WIRELESS CAPACITY WITH OBLIVIOUS POWER IN GENERAL METRICS

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ABSTRACT. The capacity of a wireless network is the maximum possible amount of simultaneous communication, taking interference into account. Formally, we treat the following problem. Given is a set of links, each a sender-receiver pair located in a metric space, and an assignment of power to the senders. We seek a maximum subset of links that are feasible in the SINR model: namely, the signal received on each link should be larger than the sum of the interferences from the other links. We give a constant-factor approximation that holds for any length-monotone, sub-linear power assignment and any distance metric.

We use this to give essentially tight characterizations of capacity maximization under power control using oblivious power assignments. Specifically, we show that the mean power assignment is optimal for capacity maximization of bi-directional links, and give a tight $\theta(\log n)$ -approximation of scheduling bi-directional links with power control using oblivious power. For uni-directional links we give a nearly optimal $O(\log n + \log\log \Delta)$ -approximation to the power control problem using mean power, where Δ is the ratio of longest and shortest links. Combined, these results clarify significantly the centralized complexity of wireless communication problems.

1. Introduction.

How much communication can occur simultaneously in any given wireless network? This basic question addressed here of the *capacity* of a wireless network has attracted tremendous attention, led by the highly-cited work on non-constructive average-case analysis by Gupta and Kumar [8]. Less is known about algorithmic results on arbitrary instances.

In the problems we consider, we are given a set of links in a metric space. In the uni-directional model, the link ℓ_v represents a directed communication request from a sender s_v to a receiver r_v , each a point in the space. In the bi-directional model, communication is two way, thus both points in the link act as sender and receiver. In either model, a power P_v is associated with each link ℓ_v (which may part of the input, or something the algorithm must find out). When communicating during the same time, two links ℓ_u and ℓ_v interfere with each other. We adopt the physical model or "SINR model" of interference, precisely defined in the Section 2. It is known to capture reality more faithfully than the graph-based models most common in the theory literature, as shown theoretically as well as experimentally [7, 13, 16].

Within this overall framework, a large number of problems can be defined. Even limited to the specific scope of this paper, at least the following dimensions in the problem space can be considered.

First, one can consider *fixed* versus *arbitrary* power. In the fixed version, the power that a link must use is a part of the input, whereas for the arbitrary power (or, "power control") case, choosing the appropriate powers to optimize capacity is a part of the problem. For ease of implementation in distributed settings, research on both fixed and arbitrary power has focused on *oblivious* power assignments, where the power of a link is a (usually simple) function of the length of the link. We study how well oblivious power assignments can handle the power control problem and provide tight characterization (upper and lower bounds) for certain cases.

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Second, there is the *capacity* problem (also known as the *single slot scheduling* problem), of finding the maximum number of links that can communicate simultaneously vs. the *scheduling* problem, which is to partition the links into the smallest number of slots such that each slot can be scheduled at once. We focus mainly on the capacity problem, but study the relationship between capacity and scheduling, specially in the context of the oblivious vs. arbitrary power dichotomy.

Finally, there is the issue of the metric space. The common assumption of the 2-dimensional Euclidean plane is natural, but for wireless transmission it is clearly a grand simplification. Antenna directionality, walls and obstructions, environmental conditions, and terrains all complicate and distort signal transmissions, questioning the geometric assumption. In this work, we are interested in exploring what can be said in the case of completely arbitrary metrics. This relates also to the issue of the path loss exponent α (defined in Section 2), a constant that defines how quickly the signal falls off from its source. Most approximation results have depended on the assumption that $\alpha > 2$ [17, 14, 2, 6, 10], while it is known that α can actually be equal to or smaller than 2 in real networks (see [18], for example).

The aim of our work is to clarify the relationships between the different problem variants, as well as obtaining improved bounds for the fundamental problems.

1.1. Our Contributions. For general metrics, we provide a O(1)-approximation algorithm for the capacity problem for all length-monotone, sub-linear power assignments (these terms are defined in Section 2). In particular, this implies constant factor approximation for uniform, mean-power and linear-power assignments. The results apply for both uni- and bi-directional links. Previously, an O(1)-approximation was known only for uniform power in the plane with $\alpha > 2$ [10].

We use the above results to study the power control problem through the use of oblivious power, specifically, mean power. For bi-directional links, we achieve a O(1)-approximation algorithm improving the previous $O(\log n)$ -approximation for metrics with bounded Assouad dimension (or doubling metrics)[9], where n is the number of links. For uni-directional links, we improve and generalize the $O(\log n \cdot \log \log \Delta)$ -approximation of [9] for doubling metrics to a $O(\log n + \log \log \Delta)$ -approximation, where Δ is the ratio between longest to the shortest linklength. This nearly matches the lower bound of $\Omega(\log \log \Delta)$ for oblivious power assignments [9].

Algorithmic results for capacity imply approximation algorithms for the corresponding scheduling problems with an extra multiplicative $O(\log n)$ factor. In all cases, these improve and/or generalize existing bounds. Among these, the $O(\log n)$ -approximation algorithm for scheduling bi-directional links with power control deserves special mention. It improves on previous poly-logarithmic bounds [12, 3], and we additionally show it to be best possible for any algorithm using oblivious power assignment. This is done by providing an example of links in a metric space that can be scheduled using some power assignment in one slot, yet require $\Omega(\log n)$ slots using oblivious power. Observe that in this construction all links are of equal length, for which uniform power is known to result in a constant-factor approximation in the 2-dimensional setting (with $\alpha > 2$) [9]. Hence, this also implies that when scheduling, general metrics are strictly harder than doubling metrics.

Another feature of our results is that they work for all positive values of the path loss exponent α , generalizing a host of results that required $\alpha > 2$. Finally, noting that the approximation factors we achieve are exponential in α (which is a constant for typical values of α), we prove an inapproximability result showing that in general, exponential dependence on the path loss exponent is necessary.

The techniques used can be viewed as a step in the evolution of SINR analysis. The key addition is to treat both the interference caused as well as interference received. We treat the pairwise affectances as a matrix, and build on sparsity properties. In addition, we extend the key technical lemma – the red-blue lemma – of [6, 10] from the plane to arbitrary distance metrics.

1.2. **Related Work.** Early work on scheduling in the SINR model was focused on heuristics and/or non-algorithmic average-case analysis (e.g. [8]). In contrast, the body of algorithmic work is mostly on graph-based models that ultimately abstract away the nature of wireless communication. The inefficiency of graph-based protocols in the SINR model is well documented and has been shown theoretically as well as experimentally [13, 16].

In seminal work, Moscibroda and Wattenhofer [15] propose the study of the *scheduling complexity* of arbitrary set of wireless links. Early work on approximation algorithms [17, 14, 2] produced approximation factors that grew with structural properties of the network.

The first constant factor approximation algorithm was obtained for capacity problem for uniform power in [6] (see also [10]) in \mathbb{R}^2 with $\alpha > 2$. Fanghänel, Kesselheim and Vöcking [4] gave an algorithm that uses at most $O(OPT + \log^2 n)$ slots for the scheduling problem with *linear* power assignment $P_v = \ell_v^{\alpha}$, that holds in general distance metrics. Recently, Kesselheim and Vöcking [12] gave a distributed algorithm that gives a $O(\log^2 n)$ -approximation for the scheduling problem with any fixed length-monotone and sub-linear power assignment. Our capacity algorithm yields an improved $O(\log n)$ -approximate centralized algorithm.

The situation with power control has been more difficult. Moscibroda, Oswald and Wattenhofer [14] showed that using either uniform or linear power assignment results in only a $\Omega(n)$ -approximation of the capacity and scheduling problems. In terms of Δ , their construction gives a $\Omega(\log \Delta)$ -lower bound. Avin, Lotker and Pignolet [1] show non-constructively that the ratio between the optimal capacity with power control and optimal capacity with uniform power is $O(\log \Delta)$ in \mathbb{R}^1 , for any $\alpha > 0$. Fanghänel et al. [3] showed that any oblivious power scheduling results in a $\Omega(n)$ -lower bound. In terms of Δ , this was shown to be $\Omega(\log \log \Delta)$ in [9]. A $O(\log \log \Delta \cdot \log n)$ -approximation algorithm was given in [9], that holds in any doubling metric where α is strictly greater than the doubling coefficient (generalizing the Euclidean setup). This was shown to lead to $O(\log \log \Delta \cdot \log^3 n)$ -approximation in general distance metrics [5].

Much of the difficulty of the power control scenario has to do with the asymmetry of the communication links. Fanghänel et al. [3] introduced a bi-directional version, where both endpoints of each link are senders and receivers (using the same power). They give an algorithm for arbitrary metric that approximates scheduling within a $O(\log^{3.5+\alpha} n)$ factor. This was improved to a $O(\log n)$ -factor for doubling metrics in [9].

In a very recent breakthrough, Kesselheim obtained a O(1)-approximation algorithm for the capacity problem with power control for doubling metrics and $O(\log n)$ for general metrics [11]. In contrast, in the power control part of our paper, we are interested in what can achieved through locally obtainable oblivious assignments, whereas the power assignment in [11] is necessarily non-oblivious.

2. Model and Preliminaries.

Given is a set $L = \{\ell_1, \ell_2, \dots, \ell_n\}$ of links, where each link ℓ_v represents a communication request from a sender s_v to a receiver r_v . The distance between two points x and y is denoted d(x,y). The asymmetric distance from link ℓ_v to link ℓ_w is the distance from v's sender to w's receiver, denoted $d_{vw} = d(s_v, r_w)$. The length of link ℓ_v is denoted simply ℓ_v . Let Δ denote the ratio between the maximum and minimum length of a link. We assume that each link has a unit-traffic demand, and model the case of non-unit traffic demands by replicating the links.

Let P_v denote the power assigned to link ℓ_v , or, in other words, s_v transmits with power P_v . We assume the path loss radio propagation model for the reception of signals, where the signal received at point y from a sender at point x with power \mathcal{P} is $\mathcal{P}/d(x,y)^{\alpha}$ where the constant $\alpha > 0$ denotes the path loss exponent. We adopt the physical model (or SINR model) of interference, in which a

node r_v successfully receives a message from a sender s_v if and only if the following condition holds:

(1)
$$\frac{P_v/\ell_v^{\alpha}}{\sum_{\ell_w \in S \setminus \{\ell_v\}} P_w/d_{wv}^{\alpha} + N} \ge \beta,$$

where N is a universal constant denoting the ambient noise, $\beta \geq 1$ denotes the minimum SINR (signal-to-interference-noise-ratio) required for a message to be successfully received, and S is the set of concurrently scheduled links in the same *slot*. We say that S is SINR-feasible (or simply feasible) if (1) is satisfied for each link in S.

Let \mathcal{P} -Capacity denote the problem of finding a maximum SINR-feasible set of links under power assignment \mathcal{P} . Let PC-Capacity denote the problem of finding a maximum cardinality set of links and a power assignment that makes these links feasible. Let $\overline{OPT}(L)$ denote the optimal capacity of a linkset L under any power assignment.

The above defines the physical model for uni-directional links. In the bi-directional setting, the asymmetry between senders and receivers disappear. The SINR constraint (1) changes only in that the definition of distance between link changes to

$$d_{wv} = \min(d(s_w, r_v), d(s_v, r_w), d(s_w, s_v), d(r_w, r_v))$$
.

With this new definition of inter-link distances, all other definitions and conditions remain unchanged. For all problems we will consider, when the problem name is prefixed with Bi-, we assume bi-directional communication.

A power assignment P is length-monotone if $P_v \geq P_w$ whenever $\ell_v \geq \ell_w$ and sub-linear if $\frac{P_v}{\ell_v^{\alpha}} \leq \frac{P_w}{\ell_w^{\alpha}}$ whenever $\ell_v \geq \ell_w$.

There are two specific power assignments we will be using throughout the paper. The first is \mathcal{U} , or the uniform power assignment, where every link transmits with the same power. The second is \mathcal{M} , or the mean power (or "square-root") assignment, where P_v is proportional to $\sqrt{\ell_v^{\alpha}}$. It is easy to verify that both \mathcal{U} and \mathcal{M} are length-monotone and sub-linear.

Affectance. We will use the notion of affectance, introduced in [6] and refined in [10], which has a number of technical advantages. The affectance $a_w^P(v)$ of link ℓ_v caused by another link ℓ_w , with a given power assignment P, is the interference of ℓ_w on ℓ_v relative to the power received, or

$$a_w^P(v) = c_v \frac{P_w/d_{wv}^{\alpha}}{P_v/\ell_v^{\alpha}} = c_v \frac{P_w}{P_v} \cdot \left(\frac{\ell_v}{d_{wv}}\right)^{\alpha},$$

where $c_v = \beta/(1 - \beta N \ell_v^{\alpha}/P_v)$ is a constant depending only on the length and power of the link ℓ_v . We will drop P when it is clear from context. Let $a_v(v) = 0$. For a set S of links and a link ℓ_v , let $a_v^P(S) = \sum_{w \in S} a_v^P(w)$ and $a_S^P(v) = \sum_{w \in S} a_w^P(v)$. Using this notation, Eqn. 1 can be rewritten as $a_S^P(v) \leq 1$, and this is the form we will use. This transforms the relatively complicated Eqn. 1 into an inequality involving a simple sum that can be manipulated more easily.

We will frequently consider the affectances arranged as a matrix $\mathbf{A} = (a_{uv})$, where $a_{uv} = a_v(u)$. Note that feasibility of a set S is equivalent to $\sum_v a_{uv} \leq 1$; $\forall u \in S$, where $\mathbf{A} = (a_{uv})$ is the appropriate affectance matrix.

We will require two simple matrix-algebraic results. Before we state them, we need a few definitions. For a matrix \mathbf{M} (vector \mathbf{v}), let $Se(\mathbf{M})$ ($Se(\mathbf{v})$) denote the sum of entries in the matrix (vector). We say that a matrix $\mathbf{M} = (m_{ij})$ is q-approximately-symmetric if, for any two indices i and j, $m_{ij} \leq q \cdot m_{ji}$ for some $q \geq 0$. For a matrix \mathbf{M} and vectors \mathbf{u} and \mathbf{v} , we will use the notation $\mathbf{M} \cdot \mathbf{u} \leq \mathbf{v}$ to mean $(\mathbf{M} \cdot \mathbf{u})_i \leq \mathbf{v}_i$ for all i.

The proof of the following two claims can be found in the appendix (Section A.1).

Claim 1. Let M be a $n \times n$ matrix that contains only non-negative entries. Assume $Se(\mathbf{M}) \leq \gamma n$ for some $\gamma \geq 0$. Then, for any $\lambda > 1$, there are at least $(1 - \frac{1}{\lambda})n$ rows (columns) \mathbf{r} of \mathbf{M} for which $Se(\mathbf{r}) \leq \gamma \cdot \lambda$.

Lemma 2. Let n be a number and \mathbf{p} be a positive n-dimensional vector. Let \mathbf{M} be a non-negative q-approximately-symmetric n-by-n matrix such that $\mathbf{Mp} \leq \mathbf{p}$. Then, $Se(\mathbf{M}) \leq (q+1)n$.

Signal-strength and robustness. A δ -signal set is one where the affectance of any link is at most $1/\delta$. A set is SINR-feasible iff it is a 1-signal set. Let $OPT_{\delta}^{P} = OPT_{\delta}^{P}(L)$ be the maximum number of links in a δ -signal set given a power assignment P (and let $OPT_{\delta}^{P,B} = OPT_{\delta}^{P,B}(L)$ be the corresponding maximum for bi-directional links). We drop δ , B and P when understood from context.

The following results are essential.

Proposition 2.1 ([3],Prop. 3). Let L be set of links and P be a power assignment for L. Then, for any $\tau \leq \delta$, $OPT_{\delta}^{P}(L) \geq \frac{OPT_{\tau}^{P}(L)}{4\delta/\tau}$ and $OPT_{\delta}^{P,B}(L) \geq \frac{OPT_{\tau}^{P,B}(L)}{8\delta/\tau}$. This is constructive.

Lemma 3 ([9]). Let ℓ_u, ℓ_v be links in a q^{α} -signal set. Then, $d_{uv} \cdot d_{vu} \geq q^2 \cdot \ell_u \ell_v$.

3. Fixed Power in General Metrics.

In this section, we will deal with the \mathcal{P} -Capacity problem for fixed power assignments. The most widely studied fixed power assignments are oblivious, where the power assigned to a link depends only on the length of the link. Our results are not restricted to oblivious power, but we will see that they do apply for popular oblivious assignments considered in the literature, such as \mathcal{U}, \mathcal{M} and linear power (where $P_v = \ell_v^{\alpha}$).

We use the simple greedy algorithm C described in Fig. 1 for \mathcal{P} -Capacity. Algorithm C is closely related to Algorithm A(c) in [10], differing primarily in that it considers also the outgoing affectances from the given link ℓ_v . Also, the last line was added to ensure feasibility in general metrics; as shown in [10], it is not needed in \mathbb{R}^2 when $\alpha > 2$.

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Algorithm \mathbf{C}(\mathsf{L},\mathsf{P})

Suppose the links L = \{\ell_1,\ell_2,\dots,\ell_n\}

are in non-decreasing order of length.

Let \gamma = 1/2

S \leftarrow \emptyset

for v \leftarrow 1 to n do

if a_S^P(v) + a_v^P(S) < \gamma

then add \ell_v to S

Output X = \{\ell_v \in S : a_S^P(v) \leq 1\}
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FIGURE 1. Algorithm for the capacity problem

3.1. Constant Factor Approximation for \mathcal{P} -Capacity. We will show the following.

Theorem 4. Algorithm C approximates P-Capacity within a constant factor for any length-monotone, sub-linear P.

The feasibility of the solution output by C is evident by virtue of the last line of the algorithm. We proceed to show that it gives constant-factor approximation. We first compare the final output to the set S.

Lemma 5. Let S and X be the sets created by the algorithm C on a given link set. Then, $|X| \ge |S|(1-\gamma)$.

Proof. Observe that by the algorithm construction, the sum $Se(\mathbf{A}) = \sum_{v,w \in S} a_v(w) = \sum_{v \in S} a_S(v)$ is upper bounded by $\gamma |S|$. Then the Lemma follows from Claim 1 by plugging in $\lambda = \frac{1}{\gamma}$.

The main task is to relate the size of the set S to the optimal solution. For this, we extend the "blue dominant centers" lemma of [6] (and "blue-shadowed lemma" of [10]) to arbitrary metric spaces and length-monotone sub-linear power assignments.

For a set S and link ℓ_v , let $S_v^- = \{\ell_w \in S : \ell_w \leq \ell_v\}$ and $S_v^+ = \{\ell_w \in S : \ell_w \geq \ell_v\}$.

Lemma 6 (Red-Blue Lemma). Let R and B be disjoint sets of links, referred to as red and blue links in a length-monotone, sub-linear power metric. If |B| > 4|R| and B is a 3^{α} -signal set, then there is a blue link ℓ_b such that $a_{R_{\bullet}^{-}}(b) + a_b(R_b^{-}) \leq 3^{\alpha}(a_B(b) + a_b(B))$.

This lemma is a consequence of the following two symmetric lemmas.

Lemma 7. Let R and B be disjoint sets of links, referred to as red and blue links in a length-monotone power metric. If |B| > 2|R| and B is a 3^{α} -signal set, then there is a set $B' \subseteq B$ of size at least |B| - 2|R| such that for all $\ell_b \in B'$, $a_{R_b^-}(b) \le 3^{\alpha}a_B(b)$.

Lemma 8. Let R and B be disjoint sets of links, referred to as red and blue links in a sub-linear power metric. If |B| > 2|R| and B is a 3^{α} -signal set, then there is a set $B' \subseteq B$ of size at least |B| - 2|R| such that for all $\ell_b \in B'$, $a_b(R_b^-) \leq 3^{\alpha}a_b(B)$.

We prove Lemma 7 below and relegate proof of the very similar Lemma 8 to the appendix.

Proof of Lemma 7. For each link $\ell_r \in R$, we will assign a set of "guards" $X_r \subseteq B$. For different ℓ_r , the X_r 's will be disjoint and each X_r will be of size at most 2. We claim that each link ℓ_b that remains in B after removing all the guard sets satisfies the lemma.

Here's how we choose the guards. We process the links in R in an arbitrary order. Initially set $B' \leftarrow B$. For each link $\ell_r \in R$, add to X_r the link $\ell_x \in B_r^+$ with the sender nearest to s_r (if one exists); also add the link $\ell_y \in B_r^+$ with the receiver nearest to s_r (if one exists; possibly, $\ell_y = \ell_x$); finally, remove the links in X_r from B' and repeat the loop.

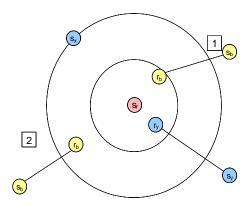


FIGURE 2. Example configuration of guards and points. Guard s_x (on left) is the nearest blue sender to red point s_r (in center) among those of links larger than ℓ_r .

Since |B| > 2|R|, $|B'| \ge |B| - 2|R| > 0$, by construction. Consider any link $\ell_b \in B'$. In what follows, we will show that the affectance of any link $l_r \in R_b^-$ on ℓ_b is comparable to the affectance of one of the guards of l_r on ℓ_b . Once we recall that the guards are blue, this implies that the overall affectance on ℓ_b from R_b^- is not much worse than that from B.

Since $\ell_b \notin X_r$, X_r is non-empty and contains guards ℓ_x and ℓ_y (potentially $\ell_x = l_y$). Consider any $\ell_r \in R_b^-$. Let d denote $d(s_x, s_r)$ and observe that since $\ell_b \notin X_r$, $d(s_b, s_r) \geq d$.

We claim that $d_{rb} \geq d/2$. Before proving the claim, let us see how this leads to the proof of the Lemma. If the claim is true, then by the triangular inequality, $d_{xb} = d(s_x, r_b) \leq d(s_x, s_r) +$ $d(s_r,r_b) = d + d_{rb} \leq 3d_{rb} \text{ . Since } \ell_x \geq \ell_r \text{ and the power metric is length-monotone, } P_x \geq P_r.$ $\text{Thus, } \frac{a_x(b)}{a_r(b)} = \frac{P_x}{P_r} \left(\frac{d_{rb}}{d_{xb}}\right)^{\alpha} \geq 3^{-\alpha} \text{ . Summing over all links in } B, \ a_B(b) \geq \sum_{\ell_r \in R_b^-} a_{X_r}(b) \geq 3^{-\alpha} \sum_{\ell_r \in R_b^-} a_r(b) = 3^{-\alpha} a_{R_b^-}(b) \text{ .}$

To prove the claim that $d_{rb} \geq d/2$, let us suppose otherwise for contradiction. Then, by the triangular inequality, $\ell_b = d(s_b, r_b) \ge d(s_b, s_r) - d(r_b, s_r) > d - d/2 = d/2$. Since ℓ_y was chosen into X_r , its receiver is at least as close to s_r as r_b , that is $d(r_y, s_r) \le d(r_b, s_r) < d/2$, and its sender is also at least far as s_x , or $d(s_y, s_r) \geq d$. So, $\ell_y \geq d(s_y, s_r) - d(s_r, r_y) > d/2$. Now, $d(r_y, r_b) \leq d/2$ $d(r_y, s_r) + d(s_r, r_b) < d \text{ and } d_{yb} \cdot d_{by} \le (\ell_y + d(r_y, r_b)) \cdot (\ell_b + d(r_y, r_b)) < (\ell_y + d) \cdot (\ell_b + d) < 9 \cdot \ell_y \ell_b.$ But since B is a 3^{α} -signal set, this is a contradiction by Lemma 3. Hence, any link ℓ_r in R_b^- satisfies $d(s_r, r_b) = d_{rb} \ge d/2.$

Lemma 9. Let S be the set found by the algorithm and $\tau = 3^{\alpha+1}/2\gamma$. Then, $|S| \geq OPT_{2\tau}/10$.

Proof. By Claim 1, there is a set $O \subseteq OPT_{2\tau}$ of size at least $OPT_{2\tau}/2$ such that for all $u \in O$,

 $a_u(O) \leq \frac{1}{\tau}$. By definition, $a_O(u) \leq \frac{1}{2\tau}$. We claim that $|S| \geq |O|/5$. Suppose otherwise. Then, $|S| < |O \setminus S|/4$. Applying Lemma 6 with $B = O \setminus S$ and R = S, we find that there is a link ℓ_b in $O \setminus S$ that satisfies $a_{S_b^-}(b) + a_b(S_b^-) \leq 1$ $3^{\alpha}(a_B(b)+a_b(B)) \leq 3^{\alpha}(\frac{1}{\tau}+\frac{1}{2\tau}) = \gamma$. The operation of the algorithm is then such that the algorithm would have added the link ℓ_b to S, which is a contradiction.

The proof of Thm. 4 is now straightforward.

Proof. We can bound $|X| \ge \frac{|S|}{2} \ge \frac{OPT_{2\tau}}{20} = \Omega(OPT)$. The first inequality is by Lemma 5, the second by Lemma 9, and the last equality is a consequence of Proposition 2.1.

We note that our algorithm applies equally in the **bi-directional** setting.

3.2. The Necessity of Exponential Dependence on α . In all results achieved in prior work as well as this work, the approximation factor depends exponentially on α . Here we show a simple approximation preserving reduction from the maximum independent set problem to prove that this dependence is necessary in general. However, this does not rule out better dependence on α for special cases, say for small values of α on the plane.

Theorem 10. For $\alpha \geq \lg n + 1$ and any fixed $\epsilon > 0$, there can be no polynomial time algorithm that approximates U-Capacity in general metrics to within a factor better than $\Omega(2^{(1-\epsilon)\alpha})$, unless P = NP.

Proof. By reduction from the maximum independent set problem in graphs. Given graph G(V, E), form a link ℓ_v for each vertex $v \in V$, with $\ell_v = 1$. If $\{u, v\} \in E$, set $d(s_v, r_u) = d(s_u, r_v) = 1$, while if $\{u,v\} \notin E$, set $d(s_v,r_u)=d(s_u,r_v)=2$. It is easy to show that the triangle inequality holds. Set assume N=0 and $\beta=1.5$. Any set of vertices $S\subseteq V$ is an independent set if and only if the corresponding set of links are SINR-feasible. To see this, first consider a set of vertices that contains at least one edge $\{u,v\}$. Then $a_S(v) \geq a_u(v) = \beta(\frac{1}{1})^{\alpha} > 1$. So, S is not feasible. On the other hand, if S is an independent set, then for all $v \in S$,

$$a_S(v) = \sum_{u \in S} a_u(v) = |S|\beta \left(\frac{1}{2}\right)^{\alpha} \le \frac{\beta}{2} \cdot |S| \left(\frac{1}{2}\right)^{\lg n} < 1.$$

The reduction is clearly approximation preserving. Since independent set is $\Omega(n^{1-\epsilon})$ -inapproximable unless P=NP (see [19]) for any fixed $\epsilon > 0$, the capacity problem is hard to approximate within a factor of $\Omega(n^{1-\epsilon}) = \Omega(2^{(1-\epsilon)\alpha})$.

4. Approximating Power Control Using Oblivious Power.

We will prove the following results.

Theorem 11. There is a $O(\log \log \Delta + \log n)$ -approximation algorithm for PC-Capacity in general metrics that uses the mean power assignment \mathcal{M} .

Theorem 12. There is a O(1)-approximation algorithm for Bi-PC-Capacity in general metrics that uses the mean power assignment \mathcal{M} .

Thus for bi-directional links, we find that mean power is essentially the best possible. For unidirectional links, oblivious power is known to be sub-optimal, but we achieve nearly tight bounds in light of lower bounds in [9].

For simplicity we assume $\beta = 1$. We shall also assume that the noise N is negligible, which is acceptable in the power control case since we can scale the powers to make the noise arbitrarily small. The following simple lemma makes this rigorous.

For any oblivious power assignment P and any number $\mathbf{s} > 0$ define the assignment $P^{\mathbf{s}}$ as $P_v^{\mathbf{s}} = \mathbf{s}P_v$. That is, $P^{\mathbf{s}}$ is a simple linear scaling of P.

Lemma 13. Assume a given link set L and ambient noise N. Then for any oblivious power assignment P, there exists a number \mathbf{s} such that the assignment $P^{\mathbf{s}}$ has the following property: If a set $S \in L$ is feasible using $P^{\mathbf{s}}$ assuming zero noise, then there is a set S' of size $\Omega(S)$ which is feasible assuming ambient noise N. Also, such an \mathbf{s} can be found efficiently.

Proof. We will be comparing the case where the noise is zero versus the case where the noise is N. We will use c_v^0 , $a_S^0(v)$, $a_u^0(V)$ etc. to denote the value of c_v and affectances in the zero noise case, and the un-superscripted versions $(c_v, a_S(v), a_u(v))$ etc.) to denote the noisy case.

First we claim that we can choose **s** such that $c_v \leq 2$ for all ℓ_v . Recall that $c_v = \beta/(1-\beta N\ell_v^{\alpha}/P_v)$ depends only on P_v . Thus we can easily choose **s** such that such that $N\ell_v^{\alpha}/P_v^{\mathbf{s}} \leq \frac{2}{\beta}$ for all ℓ_v . Thus $c_v \leq 2\beta \leq 2$ (since we have assumed $\beta = 1$).

On the other hand, if the noise level is zero then $c_v^0 = \beta = 1$. Consider a set S that is feasible under $P^{\mathbf{s}}$ assuming zero noise. Then $a_S^0(v) \leq 1$ for all ℓ_v . Now,

$$a_S(v) = \sum_{u \in S} a_u(v) = c_v \sum_{u \in S} \frac{P_u^s}{P_v^s} \left(\frac{\ell_v}{d_{uv}}\right)^{\alpha}$$

$$\leq 2c_v^0 \sum_{u \in S} \frac{P_u^s}{P_v^s} \left(\frac{\ell_v}{d_{uv}}\right)^{\alpha} = 2\sum_{u \in S} a_u^0(v) = 2a_S^0(v)$$

$$\leq 2.$$

The statement of the Lemma now follows from Prop. 2.1.

4.1. Uni-directional Capacity with Power Control via Oblivious Power. In [9], Halldórsson achieved a $O(\log n \cdot \log \log \Delta)$ approximation factor for uni-directional capacity with power control for doubling metrics via \mathcal{M} . Using tools developed in this paper, the same approximation can be achieved for general metrics in a straightforward manner. We additionally improve the bound to $O(\log n + \log \log \Delta)$.

First we need an weaker result on a class of power assignments we call Lipschitz power assignments.

Definition 14. A power assignment \mathcal{P} is Lipschitz if there is some constant c such that $\mathcal{P}(\ell_v) \leq c \cdot \mathcal{P}(\ell_u)$, for any two links ℓ_u, ℓ_v with $\ell_u/2 \leq \ell_v \leq 2\ell_u$.

We prove the following.

Theorem 15. Let \mathcal{P} be any Lipschitz power assignment. Then, for any linkset L, $\overline{OPT}(L) = O(\log \Delta)OPT_{\mathcal{P}}(L)$.

For Lipschitz power assignments, the above result is the best possible, given the matching lower bound for both uniform and linear power [14, 1].

Before proving Thm. 11, we need a weaker result for Lipschitz power assignments.

Theorem 16. A set of links L is said to contain nearly equilength links if for all $u, v \in L$, $\frac{\ell_v}{2} \le \ell_u \le 2\ell_v$. Let \mathcal{P} be any Lipschitz power assignment. Then, for any linkset L of nearly equilength links, $\overline{OPT}(L) = O(OPT_{\mathcal{P}}(L))$.

Proof. First note that for nearly equilength links, the affectance for any Lipschitz assignment \mathcal{P} varies from that of uniform power \mathcal{U} by no more than a constant factor. Thus by the signal strengthening property (Proposition 2.1) we safely assume $\mathcal{P} = \mathcal{U}$.

Let S be an optimal 6^{α} -signal subset of L. Let P be a power assignment for which S is a 6^{α} -signal set. Let $a_v^{\mathcal{U}}(u) = (\ell_u/d_{vu})^{\alpha}$ be the affectance of link ℓ_v on ℓ_u under uniform power. Then, $a_v^P(u) = a_v^{\mathcal{U}}(u)P_v/P_u$ is the affectance under P. Viewed as a matrix $\mathbf{A} = (a_{uv})$ where $a_{uv} = a_v^{\mathcal{U}}(u)$; $\forall u, v \in S$, feasibility of S using P is equivalent to $\mathbf{Ap} \leq \mathbf{p}$ where $\mathbf{p} = [P_1, P_2 \dots]^T$.

Then, $u_v(u) = u_v(u) I_v I_u$ is the affectance under I. Viewed as a matrix $I = (a_{uv})$ where $a_{uv} = a_v^{\mathcal{U}}(u)$; $\forall u, v \in S$, feasibility of S using P is equivalent to $\mathbf{Ap} \leq \mathbf{p}$ where $\mathbf{p} = [P_1, P_2, \dots]^T$. We claim that $d_{uv} \leq 2d_{vu}$ for all u, v. To show this assume $P_u \leq P_v$ and we will prove the inequality in both directions. Since $\frac{P_v}{P_u} \left(\frac{\ell_u}{d_{vu}}\right)^{\alpha} \leq 6^{-\alpha}$, we get $d_{vu} \geq 6\ell_u \geq 2(\ell_u + \ell_v)$. By the triangular inequality $d_{uv} \geq d_{vu} - (\ell_u + \ell_v) \geq d_{vu}/2$ and $d_{vu} \geq \max(2(\ell_u + \ell_v), d_{uv} - (\ell_u + \ell_v)) \geq 2d_{uv}/3$.

Then, $a_u^{\mathcal{U}}(v) = (\ell_v/d_{uv})^{\alpha} \leq (2\ell_u/\frac{1}{2}d_{vu})^{\alpha} \leq 4^{\alpha}a_v^{\mathcal{U}}(u)$. Hence, **A** is 4^{α} -nearly symmetric. By Lemma 2, the set S' of links with affectance at most $2 \cdot 4^{\alpha}$ is of size at least |S|/2. By signal strengthening (Prop. 2.1), applied both to S' and S, there is a 1-signal subset S'' in S' of size at least $|S'|/(8 \cdot 4^{\alpha})$, while $S \geq \overline{OPT}/(4 \cdot 6^{\alpha})$.

Thm. 16 can be used to achieve Thm. 15 (a generalization of a result from [9] to arbitrary distance metrics) in a straightforward manner.

Proof of Thm. 15. Given an arbitrary linkset, we can divide it into $\log \Delta$ nearly-equilength groups. Let these groups be L_i for $i = 1 \dots \log \Delta$. Then by Thm. 16, $\overline{OPT}(L_i) = O(OPT_{\mathcal{P}}(L_i))$. Let, $r = \arg \max \overline{OPT}(L_r)$. Now,

$$\overline{OPT}(L) \le \log \Delta \cdot \overline{OPT}(L_r)$$

$$= O(\log \Delta \cdot OPT_{\mathcal{P}}(L_r)) = O(\log \Delta \cdot OPT_{\mathcal{P}}(L)) .$$

We now combine the use of mean power \mathcal{M} with Thm. 15 to get an algorithm with better approximation ratio in terms of the dependence on Δ . To to do this, we modify ideas from [9]. A key idea in [9] was to partition the input links into length based classes. We adopt this idea but employ a more efficient partitioning of the input links and combine it with Thm. 15 for our result.

For this we will need the following lemma, a slight variation of Lemma 4.2 in [9].

Lemma 17 ([9]). Let Q be a set of links that are SINR-feasible using some power assignment, and let ℓ_v be a link that is shorter than the links in Q by a factor of at least $n^{2/\alpha}$. Suppose $\max(a_v^{\mathcal{M}}(w), a_w^{\mathcal{M}}(v)) \geq \frac{1}{2n}; \forall \ell_w \in Q$. Then $|Q| = O(\log \log \Delta)$.

Partitioning the input into levels. For an input set L, assume the shortest link has length ℓ_{\min} . Define non-overlapping sets, $L_k = \{u \in L : \ell_{\min} D^{k+1} > \ell_u \geq \ell_{\min} D^k\}$ for all $k \geq 1$, where $D = 8n^{2/\alpha}$. Let us call these sets levels. We partition L into M_o and M_e , where $M_o = \bigcup_{k \text{ is odd}} L_k$ and $M_e = \bigcup_{k \text{ is even}} L_k$. M_o (and M_e) is a union of levels such that intra-level distances vary by no more than D while inter-level distances vary by at least D. In what follows, we will present an $O(\log n + \log\log \Delta)$ -approximation algorithm that takes as input such a set. Since, $L = M_o \cup M_e$, we can claim the same approximation factor for any input set with additional multiplicative factor of 2.

Algorithm and analysis. We will adopt the convention that the levels are sorted in increasing order, thus if r > s, links in L_r are longer than links in L_s .

```
\begin{split} \mathbf{PC}(M_x &= \cup_{r \geq 0} L_r) \\ S &\leftarrow \emptyset \\ \text{for } r \leftarrow 0 \dots \text{ do} \\ \text{Divide } L_r &= \cup Q_p \text{ such that lengths} \\ \text{ of links within } Q_p \text{ varies by a} \\ \text{ factor of at most } 2 \\ \text{ for each } Q_p \text{ do} \\ S_p &\leftarrow \emptyset \\ \text{ for } \ell_v \in Q_p \text{ in increasing order do} \\ \text{ add } \ell_v \text{ to } S_p \text{ if both:} \\ 1. \ a_{S_p}(v) + a_v(S_p) < \gamma, \text{ and} \\ 2. \ S \text{ contains no link } \ell_w \text{ s.t.} \\ \text{ max } (a_v^{\mathcal{M}}(w), a_w^{\mathcal{M}}(v)) \geq \frac{1}{2n} \\ q \leftarrow \arg_p \max |S_p| \\ \text{ Set } S \leftarrow S \cup S_q \\ \text{ Output } X = \{\ell_v \in S : a_S(v) \leq 1\} \end{split}
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FIGURE 3. Algorithm for PC-capacity

Lemma 18. Consider the set S selected by PC(L). Then $S = \Omega(\frac{\overline{OPT}}{\log n + \log \log \Delta})$.

Proof. We partition both \overline{OPT} and S into levels: $\overline{OPT} = \cup_r O_r$ where $O_r = \overline{OPT} \cap L_r$ and $S = \cup_r S_r$ where $S_r = S \cap L_r$. Also let $o_r = |O_r|, s_r = |S_r|$. We aim to prove $S = \Omega(\frac{\overline{OPT}}{\log n + \log \log \Delta})$ or equivalently $\sum_r s_r = \Omega(\frac{1}{\log n + \log \log \Delta}) \sum_r o_r$.

Now let us define $T_{qr} \subseteq O_r$ for r > q as the set of links in O_r that fail condition 2 due to links selected from level q < r. In other words,

$$T_{qr} = \{ w \in O_r : \max(a_v^{\mathcal{M}}(w), a_w^{\mathcal{M}}(v)) \ge \frac{1}{2n},$$

for some $v \in S \cap L_q \}$.

Let $t_{qr} = |T_{qr}|$. By Lemma 17,

(2)
$$\sum_{r>a} t_{qr} \le c_1 s_q \log \log \Delta ,$$

for some constant c_1 .

With this in mind let us investigate what **PC** does for a single L_r . We find that **PC** runs on L_r the algorithm suggested by Thm. 15. This is done by partitioning the links in to nearly equilength sets Q_p , running algorithm **C** separately on each Q_p and choosing the largest feasible set found. The

only difference is due to condition 2, in other words, the $O(\log \Delta)$ approximation factor achieved is not in relation to O_r but rather $O_r \setminus \sum_{q < r} T_{qr}$. Since for any L_r , $\log \Delta = \theta(\log n)$ by construction, we can claim that

$$s_r \ge \frac{1}{c \log n} (o_r - \sum_{q < r} t_{qr}) ,$$

for some constant c. Using Eqn. 2, this implies that

$$c \log n \sum_{r} s_r \ge \sum_{r} o_r - \sum_{r} \sum_{q < r} t_{qr}$$

$$= \sum_{r} o_r - \sum_{q} \sum_{r > q} t_{qr}$$

$$\ge \sum_{r} o_r - \sum_{q} c_1 s_q \log \log \Delta ,$$

which yields that $(c \log n + c_1 \log \log \Delta)S \ge \overline{OPT}$, as desired.

To complete the proof of Thm. 11, all we need is to relate X to S. For S, the average affectance from links at the same level is bounded by construction. Additionally, also by the design of the algorithm, one can observe that the total affectance from links at other levels is at most $\frac{1}{2n} \times n = \frac{1}{2}$. Now arguing that X is within a constant factor of S takes the same route we have already seen via Lemma 1 and Proposition 2.1.

4.2. Bi-Directional Capacity with Power Control. Let $OPT_P^B(L)$ denote the optimal bi-directional capacity with respect to power assignment P and $\overline{OPT}^B(L)$ the optimal bi-directional capacity with respect to any power assignment.

We show that \mathcal{M} is optimal for bi-directional capacity modulo constant factors.

Theorem 19. For any linkset L, $OPT_{\mathcal{M}}^{B}(L) = O(\overline{OPT}^{B}(L))$.

Proof. Let P be a power assignment with which the optimal capacity $\overline{OPT}^B(L)$ is attained, and let S be the feasible subset in L of cardinality $\overline{OPT}^B(L) = OPT_P^B(L)$. Let P' be a function defined by $P(v) = P'(v) \cdot \mathcal{M}(\ell_v) = P'(v) \cdot \sqrt{\ell_v^{\alpha}}$. Recall that in the bi-directional setting, $d_{uv} = d_{vu}$, and affectance under \mathcal{M} is symmetric and given by $a_v^{\mathcal{M}}(w) = \left(\frac{\sqrt{\ell_v \ell_w}}{d_{vw}}\right)^{\alpha} = a_w^{\mathcal{M}}(v)$. Observe that

$$a_v^P(u) = \frac{P(v)}{P(u)} \cdot \left(\frac{\ell_u}{d_{vu}}\right)^{\alpha}$$
$$= \frac{P'(v)\ell_v^{\alpha/2}}{P'(u)\ell_u^{\alpha/2}} \cdot \left(\frac{\ell_u}{d_{vu}}\right)^{\alpha}$$
$$= \frac{P'(v)}{P'(u)} \cdot a_v^{\mathcal{M}}(u) .$$

Since S is feasible w.r.t. P, it holds that for each link ℓ_u in S, $\sum_{\ell_v \in S} a_v^P(u) \leq 1$, or $\sum_{\ell_v \in S} P'(v) a_v^{\mathcal{M}}(u) \leq P'(u)$. Thus, if $\mathbf{A}' = (a'_{uv})$ where $a'_{uv} = a_v^{\mathcal{M}}(u)$ is the affectance matrix for \mathcal{M} , we have that $\mathbf{A}'\mathbf{p}' \leq \mathbf{p}'$, where \mathbf{p}' is the vector of the values P'(u) for all u. Since \mathbf{A}' is symmetric, $Se(\mathbf{A}') \leq 2n$, by Lemma 2, and by Claim 1 there is a subset S' of S with at least |S|/2 links such that $\sum_{v \in S'} a'_{uv} \leq 4$ for each link $\ell_u \in S'$. Namely, for each link $\ell_u \in S'$,

$$a_{S'}^{\mathcal{M}}(u) = \sum_{v \in S'} a_v^{\mathcal{M}}(u) = \sum_{v \in S'} a'_{uv} \le 4.$$

Hence, S' is a 1/4-signal set under \mathcal{M} . Finally, by signal-strengthening, there is a 1-signal subset S'' in S' of size $|S'|/32 \ge |S|/64 = \Omega(\overline{OPT}^B(L))$.

In light of the fact that C(L) is a constant factor algorithm for \mathcal{M} by Thm. 4 (which works for bi-directional links as well), Thm. 12 is a simple consequence of Thm. 19.

5. Scheduling vs. Capacity.

Our capacity results immediately imply results for the corresponding scheduling problems, within a logarithmic factor. We state these below. Recall that the scheduling problem is to the capacity problem as what the graph coloring problem is to the independent set problem.

Corollary 20. There is an $O(\log n)$ -approximation algorithm for link scheduling under any fixed length-monotone, sub-linear power assignment \mathcal{P} . Further, for scheduling with power control, by using square-root power, we get both $O(\log \Delta \log n)$ - and $O((\log \log \Delta + \log n) \log n)$ -approximation for uni-directional case and $O(\log n)$ -approximation for bi-directional case. All results apply in general distance metrics and for all fixed $\alpha > 0$.

The relationship between capacity and scheduling has remained an open question for some time. For instance, while a constant factor approximation for capacity with uniform power (in *doubling* metrics) was obtained in [6], a corresponding result for scheduling has been elusive, with a faulty claim in [10]. We show here that scheduling can indeed be more difficult by a logarithmic factor.

5.1. **A Lower Bound.** In this section we provide bounds on how well oblivious power assignments can do in a general metric space. Lower bounds of this type have been proved previously for unidirectional links on the line [3, 9], giving $\Omega(n)$ and $\Omega(\log \log \Delta)$ lower bounds. However, these arguments don't say anything about bi-directional scheduling.

We show a construction of equilength links that are feasible with some power assignment, but using oblivious power, one needs $\Omega(\log n)$ slots to schedule the links. The reader may recall that for the capacity problem, in contrast, O(1)-approximation is possible for the bi-directional case via oblivious power (Thm. 19). Since the links have equal lengths, the only possible oblivious assignment is \mathcal{U} . The construction works for both uni-directional and bi-directional cases.

We start with a single link ℓ_1 and define the other links in the link set L in relation to ℓ_1 . For some suitably large K, define sets S_k of links for k=1...K, such that $|S_k|=2^{2(k-1)}$, and for all $i \in S_k$, $d_{i1}=(\gamma|S_k|)^{\frac{1}{\alpha}}$, where γ is a fixed constant to be defined later. Now, $L=\{\ell_1\}\cup(\cup_k S_k)$, and the total number of links $n=|L|=1+\sum_{k=1...K}|S_k|$. It is easy to see that

(3)
$$K = \theta(\log n) .$$

Furthermore, let $\ell_i = 1, \forall i$ and $d_{ij} = d_{ji} = d_{i1} + d_{j1}, \forall i, j \neq 1$. All other distances are defined by transitivity. Let N = 0 and $\beta = 1$.

Theorem 21. There exists a power-assignment P such that L is SINR-feasible using P, whereas L cannot be partitioned into $o(\log n)$ SINR-feasible sets using uniform power U.

The following two technical lemmas imply the two claims in Thm. 21. Let the $n \times n$ matrix $\mathbf{A} = a_{ij}$ be defined by entries $a_{ii} = 0$ and $a_{ij} = a_j^{\mathcal{U}}(i)$, and the vector \mathbf{p} be defined by $\mathbf{p}_i = P_i$.

Lemma 22. There exists a power assignment P such that $(\mathbf{Ap})_i \leq \mathbf{p}_i, \forall i \in L$.

Lemma 23. Take any partition of L into $r = o(\log n)$ sets Q_i (i = 1...r). Then there exists a link j in some Q_i such that $\sum_{\hat{j} \in Q_i} a_{j\hat{j}} > 1$, thus proving that Q_i is not feasible using \mathcal{U} .

Recalling that SINR condition is equivalent to $a_L^P(i) = \frac{1}{\mathbf{p}_i} \sum_{j \neq i} a_{ij} \mathbf{p}_j \leq 1$, we see that Lemma 22 shows that the links are SINR-feasible with power assignment P.

Proof of Lemma 22. We will use the power assignment P defined by, $P_1 = 1$ and $P_i = \frac{1}{2^k}, \forall i \in S_k$. Note that $a_{ij} = \left(\frac{\ell_i}{d_{ji}}\right)^{\alpha} = \frac{1}{d_{ji}^{\alpha}}$. For link ℓ_1 ,

$$(\mathbf{Ap})_{1} = \sum_{i \in L} a_{1i} P_{i} = \sum_{k} \sum_{i \in S_{k}} P_{i} a_{1i}$$
$$= \sum_{k} \sum_{i \in S_{k}} \frac{1}{2^{k}} \frac{1}{\gamma |S_{k}|} = \sum_{k} \frac{1}{2^{k}} \frac{1}{\gamma} \le \frac{1}{\gamma} \le 1 = P_{1},$$

for $\gamma \geq 1$. The first equality is simply the vector multiplication written out explicitly, the second equality divides the sum into the sums for sets S_k . The third equality uses the explicit values of the power assignment and $a_{1j} = \frac{1}{\gamma |S_k|}$ for $j \in S_k$. The first inequality follows from the fact that $\sum_{k>1} \frac{1}{2^k} = 1$.

Now consider $j \in S_{\tilde{k}}$ for some \tilde{k} . We write

(4)
$$(\mathbf{Ap})_j = \sum_{k > \tilde{k}} \sum_{i \in S_k} P_i a_{ji} + \sum_{k \leq \tilde{k}} \sum_{i \in S_k} P_i a_{ji} + P_1 a_{j1}$$

and bound these three terms separately. The last term is $P_1 a_{j1} = \frac{1}{\gamma |S_k|} = \frac{1}{\gamma 2^{2(k-1)}} \leq \frac{4}{\gamma 2^k} = \frac{4P_j}{\gamma}$.

Recall that if $k > \tilde{k}$, then $d_{ij} \geq d_{i1} \geq (\gamma |S_k|)^{1/\alpha}$ for $i \in S_k$, while if $k \leq \tilde{k}$, then $d_{ij} \geq (\gamma |S_{\tilde{k}}|)^{1/\alpha}$ for $i \in S_k$. The first term in Eqn. 4 is bounded by

$$\sum_{k>\tilde{k}} \sum_{i \in S_k} P_i a_{ji} \le \sum_{k>\tilde{k}} \sum_{i \in S_k} \frac{1}{2^k} \frac{1}{\gamma |S_k|} = \sum_{k>\tilde{k}} \frac{1}{2^k} \frac{1}{\gamma}$$
$$\le \frac{1}{\gamma} \frac{1}{2^{\tilde{k}}} \sum_{m>1} \frac{1}{2^m} = \frac{1}{\gamma} \frac{1}{2^{\tilde{k}}} = \frac{1}{\gamma} P_j.$$

The first inequality is simply by writing out the explicit values of P_i and the bound $a_{ji} = \frac{1}{d_{ij}^{\alpha}} \leq \frac{1}{\gamma |S_k|}$. Everything else is manipulation, the final equality is a consequence of the definition of P_j . Finally, the second term in Eqn. 4 is bounded by

$$\sum_{k \leq \tilde{k}} \sum_{i \in S_k} P_i a_{ji}$$

$$\leq \sum_{k \leq \tilde{k}} |S_k| \frac{1}{2^k} \frac{1}{\gamma |S_{\tilde{k}}|} = \frac{1}{\gamma} \sum_{k \leq \tilde{k}} 2^{2(k-1)-k-2(\tilde{k}-1)}$$

$$= \frac{1}{\gamma 2^{\tilde{k}}} \sum_{k \leq \tilde{k}} 2^{k-\tilde{k}} \leq \frac{2}{\gamma 2^{\tilde{k}}} = \frac{2}{\gamma} P_j.$$

The first inequality follows from the definition of P_i and $a_{ji} = \frac{1}{d_{ij}^{\alpha}} \leq \frac{1}{\gamma |S_{\tilde{k}}|}$. The first equality follows from the value of $|S_k|$. The rest are consequences of basic facts of the geometric series, and the values of $|S_{\tilde{k}}|$ and P_j .

Setting $\gamma = 14$, $(\mathbf{Ap})_j \leq \frac{4}{\gamma}P_j + \frac{2}{\gamma}P_j + \frac{1}{\gamma}P_j \leq \frac{1}{2}P_j$. This completes the proof of the Lemma.

To prove Lemma 23, first we need a simple lemma about partitions of sets.

Lemma 24. Consider a set $S = \bigcup S_k$ for $k = 1 \dots K$ where the S_k are disjoint. Consider any arbitrary partition of S into r parts labelled $Q_1 \dots Q_r$ where $S = \bigcup_{i \leq r} Q_i$. Define $f_{ik} = \frac{|S_k \cap Q_i|}{|S_k|}$ to be the fraction of S_k that falls in Q_i . Then there exists $i \leq r$ such that $\sum_k f_{ik} \geq \frac{K}{r}$.

Proof. From the definition of f_{ik} we see that $\sum_{i \leq r} f_{ik} = 1$. Then $\sum_{i \leq r} \sum_{k} f_{ik} = \sum_{k} \sum_{i \leq r} f_{ik} = \sum_{k} \sum_{i \leq r} f_{ik}$ $\sum_{k} 1 = K$. The claim now follows.

Proof of Lemma 23. Let us invoke Lemma 24 and consider the set Q_i for which the claim of the lemma holds. Thus, $\sum_{k} f_{ik} \geq \frac{K}{r} = \Omega\left(\frac{\log n}{r}\right)$ (the last equality follows from Eqn. (3)). Since the set Q_i is clearly non-empty, there is some minimum value \tilde{k} for which an element of $S_{\tilde{k}}$ exists in Q_i . Then, $\sum_{k>\tilde{k}} f_{ik} \geq \Omega\left(\frac{\log n}{r}\right) - 1$. Consider an arbitrary link $j \in S_{\tilde{k}} \cap Q_i$. Note that for $\hat{j} \in S_k$ where $k > \tilde{k}$, $d_{\hat{j}j} = (\gamma |S_k|)^{1/\alpha} + (\gamma |S_{\tilde{k}}|)^{1/\alpha} \leq 2(\gamma |S_k|)^{1/\alpha}$. Now,

$$\begin{split} \sum_{\hat{j} \in Q_i} a_{j\hat{j}} &\geq \sum_{k > \tilde{k}} \sum_{\hat{j} \in Q_i \cap S_k} a_{j\hat{j}} \\ &\geq \frac{1}{\gamma 2^{\alpha}} \sum_{k > \tilde{k}} \sum_{\hat{j} \in Q_i \cap S_k} \frac{1}{|S_k|} \\ &= \frac{1}{\gamma 2^{\alpha}} \sum_{k > \tilde{k}} f_{ik} |S_k| \frac{1}{|S_k|} \\ &= \frac{1}{\gamma 2^{\alpha}} \sum_{k > \tilde{k}} f_{ik} \geq \frac{1}{\gamma 2^{\alpha}} \left(\Omega \left(\frac{\log n}{r} \right) - 1 \right) \\ &> 1 \ , \end{split}$$

for $r = o(\log n)$. The first inequality comes from dividing the sum in to sums for different S_k 's. The second inequality comes from the upper bound on d_{ij} . The first equality follows from the definition of f_{ik} . The third inequality follows from the choice of i.

This proves that one cannot schedule the links in $o(\log n)$ slots using uniform (and thus, oblivious) power.

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APPENDIX A. MISSING PROOFS

A.1. Matrix-algebraic Claims.

Proof of Claim 1. Suppose otherwise. Then, there are at least $\frac{n}{\lambda}$ rows \mathbf{r} such that $Se(\mathbf{r}) > \gamma \cdot \lambda$. Call the set of these rows X. Then

$$Se(\mathbf{M}) \ge \sum_{r \in X} Se(\mathbf{r}) > \frac{n}{\lambda} \cdot \gamma \cdot \lambda = \gamma n$$
,

which is a contradiction. The proof for columns is nearly identical.

Proof of Lemma 2. The claim is proved by induction. The case n = 1 is immediate since $m_{11}\mathbf{p}_1 \le \mathbf{p}_1$, so $m_{11} \le 1 = n \le (q+1)n$.

Suppose the claim holds for n = k - 1, and consider the case n = k. Assume without loss of generality that \mathbf{p}_n is the smallest value in \mathbf{p} ; we otherwise rearrange the columns and rows accordingly.

By the matrix property, $\sum_i m_{ni} \mathbf{p}_i \leq \mathbf{p}_n$. Since $\mathbf{p}_i \geq \mathbf{p}_n$ by assumption, we have that $\sum_i m_{ni} \leq \sum_i m_{ni} \frac{\mathbf{p}_i}{\mathbf{p}_n} \leq 1$. By q-approximate symmetry, $\sum_i m_{in} \leq q \sum_i m_{ni} \leq q$. It follows by the inductive hypothesis that $\sum_{ij} m_{ij} = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} m_{ij} + \sum_{i=1}^{n} m_{in} + \sum_{i=1}^{n} m_{ni} - m_{nn} \leq (q+1)(n-1) + q + 1 = (q+1)n$, establishing the claim.

A.2. **Proof of Lemma 8.** To prove this part of the result, we essentially switch the role of senders and receivers, and then the proof follows that of Lemma 7.

For each link $\ell_t \in R$, we will assign a set of "guards" $X_t \in B$. For different t, the X_t 's will be disjoint and each X_t will be of size at most 2. We claim that each link ℓ_b that remains in B after removing all the guard sets satisfies the lemma.

We process the links in R in an arbitrary order. Initially set $B' \leftarrow B$. For each link $\ell_t \in R$, add to X_t the link $\ell_y \in B_t^+$ with the sender nearest to r_t (if one exists); also add the link $\ell_z \in B_t^+$ with the receiver nearest to r_t (if one exists; possibly, $\ell_y = \ell_z$); finally, remove the links in X_t from B' and repeat the loop.

Since |B| > 2|R|, $B' \ge |B| - 2|R| > 0$, by construction. Consider any link $\ell_b \in B'$. In what follows, we will show that the affectance of ℓ_b on any link $\ell_r \in R_b^-$ is comparable to the affectance of ℓ_b on one of the guards of ℓ_r . Once we recall that the guards are blue, this implies that the overall affectance on R_b^- from ℓ_b is not much higher than that on B.

Consider any $\ell_r \in R_b^-$. Since $\ell_b \notin X_r$, X_r contains two guards, ℓ_y and ℓ_z (possibly the same). Let d denote $d(r_z, r_t)$ and observe that since $\ell_b \notin X_r$, $d(r_b, r_t) \ge d$.

We claim that $d_{bt} \geq d/2$. Before proving the claim, let us see how this leads to the proof of the Lemma. If the claim is true, then by the triangular inequality,

(5)
$$d_{bz} = d(s_b, r_z) \le d(s_b, r_t) + d(r_t, r_z)$$
$$= d_{bt} + d \le 3d_{bt}.$$

Thus,

$$\frac{a_b(z)}{a_b(t)} = \frac{c_z}{c_t} \frac{P_b/d_{bz}^{\alpha}}{P_z/\ell_z^{\alpha}} \cdot \frac{P_t/\ell_t^{\alpha}}{P_b/d_{bt}^{\alpha}}
= \frac{P_t/\ell_t^{\alpha}}{P_z/\ell_z^{\alpha}} \cdot \left(\frac{d_{bt}}{d_{bz}}\right)^{\alpha} \frac{c_z}{c_t} \ge \left(\frac{d_{bt}}{d_{bz}}\right)^{\alpha} \ge 3^{-\alpha} .$$

The first inequality follows from sub-linearity, and the second from Eqn. 5. Summing over all links in B,

$$a_b(B) \ge \sum_{t \in R_b^-} a_b(X_t) \ge 3^{-\alpha} \sum_{\ell_t \in R_b^-} a_b(t) = 3^{-\alpha} a_b(R_b^-)$$
.

To prove the claim that $d_{bt} \geq d/2$, let us suppose otherwise for contradiction. Then, by the triangular inequality,

$$\ell_b = d(s_b, r_b) \ge d(s_b, s_t) - d(s_t, r_b) > d - d/2 = d/2$$
.

By the choice of ℓ_y , its sender is at least as close to r_t as b, that is $d(s_y, r_t) \leq d(s_b, r_t) < d/2$, and its receiver is also at least far as r_z , or $d(r_y, r_t) \geq d$. So, $\ell_y \geq d(r_y, r_t) - d(r_t, s_y) > d/2$. But, that implies ℓ_y and ℓ_b are too close together, because $d(s_y, s_b) \leq d(s_y, r_t) + d(s_b, r_t) < d$ and

$$d_{yb} \cdot d_{by} \le (\ell_b + d(s_y, s_b)) \cdot (\ell_y + d(s_y, s_b)) < (\ell_b + d) \cdot (\ell_y + d) < 9 \cdot \ell_y \ell_b.$$

Since B is a 3^{α} -signal set, b and y cannot coexist in B by Lemma 3, which contradicts our assumption. Hence, any link $\ell_t \in R_b^-$ satisfies $d(s_b, r_t) = d_{bt} \ge d/2$.

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