d}s \right\| \\
&\leqslant \|\mathcal{G}_h(t)u_0\| + \|\mathcal{H}_h(t)f(0)\| + \int_0^t \|\mathcal{H}_h(t-s)\partial_s f(s)\| \,\mathrm{d}s \\
&\leqslant ch^2\left(t^{-1}\|u_0\| + \|f(0)\| + \int_0^t \|\partial_s f(s)\| \,\mathrm{d}s\right),
\end{aligned} \tag{3.33}$$

and

$$
\begin{aligned}
h\|\nabla e_h\| &\leqslant h\bigg(\|\nabla\mathcal{G}_h(t)u_0\| + \|\nabla\mathcal{H}_h(t)f(0)\| + \int_0^t \|\nabla\mathcal{H}_h(t-s)\partial_s f(s)\|\,\mathrm{d}s\bigg) \\
&\leqslant ch^2\bigg(t^{-1}\|u_0\| + \|f(0)\| + \int_0^t \|\partial_s f(s)\|\,\mathrm{d}s\bigg).
\end{aligned}
\tag{3.34}
$$

□

4. Temporal semi-discrete approximation

In this section, based on the ideas in the references [14,16,18], we provide a time-stepping scheme for Eq. (1.1) and analyze its error.

4.1. **Derivation of the scheme**

Recall that the solution $u(t)$ satisfies the following equation in Laplace space,

$$
\kappa z\widehat{u}(z) + \phi(z)\widehat{u}(z) + A\widehat{u}(z) = \big(\kappa + \phi(z)z^{-1}\big)u_0 + z^{-1}f(0) + \widehat{R}(z). \tag{4.1}
$$

Let $t_n = n\tau$, $n = 0, 1, \cdots, N$, be a uniform partition of the time interval $[0, T]$ with step size $\tau = T/N$. Denote $u^n(x)$ $(n \geqslant 1)$ be the approximation of $u(x, t_n)$ and $u^0 = u_0$. Denoting $\delta_\tau(\zeta) = (1-\zeta)/\tau$, $\zeta = \mathrm{e}^{-\tau z}$, $f^n = f(t_n)$ and $R^n = R(t_n) = f^n - f^0$, we approximate z, $\widehat{u}$, u_0, $f(0)$ and $\widehat{R}$ in Eq. (4.1) by $\delta_\tau(\zeta)$, $\tau\sum_{n=1}^\infty u^n\zeta^n$, ζu_0, $\zeta f(0)$, and $\tau\sum_{n=1}^\infty R^n\zeta^n$, respectively. That is,

$$
(\kappa\delta_\tau(\zeta) + \phi(\delta_\tau(\zeta)) + A)\sum_{n=1}^\infty \zeta^n u^n = \big(\kappa + \delta_\tau^{-1}(\zeta)\phi(\delta_\tau(\zeta))\big)\frac{\zeta}{\tau}u^0 + \delta_\tau^{-1}(\zeta)\frac{\zeta}{\tau}f^0 + \sum_{n=1}^\infty \zeta^n R^n. \tag{4.2}
$$

Denote $\{\omega_n\}$ as the coefficients of power expansion of function $\phi(\delta_\tau(\zeta))$,

$$
\phi(\delta_\tau(\zeta)) = \sum_{n=0}^\infty \omega_n\zeta^n, \tag{4.3}
$$

where

$$
\omega_0 = \tau^{-\alpha}((1+\tau\mu)^\alpha - (\tau\mu)^\alpha), \quad \omega_n = (-1)^n\tau^{-\alpha}\binom{\alpha}{n}(1+\tau\mu)^{\alpha-n},\ n \geqslant 1. \tag{4.4}
$$

By the way, ω_n, $n = 0, 1, \cdots$, satisfy the following iteration relation:

$$
\omega_{n+1} = \frac{1}{1+\tau\mu}\bigg(1 - \frac{\alpha+1}{n+1}\bigg)\omega_n, \quad n \geqslant 0. \tag{4.5}
$$

Then, by the straight calculations, we have

$$
\delta_\tau(\zeta)\sum_{n=1}^\infty \zeta^n u^n - \frac{\zeta}{\tau}u^0 = \frac{1}{\tau}\sum_{n=1}^\infty \zeta^n\big(u^n - u^{n-1}\big), \tag{4.6}
$$

$$
\begin{aligned}
&\phi(\delta_\tau(\zeta))\sum_{n=1}^{\infty}\zeta^n u^n-\delta_\tau^{-1}(\zeta)\phi(\delta_\tau(\zeta))\frac{\zeta}{\tau}u^0\\
=&\left(\sum_{n=0}^{\infty}\omega_n\zeta^n\right)\left(\sum_{n=1}^{\infty}\zeta^n u^n\right)-\left(\sum_{n=0}^{\infty}\omega_n\zeta^n\right)\left(\sum_{n=1}^{\infty}\zeta^n\right)u^0\\
=&\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\omega_{n-k}\left(u^k-u^0\right)\right)\zeta^n,
\end{aligned}
\tag{4.7}
$$

and

$$
\delta_\tau^{-1}(\zeta)\frac{\zeta}{\tau}f^0+\sum_{n=1}^{\infty}\zeta^n R^n=\sum_{n=1}^{\infty}\zeta^n f^0+\sum_{n=1}^{\infty}\zeta^n\left(f^n-f^0\right)=\sum_{n=1}^{\infty}f^n\zeta^n.
\tag{4.8}
$$

Substituting Eqs. (4.6) to (4.8) into Eq. (4.2) reduces that

$$
\frac{\kappa}{\tau}\sum_{n=1}^{\infty}\left(u^n-u^{n-1}\right)\zeta^n+\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\omega_{n-k}\left(u^k-u^0\right)\right)\zeta^n+A\sum_{n=1}^{\infty}u^n\zeta^n=\sum_{n=1}^{\infty}f^n\zeta^n.
\tag{4.9}
$$

To obtain the time–stepping method for Eq. (1.1), we only need to consider the coefficients of ζ^n for each n in the above equation, i.e.,

$$
\kappa\frac{u^n-u^{n-1}}{\tau}+\sum_{k=1}^{n}\omega_{n-k}\left(u^k-u^0\right)+Au^n=f^n,\quad \forall n\geqslant 1.
\tag{4.10}
$$

Denote

$$
\bar{\partial}_\tau^{\alpha,\mu}u^n=\sum_{k=1}^{n}\omega_{n-k}\left(u^k-u^0\right).
\tag{4.11}
$$

We provide some properties of operator $\bar{\partial}_\tau^{\alpha,\mu}$ and its coefficients $\{\omega_n\}$ in the following part.

4.2. Properties of operator $\bar{\partial}_\tau^{\alpha,\mu}$ and its coefficients

The following lemma is very useful to the estimations.

Lemma 4.1 ([18]). *For $z\in\Sigma_\theta$, $\pi/2<\theta<\pi$, $|\operatorname{Im} z|\leqslant\pi/\tau$, the following estimates hold:*

$$
c_1|z|\leqslant\left|\delta_\tau\left(\mathrm{e}^{-\tau z}\right)\right|\leqslant c_2|z|,
\tag{4.12}
$$

$$
\left|\delta_\tau\left(\mathrm{e}^{-\tau z}\right)-z\right|\leqslant c\tau|z|^2,
\tag{4.13}
$$

and

$$
\delta_\tau\left(\mathrm{e}^{-\tau z}\right)\in\Sigma_\theta.
\tag{4.14}
$$

Following above lemma, we have following estimations.

Lemma 4.2. *For $z\in\Sigma_\theta$, $\pi/2<\theta<\pi$, $|\operatorname{Im} z|\leqslant\pi/\tau$, the following statements hold.*

(1) $\phi(\delta_\tau(\mathrm{e}^{-\tau z}))$, $\eta(\delta_\tau(\mathrm{e}^{-\tau z}))\in\Sigma_\theta$.

(2) For sufficiently large σ (depending on μ) and θ sufficiently close to $\pi/2$, if $z\in\Sigma_\theta\cap\{z\in\mathbb{C}:|z|\geqslant\sigma\}$, we have

$$
c|z|^\alpha\leqslant\left|\phi\left(\delta_\tau\left(\mathrm{e}^{-\tau z}\right)\right)\right|\leqslant c|z|^\alpha,
\tag{4.15}
$$

$$
c|z|\leqslant\left|\eta\left(\delta_\tau\left(\mathrm{e}^{-\tau z}\right)\right)\right|\leqslant c|z|,
\tag{4.16}
$$

and

$$
\left|\eta\left(\delta_\tau\left(\mathrm{e}^{-\tau z}\right)\right)-\eta(z)\right|\leqslant c\tau|z|^2.
\tag{4.17}
$$

Proof. Combining Lemma 2.8 and Lemma 2.9 with Eq. (4.14), one can see that the first item of this lemma holds. Furthermore, Lemma 4.2 yields that the modulus of $\delta_\tau(\mathrm{e}^{-\tau z})$ is dominated by the modulus of z. Along with the estimations of $\phi(z)$ and $\eta(z)$ (Eqs. (2.5) to (2.7)), we can obtain the bounds of $|\phi(\delta_\tau(\mathrm{e}^{-\tau z}))|$ and $|\eta(\delta_\tau(\mathrm{e}^{-\tau z}))|$ (Eqs. (4.15) and (4.16)).

We now turn to estimate the bound of $|\eta(\delta_\tau(\mathrm{e}^{-\tau z}))-\eta(z)|$. By the expression $\eta(z)=\kappa z+\phi(z)$, one has

$$\begin{aligned}\left|\eta\big(\delta_\tau\big(\mathrm{e}^{-\tau z}\big)\big)-\eta(z)\right| &= \left|\kappa\delta_\tau\big(\mathrm{e}^{-\tau z}\big)+\phi\big(\delta_\tau\big(\mathrm{e}^{-\tau z}\big)\big)-\kappa z-\phi(z)\right| \\ &\leqslant \kappa\left|\delta_\tau\big(\mathrm{e}^{-\tau z}\big)-z\right|+\left|\phi\big(\delta_\tau\big(\mathrm{e}^{-\tau z}\big)\big)-\phi(z)\right| \\ &\leqslant c\tau|z|^2+\left|\phi\big(\delta_\tau\big(\mathrm{e}^{-\tau z}\big)\big)-\phi(z)\right| \quad (\text{by Eq. (4.13)}) \\ &= c\tau|z|^2+\left|\big(\delta_\tau\big(\mathrm{e}^{-\tau z}\big)+\mu\big)^\alpha-(z+\mu)^\alpha\right|.\end{aligned} \tag{4.18}$$

Using the Taylor expansion

$$\mathrm{e}^{-\tau z}=1-\tau z+\frac{1}{2}\tau^2z^2\int_0^1\mathrm{e}^{-\theta\tau z}(1-\theta)\,\mathrm{d}\theta, \tag{4.19}$$

and denoting $w(z)=z+\mu$, we have

$$\begin{aligned}\left|(z+\mu)^\alpha-\big(\delta_\tau\big(\mathrm{e}^{-\tau z}\big)+\mu\big)^\alpha\right| &= \left|(z+\mu)^\alpha-\left(\frac{1-\mathrm{e}^{-\tau z}}{\tau}+\mu\right)^\alpha\right| \\ &= \left|w(z)^\alpha-\left(w(z)-\frac{1}{2}\tau z^2\int_0^1\mathrm{e}^{-\theta\tau z}(1-\theta)\,\mathrm{d}\theta\right)^\alpha\right| \\ &= |w(z)|^\alpha\left|1-\left(1-\frac{1}{2}\tau z^2w(z)^{-1}\int_0^1\mathrm{e}^{-\theta\tau z}(1-\theta)\,\mathrm{d}\theta\right)^\alpha\right|.\end{aligned} \tag{4.20}$$

If $\tau\left|z^2w(z)^{-1}\right|<1/2$, then there holds the following Taylor [18, Section 3.5, p.3263]

$$\left(1-\frac{1}{2}\tau z^2w(z)^{-1}\int_0^1\mathrm{e}^{-\theta\tau z}(1-\theta)\,\mathrm{d}\theta\right)^\alpha=1+O\big(\tau\left|z^2w(z)^{-1}\right|\big). \tag{4.21}$$

In this case, we obtain

$$\left|(z+\mu)^\alpha-\big(\delta_\tau\big(\mathrm{e}^{-\tau z}\big)+\mu\big)^\alpha\right|\leqslant c\tau|w(z)|^{\alpha-1}|z|^2=c\tau|z+\mu|^{\alpha-1}|z|^2. \tag{4.22}$$

For $z\in\Sigma_\theta\cap\{z\in\mathbb{C}:|z|\geqslant\sigma\}$ with σ sufficiently large and $\theta\in(\pi/2,\pi)$ sufficiently close to $\pi/2$, the Proposition 3.1 in [18] implies that

$$c|z|\leqslant|z+\mu|\leqslant c|z|. \tag{4.23}$$

Therefore, we have

$$\left|(z+\mu)^\alpha-\big(\delta_\tau\big(\mathrm{e}^{-\tau z}\big)+\mu\big)^\alpha\right|\leqslant c\tau|z|^{1+\alpha}. \tag{4.24}$$

If $\tau\left|z^2w(z)^{-1}\right|\geqslant 1/2$, then we have

$$\frac{1}{2}\leqslant\tau|z|^2|z+\mu|^{-1}\leqslant c\tau|z|, \tag{4.25}$$

for $z\in\Sigma_\theta\cap\{z\in\mathbb{C}:|z|\geqslant\sigma\}$ with σ sufficiently large and $\theta\in(\pi/2,\pi)$ sufficiently close to $\pi/2$. Therefore, by Eqs. (2.5) and (4.15), we obtain

$$\left|\phi(z)-\phi\big(\delta_\tau\big(\mathrm{e}^{-\tau z}\big)\big)\right|\leqslant|\phi(z)|+\left|\phi\big(\delta_\tau\big(\mathrm{e}^{-\tau z}\big)\big)\right|\leqslant c|z|^\alpha\leqslant c\tau|z|^{1+\alpha}. \tag{4.26}$$

In either case, we have

$$\left|\phi(z)-\phi\big(\delta_\tau\big(\mathrm{e}^{-\tau z}\big)\big)\right|\leqslant c\tau|z|^{1+\alpha}, \tag{4.27}$$

which reduces that
$$\left|\eta(z)-\eta\big(\delta_\tau\big(\mathrm{e}^{-\tau z}\big)\big)\right| \leqslant c\tau\Big(|z|^{1+\alpha}+|z|^2\Big) \leqslant c\tau|z|^2, \tag{4.28}$$
for $z \in \Sigma_\theta \cap \{z \in \mathbb{C} : |z| \geqslant \sigma\}$ with σ sufficiently large and $\theta \in (\pi/2, \pi)$ sufficiently close to $\pi/2$. The proof is complete. □

Theorem 4.3. *For $\{\omega_n\}$, the following estimates hold.*

(1) $\omega_0 > 0$, and $\omega_n < 0$ for $n \geqslant 1$.
(2) $\sum_{n=0}^{\infty} \omega_n = 0$, and $\sum_{n=0}^{m} \omega_n > 0$ for $m \geqslant 0$.
(3) $\tau^\alpha|\omega_n| \leqslant c(n+1)^{-\alpha-1}$, and $\tau^\alpha|\sum_{k=0}^{n}\omega_k| \leqslant c(n+1)^{-\alpha}$ for $n \geqslant 0$.

Proof. By definition, the Cauchy integral formula and the Cauchy integral theorem, we have
$$\begin{aligned}
\omega_n &= \frac{1}{2\pi\mathbf{i}}\int_{|\zeta|=\varrho_\delta} \zeta^{-1-n}\phi(\delta_\tau(\zeta))\,\mathrm{d}\zeta \\
&= \frac{\tau}{2\pi\mathbf{i}}\int_{\Gamma^\tau} \mathrm{e}^{t_n z}\phi\big(\delta_\tau\big(\mathrm{e}^{-z\tau}\big)\big)\,\mathrm{d}z \\
&= \frac{\tau}{2\pi\mathbf{i}}\int_{\Gamma^\tau_{\delta,\theta}} \mathrm{e}^{t_n z}\phi\big(\delta_\tau\big(\mathrm{e}^{-z\tau}\big)\big)\,\mathrm{d}z,
\end{aligned} \tag{4.29}$$

where $\delta > 0$ and $\theta \in (\pi/2, \pi)$ satisfy the conditions in Lemma 4.2, $\varrho_\delta = \mathrm{e}^{-(\delta+1)\tau}$,
$$\Gamma^\tau = \Big\{z = \delta + 1 + \mathbf{i}y : y \in \mathbb{R} \text{ and } |y| \leqslant \frac{\pi}{\tau}\Big\}, \tag{4.30}$$

and
$$\Gamma^\tau_{\delta,\theta} = \big\{z = \delta\mathrm{e}^{\mathbf{i}\varphi} : |\varphi| \leqslant \theta\big\} \cup \Big\{z = r\mathrm{e}^{\pm\mathbf{i}\theta} :\ \delta \leqslant r \leqslant \frac{\pi}{\tau\sin\theta}\Big\}, \tag{4.31}$$
oriented with an increasing imaginary part. Then, for $n \geqslant 1$, using resolvent estimate and Lemma 4.1, one has
$$\begin{aligned}
|\omega_n| &\leqslant c\tau\int_{\Gamma^\tau_{\delta,\theta}} \mathrm{e}^{t_n\,\mathrm{Re}\,z}|z|^\alpha\,\mathrm{d}z \\
&\leqslant c\tau\left(\int_\delta^{\frac{\pi}{\tau\sin\theta}} \mathrm{e}^{t_n r\cos\theta} r^\alpha\,\mathrm{d}r + \int_{-\theta}^{\theta} \mathrm{e}^{t_n\delta\cos\varphi}\delta^{\alpha+1}\,\mathrm{d}\varphi\right) \\
&\leqslant c\tau t_n^{-1-\alpha} \leqslant c\tau^{-\alpha}n^{-1-\alpha} \leqslant c\tau^{-\alpha}(n+1)^{-\alpha-1}.
\end{aligned} \tag{4.32}$$

Furthermore, denote $q_n = \sum_{k=0}^{n}$. Then,
$$\sum_{n=0}^{\infty} q_n\zeta^n = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\omega_k\right)\zeta^n = \frac{1}{1-\zeta}\phi(\delta_\tau(\zeta)). \tag{4.33}$$

Therefore, we have
$$q_n = \frac{1}{2\pi\mathbf{i}}\int_{\Gamma^\tau_{\delta,\theta}} \mathrm{e}^{t_n z}\big(\delta_\tau\big(\mathrm{e}^{-z\tau}\big)\big)^{-1}\phi\big(\delta_\tau\big(\mathrm{e}^{-z\tau}\big)\big)\,\mathrm{d}z. \tag{4.34}$$

This leads to
$$\left|\sum_{k=0}^{n}\omega_k\right| \leqslant c\int_{\Gamma^\tau_{\delta,\theta}} \mathrm{e}^{t_n\,\mathrm{Re}\,z}|z|^{\alpha-1}\,|\mathrm{d}z| \leqslant c\tau^{-\alpha}(n+1)^{-\alpha},\ n \geqslant 1. \tag{4.35}$$
When $n = 0$, $\tau^\alpha\omega_0 = \tau^\alpha\phi(1/\tau) \leqslant c$.

In addition, since ϕ is a Bernstein function, we have

$$\omega_0 = \phi\left(\frac{1}{\tau}\right) > 0, \text{ and } \omega_n = \frac{\mathrm{d}^n}{\mathrm{d}\zeta^n}\phi\left(\frac{1-\zeta}{\tau}\right) = \tau^{-n}(-1)^n\phi^{(n)}\left(\frac{1}{\tau}\right) < 0. \tag{4.36}$$

Besides, it holds

$$\sum_{n=0}^{\infty}\omega_n = \phi(0) = 0, \tag{4.37}$$

so

$$\sum_{k=0}^{n}\omega_k > 0. \tag{4.38}$$

The proof is completed. □

Theorem 4.4. *Denote $p(t) = t$. Then we have*

$$\left|\partial_t^{\alpha,\mu}p(t_n) - \bar{\partial}_\tau^{\alpha,\mu}p(t_n)\right| \leqslant ct_n^{-\alpha}\tau. \tag{4.39}$$

Proof. Using Laplace transform, we have

$$\widehat{\partial_t^{\alpha,\mu}p(t)}(z) = \phi(z)z^{-2}. \tag{4.40}$$

So inverse Laplace transform implies that

$$\partial_t^{\alpha,\mu}p(t_n) = \frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{\delta,\theta}} \mathrm{e}^{t_n z}\phi(z)z^{-2}\,\mathrm{d}z. \tag{4.41}$$

On the other hand, denote

$$q_n = \bar{\partial}_\tau^{\alpha,\mu}p(t_n) = \sum_{k=1}^{n}\omega_{n-k}p(t_k). \tag{4.42}$$

Since $p(t_0) = 0$, one has

$$\sum_{n=0}^{\infty}q_n\zeta^n = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\omega_{n-k}p(t_k)\right)\zeta^n = \left(\sum_{n=0}^{\infty}\omega_k\zeta^n\right)\left(\sum_{n=0}^{\infty}p(t_n)\zeta^n\right). \tag{4.43}$$

Notice that

$$\sum_{n=0}^{\infty}p(t_n)\zeta^n = \tau\sum_{n=0}^{\infty}n\zeta^n = \tau\zeta\frac{\mathrm{d}}{\mathrm{d}\zeta}\frac{1}{1-\zeta} = \frac{\tau\zeta^2}{(1-\zeta)^2}, \tag{4.44}$$

and

$$\sum_{n=0}^{\infty}\omega_n\zeta^n = \phi(\delta_\tau(\zeta)). \tag{4.45}$$

Then, we obtain

$$\bar{\partial}_\tau^{\alpha,\mu}p(t_n) = \frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{\delta,\theta}^\tau} \mathrm{e}^{t_n z}\phi\big(\delta_\tau\big(\mathrm{e}^{-z\tau}\big)\big)\mathrm{e}^{-2z\tau}\big(\delta_\tau\big(\mathrm{e}^{-z\tau}\big)\big)^{-2}\,\mathrm{d}z. \tag{4.46}$$

Comparing the representations of $\partial_t^{\alpha,\mu}p(t_n)$ and $\bar{\partial}_t^{\alpha,\mu}p(t_n)$, we have

$$\begin{aligned}\partial_t^{\alpha,\mu}p(t_n)-\bar{\partial}_t^{\alpha,\mu}p(t_n)=&\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{\delta,\theta}\backslash\Gamma_{\theta.\delta}^{\tau}}\mathrm{e}^{t_nz}\phi(z)z^{-2}\,\mathrm{d}z\\&+\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{\delta,\theta}^{\tau}}\mathrm{e}^{t_nz}\big(\phi(z)-\phi\big(\delta_\tau\big(\mathrm{e}^{-z\tau}\big)\big)\big)z^{-2}\,\mathrm{d}z\\&+\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{\delta,\theta}^{\tau}}\mathrm{e}^{t_nz}\phi\big(\delta_\tau\big(\mathrm{e}^{-z\tau}\big)\big)\Big(z^{-2}-\big(\delta_\tau\big(\mathrm{e}^{-z\tau}\big)\big)^{-2}\Big)\,\mathrm{d}z\\&+\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{\delta,\theta}^{\tau}}\mathrm{e}^{t_nz}\phi\big(\delta_\tau\big(\mathrm{e}^{-z\tau}\big)\big)\big(\delta_\tau\big(\mathrm{e}^{-z\tau}\big)\big)^{-2}\big(1-\mathrm{e}^{-2z\tau}\big)\,\mathrm{d}z\\=&:I_1+I_2+I_3+I_4.\end{aligned}\tag{4.47}$$

Choosing $\theta\in(\pi/2,\pi)$ sufficiently close to $\pi/2$ and δ sufficiently large, we have

$$\begin{aligned}|I_1|&\leqslant c\int_{\Gamma_{\delta,\theta}\backslash\Gamma_{\delta,\theta}^{\tau}}\mathrm{e}^{t_n\operatorname{Re}z}|z|^{-2+\alpha}\,|\mathrm{d}z|\leqslant c\int_{\frac{\pi}{\tau\sin\theta}}^{\infty}\mathrm{e}^{rt_n\cos\theta}r^{-2+\alpha}\,\mathrm{d}r\\&\leqslant ct_n^{-\alpha}\tau\int_0^\infty\mathrm{e}^{s\cos\theta}s^{\alpha-1}\,\mathrm{d}r\leqslant ct_n^{-\alpha}\tau.\end{aligned}\tag{4.48}$$

For I_2, according to Lemma 4.2, one has

$$\begin{aligned}|I_2|&\leqslant c\tau\int_{\Gamma_{\delta,\theta}^{\tau}}\mathrm{e}^{t_n\operatorname{Re}z}|z|^{\alpha-1}\,\mathrm{d}z\\&\leqslant c\tau\left(\int_{\delta}^{\frac{\pi}{\tau\sin\theta}}\mathrm{e}^{rt_n\cos\theta}r^{\alpha-1}\,\mathrm{d}r+\int_{-\theta}^{\theta}\mathrm{e}^{\delta t_n\cos\varphi}\delta^{\alpha}\,\mathrm{d}\varphi\right)\\&\leqslant ct_n^{-\alpha}\tau.\end{aligned}\tag{4.49}$$

In this way, one can also get

$$|I_3|,\ |I_4|\leqslant ct_n^{-\alpha}\tau.\tag{4.50}$$

Finally, we obtain

$$\left|\partial_t^{\alpha,\mu}p(t_n)-\bar{\partial}_\tau^{\alpha,\mu}p(t_n)\right|\leqslant ct_n^{-\alpha}\tau.$$

□

Then, for smooth function $F(t)$, we have the following estimate.

Theorem 4.5. *Let $F(t)\in\mathrm{C}^2([0,T])$ with $F'(0)=0$. Then, there holds*

$$\left|\partial_t^{\alpha,\mu}F(t)-\bar{\partial}_\tau^{\alpha,\mu}F(t_n)\right|\leqslant ct_n^{-\alpha}\tau.\tag{4.51}$$

Proof. Since $\partial_t^{\alpha,\mu}(F(t)-F(0))=\partial_t^{\alpha,\mu}F(t)$ and $\bar{\partial}_\tau^{\alpha,\mu}(F(t_n)-F(0))=\bar{\partial}_\tau^{\alpha,\mu}F(t_n)$, we only consider the function F with $F(0)=0$. By Taylor expansion, we have

$$F(t)=p(t)*F''(t),\tag{4.52}$$

where $p(t)=t$, and notation "$*$" denotes the convolution operator. For convenience, denote $g(t)=F''(t)$. Then, we get

$$\partial_t^{\alpha,\mu}F(t)=\partial_t^{\alpha,\mu}(p*g)(t)=(\partial_t^{\alpha,\mu}p*g)(t).\tag{4.53}$$

For the discrete form, we have

$$\bar{\partial}_\tau^{\alpha,\mu} F(t_n) = \sum_{k=0}^{n} \omega_{n-k} f(t_k) = (\mathcal{E}_\tau^{\alpha,\mu} * p * g)(t_n), \tag{4.54}$$

where

$$\mathcal{E}_\tau^{\alpha,\mu}(t) = \sum_{n=0}^{\infty} \omega_n \delta(t - t_n). \tag{4.55}$$

Thus,

$$\left|\partial_t^{\alpha,\mu} F(t_n) - \bar{\partial}_\tau^{\alpha,\mu} F(t_n)\right| = |((\partial_t^{\alpha,\mu} p - \mathcal{E}_\tau^{\alpha,\mu} * p) * g)(t_n)|. \tag{4.56}$$

Note that

$$(\mathcal{E}_\tau^{\alpha,\mu} * p)(t_n) = \bar{\partial}_\tau^{\alpha,\mu} p(t_n). \tag{4.57}$$

According to Theorem 4.4, we know that

$$(\partial_t^{\alpha,\mu} p - \mathcal{E}_\tau^{\alpha,\mu} * p)(t_n) \leqslant c t_n^{-\alpha} \tau. \tag{4.58}$$

So we claim that

$$(\partial_t^{\alpha,\mu} p - \mathcal{E}_\tau^{\alpha,\mu} * p)(t) \leqslant c t^{-\alpha} \tau, \ \forall t \in (t_{n-1}, t_n]. \tag{4.59}$$

In fact, by Laplace transform, we have

$$\partial_t^{\alpha,\mu} p(t) = (\mathcal{F} * 1)(t), \tag{4.60}$$

where

$$\mathcal{F}(t) = \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_{\delta,\theta}} \mathrm{e}^{zt} \phi(z) z^{-1} \, \mathrm{d}z. \tag{4.61}$$

Therefore, using Taylor expansion, one has

$$\partial_t^{\alpha,\mu} p(t) = \partial_t^{\alpha,\mu} p(t_n) + \int_{t_n}^{t} \mathcal{F}(s) \, \mathrm{d}s, \tag{4.62}$$

and

$$(\mathcal{E}_\tau^{\alpha,\mu} * p)(t) = \mathcal{E}_\tau^{\alpha,\mu} * p(t_n) + (t - t_n)(\mathcal{E}_\tau^{\alpha,\mu} * 1)(t) + \int_{t_n}^{t} (t - s) \mathcal{E}_\tau^{\alpha,\mu}(s) \, \mathrm{d}s. \tag{4.63}$$

Then, similar with the above arguments, we have

$$|\mathcal{F}(t)| \leqslant c \int_{\Gamma_{\delta,\theta}} \mathrm{e}^{t \operatorname{Re} z} |z|^{\alpha - 1} \, |\mathrm{d}z| \leqslant c t^{-\alpha}, \tag{4.64}$$

$$|(\mathcal{E}_\tau^{\alpha,\mu} * 1)(t)| \leqslant c \left| \sum_{k=0}^{n} \omega_k \right| \leqslant c \tau^{-\alpha} n^{-\alpha} \leqslant c t^{-\alpha}, \tag{4.65}$$

and

$$\left| \int_{t_n}^{t} (t - s) \mathcal{E}_\tau^{\alpha,\mu}(s) \, \mathrm{d}s \right| \leqslant c \left| \sum_{k=0}^{n} \omega_k \right| \leqslant c \tau^{-\alpha} n^{-\alpha} \leqslant c t^{-\alpha}. \tag{4.66}$$

□

4.3. Error analysis of time–stepping scheme

Now, we turn to analyze the error of time–stepping scheme Eq. (4.10). For convenience, we define

$$\mathcal{I}_\tau^{(1)} = \big(\eta\big(\delta_\tau\big(\mathrm{e}^{-\tau z}\big)\big) - \eta(z)\big)\delta_\tau^{-1}\big(\mathrm{e}^{-\tau z}\big)\big(\eta\big(\delta_\tau\big(\mathrm{e}^{-\tau z}\big)\big) + A\big)^{-1},$$

$$\begin{aligned}
\mathcal{I}_\tau^{(2)} &= \eta(z)\big(\delta_\tau^{-1}\big(\mathrm{e}^{-\tau z}\big) - z^{-1}\big)\big(\eta\big(\delta_\tau\big(\mathrm{e}^{-\tau z}\big)\big) + A\big)^{-1},\\
\mathcal{I}_\tau^{(3)} &= \eta(z)z^{-1}\Big[\big(\eta\big(\delta_\tau\big(\mathrm{e}^{-\tau z}\big)\big) + A\big)^{-1} - (\eta(z) + A)^{-1}\Big],\\
\mathcal{I}_\tau^{(4)} &= \eta(z)z^{-1}(1 - \mathrm{e}^{\tau z})(\eta(z) + A)^{-1},\\
\mathcal{J}_\tau^{(1)} &= \big(\delta_\tau^{-1}\big(\mathrm{e}^{-\tau z}\big) - z^{-1}\big)\big(\eta\big(\delta_\tau\big(\mathrm{e}^{-\tau z}\big)\big) - A\big)^{-1},\\
\mathcal{J}_\tau^{(2)} &= z^{-1}\Big[\big(\eta\big(\delta_\tau\big(\mathrm{e}^{-\tau z}\big)\big) + A\big)^{-1} - (\eta(z) + A)^{-1}\Big],
\end{aligned}$$

and

$$\mathcal{J}_\tau^{(3)} = z^{-1}(1 - \mathrm{e}^{\tau z})(\eta(z) + A)^{-1}.$$

Lemma 4.6. *For $z \in \Sigma_\theta \cap \{z \in \mathbb{C} : |z| \geqslant \sigma\}$ with σ sufficiently large and $\theta \in (\pi/2, \pi)$ sufficiently close to $\pi/2$, we have*

$$\Big\|\mathcal{I}_\tau^{(1)}\Big\|_{\mathrm{L}^2(\Omega)\to\mathrm{L}^2(\Omega)},\ \Big\|\mathcal{I}_\tau^{(2)}\Big\|_{\mathrm{L}^2(\Omega)\to\mathrm{L}^2(\Omega)},\ \Big\|\mathcal{I}_\tau^{(3)}\Big\|_{\mathrm{L}^2(\Omega)\to\mathrm{L}^2(\Omega)},\ \Big\|\mathcal{I}_\tau^{(4)}\Big\|_{\mathrm{L}^2(\Omega)\to\mathrm{L}^2(\Omega)} \leqslant c\tau, \tag{4.67}$$

and

$$\Big\|\mathcal{J}_\tau^{(1)}\Big\|_{\mathrm{L}^2(\Omega)\to\mathrm{L}^2(\Omega)},\ \Big\|\mathcal{J}_\tau^{(2)}\Big\|_{\mathrm{L}^2(\Omega)\to\mathrm{L}^2(\Omega)},\ \Big\|\mathcal{J}_\tau^{(3)}\Big\|_{\mathrm{L}^2(\Omega)\to\mathrm{L}^2(\Omega)} \leqslant c|z|^{-1}\tau. \tag{4.68}$$

Proof. For $\mathcal{I}_\tau^{(1)}$, according to Lemma 4.2, we have

$$\Big\|\mathcal{I}_\tau^{(1)}\Big\|_{\mathrm{L}^2(\Omega)\to\mathrm{L}^2(\Omega)} \leqslant c\tau|z|^2|z|^{-1}\big|\eta\big(\delta_\tau\big(\mathrm{e}^{-\tau z}\big)\big)\big|^{-1} \leqslant c\tau. \tag{4.69}$$

In this way, one can also get the estimates of operators $\mathcal{I}_\tau^{(2)}$, $\mathcal{I}_\tau^{(4)}$, $\mathcal{J}_\tau^{(1)}$ and $\mathcal{J}_\tau^{(3)}$.

For operators $\mathcal{I}_\tau^{(3)}$ and $\mathcal{J}_\tau^{(2)}$, we use the first resolvent identity, that is,

$$\mathcal{I}_\tau^{(3)} = \eta(z)z^{-1}\big(\eta\big(\delta_\tau\big(\mathrm{e}^{-\tau z}\big)\big) + A\big)^{-1}(\eta(z) + A)^{-1}\big(\eta(z) - \eta\big(\delta_\tau\big(\mathrm{e}^{-\tau z}\big)\big)\big), \tag{4.70}$$

and

$$\mathcal{J}_\tau^{(2)} = z^{-1}\big(\eta\big(\delta_\tau\big(\mathrm{e}^{-\tau z}\big)\big) + A\big)^{-1}(\eta(z) + A)^{-1}\big(\eta(z) - \eta\big(\delta_\tau\big(\mathrm{e}^{-\tau z}\big)\big)\big). \tag{4.71}$$

Then, we have

$$\Big\|\mathcal{I}_\tau^{(3)}\Big\|_{\mathrm{L}^2(\Omega)\to\mathrm{L}^2(\Omega)} \leqslant c|\eta(z)||z|^{-1}\big|\eta\big(\delta_\tau\big(\mathrm{e}^{-\tau z}\big)\big)\big|^{-1}|\eta(z)|^{-1}|z|^2\tau \leqslant c\tau, \tag{4.72}$$

and

$$\Big\|\mathcal{J}_\tau^{(2)}\Big\|_{\mathrm{L}^2(\Omega)\to\mathrm{L}^2(\Omega)} \leqslant c|z|^{-1}|\eta(z)|^{-1}\big|\eta\big(\delta_\tau\big(\mathrm{e}^{-\tau z}\big)\big)\big|^{-1}|z|^2\tau \leqslant c|z|^{-1}\tau. \tag{4.73}$$

This completes the proof. □

Furthermore, denote

$$\mathcal{E}(t) = \frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{\delta,\theta}} \mathrm{e}^{zt}(\eta(z) + A)^{-1}\,\mathrm{d}z, \tag{4.74}$$

$$\mathcal{E}_\tau^n = \frac{\tau}{2\pi\mathbf{i}}\int_{\Gamma_{\delta,\theta}^\tau} \mathrm{e}^{zt_n}\big(\eta\big(\delta_\tau\big(\mathrm{e}^{-z\tau}\big)\big) + A\big)^{-1}\,\mathrm{d}z, \tag{4.75}$$

and

$$\mathcal{E}_\tau(t) = \sum_{n=0}^{\infty}\mathcal{E}_\tau^n\delta(t - t_n). \tag{4.76}$$

Lemma 4.7.
$$\|((\mathcal{E}-\mathcal{E}_\tau)*1)(t)\|_{\mathrm{L}^2(\Omega)\to\mathrm{L}^2(\Omega)} \leqslant c\tau. \tag{4.77}$$

Proof. Since
$$(\mathcal{E}*1)(t_n) = \frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{\delta,\theta}} \mathrm{e}^{t_n z}(\eta(z)+A)^{-1}z^{-1}\,\mathrm{d}z, \tag{4.78}$$
and
$$(\mathcal{E}_\tau*1)(t_n) = \frac{1}{2\pi\mathbf{i}}\int_{\Gamma^\tau_{\delta,\theta}} \mathrm{e}^{t_n z}\big(\eta\big(\delta_\tau\big(\mathrm{e}^{-z\tau}\big)\big)+A\big)^{-1}\big(\delta_\tau\big(\mathrm{e}^{-z\tau}\big)\big)^{-1}\,\mathrm{d}z, \tag{4.79}$$
then, by Lemma 4.6, we have
$$\|((\mathcal{E}-\mathcal{E}_\tau)*1)(t_n)\|_{\mathrm{L}^2(\Omega)\to\mathrm{L}^2(\Omega)} \leqslant c\tau. \tag{4.80}$$
Similar with the arguments in Lemma 4.5, we can complete the proof. □

Now, we plan to analyze the error between $u(t_n)$ and u^n. On the one hand, the solution $u(t_n)$ to Eq. (1.1) reads
$$\begin{aligned} u(t_n) = &\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{\delta,\theta}} \mathrm{e}^{t_n z}z^{-1}\eta(z)(\eta(z)+A)^{-1}u_0\,\mathrm{d}z\\ &+\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{\delta,\theta}} \mathrm{e}^{t_n z}z^{-1}(\eta(z)+A)^{-1}f(0)\,\mathrm{d}z\\ &+\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{\delta,\theta}} \mathrm{e}^{t_n z}(\eta(z)+A)^{-1}\widehat{R}(z)\,\mathrm{d}z. \end{aligned} \tag{4.81}$$
On the other hand, according to Eq. (4.2), the Cauchy integral formula and the Cauchy integral theorem, as argumented in Eq. (4.29), the approximate solution u^n to Eq. (4.10) can be written as
$$\begin{aligned} u^n =&\frac{1}{2\pi\mathbf{i}}\int_{\Gamma^\tau_{\delta,\theta}} \mathrm{e}^{t_n z}\mathrm{e}^{-\tau z}\delta_\tau^{-1}\big(\mathrm{e}^{-\tau z}\big)\eta\big(\delta_\tau\big(\mathrm{e}^{-\tau z}\big)\big)\big(\eta\big(\delta_\tau\big(\mathrm{e}^{-\tau z}\big)\big)+A\big)^{-1}u^0\,\mathrm{d}z\\ &+\frac{1}{2\pi\mathbf{i}}\int_{\Gamma^\tau_{\delta,\theta}} \mathrm{e}^{t_n z}\mathrm{e}^{-\tau z}\delta_\tau^{-1}\big(\mathrm{e}^{-\tau z}\big)\big(\eta\big(\delta_\tau\big(\mathrm{e}^{-\tau z}\big)\big)+A\big)^{-1}f^0\,\mathrm{d}z\\ &+\frac{1}{2\pi\mathbf{i}}\int_{\Gamma^\tau_{\delta,\theta}} \mathrm{e}^{t_n z}\big(\eta\big(\delta_\tau\big(\mathrm{e}^{-\tau z}\big)\big)+A\big)^{-1}\tau\sum_{m=1}^{\infty}R^m\mathrm{e}^{-t_m z}\,\mathrm{d}z. \end{aligned} \tag{4.82}$$

Theorem 4.8. *Let $u(t_n)$ and u^n be the solutions to Eq. (1.1) and Eq. (4.10), respectively. If $u_0\in\mathrm{L}^2(\Omega)$, $f(t)$ is absolutely continuous, $f(0)\in\mathrm{L}^2(\Omega)$, and $\int_0^t\|\partial_s f(s)\|\,\mathrm{d}s<\infty$ for all $t\in[0,T]$, then we have*
$$\|u(t_n)-u^n\| \leqslant c\tau\bigg(t_n^{-1}\|u_0\|+\|f(0)\|+\int_0^{t_n}\|\partial_s f(s)\|\,\mathrm{d}s\bigg). \tag{4.83}$$

Proof. Denoting $e_n=u(t_n)-u^n$, similar with the arguments in [26, Lemma 3.12], we have
$$\begin{aligned} \|e_n\| \leqslant\ & c\int_{\Gamma^\tau_{\delta,\theta}} \mathrm{e}^{t_n\operatorname{Re}z}\Bigg(\sum_{j=1}^{4}\Big\|\mathcal{I}_\tau^{(j)}u_0\Big\|+\sum_{j=1}^{3}\Big\|\mathcal{J}_\tau^{(j)}f(0)\Big\|\Bigg)|\mathrm{d}z|+c\|\mathcal{K}_\tau\|\\ &+c\int_{\Gamma_{\delta,\theta}\backslash\Gamma^\tau_{\delta,\theta}} \mathrm{e}^{t_n\operatorname{Re}z}\Big(|z|^{-1}\|u_0\|+|z|^{-2}\|f(0)\|\Big)|\mathrm{d}z|\\ &+c\int_0^{t_n}\int_{\Gamma_{\delta,\theta}\backslash\Gamma^\tau_{\delta,\theta}} \mathrm{e}^{(t_n-s)\operatorname{Re}z}|z|^{-2}\,|\mathrm{d}z|\,\|\partial_s f(s)\|\,\mathrm{d}s\\ =:&\ I_1+I_2+I_3+I_4, \end{aligned} \tag{4.84}$$

where

$$\mathcal{K}_\tau = (((\mathcal{E}-\mathcal{E}_\tau)*1)*\partial_t f)(t_n). \tag{4.85}$$

Then, we should estimate the four terms in the above equation one by one.

According to Lemma 4.6, we have

$$I_1 \leqslant c\tau\bigg(\int_{\Gamma^\tau_{\delta,\theta}} \mathrm{e}^{t_n \operatorname{Re} z}\,|\mathrm{d}z|\|u_0\| + \int_{\Gamma^\tau_{\delta,\theta}} \mathrm{e}^{t_n \operatorname{Re} z}|z|^{-1}\,|\mathrm{d}z|\|f(0)\|\bigg). \tag{4.86}$$

Since

$$\int_{\Gamma^\tau_{\delta,\theta}} \mathrm{e}^{t_n \operatorname{Re} z}\,|\mathrm{d}z| \leqslant c\bigg(\int_\delta^{\frac{\pi}{\tau\sin\theta}} \mathrm{e}^{t_n r\cos\theta}\,\mathrm{d}r + \int_{-\theta}^{\theta} \mathrm{e}^{t_n\delta\cos\varphi}\delta\,\mathrm{d}\varphi\bigg) \leqslant ct_n^{-1}, \tag{4.87}$$

and

$$\int_{\Gamma^\tau_{\delta,\theta}} \mathrm{e}^{t_n \operatorname{Re} z}|z|^{-1}\,|\mathrm{d}z| \leqslant c\bigg(\int_\delta^{\frac{\pi}{\tau\sin\theta}} \mathrm{e}^{t_n r\cos\theta}r^{-1}\,\mathrm{d}r + \int_{-\theta}^{\theta} \mathrm{e}^{t_n\delta\cos\varphi}\,\mathrm{d}\varphi\bigg) \leqslant c, \tag{4.88}$$

we get

$$I_1 \leqslant c\tau\big(t_n^{-1}\|u_0\| + \|f(0)\|\big). \tag{4.89}$$

Then,

$$\int_{\Gamma_{\delta,\theta}\backslash\Gamma^\tau_{\delta,\theta}} \mathrm{e}^{t_n \operatorname{Re} z}|z|^{-1}\,|\mathrm{d}z| \leqslant c\tau\int_{\frac{\pi}{\tau\sin\theta}}^{\infty} \mathrm{e}^{t_n r\cos\theta}\,\mathrm{d}r, \tag{4.90}$$

and

$$\int_{\Gamma_{\delta,\theta}\backslash\Gamma^\tau_{\delta,\theta}} \mathrm{e}^{t_n \operatorname{Re} z}|z|^{-2}\,|\mathrm{d}z| \leqslant c\tau\int_{\frac{\pi}{\tau\sin\theta}}^{\infty} \mathrm{e}^{t_n r\cos\theta}r^{-1}\,\mathrm{d}r, \tag{4.91}$$

Changing variable $s = rt_n$, one has

$$\int_{\frac{\pi}{\tau\sin\theta}}^{\infty} \mathrm{e}^{t_n r\cos\theta}\,\mathrm{d}r = t_n^{-1}\int_{\frac{\pi t_n}{\tau\sin\theta}}^{\infty} \mathrm{e}^{s\cos\theta}\,\mathrm{d}s \leqslant ct_n^{-1}, \tag{4.92}$$

and

$$\int_{\frac{\pi}{\tau\sin\theta}}^{\infty} \mathrm{e}^{t_n r\cos\theta}r^{-1}\,\mathrm{d}r = \int_{\frac{\pi t_n}{\tau\sin\theta}}^{\infty} \mathrm{e}^{s\cos\theta}s^{-1}\,\mathrm{d}s \leqslant c. \tag{4.93}$$

This leads to

$$I_3 \leqslant c\tau\big(t_n^{-1}\|u_0\| + \|f(0)\|\big), \tag{4.94}$$

and

$$I_4 \leqslant c\tau\int_0^{t_n} \|\partial_s f(s)\|\,\mathrm{d}s. \tag{4.95}$$

Finally, according to Lemma 4.7, one gets

$$\|\mathcal{K}_\tau\| \leqslant c\tau\int_0^{t_n} \|\partial_s f(s)\|\,\mathrm{d}s. \tag{4.96}$$

The proof is completed. □

5. Fully discrete scheme

Combining the FEM scheme (3.5) and the backward Euler time–stepping scheme (4.10), we get a fully discrete scheme as follows:

$$\kappa\frac{u_h^n - u_h^{n-1}}{\tau} + \sum_{k=1}^{n}\omega_{n-k}\big(u_h^k - u_h^0\big) + A_h u_h^n = f_h^n, \quad u_h^0 = P_h u_0. \tag{5.1}$$

Theorem 5.1. *Let u_h^n be the solution to Eq. (5.1). Then, there holds*

$$\|u_h^n\| \leqslant \big\|u_h^0\big\| + \frac{n\tau}{\kappa}\max_{t\in[0,T]}\|f_h(t)\| \tag{5.2}$$

for $n \geqslant 1$.

Proof. From Eq. (5.1), we have

$$\frac{\kappa}{\tau}\big(u_h^n - u_h^{n-1},\, u_h^n\big) + \sum_{k=1}^{n}\omega_{n-k}\big(u_h^k - u_h^0,\, u_h^n\big) + (A_h u_h^n, u_h^n) = (f_h^n, u_h^n), \tag{5.3}$$

or equivalently,

$$\begin{aligned}&\Big(\frac{\kappa}{\tau} + \omega_0\Big)\|u_h^n\|^2 + \|\nabla u_h^n\|^2\\ =&\frac{\kappa}{\tau}\big(u_h^n,\, u_h^{n-1}\big) - \sum_{k=1}^{n-1}\omega_k\big(u_h^n,\, u_h^{n-k}\big) + \Bigg(\sum_{k=0}^{n-1}\omega_k\Bigg)\big(u_h^n,\, u_h^0\big) + \big(u_h^n,\, f_h^n\big).\end{aligned} \tag{5.4}$$

By the Cauchy-Schwartz inequality, one has

$$\Big(\frac{\kappa}{\tau} + \omega_0\Big)\|u_h^n\| \leqslant \frac{\kappa}{\tau}\big\|u_h^{n-1}\big\| - \sum_{k=1}^{n-1}\omega_k\big\|u_h^{n-k}\big\| + \Bigg(\sum_{k=0}^{n-1}\omega_k\Bigg)\big\|u_h^0\big\| + \|f_h^n\|, \quad n \geqslant 1, \tag{5.5}$$

due to $\omega_n < 0$ and $\sum_{k=0}^{n}\omega_k > 0$ for $n \geqslant 1$ (Theorem 4.3). Then, we use induction to prove our result.

For $n = 1$, Eq. (5.5) reduces that

$$\Big(\frac{\kappa}{\tau} + \omega_0\Big)\big\|u_h^1\big\| \leqslant \frac{\kappa}{\tau}\big\|u_h^0\big\| + \omega_0\big\|u_h^0\big\| + \big\|f_h^1\big\|. \tag{5.6}$$

Because

$$\frac{1}{\kappa/\tau + \omega_0} = \frac{\tau}{\kappa + \tau^{1-\alpha}[(1+\tau\mu)^{\alpha} - \tau^{\alpha}\mu^{\alpha}]} < \frac{\tau}{\kappa}, \tag{5.7}$$

we obtain

$$\big\|u_h^1\big\| \leqslant \big\|u_h^0\big\| + \frac{\tau}{\kappa}\big\|f_h^1\big\| \leqslant \big\|u_h^0\big\| + \frac{\tau}{\kappa}\max_{t\in[0,T]}\|f_h(t)\|. \tag{5.8}$$

Suppose that Eq. (5.2) holds for $n = 2, \cdots, m-1$. Then, for $n = m$, we have

$$\begin{aligned}\|u_h^m\| &\leqslant \big\|u_h^0\big\| + \frac{\Big(\kappa/\tau - \sum_{k=1}^{m-1}\omega_k\Big)(m-1)}{\kappa/\tau + \omega_0}\frac{\tau}{\kappa}\max_{t\in[0,T]}\|f_h(t)\| + \frac{\tau}{\kappa}\|f_h^m\|\\ &\leqslant \big\|u_h^0\big\| + \frac{m\tau}{\kappa}\max_{t\in[0,T]}\|f_h(t)\|.\end{aligned} \tag{5.9}$$

□

Similar with Theorem 4.8, we have the following theorem.

Theorem 5.2. *Let $u_h(t_n)$ and u_h^n be the solutions to Eq. (3.5) and Eq. (5.1), respectively. If $u_0 \in \mathrm{L}^2(\Omega)$, $f(t)$ is absolutely continuous, $f(0) \in \mathrm{L}^2(\Omega)$, and $\int_0^t \|\partial_s f(s)\| \,\mathrm{d}s < \infty$ for all $t \in [0,T]$, then we have*

$$\|u_h(t_n) - u_h^n\| \leqslant c\tau\left(t_n^{-1}\|u_0\| + \|f(0)\| + \int_0^{t_n} \|\partial_s f(s)\| \,\mathrm{d}s\right). \tag{5.10}$$

Combining the results of Theorem 3.4 and Theorem 5.2 and using the triangle inequality, we obtain the following error estimation of fully discrete scheme.

Theorem 5.3. *Let $u(t_n)$ and u_h^n be the solutions to Eq. (1.1) and Eq. (5.1), respectively. If $u_0 \in \mathrm{L}^2(\Omega)$, $f(t)$ is absolutely continuous, $f(0) \in \mathrm{L}^2(\Omega)$, and $\int_0^t \|\partial_s f(s)\| \,\mathrm{d}s < \infty$ for all $t \in [0,T]$, then we have*

$$\|u(t_n) - u_h^n\| + h\|\nabla(u(t_n) - u_h^n)\| \leqslant c\big(\tau + h^2\big)\left(t_n^{-1}\|u_0\| + \|f(0)\| + \int_0^{t_n} \|\partial_s f(s)\| \,\mathrm{d}s\right). \tag{5.11}$$

6. Numerical experiments

Consider the following 1-dimensional diffusion equation.

$$\begin{cases} \partial_t u(x,t) + \partial_t^{\alpha,\mu} u(x,t) + Au(x,t) = f(x,t), & (x,t) \in (a,b) \times (0,T], \\ u(a,t) = u(b,t) = 0, & t \in (0,T], \\ u(x,0) = (x-a)(b-x), & x \in (a,b). \end{cases} \tag{6.1}$$

Since we do not know the explicit solution, we use following formula

$$E_\tau = \left\| u_\tau^N - u_{\tau/2}^N \right\| \tag{6.2}$$

to measure temporal error, where u_τ^N means the numerical solution at time T with step size τ. Then the corresponding convergence rate can be calculated by

$$\frac{\log\left(E_\tau / E_{\tau/2}\right)}{\log 2}. \tag{6.3}$$

To investigate the convergence in time and eliminate the influence from spatial discretization, we set $h = 0.01$. Table 1 and Table 2 show the errors and convergence rates when $\alpha = 0.3$ and 0.7 for (in)homogeneous problems. As we can see, the results validate Theorem 5.3.

7. Discussion

In this paper, we discuss the numerical scheme for Eq. (1.1) with the help of complete Bernstein function. In fact, we can also extend our arguments in this paper to the more generalized problems. From Lévy-Khintchine formula of Lévy subordinator and the integral representation of Bernstein function, we see that there is one-to-one relation in between. Therefore, one can consider the following problem

$$\begin{cases} \kappa\partial_t u(x,t) + \partial_t^w u(x,t) + Au(x,t) = f(x,t), & (x,t) \in \Omega \times (0,T], \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T], \end{cases} \tag{7.1}$$

where $\kappa \geqslant 0$, and ∂_t^w is called the generalized time-fractional derivative operator, defined as

$$\partial_t^w u(t) = \frac{\partial}{\partial t}\int_0^t w(t-s)(u(t)-u(0)) \,\mathrm{d}s, \quad \widehat{w}(z) = \frac{\phi(z)}{z} \tag{7.2}$$

TABLE 1. Temporal errors and convergence rates, $a=0$, $b=1$, $T=5$, $h=0.01$, $f=0$.

μ	$\alpha \backslash \tau$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{40}$	$\frac{1}{80}$	$\frac{1}{160}$
1	0.3	1.4446E-7	6.5749E-8	3.1315E-8	1.5276E-8	7.5432E-9
		Rate	1.1357	1.0701	1.0356	1.0180
	0.7	1.0487E-7	4.6499E-8	2.1858E-8	1.0593E-8	5.2136E-9
		Rate	1.1734	1.0890	1.0451	1.0227
0.1	0.3	2.4915E-6	1.2366E-6	6.1606E-7	3.0747E-7	1.5359E-7
		Rate	1.0106	1.0053	1.0026	1.0013
	0.7	1.8560E-6	9.1311E-7	4.5291E-7	2.2555E-7	1.1255E-7
		Rate	1.0233	1.0116	1.0058	1.0029

TABLE 2. Temporal errors and convergence rates, $a=0$, $b=1$, $T=5$, $h=0.01$, $f=x^2+t^3$.

μ	$\alpha \backslash \tau$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{40}$	$\frac{1}{80}$	$\frac{1}{160}$
1	0.3	0.0013	6.5881E-4	3.3030E-4	1.6538E-4	8.2745E-5
		Rate	0.9921	0.9961	0.9980	0.9990
	0.7	0.0017	8.5316E-4	4.2776E-4	2.1417E-4	1.0716E-4
		Rate	0.9920	0.9960	0.9980	0.9990
0.1	0.3	0.0015	7.6557E-4	3.8377E-4	1.9213E-4	9.6126E-5
		Rate	0.9926	0.9963	0.9982	0.9991
	0.7	0.0019	9.5484E-4	4.7873E-4	2.3969E-4	1.1993E-4
		Rate	0.9921	0.9960	0.9980	0.9990

with ϕ being a Bernstein function.

Assumption 7.1.

(1) $\phi(\lambda)$ is a complete Bernstein function.

(2) There exist $0<\alpha\leqslant\beta<1$ and c_1, $c_2>0$ such that

$$c_1|z|^\alpha \leqslant |\phi(z)| \leqslant c_2|z|^\beta \tag{7.3}$$

for any $z\in\mathbb{C}\backslash(-\infty,0]$.

Under the second condition in Assumption 7.1, the generalized time-fractional derivative operator ∂_t^w is well defined, see, e.g., [20, 21]. The temporal semi-discrete time–stepping scheme, FEM semi-discrete scheme and fully discrete scheme of Eq. (7.2) are

$$\kappa\frac{u^n-u^{n-1}}{\tau}+\bar\partial_\tau^w u^n+Au^n=f^n, \quad u^0=u_0, \tag{7.4}$$

$$(\kappa\partial_t+\partial_t^{\alpha,\mu})u_h(t)+A_hu_h(t)=P_hf(t), \quad u_h(0)=P_hu_0, \tag{7.5}$$

and

$$\kappa\frac{u_h^n-u_h^{n-1}}{\tau}+\bar\partial_\tau^w u_h^n+A_hu_h^n=f_h^n, \quad u_h^0=P_hu_0, \tag{7.6}$$

respectively, where

$$\bar{\partial}_\tau^w u^n = \sum_{k=1}^{n} \omega_{n-k}\big(u^k - u^0\big), \tag{7.7}$$

where the coefficients are defined by Eq. (4.3).

In the case of $\kappa > 0$, the regularity and error estimations of finite element scheme above, Theorems 2.11 and 3.4, are compatible with problem (7.1) as well.

In the case of $\kappa = 0$, we have the following alternative results.

Theorem 7.2. *Let $u(t)$ be the solution to problem Eq. (7.1). Assume that $f(t)$ is absolutely continuous, $f(0) \in \mathrm{L}^2(\Omega)$ and $\int_0^t \|\partial_s f(s)\|\,\mathrm{d}s < \infty$ for all $t \in [0,T]$. Then, for $0 \leqslant r \leqslant 1$, the following statements hold.*

(1) If $u_0 \in \mathrm{L}^2(\Omega)$, then

$$\|u(t)\|_{\dot{\mathrm{H}}^{2r}} \leqslant ct^{(1-r)\alpha}\bigg(t^{-\alpha-(\beta-\alpha)r}\|u_0\| + \|f(0)\| + \int_0^t \|\partial_s f(s)\|\,\mathrm{d}s\bigg). \tag{7.8}$$

(2) If $u_0 \in \dot{\mathrm{H}}^2(\Omega)$, then

$$\|u(t)\|_{\dot{\mathrm{H}}^{2r}(\Omega)} \leqslant ct^{(1-r)\alpha}\bigg(\|u_0\|_{\dot{\mathrm{H}}^2(\Omega)} + \|f(0)\| + \int_0^t \|\partial_s f(s)\|\mathrm{d}s\bigg). \tag{7.9}$$

Theorem 7.3. *Let $u(t)$ and $u_h(t)$ be the solutions to Eq. (7.1) and Eq. (7.5), respectively. If $u_0 \in \mathrm{L}^2(\Omega)$, $f(t)$ is absolutely continuous, $f(0) \in \mathrm{L}^2(\Omega)$ and $\int_0^t \|\partial_s f(s)\|\,\mathrm{d}s < \infty$ for all $t \in [0,T]$, then we have*

$$\|u(t) - u_h(t)\| + h\|\nabla(u(t) - u_h(t))\| \leqslant ch^2\bigg(t^{-\beta}\|u_0\| + \|f(0)\| + \int_0^t \|\partial_s f(s)\|\,\mathrm{d}s\bigg). \tag{7.10}$$

However, for both cases, the error of time–stepping scheme (7.4) is hard to estimate. Specifically speaking, we need get the similar estimations like Eqs. (4.17) and (4.27).

Appendix A. Physical model

Let $B(t)$ be the n-dimensional Brownian motion with infinitesimal generator Δ, and $S(t)$ be the tempered α-stable subordinator. According to Lévy-Khintchine formula [27], there exists

$$\mathbb{E}\Big[\mathrm{e}^{-\lambda S(t)}\Big] = \mathrm{e}^{-t\eta(\lambda)}, \tag{A.1}$$

where

$$\eta(\lambda) = \kappa\lambda + \int_0^\infty \big(1 - \mathrm{e}^{-\lambda t}\big)\sigma(\mathrm{d}t) \tag{A.2}$$

with

$$\sigma(\mathrm{d}t) = \frac{\mathrm{e}^{-\mu t}}{t^{1+\alpha}}\mathrm{d}t \tag{A.3}$$

being a Lévy measure. Here, κ, $\mu \geqslant 0$, and $0 < \alpha < 1$. If $\kappa \neq 0$, we say $S(t)$ is with drift. Denote $E(t)$ as the inverse process of $S(t)$, namely,

$$E(t) = \inf\{u > 0 : S(u) > t\}. \tag{A.4}$$

Here, we consider the subordinated process $B(E(t))$ with B and E being independent. Then the probability density function (PDF) of $B(E(t))$ reads

$$P(x,t) = \int_0^\infty G(x,s)h(s,t)\,\mathrm{d}s, \tag{A.5}$$

where G and h are the PDFs of B and E, respectively. For PDF $G(x,t)$, it solves the diffusion equation

$$\partial_t G(x,t) = \Delta G(x,t). \tag{A.6}$$

As for the PDF h, there exists

$$h(u,t) = -\partial_u \int_0^t g(y,u)\,\mathrm{d}y, \tag{A.7}$$

where g is the PDF of $S(t)$. Taking Laplace transform on the both sides of the above equation with respect to t leads to

$$\widehat{h}(u,\lambda) = -\partial_u \widehat{g}(\lambda,u)\lambda^{-1} = \frac{\kappa\lambda+\phi(\lambda)}{\lambda}\mathrm{e}^{-u(\kappa\lambda+\phi(\lambda))}, \tag{A.8}$$

where $\phi(\lambda) = \eta(\lambda) - \kappa\lambda = (\lambda+\mu)^\alpha - \mu^\alpha$. Consequently, one has

$$\partial_u \widehat{h}(u,\lambda) = -(\kappa\lambda+\phi(\lambda))\widehat{h}(u,\lambda). \tag{A.9}$$

Denoting

$$\partial_t^{\alpha,\mu} u(t) = \frac{\partial}{\partial t}\int_0^t w(t-s)(u(s)-u(0))\,\mathrm{d}s \quad \text{with } \widehat{w}(z) = \phi(z)z^{-1}, \tag{A.10}$$

inverting the above equation, and using $h(u,0) = \delta(u)$ yield

$$-\partial_u h(u,t) = \kappa\partial_t h(u,t) + \partial_t^{\alpha,\mu} h(u,t) + \kappa\delta(u)\delta(t) + w(t)\delta(u). \tag{A.11}$$

Finally, we have

$$\begin{aligned}
&(\kappa\partial_t + \partial_t^{\alpha,\mu})P(x,t)\\
=&-\int_0^\infty G(x,s)(\partial_s h(s,t) - \kappa\delta(s)\delta(t) - w(t)\delta(s))\,\mathrm{d}s\\
=&\int_0^\infty (\partial_s G(x,s))h(s,t)\,\mathrm{d}s\\
=&\Delta P(x,t),
\end{aligned} \tag{A.12}$$

that is,

$$\kappa\partial_t P(x,t) + \partial_t^{\alpha,\mu} P(x,t) = \Delta P(x,t). \tag{A.13}$$

Here, we use diffusion equation (A.6), $h(\infty,t) = 0$ and $h(0,t) = \kappa\delta(t) + w(t)$ because of Eq. (A.8).

Next, we analyze the asymptotic behaviors of the mean-squared displacement (MSD) of $B(E(t))$, being defined as

$$D(t) = \left\langle (B(E(t)))^2 \right\rangle = \int_{\mathbb{R}^n} |x|^2 P(x,t)\,\mathrm{d}x \tag{A.14}$$

with $\langle\cdots\rangle$ denoting the ensemble average. Then,

$$D(t) = \int_0^\infty h(s,t)\int_{\mathbb{R}^n} |x|^2 G(x,s)\,\mathrm{d}x\mathrm{d}s = 2n\langle E(t)\rangle. \tag{A.15}$$

In fact,

$$\widehat{\langle E(t)\rangle}(s) = \int_0^\infty u\widehat{h}(u,s)\,\mathrm{d}u = \frac{1}{s(\kappa s+\phi(s))} \simeq \begin{cases} \left(\kappa+\mu^{\alpha-1}\alpha\right)^{-1}s^{-2}, & s\to 0,\\ \kappa^{-1}s^{-2}, & s\to\infty. \end{cases} \tag{A.16}$$

Finally, using Tauberian theorem leads to

$$D(t) \simeq \begin{cases} 2n\kappa^{-1}t, & t\to 0,\\ 2n\left(\kappa+\mu^{\alpha-1}\alpha\right)^{-1}t, & t\to\infty. \end{cases} \tag{A.17}$$

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